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ANALYTIC WEIGHTED BESOV SPACES ON THE UNIT DISK

1. Introduction

Given $-1 < \lambda < \infty$, $1 < p < \infty$, define the weighted Besov space $B_p^\lambda(\mathbb{D})$ on the unit disc $\mathbb{D}$ to consist of analytic in $\mathbb{D}$ functions f such that

$$\int_{\mathbb{D}} (1-|z|^2)^{p-2}|f'(z)|^p d\mu_\lambda(z) < \infty,$$

where $d\mu_\lambda(z) = (\lambda+1)(1-|z|^2)^\lambda d\mu(z)$, and $d\mu(z) = \frac{1}{\pi}dxdy$.

In the paper [1] the unweighted Besov spaces on $\mathbb{D}$ $(B_p(\mathbb{D}) = B_p^0(\mathbb{D}))$ were studied. Further, these results were extended to the case of bounded symmetric domain ([2], [3]). The characterization of functions from the Besov spaces are given in these papers in various terms, including mean oscillation in the Bergman metric, Bergman projection, etc. In the papers [2], [3] the weighted Bergman projection is used as well as some analogues of fractional differentiation.

Here we study the weighted Besov spaces $B_p^\lambda(\mathbb{D})$. Main results of the paper are Theorems 2.1, 2.2. The ideas of proofs are taken from [1], though these results cannot be immediately derived from the unweighted case. The choice of the particular weight $(1-|z|^2)^\lambda$ is motivated by many links to applications and also immediate connection to the hyperbolic Bergman distance in the unit disc.

2. Auxiliaries

We will use notations from [4]. Let $\alpha_z(w) = \frac{z-w}{1-\overline{z}w}$ be the Moebius transform of the unit disc to itself that maps $w = 0$ to $w = z$. The hyperbolic Bergman metric in $\mathbb{D}$ is given by the formula

$$\beta(z,w) = \frac{1}{2}\ln\frac{1+|\alpha_z(w)|}{1-|\alpha_z(w)|} = \frac{1}{2}\ln\frac{|1-z\overline{w}|+|z-w|}{|1-z\overline{w}|-|z-w|}, \quad z,w \in \mathbb{D}.$$

For $z \in \mathbb{D}$ and $r > 0$ set $D(z,r) = \{w \in \mathbb{D} : \beta(z,w) < r\}$, and $|D(z,r)|_\lambda = \int_{D(z,r)} d\mu_\lambda(w)$.

Given a locally summable on $\mathbb{D}$ function f define its oscillation in the Bergman metric as follows

$$\omega_r(f)(z) = \sup\{|f(z)-f(w)| : w \in D(z,r)\}.$$

The mean oscillation of f in the Bergman metric is then defined to be

$$\mathrm{MO}_{r,\lambda}(f)(z) = \frac{1}{|D(z,r)|_\lambda}\int_{D(z,r)} |f(w) - \widehat{f}_{r,\lambda}(z)|d\mu_\lambda(w),$$

where

$$\widehat{f}_{r,\lambda}(z) = \frac{1}{|D(z,r)|_\lambda}\int_{D(z,r)} f(w)d\mu_\lambda(w).$$

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Introduce $L^p(\mathbb{D}, d\mu) = \{f : \|f\|_{L^p(\mathbb{D},d\mu)} = (\int\limits_{\mathbb{D}} |f(z)|^p d\mu(z))^{1/p}\}$, $d\nu(z) = \frac{d\mu(z)}{(1-|z|^2)^2}$, $d\nu_\lambda(z) = (\lambda+1)(1-|z|^2)^\lambda d\nu(z)$. By the definition, $\|f\|_{B_p^\lambda(\mathbb{D})} = \|(1-|z|^2)f'\|_{L^p(\mathbb{D},d\nu_\lambda)}$. The weighted Bergman space $\mathcal{A}^2_\lambda(\mathbb{D})$ on the unit disc consists of analytic $L^2(\mathbb{D}, d\mu_\lambda)$ - functions, $\lambda > -1$, and the corresponding Bergman projection $P^\lambda_{\mathbb{D}} : L^2(\mathbb{D}, d\mu_\lambda) \to \mathcal{A}^2_\lambda(\mathbb{D})$ is defined as follows

$$P^\lambda_{\mathbb{D}} f(z) = \int\limits_{\mathbb{D}} f(w) K_\lambda(z,w) d\mu_\lambda(w) = \int\limits_{\mathbb{D}} \frac{f(w)}{(1-z\overline{w})^{2+\lambda}} d\mu_\lambda(w).$$

That is, for a function $f \in \mathcal{A}^2_\lambda(\mathbb{D})$,

$$f(z) = \int\limits_{\mathbb{D}} \frac{f(w)}{(1-z\overline{w})^{2+\lambda}} d\mu_\lambda(w),$$

and by density this formula is valid for analytic summable with the measure $d\mu_\lambda$ in $\mathbb{D}$ functions as well. The following theorem characterizes functions in $B_p^\lambda(\mathbb{D})$ in terms related to weighted Bergman projection.

