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ON SOME INTEGRAL EQUATIONS FOR SURFACES WITH A CONICAL POINT

As is known, the classical method of proving the theorems on the existence of a solution in different boundary value problems of mathematical physics consists in the reduction of a solution of such problems to the solution of integral equations.

For elliptic equations, the boundary value problems, in particular, different boundary value problems of three-dimensional theory of elasticity in a domain with a smooth boundary have been studied thoroughly in the monograph "Three-Dimensional Problems of the Mathematical Theory of Elasticity" by V. Kupradze, T. Gegelia, M. Basheleishvili and T. Burchuladze ([1]) by the method of integral equations.

A great number of works dealing with elliptic equations in the case of a non-smooth surface appeared during the last decade (see, for example, [3], [4], [5], [6] and references therein).

The aim of our paper is to extend the method of integral equations to the case of a surface with a conical point.

Let S^k be a curvilinear conical surface with maximal angle, less than π, with vertex at the origin of the coordinates system $Ox_1x_2x_3$. The conical surface is given by the equation $x_3 = \phi(x_1, x_2)$, where $\phi(x_1, x_2)$ is the homogeneous function of the first order. When $\sqrt{x_2^2 + x_2^2} > a$, $a > 0$ we have $\frac{\partial\phi(x_1,x_2)}{\partial x_k} \in C^{(0,\lambda)}$, $\lambda \leq 1$, $k = 1, 2$. Obviously, $\frac{\partial\phi}{\partial x_k}$ is the homogeneous function of zero order. The domain bounded by S^k is denoted by D^k.

Let D be the domain bounded by the surface S, everywhere smooth, of the class $\mathcal{L}_1(\lambda)$, outside the neighbourhood of the point 0 and coinciding with S^k at $C(0, A)$, $A > 0$, $C(0, A)$ is a sphere of radius A and center at 0 and $D \cap C(0, A) = D^k \cap C(0, A)$.

Consider the integral equation

$$\varphi(x) - \sigma \int\limits_S K(x, y)\varphi(y)dS_y = f(x), \quad x \in S - \{0\}. \tag{1}$$

The kernel K satisfies the following conditions:

1. $K(x, y) = \frac{\psi(x,y)}{|x-y|^2}$, $x, y \in S - \{0\}$, $|\psi| < M$ is the homogeneous function of zero order on S^k.

2. $|\psi(x, y)| < M|x - y|^\lambda$, $x, y \in S - \{0\}$, $|x| > a$, $|y| > a$, $0 < a < \frac{A}{2}$.

3. On a smooth portion of the surface S, $\varphi(x, y)$ is of the class $C^{(0,\gamma)}$ with respect to x and y.

We call such a kernel of the class B. The kernel of the double layer potential and of a normal derivative of a single layer potential belongs to B [2].

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We approach the surface S by some smooth surface S_h, $0 < h < A$. Let, for example,

$$x_{3h} = \begin{cases} \frac{\phi^2(x_1,x_2)}{2h} + \frac{h}{2}, & \text{if } \phi(x_1,x_2) < h \\ \phi(x_1,x_2), & \text{if } \phi(x_1,x_2) \geq h. \end{cases}$$

For $h > a$, $a > 0$ S_h is the Ljapunov surface. For every h we define the kernel $K_h(x,y)$ in such a way that at the general part of the surfaces S_h and S it coincides with K and satisfies the conditions analogous to 1, 2 and 3.

$1'$. $K_h(x,y) = \frac{\psi_h(x,y)}{|x-y|^2}$, $|\psi_h| < M$, ψ_h is the homogeneous function of zero order.

$2'$. If $x, y \in S_h$, $h > a > 0$ then $|\psi_h| < M|x-y|^\lambda$.

$3'$. On S_h, $h > a$, $\psi_h(x,y) \in C^{(0,\lambda)}$ with respect to x and y.

Let, for example, K be the kernel of the double layer potential. Assume $K_h(x,y) = \frac{\cos(x-y, n_h(y))}{|x-y|^2}$, x, $y \in S_h$. The kernel defined in such a manner satisfies the conditions $1'$, $2'$, $3'$. On the general part of the surfaces S and S_h $K(x,y) = K_h(x,y)$.

Theorem 1. *We can represent $K_h(x,y)$ as follows:*

$$K_h(x,y) = \frac{\psi_h(x,y)}{|x|^y|y|^\beta|x-y|^{2-\alpha-\beta}}, \quad \alpha \geq 0, \quad \beta \geq 0, \quad \alpha + \beta \leq \lambda, \quad |\psi_h| < M,$$

where M does not depend on h.

