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UNIFORM CONVERGENCE OF THE GENERALIZED BIEBERBACH POLYNOMIALS IN CLOSED SMOOTH DOMAINS

1. Introduction and main results

Let G be a finite domain in the complex plane $\mathbb{C}$, bounded by a rectifiable Jordan curve L, and let $G^{-} := \operatorname{Ext} L$. Further let $\mathbb{T} : = \{w \in \mathbb{C} : |w| = 1\}$, $\mathbb{D} := \operatorname{Int} \mathbb{T}$ and $\mathbb{D}^{-} := \operatorname{Ext} \mathbb{T}$. By the Riemann conformal mapping theorem, there exists a unique conformal mapping $w = \varphi(z)$ of G onto $\mathbb{D}(0, r_0) := \{w :| w |< r_0\}$ with the normalization $\varphi(z_0) = 0$, $\varphi'(z_0) = 1$. Without loss of generality, we may assume that $r_0 = 1$. The inverse mapping of φ we denote by ψ.

Let also φ_0 be the conformal mapping of G^{-} onto $\mathbb{D}^{-}$ normalized by

$$\varphi_0(\infty) = \infty, \qquad \lim_{z\to\infty} \varphi_0(z)/z > 0,$$

and $\psi_0 := \varphi_0^{-1}$. For an arbitrary function f given on G we set

$$\| f \|^p_{L_p(G)} := \iint\limits_G | f(z) |^p \, d\sigma_z, \quad p > 0.$$

It is well known (see [11], p. 433) that the function

$$\varphi_p(z) := \int\limits_{z_0}^{z} \left[\varphi'(\zeta)\right]^{2/p} d\zeta, \quad z \in G, \quad p > 0$$

minimizes the integral $\| f' \|^p_{Lp(G)}$ $(p > 0)$ in the class of all functions analytic in G with the normalization $f(z_0) = 0$, $f'(z_0) = 1$. In this work we study the approximation of φ_p by the extremal polynomials defined below.

Let Π_n be the class of all polynomials p_n of degree at most n satisfying the conditions $p_n(z_0) = 0$, $p'_n(z_0) = 1$. Then we can prove that the integral $\| \varphi'_p - p'_n \|^p_{Lp(G)}$ $(p > 1)$ is minimized in Π_n by an unique polynomial $\pi_{n,p}$. We call [6] these extremal polynomials $\pi_{n,p}$ the *generalized Bieberbach polynomials* for the pair (G, z_0). In case of $p = 2$ they are the usual Bieberbach polynomials π_n, $n = 1, 2, \dots$. The approximation problems for $\varphi_2 = \varphi$ in closed domains with various boundary conditions, where approximation is conducted by the usual Bieberbach polynomials were intensively studied in [9, 10, 13, 12, 2, 3, 7, 8].

Similar problems for φ_p $(p > 1)$ using the generalized Bieberbach polynomials were investigated in [6, 1]. In the above cited works the rate of convergence to zero of the quantity $\| \varphi_p - \pi_{n,p} \|_{\overline{G}}$ $(n \to \infty)$ has been estimated by means of the geometric properties of G.

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One of the interesting problem in this direction is the problem connected with a conjecture due to S. N. Mergelyan, who in [10] showed that the Bieberbach polynomials satisfy

$$\| \varphi - \pi_n \|_{\overline{G}} \leq c(\varepsilon) \diagup n^{1/2} \tag{1}$$

for every $\varepsilon > 0$, whenever L is a smooth Jordan curve and stated it as a conjecture that the exponent $1/2 - \varepsilon$ in (1) could be replaced by $1 - \varepsilon$.

In [7] it has been possible for us to obtain some improvement of the above cited Mergelyan's estimation (1). For its formulation, it is necessary to give some definitions as follows.

