Proc. A. Razmadze Math. Inst. 143(2007), 135–137

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THE ESTIMATION OF HIGH ORDER GENERALIZED MODULUS OF COTINUITY IN L^p_w

(Reported on 14.02.2007)

In this paper we deal with the estimation of the best approximation and generalized modulus of continuity of derivatives of periodic functions in weighted reflexive Lebesgue spaces. In unweighted Lebesgue spaces the inequalities for classical modulus of continuity and best approximations of derivatives were derived in the papers [1], [2].

1. Some definitions

Let T denote the interval $(-\pi,\pi)$. A positive almost everywhere, integrable function $w : T \to (0,\infty)$ is called as a weight function. With any given weight w we associate the w-weighted Lebesgue space $L^p_w(T)$ consisting of all measurable functions f on T such that

$$\|f\|_{L^p_w(T)} = \|fw\|_{L^p(T)} < \infty.$$

Let $1 < p < \infty$ and $1/p + 1/q = 1$. A weight function w belongs to the Muckenhoupt class $A_p(T)$ if

$$\left(\frac{1}{|I|}\int_I w^p(x)dx\right)^{1/p}\left(\frac{1}{|I|}\int_I w^{-q}(x)dx\right)^{1/q} \le C$$

with a finite constant C independent of I, where I is any subinterval of T and $|I|$ denotes the length of I.

Let $1 < p < \infty$ and $w \in A_p(T)$. We define an operator on $L^p_w(T)$ by

$$\sigma_h(g)(x) = \frac{1}{2h}\int_{x-h}^{x+h} g(t)dt, \qquad 0 < h < \pi$$

It is known that the operator σ_h is bounded uniformly with respect to h in $L^p_w(T)$, when $w \in A_p(T)$, $1 < p < \infty$. The modulus of continuity $\Omega_s(g,\cdot)_{L^p_w}$ of $g \in L^p_w(T)$ is defined by

$$\Omega_s(g,\delta)_{L^p_w} = \sup_{\substack{0<h_i<\delta\\ 1\le i\le s}} \left\| \prod_{i=1}^{s} \left(I - \sigma_{h_i}\right) g \right\|_{L^p_w}.$$

The best approximation of $f \in L^p_w(T)$ in the class Π_n of trigonometric polynomials of degree not exceeding n is defined by

$$E_n(f)_{L^p_w} = \inf\left\{ \|f - T_n\|_{L^p_w} : T_n \in \Pi_n \right\}.$$

2000 *Mathematics Subject Classification:* 41A10, 42A10.

Key words and phrases. Weighted Lebesgue space, modulus of continuity, best approximation.

2. Main results

Theorem 1. *Let $f \in L^p_w(T)$, $1 < p < \infty$, and $w \in A_p(T)$. If*

$$\sum_{k=1}^{\infty} k^{r\gamma-1} E_k^{\gamma}(f)_{L^p_w} < \infty$$

for some natural number r and $\gamma = \min\{2, p\}$, then there exists the absolutely continuous derivative $f^{(r-1)}(x)$, $f^{(r)} \in L^p_w(T)$ and the estimate

$$\Omega_s(f^{(r)}, 1/n)_{L^p_w} \leq \frac{c}{n^{2s}} \left(\sum_{k=1}^{n} k^{(r+2s)\gamma-1} E_{k-1}^{\gamma}(f)_{L^p_w} \right)^{1/\gamma} +$$

$$+ c \left(\sum_{k=n+1}^{\infty} k^{r\gamma-1} E_k^{\gamma}(f)_{L^p_w} \right)^{1/\gamma}$$

holds with a constant c independent of n and f.

Theorem 2. *Let $f \in L^p_w(T)$, $1 < p < \infty$, and $w \in A_p(T)$. If*

$$\sum_{k=1}^{\infty} k^{r\gamma-1} E_k^{\gamma}(f)_{L^p_w} < \infty$$

for some natural number r and $\gamma = \min\{2, p\}$, then there exists the absolutely continuous derivative $f^{(r-1)}(x)$, $f^{(r)} \in L^p_w(T)$ and the estimate

$$E_n(f^{(r)})_{L^p_w} \leq c \left\{ n^r E_n(f)_{L^p_w} + \left(\sum_{k=n+1}^{\infty} k^{r\gamma-1} E_k^{\gamma}(f)_{L^p_w} \right)^{1/\gamma} \right\}$$

holds with a constant c independent of n and f.

Corollary. *If*

$$E_k(f)_{L^p_w} = O\left(\frac{1}{k^{r+2s}} \right)$$

then for $\gamma = \min\{2, p\}$

$$\Omega(f^{(r)}, 1/n)_{L^p_w} = O\left(\frac{(\ln n)^{1/\gamma}}{n^{2s}} \right).$$

Acknowledgement

Part of this research was supported by TÜBİTAK-The Scientific and Technological Research Council of Turkey. The first author was supported also by the Grant INTAS No. 05-1000003-8157.

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