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ON THE MEANS OF FOURIER INTEGRALS AND BERNSTEIN INEQUALITY IN TWO-WEIGHTED SETTING

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1. Introduction

Let $w : \mathbb{R} \to \mathbb{R}$ be a weight function, that is, a nonnegative integrable function. We denote by $L_w^p(\mathbb{R})$, $1 \le p < \infty$ the Banach function space of all measurable functions f, for which

$$\|f\|_{p,w} = \left(\int_{-\infty}^{\infty} |f(x)|^p w(x)\,dx \right)^{1/p} < \infty.$$

The Fourier integral of a function $f \in L^1(\mathbb{R})$ is defined by

$$\widehat{f}(x) = \int_{-\infty}^{\infty} f(t)\, e^{ixt} dt,$$

and its Cesaro mean is can be represented as

$$S_R(x,f) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t)\, F_R(x-t)\,dt, \quad -\infty < x < \infty,$$

where

$$F_R(t) = \frac{1-\cos Rt}{Rt^2}$$

is the Fejer kernel (See [1, pp. 15]). Let also

$$f(t,x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{t f(y)}{t^2 + (x-y)^2} dy, \quad t > 0$$

be the Abel-Poisson mean of the function $f \in L^1(\mathbb{R})$.

The general information about Cesaro means of Fourier integrals and Abel-Poisson means of functions can be found in [1] and [8], respectively.

For $2\pi-$periodic functions, the problem of boundedness and estimation of norms of the Cesaro and Abel-Poisson means in weighted Lebesgue spaces in the two-weighted setting was investigated in [6], [2] and [3].

In the present paper, we investigated the above mentioned problems for non periodic functions in weighted Lebesgue spaces. We also obtained generalizations of the Bernstein inequality for integral functions of one and two variables.

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2. One-Dimensional Case

Definition 2.1. Let $1 < p \le q < \infty$. A pair of weight functions (v, w) is said to be of class $\mathcal{A}_{p,q}(\mathbb{R})$ if the $A_{p,q}$-condition

$$\sup_I \left(\frac{1}{|I|} \int_I v(x)\, dx \right)^{1/q} \left(\frac{1}{|I|} \int_I w^{1-p'}(x)\, dx \right)^{1/p'} < \infty. \tag{1}$$

holds, where $p' = p/(p-1)$ and the supremum is taken over all intervals with $I \subset \mathbb{R}$.

The following statements are true.

Theorem 2.2. Let $1 < p \le q < \infty$. The inequality

$$\|S_R(\cdot, f)\|_{q,v} \le c\, R^{\frac{1}{p} - \frac{1}{q}} \|f\|_{p,w} \tag{2}$$

holds for arbitrary $f \in L^p_w(\mathbb{R})$, where the constant c does not depend on R and f, if and only if $(v, w) \in \mathcal{A}_{p,q}(\mathbb{R})$.

Theorem 2.3. Let $1 < p \le q < \infty$. The necessary and sufficient condition for the validity of the inequality

$$\|f(t, \cdot)\|_{q,v} \le c\, t^{\frac{1}{q} - \frac{1}{p}} \|f\|_{p,w} \tag{3}$$

for arbitrary $f \in L^p_w(\mathbb{R})$, where the constant c does not depend on t and f, is $(v, w) \in \mathcal{A}_{p,q}(\mathbb{R})$.

In the case $p = q$, Theorem 2.3 was proved in [6]. For $2\pi-$periodic functions the analogues of Theorem 2.2 and Theorem 2.3 were proved in [3].

3. Two-Dimensional Case

Let $\mathbb{J}$ denote the set of all rectangles with the sides parallel to the coordinate axes.

Definition 3.1. The pair (v, w) is said to belongs to the class $\mathcal{A}_{p,q}(\mathbb{R}^2, \mathbb{J})$ if the condition

$$\sup_{J \in \mathbb{J}} \left(\frac{1}{|J|} \int_J v(x, y)\, dxdy \right)^{1/q} \left(\frac{1}{|J|} \int_J w^{1-p'}(x, y)\, dxdy \right)^{1/p'} < \infty \tag{4}$$

holds.

Let $S_{R_1 R_2}(\cdot, \cdot, f)$ and $f(t, \cdot, \cdot)$ be the Cesaro and Abel-Poisson means of the function $f \in L^1(\mathbb{R}^2)$, respectively:

$$S_{R_1 R_2}(x, y, f) = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t, s)\, F_{R_1}(x - t)\, F_{R_2}(y - s)\, dtds$$

and

$$f(t, x, y) = \frac{\Gamma(3/2)}{\pi^{3/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{tf(u, v)}{\left[t^2 + (x - u)^2 + (y - v)^2\right]^{3/2}} dudv.$$

We have the two-dimensional analogues of Theorem 2.2 and Theorem 2.3.

Theorem 3.2. Let $1 < p \le q < \infty$. The condition $(v, w) \in \mathcal{A}_{p,q}(\mathbb{R}^2, \mathbb{J})$ is necessary and sufficient for validity of the inequality

$$\left\|S_{R_1 R_2}(\cdot, \cdot, f)\right\|_{q,v} \le c\, (R_1 R_2)^{\frac{1}{p} - \frac{1}{q}} \|f\|_{p,w} \tag{5}$$

for every $f \in L^p_w(\mathbb{R}^2)$, where the constant c is independent of R_1, R_2 and f.

Theorem 3.3. Let $1 < p \leq q < \infty$. The inequality

$$\|f(t,\cdot,\cdot)\|_{q,v} \leq ct^{2\left(\frac{1}{q}-\frac{1}{p}\right)} \|f\|_{p,w} \tag{6}$$

holds for arbitrary $f \in L_w^p(\mathbb{R}^2)$, where the constant c does not depend on t and f, if and only if $(v,w) \in \mathcal{A}_{p,q}(\mathbb{R}^2, \mathbb{J})$.

4. Bernstein Inequalities in Two-Weighted Setting

By Theorem 2.2 and Theorem 2.3 we obtain the generalization of Bernstein's inequality for integral functions of one and two variables.

Theorem 4.1. Let $1 < p \leq q < \infty$, $(v,w) \in \mathcal{A}_{p,q}(\mathbb{R})$ and $G_\sigma(x)$ be an integral function of exponential type of degree $\leq \sigma$, bounded on the whole real axis. Then the inequality

$$\|G'_\sigma\|_{q,v} \leq c\sigma^{1+\frac{1}{p}-\frac{1}{q}} \|G_\sigma\|_{p,w}$$

holds, where the constant c is independent of σ.

Theorem 4.2. Let $1 < p \leq q < \infty$, $(v,w) \in \mathcal{A}_{p,q}(\mathbb{R}^2, \mathbb{J})$ and $G_{\sigma_1,\sigma_2}(x)$ be an integral function of exponential type of degree $\leq \sigma_1$ with respect to x and of degree $\leq \sigma_2$ with respect to y. Then the inequality

$$\left\|\frac{\partial^2 G_{\sigma_1,\sigma_2}}{\partial x \partial y}\right\|_{q,v} \leq c(\sigma_1\sigma_2)^{1+\frac{1}{p}-\frac{1}{q}} \|G_{\sigma_1,\sigma_2}\|_{p,w}$$

holds.

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