Proc. A. Razmadze Math. Inst. 139(2005), 89–92

R. Akgun and D. M. Israfilov

POLYNOMIAL APPROXIMATION IN WEIGHTED SMIRNOV-ORLICZ SPACE

Introduction and Main Results

Let $\Gamma \subset \mathbb{C}$ be a closed, bounded rectifiable Jordan curve in the complex plane $\mathbb{C}$. Γ separates the plane $\mathbb{C}$ into two domains $G := \operatorname{int}\Gamma$, $G^- := \operatorname{ext}\Gamma$. Without loss of generality we may assume $0 \in G$. Let $\mathbb{D} := \{w \in \mathbb{C} : |w| < 1\}$ be the unit disc, $\mathbb{T} := \partial\mathbb{D}$, $\mathbb{D}^- := \operatorname{ext}\mathbb{T}$ and $w = \varphi(z)$ be the conformal mapping of G^- onto $\mathbb{D}^-$ normalized by the conditions

$$\varphi(\infty) = \infty, \qquad \lim_{z\to\infty} \frac{\varphi(z)}{z} > 0,$$

and let $\psi := \varphi^{-1}$ be the inverse mapping of φ.

By $E^p(G)$, $0 < p < \infty$, we denote the *Smirnov class* of analytic functions in G.

Let h be a continuous function on $[0, 2\pi]$. Its modulus of continuity is defined by

$$\omega(t, h) := \sup_{t_1, t_2 \in [0,2\pi],\ |t_1 - t_2| \le t} |h(t_1) - h(t_2)|, \ \ t \ge 0.$$

The curve Γ is called *Dini-smooth* if it has a parametrization

$$\Gamma : \varphi_0(\tau), \qquad 0 \le \tau \le 2\pi$$

such that $\varphi_0'(\tau)$ is Dini-continuous, i. e.

$$\int_0^\pi \frac{\omega(t, \varphi_0')}{t}\, dt < \infty$$

and $\varphi_0'(\tau) \neq 0$ [6, p. 48].

A continuous and convex function $M : [0, \infty) \to [0, \infty)$ which satisfies the conditions

$$M(0) = 0; \quad M(x) > 0 \qquad \text{for } x > 0;$$

$$\lim_{x\to 0} \frac{M(x)}{x} = 0; \quad \lim_{x\to\infty} \frac{M(x)}{x} = \infty;$$

is called an N-function.

The complementary N-function to M is defined by

$$N(y) := \max_{x \ge 0} (xy - M(x)), \qquad y \ge 0.$$

We denote by $L_M(\Gamma)$ the linear space of Lebesgue measurable functions $f : \Gamma \to \mathbb{C}$ satisfying the condition

$$\int_\Gamma M[\alpha |f(z)|]\, |dz| < \infty$$

for some $\alpha > 0$.

2000 *Mathematics Subject Classification:* 30E10, 46E30, 41A10.

Key words and phrases. Orlicz space, Faber polynomial, weighted Smirnov-Orlicz space, direct approximation theorem.

The space $L_M(\Gamma)$ becomes a Banach space with the *Orlicz norm*

$$\|f\|_{L_M(\Gamma)} := \sup_{\rho(g;N)\le 1} \int_\Gamma |f(z)\, g(z)|\, |dz|,$$

where $g \in L_N(\Gamma)$, N is the complementary N -function to M and

$$\rho(g;N) := \int_\Gamma N[|g(z)|]\, |dz|.$$

The Banach space $L_M(\Gamma)$ is called Orlicz space.

ω is called a *weight* on Γ, if $\omega : \Gamma \to [0,\infty]$ measurable and $\omega^{-1}(\{0,\infty\})$ has measure zero with respect to Lebesgue measure.

The class of measurable functions f defined on Γ satisfying the condition $\omega f \in L_M(\Gamma)$ is called *weighted Orlicz space* and denoted by $L_M(\Gamma,\omega)$. Its norm is defined as

$$\|f\|_{L_M(\Gamma,\omega)} := \|f\omega\|_{L_M(\Gamma)}, \qquad f \in L_M(\Gamma).$$

For $z \in \Gamma$ and $\epsilon > 0$ let $\Gamma(z,\epsilon) := \{t \in \Gamma : |t - z| < \epsilon\}$. For fixed $p \in (1,\infty)$, we define $q \in (1,\infty)$ by $p^{-1} + q^{-1} = 1$. The set of all weights $\omega : \Gamma \to [0,\infty]$ satisfying

$$\sup_{t\in\Gamma}\sup_{\epsilon>0}\left(\frac{1}{\epsilon}\int_{\Gamma(z,\epsilon)} \omega(\tau)^p\, |d\tau|\right)^{1/p}\left(\frac{1}{\epsilon}\int_{\Gamma(z,\epsilon)} \omega(\tau)^{-q}\, |d\tau|\right)^{1/q} < \infty,$$

is denoted $A_p(\Gamma)$.