For a weight function ω given on L, and $p > 1$ we set

$$L^p(L, \omega) := \{f \in L^1(L) : | f |^p \omega \in L^1(L)\},$$
$$E^p(G, \omega) := \{f \in E^1(G) : f \in L^p(L, \omega)\}.$$

Let $g \in L^p(\mathbb{T}, \omega)$ and let $g_h(w)$ be the mean value function for g defined as:

$$g_h(w) := \frac{1}{2h} \int_{-h}^{h} g(we^{it})dt, \qquad 0 < h < \pi, \qquad w \in T.$$

The function $\Omega_{p,\omega}(g, \cdot) : [0, \infty) \to [0, \infty)$ defined by

$$\Omega_{p,\omega}(g, \delta) := \sup\{\| g - g_h \|_{L^p(T,\omega)}, h \leq \delta\}, \qquad 1 < p < \infty$$

is called the integral modulus of continuity in $L^p(\mathbb{T}, \omega)$ for g.

The improvement obtained in [7] can be formulated in the following way: If the boundary L is a smooth Jordan curve, then

$$\| \varphi_0 - \pi_n \|_{\overline{G}} \leq \text{const} \left(\frac{\ln n}{n}\right)^{1/2} \Omega_{2,|\psi_0'|}\left((\varphi' \circ \psi_0) \cdot (\psi_0')^{1/2}, 1/n\right), \tag{2}$$

for $n \geq 2$, where $\Omega_{2,|\psi_0'|}(\cdot, 1/n)$ is the integral modulus of continuity in $L^2(\mathbb{T}, | \psi_0' |)$ for $\varphi' \circ \psi_0) \cdot (\psi_0')^{1/2}$. From this result in particular it follows that if G is a finite domain with a smooth Jordan boundary, then

$$\| \varphi - \pi_n \|_{\overline{G}} \leq \text{const} \left(\frac{\ln n}{n}\right)^{1/2}, \qquad n \geq 2 \tag{3}$$

which improves the estimation (1).

In this work, developing the idea used in [6, 7, 8] we shall obtain a new estimation for the approximation of φ_p, $1 < p < \infty$, by means of the generalized Bieberbach polynomials $\pi_{n,p}$. This estimation in case of $p = 2$ appears simpler than (2).

We begin with the following definition.

Definition 1. Let G be a domain with a smooth boundary L, $p > 1$, and let $\Phi_p := (\varphi_p' \circ \psi_0)$ on $\mathbb{T}$. The function

$$\omega_r^*(\varphi_p', \delta) := \sup_{|h| \leq \delta} \| \Phi_p(we^{ih}) - \Phi_p(w) \|_{L^r(\mathbb{T})} =: \omega_r(\Phi_p, \delta), \qquad r > 1$$

is called the generalized integral modulus of continuity for $\varphi_p' = (\varphi')^{2/p} \in E^r(G)$.

This definition is correct, because by [14] for the smooth domains φ', $\varphi_0' \in L^r(L)$ and ψ', $\psi_0' \in L^r(\mathbb{T})$, and hence $\Phi_p \in L^r(\mathbb{T})$ for every $r > 1$.

Our main results are the following.

Theorem 1. *Let G be a finite domain with a smooth boundary L and let $p>1$. Then*

$$\| \varphi_p - \pi_{n,p} \|_{\overline{G}} \leq \begin{cases} c_1\sqrt{\frac{\ln n}{n}}\omega^*_{2+\varepsilon}(\varphi', 1/n), & p=2; \\ c_2 n^{-1/p}\omega^*_{p+\varepsilon}(\varphi'_p, 1/n), & p>2; \\ c_3 n^{1/p-1+\varepsilon}\omega^*_{p+\varepsilon}(\varphi'_p, 1/n), & 1<p<2, \end{cases}$$

for every $\varepsilon>0$ and with the constants $c_i = c_i(\varepsilon)>0$, $i=1,2,3$.

In spite of the fact that the function φ_p is defined on G it has [6] a continuous extension to $\overline{G}$. Therefore, the uniform norm in the above inequality is well defined.

From this theorem in case of $p=2$ we have the following result.

