Proc. A. Razmadze Math. Inst. 139(2005), 100–102

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ON SOME INTEGRALS ON THE SURFACE WITH A CONICAL POINT

Let us consider some integrals on a surface with a conical point, which are encountered when considering integral equations.

Let S^k be a curvilinear conical surface with maximal angle opening, less than π, with vertex at the origin of the coordinate system $Ox_1x_2x_3$. The conical surface is given by the equation $x_3 = \phi(x_1, x_2)$, where $\phi(x_1, x_2)$ is the homogeneous function of the first order. When $\sqrt{x_1^2 + x_2^2} > a$, $a > 0$ we have $\frac{\partial\phi(x_1,x_2)}{\partial x_k} \in C^{(0,\lambda)}$, $\lambda \le 1$, $k = 1, 2$. Obviously, $\frac{\partial\phi}{\partial x_k}$ is the homogeneous function of zero order. The domain bounded by S^k is denoted by D^k.

Let D be the domain bounded by the surface S, everywhere smooth, of the class $\mathcal{L}_1(\lambda)$ (Ljapunov's surface) outside every neighbourhood of the point 0 and coinciding with S^k at $C(0, A)$, $A > 0$, $C(0, A)$ is a sphere of radius A and center at 0.

We use the following classes of functions: the Hölder class $C^{(0,\gamma)}$, the class $C_\alpha : \varphi \in C_\alpha$, if $\varphi(x) = \frac{\psi(x)}{|x|^\alpha}$, $|\psi| < C$, $|x|$ is the distance from x to the conical point 0, the Hölder class with weight $C^{(0,\gamma)}_{\alpha,\gamma} : \varphi \in C^{(0,\gamma)}_{\alpha,\gamma}$, if $\varphi \in C_\alpha$ and $|x|^{\alpha+\gamma}\varphi(x) \in C^{(0,\gamma)}$, the class of differentiable functions with weight $C^{(1,\gamma)}_{\alpha,\gamma} : \varphi \in C^{(1,\gamma)}_{\alpha,\gamma}$ in the domain D bounded by the surface S, if $\frac{\partial\varphi}{\partial x_k} \in C^{(0,\gamma)}_{\alpha,\gamma}$ in D, $\varphi \in C^{(1,\gamma)}_{\alpha,\gamma}$ in $D + S - \{0\}$, if $\varphi \in C^{(1,\gamma)}_{\alpha,\gamma}$ in D, there exist on $S - \{0\}$ the limiting inside values of partial derivatives, and $\frac{\partial\varphi}{\partial x_k} \in C^{(0,\gamma)}_{\alpha,\gamma}$ in $D + S - \{0\}$. We call φ quasi-regular in $D + S$, if it is regular ([1], p.22) outside the conical point, i.e. on $\{x \in D + S\big| \cdot |x| \ge a\}$, $\forall a > a$ and $\varphi \in C^{(1,\gamma)}_{\alpha,\gamma}$ in $D + S - \{0\}$.

Consider the integral equation

$$\varphi(x) - \sigma \int\limits_S K(x, y)\varphi(y)dS_y = f(x), \quad x \in S - \{0\};$$

the kernel K satisfies the following conditions:

1. $K(x, y) = \frac{\psi(x,y)}{|x-y|^2}$, $x, y \in S - \{0\}$, $|\psi| < M$, ψ is the homogeneous function of zero order on S^k.
2. $|\psi(x, y)| < M|x - y|^\lambda$, $x, y \in S$, $|x| > a$, $|y| > a$, $0 < a < \frac{A}{2}$.
3. For $|x| > a$, $|y| > a$, $\psi(x, y) \in C^{(0,\lambda)}$ with respect to x and y.

Such a kernel we call a kernel of the class B. For example, the kernels of the double layer potential and of a normal derivative of a single layer potential are the kernels of the class B. On a smooth portion of S this kernel has weak singularity ([2]). What is the situation when approaching to 0?

Theorem 1. *The kernel $K(x, y)$ can be represented as follows:*

$$K(x, y) = \frac{\psi(x, y)}{|x - y|^{2-\alpha-\beta}|x|^\alpha|y|^\beta}, \quad \forall\alpha \ge 0, \quad \beta \ge 0, \quad \alpha + \beta \le \lambda.$$

2000 *Mathematics Subject Classification:* 35M10, 35B40, 35B60, 35B65.

Key words and phrases. Ljapunov's surface, surface with a conical point, homogeneous function, kernel of integral equation, single– and double-layer potentials, potentials of the theory of elasticity.

$\psi(x, y)$ is uniformly bounded with respect to x, y, α, β.

Thus we can vary α and β but in such a way that their sum would not exceed λ.

We study differential properties of the kernel K. The case, when K is the kernel of the double layer potential, is worth mentioning.

Theorem 2. *If K is the kernel of the double layer potential, then $K = \frac{|x|^\delta}{|y|^\delta} K^*(x, y)$, $\delta \le \lambda$, $K^* \in B$.*

The properties of compositions are studied. It is proved that

$$K_2(x, y) = \int_S K(x, t)K(t, y)dS_t = \frac{\psi_2(x, y)}{|x|^\alpha |y|^\beta |x - y|^{2-\alpha-\beta}},$$
$$\forall\alpha > 0, \quad \beta > 0, \quad |\psi_2| < C, \quad \alpha + \beta < 2, \quad \alpha + \beta \le 2\lambda$$
$$K_3(x, y) = \frac{\psi_3(x, y)}{|x|^\alpha |y|^\beta |x - y|^{2-\alpha-\beta}},$$
$$\forall\alpha > 0, \quad \beta > 0, \quad |\psi_3| < C, \quad \alpha + \beta < 2, \quad \alpha + \beta \le 3\lambda.$$

