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APPROXIMATION IN WEIGHTED BERGMAN SPACES ON INFINITE DOMAINS

1. Introduction and some auxiliary results

Let G be a simple connected domain in the complex plane $\mathbb{C}$ and let ω be a weight function given on G. For functions f analytic in G we set

$$A^1(G) := \left\{ f : \iint_G |f(z)|\, d\sigma_z < \infty \right\}$$

and

$$A^2(G,\omega) := \left\{ f : \iint_G |f(z)|^2 \, \omega(z) d\sigma_z < \infty \right\},$$

where $d\sigma_z$ denotes the Lebesgue measure in the complex plane $\mathbb{C}$.

If $\omega = 1$, we denote $A^2(G) := A^2(G,1)$. The space $A^2(G)$ is called the Bergman space on G. We refer to the spaces $A^2(G,\omega)$ as "weighted Bergman spaces". It becomes a normed spaces if we define

$$\|f\|_{A^2(G,\omega)} := \left(\iint_G |f(z)|^2 \, \omega(z) d\sigma_z \right)^{1/2}.$$

Now, let L be a finite quasiconformal curve in the complex plane $\mathbb{C}$. We recall that L is called a quasiconformal curve if there exists a quasiconformal homeomorphism of the complex plane onto itself that maps a circle onto L. We denote by G_1 and G_2 the bounded and unbounded complements of $\mathbb{C} \setminus L$, respectively. It is clear that if $f \in A^2(G_2)$, then it has zero in ∞ at least second order. As in the bounded case [7], p. 5. $A^2(G_2)$ is a Hilbert space with the inner product

$$\langle f, g\rangle := \iint_{G_2} f(z)\overline{g(z)} d\sigma_z,$$

which can be easily verified. Moreover, the set of polynomials of $1/z$ are dense in $A^2(G_2)$ with respect to the norm

$$\|f\|_{A^2(G_2)} := (\langle f, f\rangle)^{1/2}.$$

Also, for $n = 1, 2, \dots$ there exists a polynomial $P_n^*(1/z)$ of $1/z$, of degree$\leq n$, such that $E_n(f, G_2) = \|f - P_n^*\|_{A^2(G_2)}$ (see, for example: [6], p. 59, Theorem 1.1.), where

$$E_n(f, G_2) := \operatorname{Inf} \left\{ \|f - P\|_{A^2(G_2)} : P \text{ is a polynomial of } 1/z, \text{ of degree } \leq n \right\}$$

denotes the minimal error of approximation of f by polynomials of $1/z$ of degree at most n. The polynomial $P_n^*(1/z)$ is called the best approximated polynomial to $f \in A^2(G_2)$.

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Let D be the open unit disc and $w = \varphi(z)$ the conformal mapping of G_1 onto $C\overline{D} := \mathbb{C} \setminus \overline{D}$, normalized by conditions

$$\varphi(0) = \infty \quad \text{and} \quad \lim_{z \to 0} z\varphi(z) > 0,$$

and let ψ be the inverse of φ. For an arbitrary fixed number $R > 1$ we put

$$L_R := \{z : |\varphi(z)| = R\}, \quad G_{2,R} := \{z : z \in G_1, 1 < |\varphi(z)| < R\} \cup \overline{G_2}$$

If a function $g(z)$ is analytic in G_1 and $g(0) > 0$, then the function $g(z)\varphi^m(z)$ has a pole of order m at the origin, i.e. this expansion holds

$$g(z)\varphi^m(z) = F_m(1/z, g) + Q_m(z, g) \qquad \text{for } z \in G_1,$$

where $F_m(1/z, g)$ denotes the polynomial of negative powers of z and the term $Q_m(z, g)$ contains non-negative powers of z; hence $Q_m(z, g)$ is a function analytic in the domain G_1. The polynomial $F_m(1/z, g)$ of negative powers of z is called the generalized Faber polynomial of order m for the domain G_2. These polynomials satisfy the following expansion

$$\frac{g\left[\psi(w)\right]\psi'(w)}{\psi(w) - z} = \sum_{m=1}^{\infty} F_m\left(1/z\right)\frac{1}{w^{m+1}}$$

