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G. Oniani

On the Strong Differentiation of Multiple Integrals Along Different Frames, II

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Notation. We call a frame in the space $\mathbb{R}^n (n \geq 2)$ a set whose elements are n mutually orthogonal straight lines passing through the origin. Denote a frame by $\theta(\theta = \{\theta_1, \dots, \theta_n\})$ and a set of all frames by $\theta(\mathbb{R}^n)$.

We call a frame of a rectangular parallelepiped P in $\mathbb{R}^n$ the frame θ for which the edges of P are parallel to the corresponding lines $\theta^j (j = 1, \dots, n)$.

For the frame θ, by $I(\theta)$ we denote a differentiation basis for which $I(\theta)(x)(x \in \mathbb{R}^n)$ consists of all rectangular parallelepipeds containing x and having the frame θ.

Differentiability of an integral with respect to $I(\theta)$ is sometimes called as a strong differentiability along the frame θ, and in the case $\theta = \{0x_1, \dots, 0x_n\}$ as a strong differentiability.

Let $G^n = (0,1)^n$. We denote by $\Phi(L)(G^n)$ a class of all measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ with the following properties: $\operatorname{supp} f \subset G^n$ and $\int_{G^n} \Phi\big(|f|\big) < \infty$.

Definition. A set $E \subset \theta(\mathbb{R}^n)$ is called an R-set if there exists a function $f \in L(G^n)$, $f \geq 0$ such that for every $\theta \in E$, $\overline{D}_{I(\theta)}(\int f, x) = \infty$ a.e. on G^n, and for every $\theta \notin E$, $\int f$ is strongly differentiable along θ.

The notion of the R-set was introduced in [1] for $n = 2$ and in [2] for any $n \geq 2$.

Denote by S_n a set of all rearrangements of the set $\{1, \dots, n\}$, i.e., a set of all bijections $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$.

The natural metric in $\theta(\mathbb{R}^n)$ is denoted as follows: $\theta_1, \theta_2 \in \theta(\mathbb{R}^n)$, $\theta_1 = \{\theta_{11}, \dots, \theta_{1n}\}$, $\theta_2 = \{\theta_{21}, \dots, \theta_{2n}\}$

$$\operatorname{dist}(\theta_1, \theta_2) = \min \left\{ \sum_{i=1}^{n} \angle\Big(\theta_{1\sigma_1(i)}, \theta_{2\sigma_2(i)}\Big) : \sigma_1, \sigma_2 \in S_n \right\}, \tag{1}$$

where $\angle(\cdot, \cdot)$ is the angle between the lines.

Below, $\theta(\mathbb{R}^n)$ is assumed to be the metric space with metric (1).

Let $1 \leq k \leq n-1$. We call a set $E \subset \theta(\mathbb{R}^n)$ an orbit of k-th order if there exist mutually orthogonal lines $l_1, \dots, l_{n-k}$ passing through the origin, such that $E = \{\theta \in \theta(\mathbb{R}^n) : l_1, \dots, l_{n-k} \in \theta\}$.

We call a set $E \subset \theta(\mathbb{R}^n)$ an orbit if E is an orbit of k-th order for some $k \in \{1, \dots, n-1\}$.

Let $\theta \in \theta(\mathbb{R}^n)$, $0 < \varepsilon < \pi/4$, $\sigma \in S_n$. By $T_{\sigma,1}(\theta, \varepsilon)$ is denoted a set of all frames $\theta' \in \theta(\mathbb{R}^n)$ such that $\angle(\theta_{\sigma(1)}, l) > \varepsilon$ for every $l \in \theta'$; if $n \geq 3$, then for $i \in \{2, \dots, n-1\}$ we denote by $T_{\sigma,i}(\theta, \varepsilon)$ a set of all frames $\theta' \in \theta(\mathbb{R}^n)$ for which

(1) $\theta_{\sigma(1)}, \dots, \theta_{\sigma(i-1)} \in \theta'$;

(2) $\angle(\theta_{\sigma(i)}, l) > \varepsilon$ for every $l \in \theta' \backslash \{\theta_{\sigma(1)}, \dots, \theta_{\sigma(i-1)}\}$.