Corollary 1. *Let G be a domain with a smooth boundary L. Then*

$$\| \varphi - \pi_n \|_{\overline{G}} \leq c_1 \left(\frac{\ln n}{n}\right)^{1/2} \omega^*_{2+\varepsilon}(\varphi', 1/n) \qquad n\geq 1$$

with a constant $c_1 = c_1(\varepsilon)>0$.

As it follows from Definition 1 the modulus of continuity $\omega^*_{2+\varepsilon}(\varphi', 1/n)$ is simpler than $\Omega_{2,|\psi_0'|}((\varphi'\circ\psi_0)\times(\psi_0')^{1/2}, 1/n)$.

If L is sufficiently smooth and $p\geq 2$, then $\varepsilon>0$ may be omitted ın Theorem 1. In particular, for domain G with a Dini-smooth boundary L the following theorem holds.

Theorem 2. *If L is Dini-smooth and $p>1$, then*

$$\| \varphi_p - \pi_{n,p} \|_{\overline{G}} \leq \begin{cases} c_4\sqrt{\frac{\ln n}{n}}\omega^*_{2}(\varphi', 1/n) & p=2; \\ c_5 n^{-1/p}\omega^*_{p}(\varphi'_p, 1/n), & p>2; \\ c_6 n^{1/p-1+\varepsilon}\omega^*_{p}(\varphi'_p, 1/n), & 1<p<2, \end{cases}$$

for every $\varepsilon>0$ and with the constants c_4, $c_5>0$, $c_6 = c_6(\varepsilon)>0$.

If $L\in C(1,\alpha)$, $1<\alpha<1$, then by virtue of the Kellogg-Warschawski theorem $\omega^*_2(\varphi', 1/n)\leq c_7/n^{\alpha}$ and hence the following result holds.

Corollary 2. *Let $L\in C(1,\alpha)$, $1<\alpha<1$. Then*

$$\| \varphi - \pi_n \|_{\overline{G}} \leq c_8\sqrt{\ln n}\big/n^{1/2+\alpha}, \quad n\geq 1$$

with a constant $c_8>0$.

Earlier, the last result was proved in [13].

2. Auxiliary Results

For the function φ_p and a weight ω we set

$$\varepsilon_n\left(\varphi_p'\right)_r := \inf_{p_n} \| \varphi_p' - p_n \|_{L_r(G)}, \qquad E_n^{\circ}\left(\varphi_p'\right)_r := \inf_{p_n} \| \varphi_p' - p_n \|_{L^r(L)},$$

$$E_n^{\circ}\left(\varphi_p', \omega\right)_r := \inf_{p_n} \| \varphi_p' - p_n \|_{L^r(L,\omega)} ,$$

where inf is taken over all polynomials p_n of degree at most n and

$$\| \varphi_p' - p_n \|_{L^r(L,\omega)} := \left(\int_L | \varphi_p'(z) - p_n(z) |^r \omega(z) \, | dz | \right)^{1/r} .$$

One of the important step in the proofs of the main results is the following theorem.

Theorem 3. *Let G be a domain with a smooth boundary L and let $p > 1$. Then*

$$\varepsilon_n\left(\varphi_p'\right)_p \leq c_9 n^{-1/p} E_n^{\circ}\left(\varphi_p',\ 1/\mid \varphi_0' \mid\right)_p$$

with a constant $c_9 > 0$.

The following theorem gives an estimation for the quantity $E_n^{\circ}\left(\varphi_p',\ 1/\mid \varphi_0' \mid\right)_p$

Theorem 4. *Let G be a domain with a smooth boundary L, and let $1 < p < \infty$. Then*

$$E_n^{\circ}\left(\varphi_p',\ 1/\mid \varphi_0' \mid\right)_p \leq c_{10}\omega_{p+\varepsilon}^{*}(\varphi_p', 1/n$$

for every $\varepsilon > 0$ and with a constant $c_{10} = c(\varepsilon)$.

