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Criteria of the Boundedness and Compactness for the Riemann-Liouville Type Discrete Operators

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In the present note we establish necessary and sufficient conditions governing the boundedness and compactness for the Riemann-Liouville type weighted discrete operator

$$(I_{\alpha,\sigma}\beta)_n = v_n \sum_{k=1}^{n} \Big(n^\sigma - k^\sigma + 1\Big)^{\alpha-1}\beta_k, \quad n \geq 1,$$

from $l_p(\mathbb{N})$ to $l_q(\mathbb{N})$, where $1 < p \leq q < \infty$, $\alpha > 1/p$, $\sigma \in \mathbb{N}$ and $\{v_n\}$ is a sequence on $\mathbb{N}$. The two-sided estimates of Schatten-von Neumann norms for the operator $I_{\alpha,\sigma}$ are also obtained.

The boundedness and compactness criteria for the weighted Riemann-Liouville operator

$$R_\alpha f(x) = v(x)\int_0^x (x-y)^{\alpha-1} f(y)\,dy, \quad x > 0,$$

from $L^p(R_+)$ to $L^q(R_+)$ ($1 < p,\ q < \infty,\ \alpha > 1/p$) were derived in [1] (see [2] for $p = q = 2$ and $\alpha > 1/2$, and also [3]).

Analogous results for the Volterra type integral operators involving the Erdelyi-Köber weighted operators

$$J_{\alpha,\sigma} f(x) = v(x)\int_0^x (x^\sigma - y^\sigma)^{\alpha-1} f(y)\,dy, \quad x > 0, \quad \sigma > 0, \quad \alpha > 1/p,$$

are obtained in [4] and [5], and necessary and sufficient conditions ensuring two-weight inequalities for the Hardy operator

$$Hf(x) = \int_0^x f(y)\,dy$$

are found in [6–8]. The analogous problem concerning the operator R_α for $\alpha > 1$ is studied in [9–10]. For the boundedness weighted criteria of the discrete Hardy transform $(H\beta)_n = \sum_{k=1}^{n} \beta_k$ the reader can be referred to [11–12] (for the weighted Riemann-Liouville discrete operator $(T_\alpha\beta)_n = \sum_{k=1}^{n}(n-k+1/2)^{\alpha-1}\beta_k$, see [3]). Finally, it should be mentioned that the two-weight (p,q) type $1 < p < q < \infty$ inequalities for integral operators with positive kernels can be found in [13].

The following statements hold:

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Theorem 1. *Let $1 < p \le q < \infty$, $\sigma \in \mathbb{N}$, $\alpha > 1/p$. Then the following conditions are equivalent:*

(i) $I_{\alpha,\sigma}$ *is bounded from* $l_p(\mathbb{N})$ *to* $l_q(\mathbb{N})$*;*

(ii)

$$B \equiv \sup_{m\ge 1} B(m) \equiv \sup_{m\ge 1}\Big(\sum_{k=m}^{+\infty}|v_k|^q k^{\sigma(\alpha-1)q}\Big)^{1/q} m^{1/p'} < \infty;$$

(iii)

$$B_1 \equiv \sup_{j\ge 0} B_1(j) \equiv \sup_{j\ge 0}\Big(\sum_{k=2^j}^{2^{j+1}}|v_k|^q k^{\sigma(\alpha-1)q+q/p'}\Big)^{1/q} < \infty.$$

Moreover, $\|I_{\alpha,\sigma}\| \approx B \approx B_1$*.*

Theorem 2. *Let $1 < p \le q < \infty$, $\sigma \in \mathbb{N}$, $\alpha > 1/p$. Then the following statements are equivalent:*

(i) $I_{\alpha,\sigma}$ *is compact from* $l_p(\mathbb{N})$ *to* $l_q(\mathbb{N})$*;*

(ii) $B < \infty$ *and* $\lim\limits_{n\to+\infty} B(n) = 0$*;*

(iii) $B_1 < \infty$ *and* $\lim\limits_{j\to+\infty} B_1(j) = 0$.

Let H be a separable Hilbert space and let $\sigma_\infty(H)$ be a class of all compact operators $T : H \to H$ which forms an ideal in the normed algebra B of all bounded linear operators in H.

To construct a Schatten-von Neumann ideal $\sigma_p(H)$ $(0 < p < \infty)$ in $\sigma_\infty(H)$, we use a sequence of singular numbers $s_j(T) \equiv (\lambda_j(T^*T))^{1/2}$, where the eigenvalues $\lambda_j(T^*T)$ of the operator T^*T are non-negative, repeated according to their multiplicity and arranged in decreasing order. A Schatten–von Neumann quasi-norm (a norm if $1 \le p < \infty$) is defined as follows:

$$\|T\|_{\sigma_p(H)} \equiv \Big(\sum_{j=0}^{+\infty} s_j^p(T)\Big)^{1/p}, \quad 0 < p < \infty,$$

with the usual modification if $p = \infty$. For the definition and properties of Schatten–von Neumann ideals we refer to [14].

The two-sided estimates for the Schatten norms of the weighted Hardy operator were established in [15–16] (see also [17]). The analogous problems for the Riemann-Liouville operator R_α are solved in [2] for $\alpha > 1/2$ and in [18] for $\alpha > 1$ in the case of two weights. Necessary and sufficient conditions for the Volterra type integral operators involving e.g., the Riemann-Liouville operators $R_\alpha(\alpha \in (1/2, 1), \sigma > 0)$ and the Erdelyi-Köber operator $J_{\alpha,\sigma}$ $(\alpha \in (1/2, 1), \sigma > 0)$ to belong to the classes $\sigma_p(L^2(R_+))$, were obtained in [19–20].

For the operator $I_{\alpha,\sigma}$ we have the following

Theorem 3. *Let $2 \le p < \infty$, $\alpha > 1/2$ and $\sigma \in \mathbb{N}$. Then $I_{\alpha,\sigma} \in \sigma_p(l_2(\mathbb{N}))$ if and only if $\{D_n\} \in l_2(\mathbb{Z}_+)$, where*

$$D_n = \Big(\sum_{k=2^n}^{2^{n+1}} v_k^2 k^{\sigma(\alpha-1)2+1}\Big)^{1/2},$$

and $\mathbb{Z}_+ \equiv \mathbb{N}\cup\{0\}$. Moreover, there exist positive constants b_1 and b_2 such that

$$b_1\|D_n\|_{l_p(\mathbb{Z}_+)} \le \|I_{\alpha,\sigma}\|_{\sigma_p(l_2(\mathbb{N}))} \le b_2\|D_n\|_{l_p(\mathbb{Z}_+)}.$$

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