Along with equation (1), let us consider an auxiliary equation

$$\varphi_h(x) - \sigma \int_{S_h} K_h(x,y)\varphi_h(y)dS_y = f_h(x), \quad f_h(x_1,x_2,x_{3h}) = f(x_1,x_2,x_3). \tag{2}$$

After $p-1$ iterations of equations (1) and (2) we obtain

$$\varphi(x) - \overline{\sigma} \int_S K_p(x,y)\varphi(y)dS_y = F(x), \quad x \in S - \{0\},$$

where $\overline{\sigma} = \sigma^p$, $F(x) = f + \sigma K f + \sigma^2 K_2 f + \cdots + \sigma^{p-1} K_{p-1} f$, $K_p = \frac{\psi_p(x,y)}{|x|^\alpha|y|^\beta}$, $\alpha > 0$, $\beta > 0$, $\alpha + \beta = 2$ and the equation with the continuous kernel

$$\varphi_h(x) - \overline{\sigma} \int_{S_h} K_{ph}(x,y)\varphi_h(y)dS_y = F_h(x),$$

$$F_h(x) = f_h(x) + \sigma K_h f_h + \cdots + \sigma^{p-1} K_{h(p-1)} f_h, \quad K_{hp} = \frac{\psi_{ph}(x,y)}{|x|^\alpha|y|^\beta},$$

$$\alpha > 0, \quad \beta > 0, \quad \alpha + \beta = 2,$$

Theorem 2. *For every p, $\lim_{h\to 0} \psi_{hp}(\overline{x},\overline{y}) = \psi_p(x,y)$, $x(x_1,x_2,x_3)$, $y(y_1,y_2,y_3) \in S - \{0\}$, $\overline{x}(x_1,x_2x_{3h})$, $\overline{y}(y_1,y_2,y_{3h}) \in S_h$.*

The tending is uniform on $\{x,y \in S||x| \geq a, \quad |y| \geq a\}$, $a > 0$.

For the continuous kernel K_{ph} we construct the Fredholm series $D_h(\overline{\sigma})$, $D_h(x,g,\overline{\sigma})$. The dependence of these functions on h is of interest.

Theorem 3. *There exists the function $\alpha(h)$, $\lim_{h\to 0} \alpha(h) = 0$, $\alpha(h) > ch^\varepsilon$, ε is an arbitrarily small number, such that the series $\alpha(h)D_h(\overline{\sigma})$ converges uniformly with respect to h, $\overline{\sigma}$, $h \leq \frac{A}{2}$.*

Theorem 4. *There exists the function $A(h)$ $\lim_{h\to 0} A(h) = 0$, $A(h) > ch^\varepsilon$, ε is an arbitrarily small number, such that the series $|x|^\alpha|y|^\beta A(h)D_{hp}(x,y,\overline{\sigma}) = |x|^\alpha|y|^\beta A(h) \cdot \sum_{n=1}^{\infty} \frac{\overline{\sigma}^n d_{hn}(x,y)}{n!}$ converges uniformly with respect to x, y, h.*

On the basis of these theorems we prove the theorem on the alternative.

Theorem 5. *If K is the kernel of the double layer potential, then either the equation has an eigne function of the class $C^{(0,\gamma)}$, $\gamma < \frac{\gamma}{2}$, or there exists the resolvent $R(x,y,\sigma)$ satisfying the equations*

$$\begin{aligned} R(x,y,\sigma) &= K(x,y) + \sigma \int\limits_S K(x,t)R(t,y,\sigma)dS_t \\ R(x,y,\sigma) &= K(x,y) + \sigma \int\limits_S R(x,t,\sigma)K(t,y)dS_t. \end{aligned} \tag{3}$$

The solution of equation (1) is unique and given by the formula

$$\varphi(x) = f(x) + \sigma \int\limits_S R(x,y,\sigma)f(y)dS_y. \tag{4}$$

If $f \in C^{(0,\gamma)}$, $\gamma < \frac{\lambda}{2}$ then $\varphi \in C^{(0,\gamma')}$, where $\forall\gamma' < \gamma$, $\gamma - \gamma'$ is an arbitrarily small number.