And so on, there exists p, such that

$$K_{p-1}(x - y) = \frac{\psi_{p-1}(x, y)}{|x|^\alpha |y|^\beta |x - y|^{2-\alpha-\beta}},$$
$$\forall\alpha > 0, \quad \beta > 0, \quad |\psi_{p-1}| < C, \quad \alpha + \beta < 2,$$
$$\alpha + \beta \le (p - 1)\lambda, \quad 2 - (p - 1)\lambda \le \lambda.$$

Then

$$K_p = \frac{\psi_p(x, y)}{|x|^\alpha |y|^\beta}, \quad \alpha + \beta = 2, \quad \alpha > 0, \quad \beta > 0, \quad |\psi_p| < C$$

Thus, outside the conical point K_p is the continuous kernel.

Differential properties of compositions are studied.

Let S be the surface with a conical point, and $S \in \mathcal{L}(\lambda)$ outside the conical point. Then for the potentials we prove the following

Theorem 3. *If $f(x) \in C^{(0,\gamma)}$ on S, $\gamma < \frac{\lambda}{2}$, then the double layer potential $Wf \in C^{(0,\gamma)}$, the Hölder coefficient is uniformly bounded with respect to $|x|$.*

Theorem 4. *Let $f \in C^{0,\gamma}$ on S, $\gamma < \frac{\lambda}{2}$. Then there exist on S the limiting inside values $W(f)(z)$, and if*

$$W_i(f)(z) = \begin{cases} W(f)(z), & z \in D \\ f(z) + W(f)(z), & z \in S - \{0\}, \end{cases}$$

then $W_i(f)(z) \in C^{(0,\gamma)}$ in $D + S$. The Hölder coefficient is uniformly bounded with respect to $|z|$.

Let $V(f) = \int_S \frac{f(y)}{|x-y|} dS_y$ be the single layer potential. Consider the integral $\int_S \frac{\partial}{\partial x_k} \frac{1}{|x-y|} \times f(y)dS_y$ which is singular and understood in the sense of the principal value.

It is proved that if S is a smooth surface and $f \in C^{(0,\gamma)}$ on S, then $\int_S \frac{\partial}{\partial x_k} \frac{1}{|x-y|} f(y)dS_y \in C^{(0,\gamma)}$, $\gamma < \lambda$, ([1], p. 231).

In the case if the surface has a conical point, we have the following

Theorem 5. *Let S be the surface with a conical point, and $S \in \mathcal{L}_1(\lambda)$ outside the conical point. If $f \in C^{(0,\gamma)}_{\alpha,\gamma}$ on $S - \{0\}$, $\gamma < \lambda$, $\alpha + \gamma < 1$ then $\int_S \frac{\partial}{\partial x_k} \frac{1}{|x-y|} f(y)dS_y \in C^{(0,\gamma)}_{\alpha,\gamma}$, $x \in S - \{0\}$.*

Theorem 6. *Let* $f \in C^{(0,\gamma)}_{\alpha,\gamma}$ *on* S, $\gamma < \lambda$, $\alpha + \gamma < 1$. *Then there exist the limiting inside values of the first derivatives of the single layer potential with respect to the Cartesian coordinates, and if*

$$\Big(\frac{\partial V(\varphi)}{\partial x_k}\Big)_i = \begin{cases} \frac{\partial V(\varphi)(x)}{\partial x_k}, & x \in D \\ \frac{\partial V(\varphi)}{\partial x_k} + n_k(x)\varphi(x), & x \in S - \{0\}, \end{cases}$$

then $\frac{\partial V(\varphi)}{\partial x_k} \in C^{(0,\gamma)}_{\alpha,\gamma}$ *in* $D + S\{0\}$.

Consider potentials of the theory of elasticity. The equation of statics of an isotropic elastic medium in the vector form is written as follows ([1], [11]):

$$\mu \Delta u + (\lambda + \mu)\operatorname{grad}\operatorname{div} u = 0, \tag{5}$$

where λ and μ are the Lame constants for the elastic medium. The stress operator has the form ([1], [17]).

$$T\Big(\frac{\partial}{\partial x}, n\Big)u = 2\mu \frac{\partial u}{\partial n} + \lambda \operatorname{div} u + \mu[n \times \operatorname{rot} u].$$

The vectors $\Gamma^j(x) = (\Gamma_{1j}, \Gamma_{2j}, \Gamma_{3j})$, $\Gamma_{kj} = \frac{\lambda' \delta_{kj}}{|x|} + \mu' \frac{x_k x_j}{|x|^3}$, $k, j = 1, 2, 3$, are the fundamental solutions of equations of statics. λ', μ' [1,33] are certain combinations of the numbers λ, μ.

Introduce the notation

$$\Gamma(x, n) = T\Big(\frac{\partial}{\partial x}, n\Big)\Gamma(x), \quad \Gamma^*(y - x, n) = \Big[T\Big(\frac{\partial}{\partial y}, n\Big)\Gamma(y - x)\Big]^*.$$

Using these singular solutions of equations of statics, we obtain the single and double layer potentials of the theory of elasticity ([1], pp. 216-217)

$$V(\varphi)(z) = \int\limits_S \Gamma(y - z)\varphi(y) dS_y, \quad W(p)(z) = \int\limits_S \Gamma^*(y - z, n(y))\varphi(y) dS_y.$$

$W(\varphi)(z)$ and $\frac{\partial V(\varphi)(z)}{\partial z_k}$ are the singular integrals which are understood in the sense of the principal value. For these potentials, the theorems, analogous to Theorems 3, 4, 5 and 6 are proved.

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