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T. Kandelaki

Algebraic and Topological Bivariant KK-Theories of C^*-Algebras and their isomorphism

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Preliminary. Let A be a separable and B a σ-unital trivially graded C^*-algebras (real or complex) with actions of a fixed compact group G. In the paper [5], in the complex case, the author constructed the C^*-category $\mathrm{Rep}(A;B)$ and it was proved that the topological K-theory groups are isomorphic to the Kasparov equivariant KK-theory groups up to a dimension shift. Thus the formula

$$KK_n^t(A,B) = K_{n+1}^t(\mathrm{Rep}(A,B)) \tag{1}$$

is a good definition of the topological equivariant KK-theory.

In the same way as in [5], for a real or complex algebra case, one can construct the C^*-category $\mathrm{Rep}(A,B)$. As it was pointed out in [5], one offers to define algebraic and topological bivariant KK-theories by the formula

$$KK_n^a(A,B) = K_{n+1}^a(\mathrm{Rep}(A,B)) \tag{2}$$

and (1) respectively, where K_n^a and K_n^t are variants of algebraic and topological K-theories respectively, which will be considered below.

Our main result says that algebraic and topological KK-theories are essentially isomorphic to the Kasparov bivariant KK-theory. This result opens new look at KK-theory and its applications.

Below we shall describe a way of the proof.

On the algebraic and topological K-theories. Our definition is a modification of some arguments from [1], [3].

Let a and a' be objects in an additive C^*-category A and let I be a closed ideal. We say that $a \leq a'$ if there exists a morphism $s : a \to a'$ such that $s^*s = 1_a$. Denote by $L(a)$ (resp. $I(a)$) the C^*-algebra $\hom_A(a,a)$ (resp $\hom_I(a,a)$). We have a correctly defined inductive system of abelian groups $\{K_*^a(L(a)), \sigma_{aa'}\}_a$ and $\{K_*^t(L(a)), \sigma_{aa'}\}_a$, where K_*^a (resp. K_*^t) are usual algebraic (resp. topological) K-theory groups of the algebra (resp. C^*-algebra) $L(a)$. We suppose that $K_*^a(A) = \lim_a K_*^a(L(a))$ (resp. $K_*^t(A) = \lim_a K_*^t(L(a))$). So defined algebraic K-groups are essentialy isomorphic to the Quillen's K-groups $K_*^Q(A)$ (with respect to the class of all split short exact sequences), when $* \geq 0$. One can generalize this definition for an ideal I which does not depend on the choice of the enveloping additive C^*-category of I. These K-theories have excision property generalizing the analogous property of algebraic and topological K-theories of C^*-algebras, [7]. There is an essential transformation $\Lambda_* : K_*^a \to K_*^t$ which is a generalization of the classical natural transformation between algebraic and topological K-theories.

The isomorphism in dimension zero. According to some arguments from [5], One can to prove the isomorphisms

$$KK_0^a(A,B) = KK_0(A,B) \ \ \textit{and} \ \ KK_0^t(A,B) = KK_0(A,B). \tag{3}$$

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where $KK_0(A,B)$ is the Kasparov KK-theory group.

Higson's homotopy invariant theorem. According to [4], we can formulate Higson's homotopy invariant theorem in the following form.

Any stable and split exact covariant (or contravariant) functor from the category (real or complex) separable C^*-algebras to the category of abelian groups is homotopy invariant.

On the excision property of bivariant KK-theories. The following excision property is a generalization and modification of some arguments from [4], [3], [6].

Let $0 \to J \to A \to A/J \to 0$ be an exact sequence of separable (real or complex)C^*-algebras. Then for any σ-unital trivially graded C^*-algebra B (real or complex) there exists a long exact sequence of algebraic (and topological) bivariant KK-theory groups

$$\cdots \to KK^a_{n+1}(J,B) \to KK^a_n(A/J,B) \to KK^a_n(A,B) \to KK^a_n(J,B) \to KK^a_{n-1}(A/J,B) \to \cdots$$

and

$$\cdots \to KK^t_{n+1}(J,B) \to KK^t_n(A/J,B) \to KK^t_n(A,B) \to KK^t_n(J,B) \to KK^t_{n-1}(A/J,B) \to \cdots$$

On the Cuntz-Bott periodicity. Here our approach is based on the Cuntz's work [2] on Bott periodicity.

Let $\{h^n\}_{n\in Z}$ be cohomology functors on the category of real or complex C^*-algebras which have the stability property. Then

$$\begin{gathered} h^{n+1}(A) \approx h^n(SA) \quad \text{in the complex case,} \\ h^{n+1}(A) \approx h^n(A \otimes C_0^R(R) \otimes C_0^R(iR)) \quad \text{in the real case,} \end{gathered} \tag{4}$$

where $C_0^R(iR)$ is the algebra of fixed elements of the complex algebra $C_0(R)$ relative to the complex conjugation $c(f)(s) = \bar{f}(-s)$.

Stability property of bivariant KK-theories. Let A and B be as above. Let $\sigma : A \to A \otimes \mathcal{K}$ be a homomorphism given by a map $a \mapsto a \otimes p$, where p is a rank one projection in $\mathcal{K}$. Then the induced homomorphism

$$\sigma^* : KK^a_*(A \otimes \mathcal{K}, B) \to KK^a_*(A,B) \tag{5}$$

is the isomorphism.

The main theorem. From the above results one deduces the following.

Theorem 1. *Let A be a separable and B be a σ-unital trivially graded C^*-algebras* (*real or complex*) *with actions of a fixed compact group G. Then the natural homomorphism*

$$\Lambda_* : KK^a_*(A,B) \to KK^t_*(A,B) \approx KK_*(A,B)$$

is the isomorphism, where $KK_*(A,B)$ are the Kasparov KK-groups.

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Author's address:
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., 380093, Tbilisi
Georgia