Proc. A. Razmadze Math. Inst. 139(2005), 107–110

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BOUNDEDNESS OF GENERALIZED MULTILINEAR FRACTIONAL INTEGRALS

Classical Riesz potential is an important technical tool in harmonic analysis, theory of functions and partial differential equations. Multilinear maximal operator and multilinear fractional integral operator and related topics have been areas of research of many mathematicians such as R. Coifman and L. Grafakos [2], L. Grafakos [3]–[4], L. Grafakos and N. Kalton [5], C. E. Kenig and E. M. Stein [7], Y. Ding and S. Lu [6] and others.

The purpose of this note is to describe several results about multilinear operators of fractional integral type. We study $L_{p_1} \times L_{p_2} \times \ldots \times L_{p_k}$ boundedness of the generalized multilinear fractional integrals. O'Neil type inequality for multilinear fractional integral is proved. We give a new proof of the Hardy-Littlevood-Sobolev multilinear fractional integration theorem, based on a pointwise estimate of the rearrangement multilinear fractional integral.

Let $\mathbb{R}^n$ is the n-dimensional Euclidean space of points $x = (x_1, \ldots, x_n)$ with norms $|x| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$, let $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$.

For $1 \le p < \infty$ let $L_p(\mathbb{R}^n)$ be the space of measurable functions g on $\mathbb{R}^n$ with finite norm

$$\|g\|_{L_p(\mathbb{R}^n)} = \left(\int\limits_{\mathbb{R}^n} |g(x)|^p dx \right)^{1/p}.$$

We shall denote by $L_0(\mathbb{R}^n)$ the class of Lebesgue measurable functions that are finite a.e. and $g^*(t) = \inf\{s : \lambda_g(s) \le t\}$ is the decreasing rearrangement of g, where $\lambda_g(s) = |\{x \in \mathbb{R}^n : |g(x)| > s\}|$ is the distribution function of g with respect to the Lebesgue measure (we refer the reader to [1] for further information about distribution functions and decreasing rearrangements).

Set

$$g^{**}(t) = \frac{1}{t} \int\limits_0^t g^*(s) ds.$$

For every $t > 0$ we have (see [1], p. 53)

$$g^{**}(t) = \sup_{|E|=t} \frac{1}{t} \int\limits_E |g(y)| dy$$

Besides, by the Hardy-Littlewood theorem (see [1], p. 44), for every $f_1,\ f_2 \in L_0(\mathbb{R}^n)$,

$$\int\limits_{\mathbb{R}^n} |f_1(x) f_2(x)| dx \le \int\limits_0^\infty f_1^*(t) f_2^*(t) dt.$$

It is well known that if $p > 1$, then $\left(\int_0^\infty (g^{**}(t))^p\, dt\right)^{1/p}$ is comparable with the $L_p(\mathbb{R}^n)$ norm of f and $g^{**}(t) = \sup\limits_{|E|=t} |E|^{-1} \int_E |g(y)| dy$.

2000 *Mathematics Subject Classification:* 2B20, 42B25, 42B35.

Key words and phrases. Lebesgue space, multilinear fractional integral, O'Neil type inequality.

For $1 \le p < \infty$ the weak L_p spaces $WL_p(\mathbb{R}^n)$ is the sets of locally integrable functions $g(x)$, $x \in \mathbb{R}^n$ with finite norms

$$\|g\|_{WL_p(\mathbb{R}^n)} = \sup_{t>0} t\,(g^*(t))^{1/p}.$$

Let $k \ge 2$ will denote an integer, θ_j $(j = 1, 2, \cdots, k)$ will be fixed, distinct, and nonzero real numbers. It is said that p is the harmonic mean of $p_1, p_2, \cdots, p_k > 1$ if $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k}$. We denote $\mathbf{f} = (f_1, f_2, \cdots, f_k)$. If $f_j \in L_{p_j}(\mathbb{R}^n)$, $j = 1, 2, \cdots, k$, then we say that $\mathbf{f} \in L_{p_1} \times L_{p_2} \times \cdots \times L_{p_k}(\mathbb{R}^n)$. Also for $\mathbf{f} = (f_1, f_2, \cdots, f_k)$ we define $\mathbf{f}^{**}(t) = (1/t)\int_0^t f_1^*(s)\dots f_k^*(s)ds$, $t > 0$.

