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APPROXIMATION PROPERTIES OF THE GENERALIZED BIEBERBACH POLYNOMIALS IN THE CLOSED DINI-SMOOTH DOMAINS

1. Introduction

Let G be a finite simply connected domain in the complex plane $\mathbb{C}$, bounded by a rectifiable Jordan curve L, and let $z_0 \in G$. By the Riemann mapping theorem, there exists a unique conformal mapping $w = \varphi_0(z)$ of G onto $D(0, r_0) := \{w : | w |< r_0\}$ with the normalization $\varphi_0(z_0) = 0$, $\varphi_0'(z_0) = 1$.

Without loss of generality, we may assume that the conformal radius of G with respect to z_0 equals 1.

Let $\psi_0(w)$ be the inverse to $w = \varphi_0(z)$. Let also $G^- := \operatorname{ext} L$, $D := D(0, 1) = \{w : | w |< 1\}$, $T := \partial D$, $D^- := \{w : \ | w |> 1\}$, and let φ be the conformal mapping of G^- onto D^-, normalized by

$$\varphi(\infty) = \infty, \ \lim_{z\to\infty} \varphi(z)/z > 0.$$

We denote by ψ the inverse mapping to φ.

For an arbitrary analytic function f given on G and $p > 0$, we set

$$\|f\|^p_{L_p(G)} := \iint\limits_G |f(z)|^p \, d\sigma_z.$$

If the analytic function f has a continuous extension to $\overline{G}$, we also use the uniform norm

$$\|f\|_{\overline{G}} := \sup\left\{|f(z)|, z \in \overline{G}\right\}.$$

It is well known that the function $\varphi_p(z) := \int\limits_{z_0}^{z} \left[\varphi_0'(\zeta)\right]^{2/p}$, $p > 0$, minimizes the integral $\| f' \|^p_{L_p(G)}$ in the class of all analytic functions in G with the normalization $f(z_0) = 0$, $f'(z_0) = 1$. On the other hand, let Π_n be the class of all polynomials p_n of degree at most n satisfying the conditions $p_n(z_0) = 0$, $p_n'(z_0) = 1$. Then the integral $\| \varphi_p' - p_n' \|_{L_p(G)}, 1 < p < \infty$, is minimized in Π_n by an unique polynomial $\pi_{n,p}$ which is called [6] the n^{th} generalized Bieberbach polynomial for the pair (G, z_0). As it is known, in case of $p = 2$ they are the usual Bieberbach polynomials π_n. By the results due to Markushevich and Farrel, if G is a Caratheodory domain, then $\| \varphi_p' - \pi_{n,p}' \|_{L_p(G)} \to 0 \, (n \to \infty)$ and this implies the convergence $\pi_{n,p}(z) \to \varphi_p(z) \, (n \to \infty)$ for $z \in G$, uniformly on compact subsets of G. The approximation properties of the polynomials $\pi_{n,p}$, $n = 1, 2, \ldots$ on the various closed domains were investigated in [12], [13], [17], [16], [15], [1], [2], [3], [5], [4], [6], [7], [8], [9], [10], [11].

In this work, we investigate the convergence of the polynomials $\pi_{n,p}$, $n = 1, 2, \ldots$ on a subclass of closed Dini-smooth domains.

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2. Definitions and Auxiliary results

Let $\psi_0(e^{it})$, $0 \le t \le 2\pi$, be the conformal parametrization of the smooth boundary L and let $\beta(t)$ be its tangent direction angle at the point $\psi_0(e^{it})$.

Definition 1. We say that L $\in \mathfrak{B}(\alpha,\mu)$ if

$$\omega(\beta,\delta) := \sup_{|h|\le\delta} \|\beta(\cdot) - \beta(\cdot + h)\|_{[0,2\pi]} \le c\delta^{\alpha}\ln^{\mu}\frac{1}{\delta}$$

for some parameters $\alpha \in (0,1]$, $\mu \in [0,\infty)$ and for a positive constant c independent of δ.