for $z \in G_2$ and $w \in C\overline{D}$, which converges absolutely and uniformly on compact subsets of $G_2 \times C\overline{D}$. Differentiation of this equality with respect to z gives

$$\frac{z^2 g\left[\psi\left(w\right)\right]\psi'(w)}{\left[\psi\left(w\right) - z\right]^2} = \sum_{m=1}^{\infty} -F_m'\left(1/z\right)\frac{1}{w^{m+1}} \tag{1}$$

for every $(z, w) \in G_2 \times C\overline{D}$, where the series converges absolutely and uniformly on compact subsets of $G_2 \times C\overline{D}$. More information for Faber and generalized Faber polynomials can be found in [11], p. 44 and p. 255 and [7], p. 42.

In [4], V. I. Belyi gave the following integral representation for the functions f analytic and bounded in the domain G_1

$$f(z) = -\frac{1}{\pi}\iint_{G_2} \frac{(f \circ y)(\zeta)}{(\zeta - z)^2} y_{\overline{\zeta}}(\zeta) d\sigma_\zeta, \qquad z \in G_1. \tag{2}$$

Here $y(z)$ is a K-quasiconformal reflection across the boundary L, i.e., a sense-reversing K-quasiconformal involution of the extended complex plane keeping every point of L fixed, such that $y(G_1) = G_2$, $y(G_2) = G_1$, $y(0) = \infty$ and $y(\infty) = 0$. Such a mapping of the plane does exist [10], p. 99. As follows from Ahlfors' theorem [1], p. 80, the reflection $y(z)$ can always be chosen canonical in the sense that it is differentiable on $\mathbb{C}$ almost everywhere, except possibly at the points of the curve L, and for any sufficiently small fixed $\delta > 0$ it satisfies the relations

$$\begin{aligned} |y_\varsigma| + |y_{\overline{\varsigma}}| &\leq c_1, & \delta < |\varsigma| < \frac{1}{\delta} \quad &\text{for } \varsigma \notin L, \\ |y_\varsigma| + |y_{\overline{\varsigma}}| &\leq c_2\,|\varsigma|^{-2}, & |\varsigma| \geq \frac{1}{\delta} \quad &\text{for } |\varsigma| \leq \delta. \end{aligned}$$

with some constants c_1 and c_2, independent of ζ.

Let g be an analytic function in G_1, $g(0) > 0$, and let

$$\iint_{G_1} |g(z)|^2\,|y_{\overline{z}}|^2\,d\sigma_z < \infty,$$

where y is a canonical reflection across the boundary L. For every such g we define a weight function ω in the following way

$$\omega(z) := \frac{1}{\left|(g \circ y)\,(z)\right|^2}, \qquad z \in G_2.$$

We denote by $W^2(G_2)$ the set all of weight functions ω defined as above.

In this work, for the first time, we obtain (Lemma 1) an integral representation on the domain G_2 for a function $f \in A^1(G_2)$. By means of this integral representation we define a generalized Faber series of a function $f \in A^1(G_2)$ to be of the form

$$\sum_{m=1}^{\infty} a_m(f,g) F'_m\left(1/z,g\right),$$

with the generalized Faber coefficients $a_m(f,g)$, $m = 1,2,\ldots$. Then, we investigate the convergence, the uniqueness and the approximation rate of this series.

For the bounded domains the results presented here were stated and proved in [9] and [5], respectively. Similar problems in $A(\overline{G})$, where $A(\overline{G})$ denotes the class of functions which are continuous in $\overline{G}$ and analytic in G were studied in [8].

Considering only the canonical quasiconformal reflections, I. M. Batchaev [3] generalized the integral representation in (2) to functions $f \in A^1(G_1)$. The accurate proof of the Batchaev's result is given in [2], p. 110, Theorem 4.4. A similar integral representation can also be obtained for functions $f \in A^1(G_2)$. The following result holds.

Lemma 1. *Let $f \in A^1(G_2)$. If $y(z)$ is a canonical quasiconformal reflection with respect to L, then*

$$f(z) = -\frac{1}{\pi}\iint\limits_{G_1} \frac{(f\circ y)(\zeta)z^2}{(\zeta-z)^2[y(\zeta)]^2} y_{\overline{\zeta}}(\zeta)d\sigma_\zeta, \qquad z \in G_2. \tag{3}$$

From now on, the reflection $y(z)$ will be a canonical K-quasiconformal reflection with respect to L.

