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To the Categorial Saks Theorem

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Let n, $n \geq 2$ be a fixed natural number and $E_n = [0,1]^n$.

Let σ denote a family of n mutually orthogonal straight lines $\{\sigma_1, \dots, \sigma_n\}$ which intersect at the origin. A set of all such families will be denoted by $\Gamma(R^n)$ and its elements will be called directions. If $\sigma \in \Gamma(R^n)$ is an arbitrary fixed direction, then by $B_{n\sigma}(x)$ we denote a family of all those n-dimensional containing the point x rectangles whose sides are parallel to straight lines from σ. In case σ is a standard direction, the use will be made of the notation $B_n(x) \equiv B_{n\sigma}(x)$.

Let $I = \prod\limits_{k=1}^{n} (a_k, b_k)$ be an arbitrary n-dimensional rectangle and let $(I)^k \equiv (a_k, b_k)$. Assume, moreover, that $\{k_1, \dots, k_n\}$ is a rearrangement of the set $\{1, 2, \dots, n\}$ such that $|(I)^{k_1}| \geq \dots \geq |(I)^{k_n}|$, and suppose $M_i(I) = |(I)^{k_i}|$, $i = \overline{1,n}$.

Let $R \in B_{n\sigma}(x)$ and let I be that n-dimensional interval by whose twisting with respect to the center of symmetry one can obtain a rectangle R. By $M_i(R)$ we denote respectively the values $M_i(I)$.

It is known ([1]) that if $f \in L(\log^+ L)^{n-1}(E_n)$, then

$$\lim_{\substack{\delta(I)\to 0\\ I\in B_n(x)}} \frac{1}{|I|} \int\limits_I f = f(x), \quad \text{a.e.} \quad x \in E_n.$$

S. Saks [2] has stated that for almost all (in the sense of Baire) categories of summable functions $f \in L(E_n)$ the integral means $1/|I| \int_I f$ diverge unboundedly to $+\infty$ everywhere. Just for almost all $f \in L(E_n)$ the relation

$$\limsup_{\substack{\delta(I)\to 0\\ I\in B_n(x)}} \frac{1}{|I|} \int\limits_I f = +\infty,$$

is fulfilled almost everywhere on E_n.

Strengthening the results obtained by J. Marstrand [3] and A. Stokolos [4], L. Melero [5] has established that if $\Psi(t)$ is the Orlicz function ([6]) such that $\Psi(t) = 0(t \lg^{n-1} t)$, $t \uparrow \infty$ and satisfies the Δ_2 condition, then for almost all functions of Orlicz space $\Psi^*(L)(E_n)$ the means $1/|R| \int\limits_R f$ diverge unboundedly to $+\infty$ in all directions almost everywhere: the relation

$$\limsup_{\substack{\delta(R)\to 0\\ R\in B_{n\sigma}(x)}} \frac{1}{|R|} \int\limits_R f = +\infty$$

is fulfilled almost everywhere on E_n for every fixed direction σ.

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Let us introduce the function $\omega(t_1,\dots,t_{n-1}) : (0,1)^{n-1} \to (0,+\infty)$,

$$\omega(t_1,\dots,t_{n-1}) = \prod_{i=1}^{n-1} \lg(1/t_i).$$

Strengthening Saks's result, Karagulyan [7] has obtained new in principle result by noting that for almost all (in a sense of Baire categories) summable functions $f \in L(E_n)$ the integral means $1/|I| \int_I f$ diverge to $+\infty$ with the maximal rate, or more precisely, for almost all $f \in L(E_n)$ the relation

$$\limsup_{\substack{\delta(I)\to 0\\ I\in B_n(x)}} \frac{1}{|I|\omega(|(I)|^1,\dots,|(I)^{n-1}|)} \int\limits_I f = +\infty$$

is fulfilled almost everywhere on E_n.

Assume that $v(t) : (0,1) \to (0,+\infty)$ is a nonincreasing function such that

$$v(t) = o(\lg(1/t)), \quad t \to 0.$$

Consider the function $\nu(t_1,\dots,t_{n-1}) : (0,1]^{n-1} \to (0,+\infty)$,

$$\nu(t_1,\dots,t_{n-1}) = \prod_{i=1}^{n-1} v(t_i).$$

It turns out that for almost all summable functions $f \in L(E_n)$ the integral means $1/|R| \int\limits_R f$ diverge to $+\infty$ in all directions with close to the maximal rate.

Theorem 1. *For almost all (in a sense of Baire categories) functions $f \in L(E_n)$ the relation*

$$\overline{\lim_{\substack{R\in B_{n\sigma}(x)\\ \delta(R)\to 0}}} \frac{1}{|R|\nu(\frac{M_n(R)}{M_1(R)},\dots,\frac{M_n(I)}{M_{n-1}(R)})} \int\limits_R f = +\infty$$

is fulfilled almost everywhere on E_n for every fixed direction $\sigma \in \Gamma(R^n)$.

