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On one Consequence of the Theorem on Weights

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Let Γ be the Jordan line and $\rho(t)$ be a positive measurable function on Γ. As usual, we denote by S_Γ a singular operator

$$\big(S_\Gamma\varphi\big)(\tau)=\frac{1}{\pi i}\int\limits_\Gamma\frac{\varphi(t)dt}{t-\tau},$$

where integration is understood in a sense of the Cauchy principal value. If the operator S_Γ is bounded in $L_p(\Gamma)$, $p>1$, then we write $\Gamma\in R$, i.e., Γ is a regular curve.

When solving the boundary value problem of linear conjugation it is very important to know some information about the functions $\rho(t)$ for which the operator $\rho S\rho^{-1}$ is bounded in $L_p(\Gamma)$, $p>1$ i.e.,

$$\Big\|\rho S_\Gamma\rho^{-1}\varphi\Big\|_{L_p(\Gamma)}\leq c\|\varphi\|_{L_p(\Gamma)}, \tag{1}$$

where $\Gamma\in R$, and c is an independent of φ constant. In this case the function $\rho(t)$ is called a weight and written $\rho\in W_\rho(\Gamma)$. Obvionsly, we can assume that $\rho(t)>0$ almost everywhere. The aim of the present work is to show one statement dealing with the weight function $\rho(t)$, which is the consequence of the well-known theorem on weights ([1], p. 52, Theorem 4.2). It makes no difficulty to obtain this consequence, but in our opinion it is interesting by its simple and clear statement and by the fact that it can be used for the solution of the boundary value problem both for closed and unclosed contours. (In the case of closed contours we have done this for some class of lines and coefficients. The case of unclosed lines will be considered in the near future by using the result of the present paper).

In the sequel, we will need some notations. As in our previous works by Γ_{ab} we denote a continuous with ends a and b and directed from a to b line. When it is necessary to note that the points a and b belong or do not belong to the arc, the use will be made of the notation $\Gamma_{(a,b)}$, $\Gamma_{[a,b]}$,$\Gamma_{[a,b)}$, $\Gamma_{(a,b]}$.

If t is an inner point on Γ, then we denote by Γ_t an open arc neighborhood of the point t on Γ, i.e., if $t=t(s)$, $0\leq s\leq\nu\Gamma$ ($\nu\Gamma$ is the length of Γ) is the equation of Γ in the arc coordinates, then $\Gamma_t\equiv\Gamma_{(t(s-h_1),\,t(s+h_2))}$, $0<h_i<\nu\Gamma$, $i=1,2$. If Γ is an unclosed line, then in what follows the end points will be assumed to belong to Γ, i.e., $\Gamma=\Gamma_{[a,b]}$, and at the points $t=a$ and $t=b$ we will have half-open intervals $\Gamma_a=\Gamma_{[a,t(h))}$, $\Gamma_b=\Gamma_{(t(\nu\Gamma-h),\,b]}$, $0<h<\nu\Gamma$.

In the sequel we will need the well-known class $A_p(\Gamma)$. To define this class for $\Gamma\in R$, we denote by $\Gamma(z;r)$, as in [1], a set of points on Γ for which $|z-t|\leq r$ and by $\nu\Gamma(z,r)$ we denote an arc measure of that set. In the sequel, we will assume that $w(t)>0$ almost everywhere.

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They say that $w(t) \in A_p(\Gamma)$, $1 < p < \infty$ if

$$\sup_{\substack{z\in\Gamma \\ 0<r<\operatorname{diam}\Gamma}} \frac{1}{\nu\Gamma(z,r)} \int\limits_{\Gamma(z,r)} w(t)\,dt \Big(\frac{1}{\nu\Gamma(z,r)} \int\limits_{\Gamma(z,r)} w^{1-q}(t)\,dt \Big)^{p-1} < \infty, \qquad (2)$$

$$q = p(p-1)^{-1}.$$

The basic theorem we are relying on is the following ([1], p. 52).

Theorem 1^0. *If $1 < p < \infty$, $\Gamma \in R$ then to fulfil the inequality*

$$\int\limits_{\Gamma} |S_\Gamma f|^p w(t)\,d\nu \le c \int\limits_{\Gamma} |f(t)|^p w(t)\,d\nu, \qquad (3)$$

where c does not depend on f, it is necessary and sufficient that $w \in A_p(\Gamma)$.

From the above theorem follows the following assertion.

Theorem 1. *If Γ is the Jordan curve, $\Gamma \in R$, and for every point $t \in \Gamma$ there exists an arc neighborhood Γ_t such that $\rho \in W_p(\Gamma_t)$, then $\rho \in W_p(\Gamma)$ (Recall that if Γ is an unclosed line, then $\Gamma = \Gamma_{[a,b]}$).*

It should be noted that if $\rho(t) = [w(t)]^{1/p}$, then condition (2) can be written as

$$\sup_{\substack{z\in\Gamma \\ 0<r<\operatorname{diam}\Gamma}} \Big(\frac{1}{\nu\Gamma(z,r)} \int\limits_{\Gamma(z,r)} \rho^p(t)\,d\nu \Big)^{1/p} \Big(\frac{1}{\nu\Gamma(z,r)} \int\limits_{\Gamma(z,r)} \rho^{-q}(t)\,d\nu \Big)^{1/q} < \infty, \qquad (2')$$

and inequality (3) will be written in terms of (1).