Denote also

$$T_\sigma(\theta, \varepsilon) = \bigcup_{i=1}^{n-1} T_{\sigma,i}(\theta, \varepsilon),$$

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and

$$T(\theta,\varepsilon)=\{T_\sigma(\theta,\varepsilon):\sigma\in S_n\}.$$

Let $n\ge 3$; $2\le k\le n-1$; $l_1,\dots,l_{n-k}$ be mutually orthogonal lines in $\mathbb{R}^n$ passing through the origin; $E=\{\theta\in\theta(\mathbb{R}^n): l_1,\dots,l_{n-k}\in\theta\}$; $0<\varepsilon<\pi/4$ and $\sigma\in S_{n-k}$. By $T_{\sigma,1}(E,\varepsilon)$ is denoted a set of all frames $\theta\in\theta(\mathbb{R}^n)$ for which $\angle(\theta_{\sigma(1)},l)>\varepsilon$ for every $l\in\theta$; if $n-k\ge 2$, then for $i\in\{2,\dots,n-k\}$ we denote by $T_{\sigma,i}(E,\varepsilon)$ a set of all frames $\theta\in\theta(\mathbb{R}^n)$ for which

(1) $l_{\sigma(1)},\dots,l_{\sigma(i-1)}\in\theta$;
(2) $\angle(l_{\sigma(i)},l)>\varepsilon$ for every $l\in\theta\backslash\{l_{\sigma(1)},\dots,l_{\sigma(i-1)}\}$.
Denote

$$T_\sigma(E,\varepsilon)=\overset{n-1}{\underset{i=1}{\cup}}T_{\sigma,i}(E,\varepsilon),$$

and

$$T(E,\varepsilon)=\{T_\sigma(E,\varepsilon):\sigma\in S_{n-k}\}.$$

Let us note that

$$T(\{\theta\},\varepsilon)=T(\theta,\varepsilon),\ \ T_\sigma(\{\theta\},\varepsilon)=T_\sigma(\theta,\varepsilon)\ \text{ and }\ T_{\sigma,i}(\{\theta\},\varepsilon)=T_{\sigma,i}(\theta,\varepsilon).$$

Let $k\in\{1,\dots,n-1\}$. Assume also that for every $i\in\mathbb{N}$, $l_1^{(i)},\dots,l_{n-k}^{(i)}$ are mutually orthogonal lines in $\mathbb{R}^n$ passing through the origin, and

$$E_i=\{\theta\in\theta(\mathbb{R}^n): l_1^{(i)},\dots,l_{n-k}^{(i)}\in\theta\}.$$

Let $l_1,\dots,l_{n-k}$ be mutually orthogonal lines in $\mathbb{R}^n$ passing through the origin and

$$E=\{\theta\in\theta(\mathbb{R}^n): l_1^{(i)},\dots,l_{n-k}^{(i)}\in\theta\}.$$

We shall say that E is a limit orbit of the set $\overset{\infty}{\underset{i=1}{\cup}}E_i$ if there are $i_1<i_2<\dots$ and $\sigma_p\in S_{n-k}(p\in\mathbb{N})$ such that

$$\lim_{p\to\infty}\angle\Big(l^{(i_p)}_{\sigma_p(1)},l_1\Big)=\dots=\lim_{p\to\infty}\angle\Big(l^{(i_p)}_{\sigma_p(n-k)},l_{n-k}\Big)=0.$$

For a set $E\subset\theta(\mathbb{R}^n)$ and $\varepsilon>0$ we denote $V[E,\varepsilon]=\{\theta\in\theta(\mathbb{R}^n):\operatorname{dist}(\theta,E)\le\varepsilon\}$.

For $\nu\in\{2,\dots,n-1\}$ and $\theta\in\theta(\mathbb{R}^n)$ we denote by $I_\nu(\theta)$ a differentiation basis in $\mathbb{R}^n$ for which $I_\nu(\theta)(x)$ $(x\in\mathbb{R}^n)$ consists of all rectangular parallelepipeds P in $\mathbb{R}^n$ with the properties

(1) $x\in P$;
(2) P has the frame θ;
(3) ν of mutually orthogonal edges of P have one and the same length.
Clear $I_1(\theta)=I(\theta)$.

Results. Theorems given below generalize the results of [2]–[4].