In the following we define the k-linear fractional integral operator by

$$I_{\Omega,\alpha}(\mathbf{f})(x) = \int_{\mathbb{R}^n} \frac{\Omega(x)}{|x|^{n-\alpha}} f_1(x - \theta_1 y)\dots f_k(x - \theta_k y)\,dy$$

and k-linear fractional type integral operator by

$$K_\alpha(\mathbf{f})(x) = \int_{\mathbb{R}^n} K_\alpha(y) f_1(x - \theta_1 y)\dots f_k(x - \theta_k y)\,dy,$$

where $K_\alpha \in WL_{n/(n-\alpha)}(\mathbb{R}^n)$.

Note that, if $K_\alpha(x) = \frac{\Omega(x)}{|x|^{n-\alpha}}$, $0 < \alpha < n$, $\Omega \in L_{n/(n-\alpha)}(\mathbb{S}^{n-1})$, then $K_\alpha^*(t) = \left(\frac{A}{nt}\right)^{(n-\alpha)/n}$, $K_\alpha^{**}(t) = \frac{n}{\alpha} K_\alpha^*(t)$, where $A = \|\Omega\|^{n/(n-\alpha)}_{L_{n/(n-\alpha)}(\mathbb{S}^{n-1})}$ and therefore $K_\alpha \in WL_{n/(n-\alpha)}(\mathbb{R}^n)$.

Lemma 1. *Let $f_1, f_2, \dots, f_k \in L_0(\mathbb{R}^n)$. Then*

$$\int_{\mathbb{R}^n} |f_1(x) f_2(x) \cdots f_k(x)|dx \le \int_0^\infty f_1^*(t) f_2^*(t) \cdots f_k^*(t)dt. \tag{1}$$

Lemma 2. *Let $f_1, f_2, \dots, f_k \in L_0(\mathbb{R}^n)$. Then*

$$\int_{\mathbb{R}^n} |f_1(x-\theta_1 y) f_2(x-\theta_2 y) \cdots f_k(x-\theta_k y)|dy \le C_\theta \int_0^\infty f_1^*(t) f_2^*(t) \cdots f_k^*(t)dt, \tag{2}$$

where $C_\theta = |\theta_1 \dots \theta_k|^{-n}$.

The analogy of O'Neil inequality (see, [8]) for k-linear integral operator by

$$(\mathbf{f}, g)(x) = \int_{\mathbb{R}^n} g(y) f_1(x - \theta_1 y) \cdots f_k(x - \theta_k y)\,dy,$$

is correct

Lemma 3. *Let $f_1, f_2, \dots, f_k \in L_0(\mathbb{R}^n)$. Then for all $0 < t < \infty$, the following inequality is valid*

$$(\mathbf{f}, g)^{**}(t) \le C_\theta \left(t\, \mathbf{f}^{**}(t) g^{**}(t) + \int_t^\infty f_1^*(t) f_2^*(t) \cdots f_k^*(t) g^*(t)dt \right). \tag{3}$$

Lemma 4. *Let $f_1, f_2, \dots, f_k \in L_0(\mathbb{R}^n)$. Then for any $t > 0$*

$$(\mathbf{f}, g)^{**}(t) \le C_\theta \int_t^\infty \mathbf{f}^{**}(t) g^{**}(t)dt. \tag{4}$$

For generalized multilinear fractional integrals $(K_\alpha, \mathbf{f})$ the following theorem is valid:

Theorem 1. *Let* $K_\alpha \in WL_{\frac{n}{n-\alpha}}(\mathbb{R}^n)$, $0<\alpha<n$. *Then*

$$(K_\alpha,\mathbf{f})^*(t) \le (K_\alpha,\mathbf{f})^{**}(t) \le$$
$$\le C_1\Big(t^{\frac{\alpha}{n}-1}\int_0^t f_1^*(s)f_2^*(s)\cdots f_k^*(s)ds + \int_t^\infty s^{\frac{\alpha}{n}-1}f_1^*(s)f_2^*(s)\cdots f_k^*(s)ds\Big), \quad (5)$$

where $C_1 = \left(\frac{n}{\alpha}\right)^2\ C_\theta\ \|K_\alpha\|_{WL_{\frac{n}{n-\alpha}}}$.