Theorem 2.1. *Suppose $1 < p < \infty$, $-1 < \lambda < \infty$ and f is analytic in $\mathbb{D}$, then the following are equivalent:*
1. $f \in B_p^\lambda(\mathbb{D})$;
2. $f \in P^\lambda_{\mathbb{D}} L^p(\mathbb{D}, d\nu_\lambda)$;
3. $(1-|z|^2)^m f^{(m)} \in L^p(\mathbb{D}, d\nu_\lambda)$, $m \geq 2$;
4. $\int\limits_{\mathbb{D}}\int\limits_{\mathbb{D}} \frac{|f(z)-f(w)|^p (1-|z|^2)^\lambda}{|1-\overline{z}w|^{4+2\lambda}} d\mu_\lambda(z) d\mu_\lambda(w) < \infty.$

Now we give characterization of functions in $B_p^\lambda(\mathbb{D})$ in terms of oscillation in the hyperbolic Bergman metric.

Theorem 2.2. *If $r > 0$, $-1 < \lambda < \infty$, $1 < p < \infty$ and f is analytic in $\mathbb{D}$, then the following are equivalent:*
1. $f \in B_p^\lambda(\mathbb{D})$;
2. $\mathrm{MO}_{r,\lambda}(f) \in L^p(\mathbb{D}, d\nu_\lambda)$;
3. $\omega_r(f) \in L^p(\mathbb{D}, d\nu_\lambda)$;
4. $|D(z,r)|_\lambda^{-1} \int\limits_{D(z,r)} |f(w) - f(z)| d\mu_\lambda(w) \in L^p(\mathbb{D}, d\nu_\lambda)$.

The following theorem characterizes Taylor coefficients of functions in $B_p^\lambda(\mathbb{D})$.

Theorem 2.3. *Suppose $f \in B_p^\lambda(\mathbb{D})$, $-1 < \lambda < \infty$, then for all $1 < p < \infty$ there is a constant C_p such that*

$$|a_n| \leq C_p (\lambda+1)^{-\frac{1}{p}} \|f\|_{B_p^\lambda(\mathbb{D})} n^{\frac{\lambda}{p} - \frac{1}{p}}, \quad n = 1, 2, 3, \ldots, \tag{2.1}$$

where a_n– coefficients of Taylor series of function f.

In conclusion we list some facts on interpolation and duality of Besov spaces. Introduce

$$I^\lambda f(z) = (1-|z|^2)^2 \int\limits_{\mathbb{D}} \frac{f(w)}{(1-z\overline{w})^{4+\lambda}} d\mu_\lambda(w).$$

Regarding the duality of Besov spaces, it is natural to consider the following invariant pairing formula

$$\langle f, g \rangle_\lambda = \int\limits_{\mathbb{D}} f'(z) \overline{g'(z)} d\mu_\lambda(z). \tag{2.2}$$

Though we can get same result using the following formula for pairing

$$\langle f, g \rangle_\lambda^\# = \int\limits_{\mathbb{D}} I^\lambda f(z) \overline{I^\lambda g(z)} d\nu_\lambda(z). \tag{2.3}$$

Theorem 2.4. *Let $1 < p < \infty$, $1/p + 1/q = 1$, $-1 < \lambda < \infty$. Under either* (2.2) *or* (2.3) *pairing formula we have the following duality*

$$(B_p^\lambda(\mathbb{D}))^* \cong B_q^\lambda(\mathbb{D}),$$

with equivalent norm.

Let as usual $[B_{p_0}^\lambda(\mathbb{D}), B_{p_1}^\lambda(\mathbb{D})]_\theta$ stand for interpolation space, obtained by the complex interpolation.

Theorem 2.5. *Let $1 < p_0$, $p_1 < \infty$, $0 < \theta < 1$ and $-1 < \lambda < \infty$. Then*

$$[B_{p_0}^\lambda(D), B_{p_1}^\lambda(D)]_\theta = B_q^\lambda(\mathbb{D}), \quad \frac{1}{q} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},$$

with equivalent norm.

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