Let $f \in A^1(G_2)$. Substituting $\zeta = \psi(w)$ in (3), for $z \in G_2$ we get

$$f(z) = -\frac{1}{\pi}\iint\limits_{C\overline{D}} \frac{(f\circ y)\left[\psi\left(w\right)\right]\overline{\psi'}(w)y_{\overline{\zeta}}\left[\psi\left(w\right)\right]}{\left[(y\circ\psi)\left(w\right)\right]^2 g\left[\psi\left(w\right)\right]}\frac{g\left[\psi\left(w\right)\right]z^2\psi'(w)}{\left[\psi\left(w\right)-z\right]^2}d\sigma_w. \tag{4}$$

Thus, if we define the coefficients $a_m(f,g)$ by

$$a_m(f,g) := \frac{1}{\pi}\iint\limits_{C\overline{D}} \frac{(f\circ y)\left[\psi\left(w\right)\right]\overline{\psi'}(w)}{w^{m+1}g\left[\psi\left(w\right)\right]\left[(y\circ\psi)\left(w\right)\right]^2} y_{\overline{\zeta}}\left[\psi\left(w\right)\right]d\sigma_w, \quad m = 1,2,\ldots$$

then, by (1) and (4), we can associate a formal series $\sum_{m=1}^{\infty} a_m(f,g)F'_m(1/z,g)$ with the function $f \in A^1(G_2)$, i.e.,

$$f(z) \sim \sum_{m=1}^{\infty} a_m(f,g)F'_m\left(1/z,g\right).$$

We call this formal series a generalized Faber series of $f \in A^1(G_2)$, and the coefficients $a_m(f,g)$ are called generalized Faber coefficients of f.

2. Main results

Theorem 1. *Let $f \in A^2(G_2,\omega)$, $\omega \in W^2(G_2)$. If*

$$\sum_{m=1}^{\infty} a_m(f,g)F'_m\left(1/z,g\right)$$

is a generalized Faber series of f, then this series converges uniformly to f on the compact subsets of G_2.

A uniqueness theorem for the series

$$\sum_{m=1}^{\infty} a_m(f,g)F'_m\left(1/z,g\right)$$

which converges to $f \in A^2(G_2,\omega)$ with respect to the norm $\|\cdot\|_{A^2(G_2,\omega)}$ is the following.

Theorem 2. *Let g be an analytic function, bounded and non-vanishing in G_1, let $\{b_m\}$ be a complex number sequence. If the series*

$$\sum_{m=1}^{\infty} b_m F'_m\left(1/z, g\right)$$

converges to a function $f \in A^2(G_2, \omega)$ in the norm $\|\cdot\|_{A^2(G_2,\omega)}$, then the b_m, $m = 1, 2, \ldots,$ are the generalized Faber coefficients of f.

Let y_R be K_R-quasiconformal reflection across the boundary L_R. Then the following theorem estimates the error of the approximation of $f \in A^2(G_{2,R})$ by the partial sums of the series

$$\sum_{m=1}^{\infty} a_m(f) F'_m\left(1/z\right)$$

in the norm $\|\cdot\|_{A^2(G_2,\omega)}$ regarding to $E_n(f, G_{2,R})$ for the special case $\omega(z) = 1/\left|z\right|^4$ of the weighted function ω given on G_2.

Theorem 3. *If $f \in A^2(G_{2,R})$ for $R > 1$, $\omega(z) := 1/\left|z\right|^4$ and*

$$S_n\left(f, 1/z\right) = \sum_{m=1}^{n+1} a_m(f) F'_m\left(1/z\right)$$

is the n-th partial sum of its generalized Faber series

$$\sum_{m=1}^{\infty} a_m(f) F'_m\left(1/z\right),$$

then for all natural numbers n

$$\|f - S_n(f, \cdot)\|_{A^2(G_2,\omega)} \leq \frac{c}{\sqrt{\left(1 - k_R^2\right)\left(R^2 - 1\right)}} \frac{E_n(f, G_{2,R})}{R^{n+1}}$$

with a constant c independent of n, where $k_R := \left(K_R - 1\right) / \left(K_R + 1\right)$.

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