Corollary 1. *Suppose that* $0<\alpha<n$, $\Omega \in L_{n/(n-\alpha)}(\mathbb{S}^{n-1})$. *Then the following inequality*

$$\left(I_{\Omega,\alpha}\mathbf{f}\right)^*(t) \le \left(I_{\Omega,\alpha}\mathbf{f}\right)^{**}(t) \le$$
$$\le C_2\Big(t^{\frac{\alpha}{n}-1}\int_0^t f_1^*(s)f_2^*(s)\cdots f_k^*(s)\ ds + \int_t^\infty s^{\frac{\alpha}{n}-1}f_1^*(s)f_2^*(s)\cdots f_k^*(s)\ ds\Big),$$

holds, where $C_2 = \left(\frac{n}{\alpha}\right)\ C_\theta\ \left(\frac{A}{n}\right)^{(n-\alpha)/n}$, $A = \|\Omega\|^{n/(n-\alpha)}_{L_{n/(n-\alpha)}(\mathbb{S}^{n-1})}$.

Analogously we have

Theorem 2. *Let* $K_\alpha \in WL_{\frac{n}{n-\alpha}}(\mathbb{R}^n)$, $0<\alpha<n$. *Then*

$$(K_\alpha,\mathbf{f})^*(t) \le (K_\alpha,\mathbf{f})^{**}(t) \le C_1\int_t^\infty s^{\frac{\alpha}{n}-1}\mathbf{f}^{**}(s)\ ds, \quad (6)$$

Corollary 2. *Suppose that* $0<\alpha<n$, $\Omega \in L_{n/(n-\alpha)}(\mathbb{S}^{n-1})$. *Then the following inequality*

$$\left(I_{\Omega,\alpha}\mathbf{f}\right)^*(t) \le \left(I_{\Omega,\alpha}\mathbf{f}\right)^{**}(t) \le C_2\int_t^\infty s^{\frac{\alpha}{n}-1}\mathbf{f}^{**}(t)ds,$$

holds.

Theorem 3. *Suppose that* $0<\alpha<n, K_\alpha \in WL_{\frac{n}{n-\alpha}}(\mathbb{R}^n)$. *Let* p *be the harmonic mean of* $p_1,p_2,\cdots,p_k>1$ *and* q *satisfy* $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$. *Then* $K_\alpha * \mathbf{f}$ *is bounded operator from* $L_{p_1}\times L_{p_2}\times\cdots\times L_{p_k}(\mathbb{R}^n)$ *to* $L_q(\mathbb{R}^n)$ *for* $n/(n+\alpha)\le p<n/\alpha$ *(equivalently* $1\le q<\infty$*) and*

$$\|(K_\alpha,\mathbf{f})\|_{L_q(\mathbb{R}^n)} \le C\prod_{j=1}^k \|f_j\|_{L_{p_j}(\mathbb{R}^n)},$$

where $C>0$ *independent of* f.

Corollary 3. *Let* $0<\alpha<n$, $\Omega \in L_{n/(n-\alpha)}(\mathbb{S}^{n-1})$, p *be the harmonic mean of* $p_1,p_2,\cdots,p_k>1$ *and* q *satisfy* $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$. *Then* $I_{\Omega,\alpha}\ \mathbf{f}$ *is bounded operator from* $L_{p_1}\times L_{p_2}\times\cdots\times L_{p_k}(\mathbb{R}^n)$ *to* $L_q(\mathbb{R}^n)$ *for* $n/(n+\alpha)\le p<n/\alpha$ *(equivalently* $1\le q<\infty$*) and*

$$\|I_{\Omega,\alpha}\ \mathbf{f}\|_{L_q(\mathbb{R}^n)} \le C\prod_{j=1}^k \|f_j\|_{L_{p_j}(\mathbb{R}^n)},$$

where $C>0$ *independent of* f.

Remark 1. Note that, in the case $\Omega\equiv 1$ Corollary 3 was proved in [3] and in the case $\Omega \in L_s(\mathbb{S}^{n-1})$, $s>n/(n-\alpha)$ Corollary 3 was proved in [6].

Acknowledgements

The author was partially supported by the grants of Azerbaijan-U. S. Bilateral Grants Program (project ANSF Award / 3102)

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