Theorem 6. *Let $K \in B$. Then either equation* (1) *has an eigne function of the class $C_{\alpha+\varepsilon}$, $\varepsilon > 0$ is an arbitrarily small number, or there exists the resolvent $R(x,y,\sigma)$ satisfying equations* (3)*, and the solution of equation* (1) *is unique and given by formula* (4)*. In this case, if $f(x) \in C^{(0,\gamma)}_{\alpha,\gamma}$, $\gamma < \lambda$ then $\varphi(x) \in C^{(c,\gamma)}_{\alpha',\gamma}$, where $\alpha' > \alpha$, $\alpha' - \alpha$ is an arbitrarily small number.*

The obtained results can be used for investigation of the boundary value problems. On the basis of Theorem 5 we have

Theorem 7. *There exists the solution $u(x)$ of the Dirichlet problem with the boundary function $f(x) \in C^{(0,\gamma)}$, $\gamma < \frac{\lambda}{2}$ and $u(x) \in C^{0,\gamma'}$, $\forall\gamma' < \gamma$, $\gamma - \gamma'$ is an arbitrarily small number, $x \in D + S$.*

The problem on the existence of a quasi-regular solutions also studied.

On the basis of Theorem 6 we obtain.

Theorem 8. *If $f(x) \in C^{(0,\gamma)}_{\alpha,\gamma}$, $\gamma < \lambda$, $\alpha < \lambda$, $\alpha + \gamma < 1$ and $\int\limits_S f(x)dx = 0$, then there exists the solution $u(x)$ of the Neumann problem with the boundary function $f(x)$ and $u(x) \in C^{(1,\gamma)}_{\alpha',\gamma}$, where $\alpha' > \alpha$, $\alpha' - \alpha$ is an arbitrarily small number.*

Let us consider the boundary problems of statics of the theory of elasticity. The equation of statics of the isotropic elastic medium written in the vector form is

$$\mu\Delta u + (\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}\,u = 0. \tag{5}$$

Problem I of Statics. Find in D a solution $u(x)$ of equation (5) by the boundary condition $u(x)|_i = f(x)$, $x \in S$.

Problem II of Statics. Find in D a solution $u(x)$ of equation (5) by the boundary condition $T\Big(\frac{\partial}{\partial x}, n\Big)u(x)|_i = f(x)$, $x \in S$.

Solutions of Problems I and II will be sought in the form of the double and single layer potentials of the theory of elasticity. We obtain the system of singular integral equations

$$\varphi(x) - \int\limits_S K(x,y)\varphi(y)dS_y = f(x) \tag{6}$$

where, for example, in the case of Problem II, $K(k,y) = K_1(x,y) + K_2(x,y)$ ([1], pp.171-181)

$$K_1 = \frac{1}{2\pi}\frac{\mu}{\lambda+2\mu}\frac{1}{|x-y|^2}\|\sigma_{kj}\|_{3\times 3}, \quad \sigma_{kj} = \frac{x_k-y_k}{|x-y|}\cos(n(y),x_j) - \frac{x_j-y_j}{|x-y|}\cos(n_y,x_k)$$

$$K_2(x,y) = \frac{1}{2\pi}\frac{1}{\lambda+2\mu}\left\|\left[\mu\delta_{jk} + 3(\lambda+\mu)\frac{(x_j-y_j)(x_k-y_k)}{|x-y|^2}\right]\frac{\partial}{\partial n_y}\frac{1}{|x-y|}\right\|_{3\times 3}.$$

In the case of a smooth surface is constructed a regularizer for (6) [1, p. 170-190], while in the case of a surface with a conical point we construct a regularizer in such a way that the kernel in the equation obtained after regularization is of the class B. For that kernel the obtained results are valid.

The theorems analogous to Theorems 7 and 8 are proved.

References

1. V. D. Kupradze, T. G. Gegelia, M. O. Basheleishvili, and T. V. Burchuladze, Three-dimensional problems in the mathematical theory of elasticity. (Russian) *Izdat. Tbilis. Univ., Tbilisi* 1968.
2. S. Sobolev, Uravneniya matematičeskoĭfiziki. (Russian) The equations of mathematical physics. *3rd Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow*, 1954.
3. V. A. Kondrat'ev, Boundary value problems for elliptic equations in domains with conical or angular points. (Russian) *Trudy Moskov. Mat. Obšč.* **16**(1967), 209–292.
4. V. G. Maz'ya, Plamenevskii B.A., Asymptotics of solutions of elliptic boundary value problems under singular perturbations of the boundary. *Tbilisi State University, Tbilisi,* 1981.
5. Ju. D.Burago and V. G. Maz'ya, Certain questions of potential theory and function theory for regions with irregular boundaries. (Russian) *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **3**(1967).
6. B. A. Plamenevskii, Algebras of pseudo-differential operators. (Russian) *Nauka, Moscow*, 1968.

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