We denote by $L^p(\Gamma,\omega)$ the set of measurable functions $f : \Gamma \to \mathbb{C}$ such that $|f|\,\omega \in L^p(\Gamma)$, $1 < p < \infty$.

Let $M^{-1} : [0,\infty) \to [0,\infty)$ be the inverse function of the N-function M. The upper and lower *indices* α_M, β_M (see, for example [1, p. 350]) of the function $\varrho : (0,\infty) \to (0,\infty]$

$$\varrho(x) := \limsup_{y\to\infty} \frac{M^{-1}(y)}{M^{-1}(y/x)}, \quad x \in (0,\infty)$$

are called the *Boyd indices* of the Orlicz space $L_M(\Gamma)$. In this case [1, p. 13]

$$\alpha_M = \lim_{x\to 0} \frac{\log\varrho(x)}{\log x}, \quad \beta_M = \lim_{x\to\infty} \frac{\log\varrho(x)}{\log x}.$$

The Boyd indices α_M, β_M are called *nontrivial* if $0 < \alpha_M$ and $\beta_M < 1$.

Definition 1. For a weight ω on Γ we denote by $E_M(G,\omega)$ the subclass of analytic functions of $E^1(G)$ whose boundary value functions belong to weighted Orlicz space $L_M(\Gamma,\omega)$.

Let $g \in L_M(\mathbb{T},\omega)$, and $\omega \in A_{\frac{1}{\alpha_M}}(\mathbb{T})$, $\omega \in A_{\frac{1}{\beta_M}}(\mathbb{T})$. We define as the shift

$$\sigma_h(g)(w) := \frac{1}{2h}\int_{-h}^{h} g\left(we^{it}\right) dt, \quad 0 < h < \pi, \quad w \in \mathbb{T}.$$

Since [3] we have

$$\|\sigma_h(g)\|_{L_M(\mathbb{T},\omega)} \le c_3\, \|g\|_{L_M(\mathbb{T},\omega)},$$

and therefore $\sigma_h(g) \in L_M(\mathbb{T},\omega)$ for any $g \in L_M(\mathbb{T},\omega)$.

Definition 2. Let $g \in L_M(\mathbb{T},\omega)$, *and* $\omega \in A_{\frac{1}{\alpha_M}}(\mathbb{T})$, $\omega \in A_{\frac{1}{\beta_M}}(\mathbb{T})$. The function

$$\Omega^r_{M,\omega}(g,\delta) := \sup_{\substack{0<h_i\le\delta \\ i=1,2,\ldots,r}} \left\| \prod_{i=1}^{r} \left(I - \sigma_{h_i}\right) g \right\|_{L_M(\mathbb{T},\omega)}, \quad \delta > 0, \quad r = 1, 2, \cdots$$

is called r-th modulus of smoothness of g, where I is identity operator.

We take $\omega_0(w) := \omega(\psi(w))$ and $f_0(w) := f(\psi(w))$, $w \in \mathbb{T}$. Then we have $f_0 \in L_M(\mathbb{T}, \omega_0)$ for $f \in L_M(\Gamma, \omega)$ and we define the modulus of continuity of $f \in L_M(\Gamma, \omega)$ as

$$\Omega^r_{\Gamma,M,\omega}(f,\delta) := \Omega^r_{M,\omega_0}(f_0,\delta), \qquad \delta > 0.$$

Let

$$E_n(f,\Gamma)_{M,\omega} := \inf_{P \in \mathcal{P}_n} \|f - P\|_{L_M(\Gamma,\omega)}$$

be the best approximation to $f \in L_M(\Gamma, \omega)$ in the class $\mathcal{P}_n$ of algebraic polynomials of degree not greater than n.

Some direct theorems of approximation theory in Smirnov-Orlicz class were proved in [2] for domains bounded by Carleson curves and in [4] for domains bounded by Dini-smooth curves. Inverse theorems in Smirnov-Orlicz class were obtained by V. M. Kokilashvili in [5] for domains with Dini-smooth boundary.

In this work we prove some direct and inverse theorems in weighted Smirnov-Orlicz class. In particular, we obtain a constructive characterization of generalized Lipschitz classes $\mathrm{Lip}^* \alpha(M, \omega)$, $\alpha > 0$. The main results of this work are the following.