Theorem 1. *Every R-set is a set of type G_δ. Moreover, for every $f\in L(G^n)$ the set*

$$R(f)=\Big\{\theta\in\theta(\mathbb{R}^n):\overline{D}_{I(\theta)}\Big(\int f,x\Big)=\infty\ \text{ a.e. on }\ G^n\Big\}$$

is of type G_δ.

Theorem 2. *At most countable union of orbits of $(n-1)$-th order is an R-set if and only if it is of type G_δ.*

Theorem 3. *At most countable union of orbits of k-th $(1\le k\le n-2)$ order, contained in some fixed orbit of $(k+1)$-th order, is an R-set if and only if it is of type G_δ.*

Theorem 4. *At most countable union of orbits of k-th* $(1 \le k \le n-1)$ *order, having at most a countable number of limit orbits, is an R-set.*

Theorem 5. *At most countable set, having at most countable number of limit frames, is an R-set.*

Theorem 6. *For every countable union of orbits* E *and for every* $\theta \notin E$ *there exists a zero measure* * *R-set* Q *such that* $E \subset Q$ *and* $\theta \notin Q$.

Theorem 7. *There exists a zero measure R-set of the second category in every ball from* $\theta(\mathbb{R}^n)$.

Theorem 8. *There exists a zero measure perfect R-set.*

Theorems 2–8 follow from Theorem 1 and the following statement.

Theorem 9. *For every sequence* $\{\varepsilon_m\}$ *with* $\varepsilon_m > 0$ *and* $\lim_{m\to\infty} \varepsilon_m = 0$ *there exists a sequence* $\{\delta_m\}$ *with* $0 < \delta_m < \varepsilon_m$ $(m \in \mathbb{N})$ *such that for every sequence of orbits* $\{E_m\}$ *and sets* $\{T_m\}$, *where* $T \in T(E_m, \varepsilon_m)$ $(m \in \mathbb{N})$, *there exists an R-set* E *such that*

$$\overline{\lim_{m\to\infty}} V[E_m, \delta_m] \subset E \subset \overline{\lim_{m\to\infty}} \Big[\theta(\mathbb{R}^n) \backslash T_m\Big].$$

The following generalization of Theorem 9 is also true.

Theorem 10. *For every function* $f \ge 0$, $f \in L\backslash L(\ln^+ L)^{n-\nu}(G^n)$ *where* $1 \le \nu \le n-1$, *and for a sequence* $\{\varepsilon_m\}$ *with* $\varepsilon_m > 0$ $(m \in \mathbb{N})$ *and* $\lim_{m\to\infty} \varepsilon_m = 0$ *there exists a sequence* $\{\delta_m\}$ *with* $0 < \delta_m < \varepsilon_m$ $(m \in \mathbb{N})$ *such that for every sequence of sets* $\{V[E_i, \delta_{m_i}]\}$, *where* $E_i (i \in \mathbb{N})$ *are orbits and* $m_1 < m_2 < \dots$, *and for every sequence of sets* $\{T_i\}$, *where* $T_i \in T(E_i, \varepsilon_i)$ $(i \in \mathbb{N})$, *there exists an equimeasurable with* f *function* $g : \mathbb{R}^n \to \mathbb{R}$, $\operatorname{supp} g \subset G^n$, *such that*

(1) *for every* $\theta \in \overline{\lim_{i\to\infty}} V[E_i, \delta_{m_i}]$

$$\overline{D}_{I(\nu)(\theta)}\Big(\int g, x\Big) = \infty \quad a.e. \text{ on } G^n;$$

(2) *for every* $\theta \notin \overline{\lim_{i\to\infty}} [\theta(\mathbb{R}^n) \backslash T_{m_i}]$

$$\int g \quad \text{is strongly differentiable along} \quad \theta;$$

(3) *for every translation invariant density basis* B *in* $\mathbb{R}^n$ *the set*

$$\Big\{x \in G^n : \overline{D}_B\Big(\int g, x\Big) > g(x)\Big\}$$

is either of zero or full measure in G^n.

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*The question is on Haar's measure in $\theta(\mathbb{R}^n)$.

The results analogous to those given above for the two-dimensional case have been proved in [5].

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Author's address:

A. Tsereteli Kutaisi State University
55, Tamar Mephe St., Kutaisi 384000
Georgia