Corollary 3. *Let G be a domain with a Dini-smooth boundary L and let $p > 1$. Then*

$$E_n^{\circ}\left(\varphi_p',\ 1/\mid \varphi_0' \mid\right)_p \leq c_{11}\omega_p^{*}(\varphi_p', 1/n)$$

with a constant $c_{11} > 0$.

The approximation properties of the polynomials $\pi_{n,p}'$, $n = 1, 2, \ldots,$ are given in the following lemmas.

Lemma 1. *Let G be a domain with a smooth boundary L and let $p > 1$. Then*

$$\| \varphi_p' - \pi_{n,p}' \|_{L_p(G)} \leq c_{12} n^{-1/p} \omega_{p+\varepsilon}^{*}(\varphi_p', 1/n), \quad n = 1, 2, \ldots$$

for every $\varepsilon > 0$ and with $c_{12} = c_{12}(\varepsilon) > 0$.

Lemma 2. *Let G be a domain with a Dini-smooth boundary L and let $p > 1$. Then*

$$\| \varphi_p' - \pi_{n,p}' \|_{L_p(G)} \leq c_{13} n^{-1/p} \omega_p^{*}(\varphi_p', 1/n), \qquad n = 1, 2, \ldots$$

with a constant $c_{13} > 0$.

The following result is a particular case of the more general result proved in [3] (for $p = 2$) and [6] (in case of $1 < p < \infty$).

Lemma 3. *Let G be a finite domain with a smooth boundary L and let p_n be any polynomial of degree $\leq n$ with $p_n(z_0) = 0$. Then*

$$\| p_n \|_{\overline{G}} \leq \begin{cases} c_{14}\sqrt{\log\ n}\, \| p_n' \|_{L_p(G)}, & p = 2; \\ c_{15} \| p_n' \|_{L_p(G)}, & p > 2; \\ c_{16} n^{2/p-1+\varepsilon} \| p_n' \|_{L_p(G)}, & 1 < p < 2 \end{cases}$$

for every $\varepsilon > 0$, and with $c_{14}, c_{15} > 0$ and $c_{16} = c_{16}(\varepsilon) > 0$.

The following lemma was proved in [3], Lemma 15 in case of $p = 2$ treating the Bieberbach polynomials π_n, $n = 1, 2, \ldots$. The proof in case of $p \in (1, \infty)$, goes similarly.

Lemma 4. *Let G be a finite simple connected domain, and let p_n be a polynomial of degree $\leq n$ satisfying the condition $p_n(z_0) = 0$. Assume that*

$$\| p_n \|_{\overline{G}} \leq c_{17}\alpha_n \| p_n' \|_{L_p(G)}$$

and

$$\| \varphi_p' - \pi_{n,p}' \|_{L_p(G)} \leq c_{18}\beta_n,$$

with some positive constants c_{20} and c_{21}, where

$$\{\alpha_n\} \nearrow, \{\beta_n\} \searrow \quad \text{and} \quad \{\gamma_n := \alpha_n \cdot \beta_n\} \searrow.$$

If in addition, there exists a sequence of indexes $\{n_k\}$ such that

$$\gamma_{n_{k+1}} \leq \varepsilon\gamma_{n_k}, \quad \alpha_{n_{k+1}} \leq c_{19}\alpha_{n_k}, \qquad k = 1, 2, \ldots$$

for some $\varepsilon \in (0, 1)$ and a constant $c_{19} \geq 1$, then

$$\| \varphi_p - \pi_{n,p} \|_{\overline{G}} \leq c_{20}\gamma_n.$$

3. Proof of Main Results

We apply the familar method of Simonenko [12] and Andrievskii [3] for $p = 2$, its modification in case of $1 < p < \infty$ given in [6] and also Theorem 3.

Proof of Theorem 1***.*** The proof goes similarly to that of the main result of [8], by using Theorems 3, 4 and Lemmas 1, 3, and with a suitable choice of α_n, β_n and n_k in Lemma 4.

Proof of Theorem 2***.*** The same method of proof is valid also in this case; merely we apply Corollary 3 and Lemma 2 instead of Theorem 4 and Lemma 1, respectively.

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