Theorem 1. *Let G be a finite, simply connected domain with the Dini-smooth boundary Γ and let $L_M(\Gamma)$ be an Orlicz space with nontrivial Boyd indices α_M, β_M, and $\omega \in A_{\frac{1}{\alpha_M}}(\mathbb{T})$, $\omega \in A_{\frac{1}{\beta_M}}(\mathbb{T})$. If $S_n(f,.) := \sum_{k=0}^{n} a_k \Phi_k$ is the n-th partial sum of the Faber expansion of $f \in E_M(G, \omega)$, then for every natural number n*

$$\| f - S_n(f,.)\|_{L_M(\Gamma,\omega)} \le c\, \Omega^r_{\Gamma,M,\omega}\left(f, \frac{1}{n+1}\right)$$

with some constant c independent of n.

Theorem 2. *Let $\Gamma := \partial G$ be a Dini-smooth curve, $f \in E_M(G, \omega)$, $\omega \in A_{\frac{1}{\alpha_M}}(\Gamma)$, and $\omega \in A_{\frac{1}{\beta_M}}(\Gamma)$. Then*

$$\Omega^r_{\Gamma,M,\omega}\left(f, \frac{1}{n}\right) \le \frac{c}{n^{2r}} \sum_{k=1}^{n} k^{2r-1} E_k(f,\Gamma)_{M,\omega}, \quad r = 1, 2, \cdots .$$

Corollary 1. *Under the condition of the Theorem 2, if*

$$E_n(f,\Gamma)_{M,\omega} = \mathcal{O}\left(n^{-\alpha}\right), \quad \alpha > 0, \quad n = 1, 2, 3, \cdots ,$$

then

$$\Omega^r_{\Gamma,M,\omega}(f,\delta) = \begin{cases} \mathcal{O}(\delta^\alpha) & ; r > \alpha/2 \\ \mathcal{O}(\delta^\alpha \log(1/\delta)) & ; r = \alpha/2 \\ \mathcal{O}\left(\delta^{2r}\right) & ; r < \alpha/2 \end{cases}$$

for $f \in L_M(\Gamma, \omega)$.

Definition 3. For $\alpha > 0$ *let* $r := [\alpha/2] + 1$. The set of functions $f \in E_M(G, \omega)$ such that

$$\Omega^r_{\Gamma,M,\omega}(f,\delta) = \mathcal{O}(\delta^\alpha), \quad \delta > 0$$

is called generalized Lipschitz class $\mathrm{Lip}^* \alpha(M, \omega)$.

According to corollary 1 we have

Corollary 2. *Under the condition of the Theorem 2, if*

$$E_n(f) = \mathcal{O}\left(n^{-\alpha}\right), \quad \alpha > 0, \quad n = 1, 2, 3, \cdots ,$$

then $f \in \mathrm{Lip}^* \alpha(M, \omega)$.

Theorem 1 and Corollary 2 imply

Theorem 3. *Let $\alpha > 0$. Then under the condition of the Theorem 2 the following conditions are equivalent:*

(a) $f \in \mathrm{Lip}^* \alpha (M, \omega)$

(b) $E_n (f) = \mathcal{O}\left(n^{-\alpha}\right), \quad n = 1, 2, 3, \ldots$.

References

1. A. Böttcher and Yu. I. Karlovich, Carleson curves, Muckenhoupt weights, and Toeplitz operators. *Progress in Mathematics,* 154. *Birkhäuser Verlag, Basel,* 1997.
2. A. Guven and D. M. Israfilov, Polynomial approximation in Smirnov–Orlicz classes. *Comput. Methods Funct. Theory* **2**(2002), No. 2, 509–517.
3. D. M. Israfilov and A. Guven, Approximation by trigonometric polynomials in weighted Orlitz space. (To appear)
4. D. M. Israfilov, B. Oktay, and R. Akgun, Approximations in Smirnov–Orlitz classes. *Glas. Mat. Ser. III* **40 (60)** (2005), 87–102.
5. V. Kokilashvili, On analytic functions of Smirnov–Orlicz classes. *Studia Math.* **31**(1968), 43–59.
6. Ch. Pommerenke, *Boundary behaviour of conformal maps.* Grundlehren der Mathematischen Wissenschaften, 299. *Springer-Verlag, Berlin,* 1992.

Authors' Address:

Balikesir University,
Faculty of art and science,
Department of mathematics
Balikesir, 10100
Turkey