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On the Fourier-Haar Series Convergence with Respect to Sets which are Homothetic to the Given Set

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Notation. Let $I^2 = (0,1)\times(0,1)$, $\mathbb{R}^2_+ = (0,\infty)\times(0,\infty)$ and for every $W\subset\mathbb{R}^2$

$$rW = \{rz : z\in W\} \quad (r\in\mathbb{R}).$$

For $W\subset\mathbb{R}^2_+$ and $f\in L(I^2)$ by $S_{r,W}(f,z)$ $(r>0)$ we denote partial sums with respect to the set W of the Fourier-Haar series of the function f, i.e.,

$$S_{r,W}(f,z) = \sum_{(i,j)\in rW} c_{ij}(f)\chi_i(x)\chi_j(y),$$

where $z=(x,y)$, χ_i are the Haar functions and $c_{ij}(f)$ are Fourier coefficients of the function f with respect to Haar's system. If there exists $\lim\limits_{r\to\infty} S_{r,W}(f,z)$, then they say that the Fourier-Haar series of the function f converges with respect to W at the point z.

The set W is said to be a set of the first type, if there are positive numbers a and b and continuous monotone functions $f:[0,a]\to\mathbb{R}$ and $g:[0,b]\to\mathbb{R}$, such that $f(0)$, $f(a)$, $g(0)$, $g(b)$ are positive, and

$$W = \{(x,y): 0<x<a,\ 0<y<f(x)\}\cup\{(x,y): 0<y<b,\ 0<x<g(y)\}.$$

The set $W\subset\mathbb{R}^2_+$ is said to be a set of the second type, if there are $\varepsilon>0$ and $n\in\mathbb{N}$, such that $\operatorname{dist}(W,0x)>\varepsilon$ and $\operatorname{dist}(W,0y)>\varepsilon$, and every horizontal or vertical cross-section of the set W consists of not more than n segments.

The set W is said to be tame, if it can be represented in terms of $W=W_1\cup W_2$, where W_1 is the set of the first and W_2 is that of the second type.

We will say that the set $W\subset\mathbb{R}^2_+$ is in the standard position, if $\partial W\cap 0x$ and $\partial W\cap 0y$ are the segments having one common edge at the origin.

We will say that the set W is a pseudo-interval, if it can be represented in terms of $W=W_1\cup W_2\cup W_3$, where W_1 and W_2 are two-dimensional intervals having standard position, and W_3 is the set of the second type.

Note that tame is:

(1) every convex open set having standard position;

(2) every polygon having standard position;

(3) a subgraph interior of every function f with the properties: the function f is defined on the segment $[0,a]$ monotonically and continuously, and $f(t)>0$ for $t\in[0,a]$;

(4) every set of the type

$$\{(x,y): 0<x<a,\ 0<y<f(x)\}\cup\{(x,y): 0<y<b,\ 0<x<g(y)\},$$

where $a,b>0$, while f and g are piecewise monotone, continuous functions on the segments $[0,a]$ and $[0,b]$, respectively, and $f(x)>0$ for $t\in[0,a]$, $g(t)>0$ for $t\in[0,b]$.

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Results. If W is a two-dimensional interval with a standard position, i.e., $W = (0,x) \times (0,y)$, then (see [1]) for every $f \in L(I^2)$

$$\lim_{r\to\infty} S_{r,W}(f,z) = f(z) \quad \text{a.e. on} \quad I^2.$$

If W is a quarter of a circle having standard position, i.e., $W = \{(x,y) \in \mathbb{R}_+^2 : x^2 + y^2 < 1\}$, then (see [2] and [3]) there exists the function $f \in L(I^2)$ such that

$$\overline{\lim_{r\to\infty}} S_{r,W}(f,z) = \infty \ \text{ a.e. on } \ I^2.$$

There naturally arises the question: what form the set W must have in order for the Fourier-Haar series of every summable function to converge almost everywhere?

A complete answer to the above question in a class of tame sets gives the following

Theorem 1. *Let W be a tame set. Then the following statements are equivalent:*
(1) *for every $f \in L(I^2)$,*

$$\lim_{r\to\infty} S_{r,W}(f,z) = f(z) \ \text{ a.e. on } \ I^2;$$

(2) *W is a pseudo-interval.*

Let a tame set W not be a pseudo-interval. What can one say about the convergence of the Fourier-Haar series with respect to W? In terms of integral classes a full answer to the question gives the following

Theorem 2. *Let the tame set W not be a pseudo-interval. Then:*
(1) *for every $f \in L\ln^+ L(I^2)$,*

$$\lim_{r\to\infty} S_{r,W}(f,z) = f(z) \ \text{ a.e. on } \ I^2;$$

(2) *for every $f \in L\backslash L\ln^+ L(I^2)$ there exists an equimeasurable with f function $g \in L(I^2)$ such that*

$$\overline{\lim_{r\to\infty}} S_{r,W}(g,z) = \infty \ \text{ a.e. on } \ I^2.$$

Thus, in the case, where the tame set W is not a pseudo-interval, the best integral class for the convergence is $L\ln^+ L(I^2)$.

