Proc. A. Razmadze Math. Inst. 139(2005), 128–131

A. Kirtadze and G. Pantsulaia

ON NULL SETS IN INFINITE-DIMENSIONAL BANACH SPACES

Let B be an infinite-dimensional separable Banach space, $e_1, e_2, \cdots \in B$, $\sum_{i\in\mathbb{N}} ||e_i|| < \infty$ and the span of $e_1, e_2, \cdots$ is dense in B.

Definition 1. A set $X \subset B$ is called an **Aronszajn null set** in B (cf.[1]), if it can be written as union of Borel sets E_n such that each E_n is null on every line in the direction e_n (i.e., for every $a \in B$, $\mu_1\{t \in \mathbb{R} : a + te_n \in E_n\} = 0$, where μ_1 is the one-dimensional standard Lebesgue measure).

We denote the class of all Aronszajn null sets in B by $A.N.S(B)$.

Definition 2. Following Mankiewicz (cf.[7]), a Borel set $X \subset B$ is called a **cube null set**, if it is null for every non-degenerate cube measure(Non-degenerate cube measures in B may be defined as distributions of the random variable of the form $a+\sum_{k\in\mathbb{N}} X_k e_k$, where $a \in B$ and $(X_k)_{k\in N}$ are uniformly distributed mutually independent random variables with values in $[0, 1]$).

We denote the class of all cube null sets in B by $C.N.S(B)$.

Definition 3. Following Phelps(cf.[10]), a Borel set is called **Gaussian null set** if it is null for every Gaussian measure on B (Gaussian measures in B may be defined as distributions of a.s.convergent sums $a + \sum_{k\in\mathbb{N}} X_k e_k$, where $a \in B$ and $(X_k)_{k\in N}$ are mutually independent standard Gaussian variables).

We denote the class of all Gaussian null sets in B by $G.N.S(B)$.

Definition 4. Let K be the class of all non-zero finite measures defined on the Borel σ-field $\mathcal{B}(B)$. We denote by $\mathcal{B}(B)^{\mu}$ the completion of $\mathcal{B}(B)$ with respect to the measure μ for $\mu \in K$. A set $E \subset B$ is called universally measurable if $E \in \cap_{\mu\in K}\mathcal{B}(B)^{\mu}$.

Definition 5. Following Christensen(cf.[3]) , a universally measurable set E is **Haar null** if there is a Borel probability measure μ on B such that every translate of E has $\overline{\mu}$-measure zero, where $\overline{\mu}$ denotes, as usual, the completion of the measure μ.

We denote the class of Haar null sets sets in B by $H.N.S(B)$.

Definition 6. Following Brian R. Hurt, Tim Sauer and James A. Yorke (cf.[6]), a set X is called **shy** if it is a subset of a Borel set X' for which $\mu(X' + v) = 0$ for every $v \in B$ and some Borel probability measure μ such that $\mu(K) = \mu(B)$ for some compact K(in this case we say also that the measure μ transvers to X) .

We denote the class of all shy sets in B by $S.S(B)$.

Remark 1. For every infinite-dimensional separable Banach space B, the relations

$$\mathbb{G}.\mathbb{N}.\mathbb{S}(B) = \mathbb{A}.\mathbb{N}.\mathbb{S}(B) = \mathbb{C}.\mathbb{N}.\mathbb{S}(B) \subset \mathbb{H}.\mathbb{N}.\mathbb{S}(B) = \mathbb{S}.\mathbb{S}(B) \subset \mathbb{T}.\mathbb{N}.\mathbb{S}(B)$$

are valid.

2000 *Mathematics Subject Classification:* 28A35,28C15,28C20.

Key words and phrases. Solovay model, translation-invariant Borel measure, Haar null set, Aronszajn null set, Gaussian null set, Cub null set, Shy set.

The result $\mathbb{H}.\mathbb{N}.\mathbb{S}(B) = \mathbb{S}.\mathbb{S}(B)$ was obtained by M.B. Stinchcombe(cf.[12]).

The assertion - $\mathbb{C}.\mathbb{N}.\mathbb{S}(B) \subset \mathbb{H}.\mathbb{N}.\mathbb{S}(B)$ was established by Y.Benyamini and J.Lindenstrauss (cf.[2]).

