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ON THE DIFFERENTIAL PROPERTIES OF FUNCTIONS OF BOUNDED VARIATION IN HARDY SENSE

1. Definitions and notation

The definitions of functions of bounded variation in various senses given below one may find in [1-3].

For $x, y \in \mathbb{R}^n$ with $x \le y$ (i.e. $x_i \le y_i$ for every $i \in \overline{1,n}$) by I_x^y denote the interval $\prod_{i=1}^{n} [x_i, y_i]$. The *mixed difference* of $f : [0,1]^n \to \mathbb{R}$ on an interval $I = I_x^y \subset [0,1]^n$ is the quantity:

$$\Delta(f, I) = \sum_{\varepsilon_1=0}^{1} \cdots \sum_{\varepsilon_n=0}^{1} (-1)^{n-\sum\limits_{i=1}^{n} \varepsilon_i} f(x_1 + \varepsilon_1(y_1 - x_1), \ldots, x_n + \varepsilon_n(y_n - x_n)).$$

A partition of $[0,1]^n$ is a finite collection of non-overlapping intervals the union of which is $[0,1]^n$.Let Π be the collection of all partitions of $[0,1]^n$.

A function $f : [0,1]^n \to \mathbb{R}$ is said to have a *bounded variation in Vitali sense* if

$$\sup_{P \in \Pi} \sum_{I \in P} |\Delta(f, I)| < \infty.$$

The class of all functions on $[0,1]^n$ of bounded variation in Vitali sense denote by $\mathbb{V}_n$.

A number of elements of a set $B \subset \overline{1,n}$ denote by $|B|$.

For $B \subset \overline{1,n}$ with $0 < |B| < n$, $t \in [0,1]^{n-|B|}$ and $\tau \in [0,1]^{|B|}$ by (t, τ, B) denote the point of $\mathbb{R}^n$ for which $(t, \tau, B)_i = t_{|\overline{1,i} \setminus B|}$ if $i \notin B$ and $(t, \tau, B)_i = \tau_{|\overline{1,i} \cap B|}$ if $i \in B$.

Let f be a function on $[0,1]^n$. For $B \subset \overline{1,n}$ with $0 < |B| < n$ and $t \in [0,1]^{n-|B|}$ by $f_{B,t}$ denote the function on $[0,1]^{|B|}$ for which

$$f_{B,t}(\tau) = f\big((t, \tau, B)\big) \ \ \big(\tau \in [0,1]^{|B|}\big).$$

Denote also $f_B = f_{B,0}$ where 0 is the zero element of $\mathbb{R}^{n-|B|}$ and $f_{\overline{1,n}} = f$.

A function $f : [0,1]^n \to \mathbb{R}$ is said to have a *bounded variation in Hardy sense* if f and its every section has a bounded variation in Vitali sense,i.e.$f \in \mathbb{V}_n$ and $f_{B,t} \in \mathbb{V}_{|B|}$ for every $B \subset \overline{1,n}$ with $0 < |B| < n$ and $t \in [0,1]^{n-|B|}$. The class of all functions on $[0,1]^n$ with bounded variation in Hardy sense denote by $\mathbb{H}_n$. Due to one result of Leonov [4]

$$f \in \mathbb{H}_n \Leftrightarrow f_B \in \mathbb{V}_{|B|} \ \ \textit{for every nonempty} \ \ B \subset \overline{1,n}.$$

A function $f : [0,1]^n \to \mathbb{R}$ is said to have a *bounded variation in Arzela sense* if the set of all sums

$$\sum_{k=1}^{m-1} |f(x_{k+1}) - f(x_k)|,$$

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where $m \in \mathbb{N}$ and $(0,\dots,0) = x_1 \le x_2 \le \dots \le x_m = (1,\dots,1)$, is bounded.Note that every function of bounded variation in Hardy sense has also a bounded variation in Arzela sense(see e.g., [1] or [5]).

Recall that a Lebesgue indefinite integral of a function $f \in L[0,1]^n$ is defined as follows

$$F_f(x) = \int\limits_{[0,x_1]\times\dots\times[0,x_n]} f(y)dy \qquad (x \in [0,1]^n).$$

From the above mentioned result of Leonov it follows evidently that a Lebesgue indefinite integral of arbitrary function $f \in L[0,1]^n$ has a bounded variation in Hardy sense.

For $n \ge 2, h \in \mathbb{R}^n$ and $i \in \overline{1,n}$ denote by $h(i)$ the point in $\mathbb{R}^n$ such that $h(i)_j = h_j$ for every $j \in \overline{1,n}\backslash\{i\}$ and $h(i)_i = 0$.

Let $n \ge 2$ and f be a function defined in a neighborhood of a point $x \in \mathbb{R}^n$. If for $i \in \overline{1,n}$ there exists the limit

$$\lim_{h\to 0} \frac{f(x+h) - f(x+h(i))}{h_i},$$

then let us call its value as *the i-th strong partial derivative of f at x* and denote it by $D_{[i]}f(x)$. If f has finite $D_{[i]}f(x)$ for every $i \in \overline{1,n}$ then following Dzagnidze [6] let us say that *there exists a strong gradient of f at x or f has a strong gradient at x.*

It is easy to check that (see [6] for details) if a function f has a strong gradient at a point x then it is differentiable at x and the converse assertion is not true:the function $f(x_1,x_2) = |x_1x_2|^{\frac{2}{3}}$ is differentiable at the point $(0,0)$, but $\overline{D}_{[1]}f(0,0) = \overline{D}_{[2]}f(0,0) = +\infty$. Thus the condition of differentiability at the fixed point is weaker then the condition of the existence of a strong gradient in the same point. Moreover, Oniani [7] for arbitrary $n \ge 2$ constructed a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ such that the set of all points at which f is differentiable but does not have a strong gradient is of full measure.

2. Result

The differential properties of functions of bounded variation in different senses was investigated by various authors. In particular, there are known following results: *Every function on $[0,1]^n$ of bounded variation in Arzela sense (and consequently in Hardy sense) is differentiable almost everywhere*(Burkill and Haslam-Jones [8]); *Every function on $[0,1]^n$ of bounded variation in Kronrod-Vitushkin sense is differentiable almost everywhere* (Kronrod [9] (for $n = 2$), Vitushkin [3,§26] (for arbitrary $n \ge 2$); see also [10,Ch.5,§5]); *There exists a function on $[0,1]^2$ with bounded variation in Tonelli sense which is non-differentiable everywhere* (Stepanoff [11]; see also [12, Ch. 9,§12]). *Note that analogous statement for functions of bounded variation in Vitali sense is obvious.*

Since an indefinite integral of arbitrary $f \in L[0,1]^n$ has a bounded variation in Hardy sense then by virtue of Burkill and Haslam-Jones' result it is differentiable almost everywhere. However, in works of Dzagnidze [6] (for $n = 2$) and Dzagnidze and Oniani [13](for arbitrary $n \ge 2$) it was proved that an indefinite integral has a stronger differential property, namely, it has a strong gradient almost everywhere. In this connection naturally arises question *whether analogous conclusion is true for every function of bounded variation in Hardy sense (in Arzela sense).*

The following theorem gives a positive answer to the first part of the question.

Theorem. *Every function on $[0,1]^n$ of bounded variation in Hardy sense has a strong gradient almost everywhere.*

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