Let

$$v^*(t) = \lg(2/t)\lg\lg(2/t)\dots(\lg\lg\dots\lg(2/t))^p, \quad p > 1 \ \ t \in (0,1).$$

Consider the function $\nu^*(t_1,\dots,t_{n-1}) : (0,1)^{n-1} \to (0,+\infty)$,

$$\nu^*(t_1,\dots,t_{n-1}) = \prod_{i=1}^{n-1} v^*(t_i).$$

The fact that the divergence of integral means in Theorem 1 has the rate close to maximal follows from the following result [8]: for any function $f \in L(E_n)$,

$$\overline{\lim_{\substack{I\in B_n(x)\\ \delta(I),r(I)\to 0}}} \frac{1}{|I|\nu^*(\frac{M_n(I)}{M_1(R)},\dots,\frac{M_n(I)}{M_{n-1}(I)})} \int\limits_I f = 0 \quad \text{a.e.},$$

where $r(I) = M_n(I)/M_1(I)$.

Let now $n \geq 3$. Denote by $\Theta(R^n)$ a set of all those directions of σ from $\Gamma(R^n)$ for which a constituent line σ_i coincides with the $0x_j$-axis if only for one pair $i, j, 1 \leq i, j \leq n$.

As the following theorem shows, for some functions the integral means diverge with the maximal rate in certain directions only.

Theorem 2. *Let $n \geq 3$. There exists a nonnegative function $f \in L(\mathbb{R}^n)$ such that*
(1) *for every direction $\sigma \in \Theta(R^n)$*

$$\overline{\lim_{\substack{R\in B_{n\sigma}(x)\\ \delta(R)\to 0}}} \frac{1}{|R|\nu(\frac{M_n(R)}{M_1(R)},\dots,\frac{M_n(R)}{M_{n-1}(R)})}\int\limits_R f = +\infty, \quad a.e.;$$

(2) *for every direction $\sigma \in \Gamma(R^n)\backslash\Theta(R^n)$*

$$\lim_{\substack{R\in B_{n\sigma}(x)\\ \delta(R)\to 0}} \frac{1}{|R|}\int\limits_R f = f(x) \quad a.e.$$

Consider now the case $n = 2$.

Let Orlicz's function $\Phi(t) = t\phi(t)$ satisfy the condition $\phi(t) = o(\lg t)$, $t \uparrow \infty$. Assume that $\mathbf{w}(t) : (0,1] \to (0,+\infty)$ is a nonincreasing function such that

$$\mathbf{w}(t) = 0\Big(\frac{\lg(1/t)}{\phi(1/t)}\Big), \quad t \to 0.$$

Consider the function

$$\mathbf{w}^*(t) = \frac{v^*(1/t)}{\phi(1/t)}, \quad t \to (0,1).$$

It has been mentioned [9] that if $\phi(t)$ is upper convex, then for any function $f \in \Phi(L)(E_2)$ (this result is formulated in somewhat different but equivalent manner)

$$\lim_{\substack{R\in B_2(x)\\ \delta(I),r(I)\to 0}} \frac{1}{|I|\mathbf{w}^*(\frac{M_2(R)}{M_1(R)})}\int\limits_I f = 0, \quad \text{a.e.}$$

It turns out that for almost all (in a sense of Baire categories) functions $f \in \Phi^*(L)(E_2)$ the integral means $1/|R|\int\limits_R f$ diverge to $+\infty$ with the rate close to maximal in all directions almost everywhere.

Theorem 3. *If $\phi(t)$ satisfies the Δ_2 condition, then for almost all (in a sense of Baire categories) functions f of Orlicz's space $\Phi(L)(E_2)$ the relation*

$$\overline{\lim_{\substack{R\in B_{n,\sigma}(x)\\ \delta(R)\to 0}}} \frac{1}{|R|\mathbf{w}(\frac{M_2(R)}{M_1(R)})}\int\limits_R f = +\infty,$$

is fulfilled almost everywhere on E_2 for every fixed direction $\sigma \in \Gamma(R^2)$.

The following statement strengthening in particular the result of [10] holds.

Theorem 4. *Given a sequence of directions $(\sigma_l)_{l=1}^{\infty}$, there exists a nonnegative function $f \in \Phi(L)(R^2)$ such that*
(1) *for every direction σ_l*

$$\overline{\lim_{\substack{R\in B_{2\sigma_l}(x)\\ \delta(R)\to 0}}} \frac{1}{|R|\mathbf{w}(\frac{M_2(R)}{M_1(R)})}\int\limits_R f = +\infty, \quad a.e. \quad x \in E_2;$$

(2) *for almost all directions $\sigma \neq \sigma_l$, $l = 1, 2, \dots$.*

$$\lim_{\substack{R\in B_{2\sigma}(x)\\ \delta(R)\to 0}} \frac{1}{|R|}\int\limits_R f = f(x), \quad a.e. \quad x \in E_2.$$

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