The coincidence of the classes $\mathbb{G}.\mathbb{N}.\mathbb{S}(B)$, $\mathbb{A}.\mathbb{N}.\mathbb{S}(B)$ and $\mathbb{C}.\mathbb{N}.\mathbb{S}(B)$ was proved by Marianna Csörnyei (cf.[5]).

The goal of our paper is to introduce new classes of null sets in the infinite-dimensional separable Banach space with a basis. We do it in the well known Solovay's model (cf.[11]) which is the following system of axioms:

$$(ZF) \ \& \ (DC) \ \& \ (\text{every subset of } \mathbb{R} \text{ is measurable in the Lebesgue sense}),$$

where (ZF) denotes the Zermelo-Fraenkel set theory and (DC) denotes the Axiom of Dependent Choices.

Lemma 1. *In Solovay's model there exists a translation-invariant measure μ on R^N such that:*
(i) $\mu([0,1]^N) = 1$;
(ii) $\mathrm{dom}(\mu) = P(R^N)$, *where $P(R^N)$ denotes the powerset of R^N.*

The proof of Lemma 1 can be found in [8],[9].

Let B be an infinite-dimensional separable Banach space with a basis $(e_k)_{k\in N}$ such that $\sum_{k\in N} ||e_k|| < \infty$.

Let a transformation $A : B \to R^N$ is defined by

$$A(z) = (x_k)_{k\in N},$$

where $z = \sum_{k\in N} x_k e_k$.

The following assertion is valid.

Lemma 2. *In Solovay's model the functional μ_A, defined by*

$$(\forall X)(X \subset B \to \mu_A(X) = \mu_N(A(X)),$$

is a translation-invariant measure given on the powerset of B and such that $\mu_A(\Delta) = 1$, where $\Delta = A^{-1}([0,1]^N)$.

Proof. Indeed, for $h \in B$ and $X \subset B$, we have

$$\mu_A(X+h) = \mu_N(A(X+h)) = \mu_N(A(X) + A(h)) = \mu_N(A(X)) = \mu_A(X).$$

Obviously,

$$\mu_A(\Delta) = \mu_N(A(A^{-1}([0,1]^N)) = \mu_N([0,1]^N) = 1.$$

This ends the proof of Lemma 2. □

Remark 2. It is reasonable to note that in Solovay's model every subset in an arbitrary Polish topological vector space is universally measurable.

Now we are able to introduce(in Solovay's model) new classes of null sets in B.

Definition 7. We say that a Borel set $X \subset B$ is **a cube null set** in B defined by a basis $\Gamma = (e_k)_{k\in N}$ if a standard cube measure ν defined by Γ transvers to X.

The class of all cube null sets in B defined a basis Γ will be denoted by $C.N.S.(B,\Gamma)$.

Definition 8. We say that $X \subset B$ is **Lebesgue null set** in B defined by a basis Γ if $\overline{\mu}_A(X) = 0$.

The class of all Lebesgue null sets in B defined by a basis Γ will be denoted by $L.N.S(B,\Gamma)$.

Definition 9. We say that a transformation $C : B \to \mathbb{R}^{\mathbb{N}}$ belongs to a class $\mathcal{C}$, if there exists a basis $e_1, e_2, \cdots$ in B such that $\sum_{i\in\mathbb{N}} ||e_i|| < \infty$ and

$$C(z) = (x_k)_{k\in N}$$

for $z = \sum_{k\in N} x_k e_k$.

If $C \in \mathcal{C}$, we set $\mu_C(X) = \mu_{\mathbb{N}}(C(X))$ for $X \subset B$.

Definition 10. We say that $X \subset B$ is **Lebesgue null set** in B if $\mu_C(X) = 0$ for arbitrary $C \in \mathcal{C}$.

The class of Lebesgue null sets in B will be denoted by $L.N.S(B)$.

Definition 11. We say that $X \subset B$ is **quasi-Lebesgue null set** in B if $\overline{\mu}_C(X) = 0$ for some $C \in \mathcal{C}$.

The class of all quasi-Lebesgue null sets in B we denote by $Q.L.N.S(B)$.

Definition 12. We say that $X \subset B$ is a **quasi-finite null set** if there exists a quasi-finite translation-invariant Borel measure ν such that $\overline{\nu}(X) = 0$, where $\overline{\nu}$ denotes the usual completion of ν.