Statements of Theorem 2 follow from the following more strong theorems.

Theorem 3. *Let W be a tame set. If the Fourier-Haar series of the function $f \in L(I^2)$ converges in Pringsheim sense to $f(z)$ for every point z from the set E, then $\lim_{r\to\infty} S_{r,W}(f,z) = f(z)$ a.e. on E.*

Remark 1. For the case $W = \{(x,y) \in \mathbb{R}_+^2 : x^2 + y^2 < 1\}$, Theorem 3 has been proved in [4].

Remark 2. From Theorem 3 follows the first proposition of Theorem 2, since, as is known (see [1]), the Fourier-Haar series of every function f from $L\ln^+ L(I^2)$ converges to $f(z)$ a.e.

Theorem 4. *For every function $f \in L\ln^+ L(I^2)$ there exist an equimeasurable with f function $g \in L(I^2)$ and a set of full measure $E \subset I^2$ such that for every tame set W which is not a pseudo-interval,*

$$\overline{\lim_{r\to\infty}} S_{r,W}(g,z) = \infty \ \text{ for every } \ z \in E.$$

Remark 3. For the case $W = \{z \in \mathbb{R}_+^2 : p(z) < 1\}$, where p is a norm in $\mathbb{R}^2$, Theorems 1–4 have been announced by us in [5].

Theorem 5. *Let the set $W \subset \mathbb{R}^2_+$ is such that $rW \subset tW$ for $0 < r < t$, and let*

$$\Delta_W = \{f \in L(I^2) : \overline{\lim_{r\to\infty}} S_{r,W}(f,z) = \infty \ \text{ a.e. on } I^2\}.$$

If the topological space satisfies the following five conditions:

(1) *$X \subset L(I^2)$, $0 \in X$ and $f + g \in X$ for $f, g \in X$;*
(2) *the topology in X is invariant with respect to a shift;*
(3) *if f_n converges to f in X, then $\lim_{n\to\infty} c_{ij}(f_n) = c_{ij}(f)$ $(i, j \in \mathbb{N})$;*
(4) *a set of all functions $f \in X$ for which $\overline{\lim_{n\to\infty}} S_{r,W}(f,z) < \infty$ a.e. on I^2, is dense everywhere in X;*
(5) *every neighborhood of zero contains a function from Δ_W;*

then $X\backslash\Delta_W$ is of the first category in X.

Taking into account Theorem 5, from Theorem 2 we easily arrive at

Corollary. *Let the tame set W not be a pseudo-interval. Then:*

(1) *$L(I^2)\backslash\Delta_W$ is of the first category in $L(I^2)$;*
(2) *for every separable Orlicz space $\Phi(L)(I^2)$, for which $\Phi(L)(I^2)\backslash L\ln^+ L(I^2) \neq \varnothing$, the set $\Phi(L)(I^2)\backslash\Delta_W$ is of the first category in $\Phi(L)(I^2)$;*
(3) *for every separable symmetrical space E regarding Lebesgue measure on I^2, for which $E\backslash L\ln^+ L(I^2) \neq \varnothing$, the set $E\backslash\Delta_W$ is of the first category in E.*

Remark 4. In case $W = \{(x,y) \in \mathbb{R}^2_+ : x^2 + y^2 < 1\}$, the first statement of the above corollary has been proved in [6].

References

1. O. Dzagnidze, On the representation of measurable functions by double Fourier series. (Russian) *Soobshch. Akad. Nauk Gruzin. SSR* **34**(1964), No. 3, 277–282.
2. G. Kemkhadze, On the divergence of spherical partial sums of double Fourier-Haar series. (Russian) *Trudy Tbiliss. Polytech. Inst.* **285**(1985), No. 3, 42–48.
3. G. Tkebuchava, On the divergence of double Fourier-Haar series by spheres. *Anal. Math.* **3**(1994), 147–153.
4. G. Kemkhadze, On the convergence of multiple Fourier-Haar series by spheres. (Russian) *Trudy Tbiliss. Mat. Inst.* **55**(1977), 27–38.
5. G. Oniani, On the convergence of Fourier-Haar series with respect to a norm. *Bull. Georgian Acad. Sci.* **163**(2001), No. 2, 215–217.
6. G. Lepsveridze, On the spherical divergence of double Fourier-Haar series. *Proc. A. Razmadze Math. Inst.* **122**(2000), 105–123.

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