In particular the class $\mathfrak{B}(\alpha,0)$, $0 < \alpha < 1$, coincides with the class of Lyapunov curves. Futhermore, it is easy to verify that if $0 < \alpha_1 < \alpha_2 \le 1$, then

$$\mathfrak{B}(\alpha_1,\mu) \supset \mathfrak{B}(\alpha_{2,}\mu),\ \ \mu \in [0,\infty)$$

and also

$$\mathfrak{B}(\alpha,\mu_1) \subset \mathfrak{B}(\alpha,\mu_2),\ \alpha \in (0,1]$$

for $0 \le \mu_1 < \mu_2 < \infty$.

It is easily seen from the definition 1 that every curve $L \in \mathfrak{B}(\alpha,\beta)$ *with* $\alpha \in (0,1]$ *and* $\beta \in [0,\infty)$ is Dini-smooth.

If L is Dini-smooth, then [14], p. 48, φ_0' has a continuous extension to $\overline{G}$. Hence the following definition is correct.

Definition 2. Let G be a domain with a smooth boundary L, and let $\Phi_p(w) := \left(\varphi_0'\right)^{2/p}(\psi(w))$. The function

$$\omega(\Phi_p,\delta) := \sup_{|h|\le\delta} \|\ \Phi_p(we^{ih}) - \Phi_p(w)\ \|_T,\ \ p > 1$$

is called the generalized integral modulus of continuity for $\left(\varphi_0'\right)^{2/p} \in E^p(G)$.

The following lemma holds.

Lemma 1. *If* $L \in \mathfrak{B}(\alpha,\mu)$ *with* $\alpha \in (0,1]$ *and* $\mu \in [0,\infty)$*, then*

$$\omega((\psi_0')^{2/p},\delta) := \sup_{|h|\le\delta}\left\|\left(\psi_0'\right)^{2/p}\left(we^{ih}\right) - \left(\psi_0'\right)^{2/p}(w)\right\|_T \le \begin{cases} c\delta^{\alpha}\ln^{\mu}\frac{1}{\delta}, & \alpha \in (0,1); \\ c\delta\ln^{\mu+1}\frac{1}{\delta}, & \alpha = 1. \end{cases}$$

Lemma 2. *If* $L \in \mathfrak{B}(\alpha,\mu)$ *with* $\alpha \in (0,1]$ *and* $\mu \in [0,\infty)$*, then*

$$\omega(\Phi_p,\delta) \le \begin{cases} c\delta^{\alpha}\ln^{\mu}\frac{1}{\delta}, & \alpha \in (0,1); \\ c\delta\ln^{\mu+1}\frac{1}{\delta}, & \alpha = 1. \end{cases}$$

We will use the following theorem which can be easily obtained from [8], Theorem 3.

Theorem 1. *Let G be a domain with a Dini-smooth boundary L, $p > 1$ and let*

$$S_n\left(\varphi_p', z\right) := \sum_{k=0}^{n} a_k\left(\varphi_p'\right)F_k(z), \qquad n = 0,1,2,\ldots$$

be the n^{th} partial sums of the Faber series of φ_p'. Then

$$\|\ \ \varphi_p' - S_n\left(\varphi_p',\cdot\right)\ \|_{L^p(L)} \le c\ \omega(\Phi_p,1/n)$$

with a some constant $c > 0$.

3. Main Result

Theorem 2. *If $L \in \mathfrak{B}(\alpha,\mu)$ with $\alpha \in (0,1]$ and $\mu \in [0,\infty)$, then for $p \geq 2$,*

$$\| \varphi_p - \pi_{n,p} \|_{\overline{G}} \leq c_1 \begin{cases} n^{-\alpha-1/p} \ln^{\mu} n, & \alpha \in (0,1); \\ n^{-1-1/p} \ln^{\mu+1} n, & \alpha = 1 \end{cases}$$

with a constant $c_1 > 0$ and for $1 < p < 2$,

$$\| \varphi_p - \pi_{n,p} \|_{\overline{G}} \leq c_2 \begin{cases} n^{-\alpha-1+\frac{1}{p}+\varepsilon} \ln^{\mu} n, & \alpha \in (0,1); \\ n^{-2+\frac{1}{p}+\varepsilon} \ln^{\mu+1} n, & \alpha = 1 \end{cases}$$

with a constant $c_2 = c_2(\varepsilon) > 0$.

This result, in case of $p = 2$, was obtained in [10] which improves the result given by Wu-Xue-Mou in [17].

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