The class of all quasi-finite null sets in B will be denoted by $Q.F.N.S(B)$.
One can prove the validity of the following propositions.

Theorem 1. *Let B be an infinite-dimensional separable Banach space with a basis $(e_k)_{k\in N}$ such that $\sum_{k\in N} ||e_k|| < \infty$. Then in Solovay's model we have*

$$\mathbb{L}.\mathbb{N}.\mathbb{S}.(B) \subset \mathbb{L}.\mathbb{N}.\mathbb{S}.(B,\Gamma) \subset \mathbb{Q}.\mathbb{L}.\mathbb{N}.\mathbb{S}.(B) \subseteq Q.F.N.S(B).$$

Theorem 2. *Let B be an infinite-dimensional separable Banach space with a basis $\Gamma = (e_k)_{k\in N}$ such that $\sum_{k\in N} ||e_k|| < \infty$. Then in Solovay's model we have*

$$\mathbb{Q}.\mathbb{F}.\mathbb{N}.\mathbb{S}(B) \subseteq S.S(B).$$

Theorem 3. *Let B be an infinite-dimensional separable Banach space with a basis $\Gamma = (e_k)_{k\in N}$ such that $\sum_{k\in N} ||e_k|| < \infty$. Then in Solovay's model we have*

$$\mathbb{L}.\mathbb{N}.\mathbb{S}(B,\Gamma) \subseteq \mathbb{C}.\mathbb{N}.\mathbb{S}.(B,\Gamma).$$

Acknowledgement

The work is partially supported by the Georgian High Schools Foundation for Scientific Research, Grant No.1.01.90

References

1. N. Aronszajn, Differentiability of Lipschitzian mappings between Banach spaces. *Studia Math.* **57**(1976), No. 2, 147–190.
2. Y. Benyamini and J. Lindenstrauss, Geometric nonlinear functional analysis. Vol. 1. *American Mathematical Society Colloquium Publications,* 48. *American Mathematical Society, Providence, RI,* 2000.
3. J. P. R. Christensen, Topology and Borel structure. *Descriptive topology and set theory with applications to functional analysis and measure theory. North-Holland Mathematics Studies,* Vol. 10. (*Notas de Matemätica,* No. 51). *North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York,* 1974.
4. J. Cichon, A. Kharazishvili, and B. Weglorz, Subsets of the real line. *Wydawnictwo Uniwersytetu Lódzkiego, Lódz,* 1995.
5. M. Csörnyei, Aronszajn null and Gaussian null sets coincide. *Israel J. Math.* **111**(1999), 191–201.
6. B. R. Hunt, T. Sauer, and J. A. Yorke, Prevalence: a translation-invariant "almost every" on infinite-dimensional spaces. *Bull. Amer. Math. Soc. (N.S.)* **27**(1992), No. 2, 217–238.
7. P. Mankiewicz, On the differentiability of Lipschitz mappings in Fréchet spaces. *Studia Math.* **45**(1973), 15–29.

A measure μ defined in E is called quasi-finite if there exists $Y_0 \subseteq E$ that $0 < \mu(Y_0) < \infty$.

8. G. Pantsulaia, On translation-invariant measures in the nonseparable Banach space ℓ^∞. *Georgian Technical Univ. Press.* **430**(2000), No. 2, 18–20.
9. G. Pantsulaia, Relations between shy sets and sets of ν_p-measure zero in Solovay's model. *Bull. Pol. Acad. Sci. Math.* **52**(2004), No. 1, 63–69.
10. R. R. Phelps, Gaussian null sets and differentiability of Lipschitz map on Banach spaces. *Pacific J. Math.* **77**(1978), No. 2, 523–531.
11. R. M. Solovay, A model of set-theory in which every set of reals is Lebesgue measurable. *Ann. of Math.* (2) **92**(1970), 1–56.
12. M. B. Stinchcombe, The gap between probability and prevalence: loneliness in vector spaces. *Proc. Amer. Math. Soc.* **129**(2001), No. 2, 451–457.

Authors' address:

Department of Mathematics
Georgian Technical University
77 M.Kostava, Tbilisi 0175
Georgia

Authors' Current address:
I. Vekua Institute of Applied Mathematics
I. Javakhishvili Tbilisi State University
University str. 2, Tbilisi 0143
Georgia.