Denote further

$$B\Big(\Gamma(z,z), \rho(t), p\Big) \equiv \Big(\frac{1}{\nu\Gamma(z,r)} \int\limits_{\Gamma(z,r)} \rho^p(t)\,d\nu \Big)^{1/p} \Big(\frac{1}{\nu\Gamma(z,r)} \int\limits_{\Gamma(z,r)} \rho^{-q}(t)\,d\nu \Big)^{1/q}.$$

Theorem 1^0 in this notation can be formulated as follows:

The necessary and sufficient condition for inequality (1) to be fulfilled, is the condition

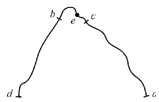
$$\sup_{\substack{z\in\Gamma \\ 0<r<\operatorname{diam}\Gamma}} B\Big(\Gamma(z,r); \rho(t); p\Big) < \infty. \qquad (4)$$

Naturally, this theorem is the same Theorem 1^0, but written in some other notation.

To prove Theorem 1 we will first prove two lemmas.

Lemma 1. *If Γ is the Jordan curve, $\Gamma = \Gamma_{[a,b]} \cup \Gamma_{[c,d]}$, $a \ne d$ and $\Gamma_{(a,b)} \cap \Gamma_{(c,d)} \ne \varnothing$, then from $\rho(t) \in W_p(\Gamma_{a,b}) \cap L_p(\Gamma_{c,d})$, $p > 1$, it follows that $w \in W_p(\Gamma)$.*

Proof. Let us take some point $e \in \Gamma_{(c,b)}$ and denote



$$\delta_1 = \inf |z-t|, \quad \text{for } z \in \Gamma_{[a,e]}, \ t \in \Gamma_{[b,d]},$$
$$\delta_2 = \inf |z-t|, \quad \text{for } z \in \Gamma_{[e,d]}, \ t \in \Gamma_{[a,c]},$$
$$\delta = \frac{1}{2}\min(\delta_1, \delta_2).$$

Let $z \in \Gamma_{[a,e]}$. Then $\Gamma(z,\delta) \subset \Gamma_{[a,b]}$. Owing to the fact that $\rho(t) \in W_p(\Gamma_{ab})$, by Theorem 1^0 we have

$$\sup_{\substack{z\in\Gamma_{[a,e]}\\ 0<r<\delta}} B\Big(\Gamma(z,r);\rho(t);p\Big) < \infty. \tag{5}$$

However, if $z \in \Gamma_{[e,d]}$, then $\Gamma(z;\delta) \subset \Gamma_{[c,d]}$, and since $\rho \in W_\rho\Gamma_{c,d}$, by Theorem 1^0 we have

$$\sup_{\substack{z\in\Gamma_{[e,d]}\\ 0<r<\delta}} B\Big(\Gamma(z,r);\rho(t);p\Big) < \infty. \tag{6}$$

Inequalities (5) and (6) together result in

$$\sup_{\substack{z\in\Gamma\\ 0<r<\delta}} B\Big(\Gamma(z,r);\rho(t);p\Big) < \infty. \tag{7}$$

Let now $\delta \le r < \operatorname{diam}\Gamma$. Then $\nu\Gamma(z,r) \ge \delta$. Moreover, $\rho \in L_p(\Gamma_{ab})$ and $\rho \in L_p(\Gamma_{cd})$, whence $\rho \in L_p(\Gamma)$. Similarly, $\rho^{-1} \in L_q(\Gamma)$. Therefore

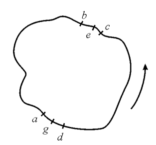
$$\sup_{\substack{z\in\Gamma\\ \delta\le r<\operatorname{diam}\Gamma}} B\Big(\Gamma(z,r);\rho(t);p\Big) < \frac{1}{\delta}\|\rho\|_{L_p(\Gamma)}\|\rho^{-1}\|_{L_q(\Gamma)}. \tag{8}$$

Inequalities (7) and (8) together give (4), whence again by Theorem 1^0 we obtain $\rho \in W_p(\Gamma)$. □

The second lemma deals with the case of a simple closed contour Γ. In this case the positive direction Γ leaves as a result of the circuit the finite domain D^+ bounded by Γ on the left. If the arc Γ_{ab} lies on the closed line Γ, then the direction on Γ_{ab} coincides with the positive direction on Γ. Then $\Gamma = \Gamma_{[a,b]} \cup \Gamma_{[b,a]}$ for any $a \in \Gamma$, $b \in \Gamma$.

Lemma 2. *If Γ is the closed Jordan line, $\Gamma \in R$, $\Gamma = \Gamma_{ab} \cup \Gamma_{cd}$ and $\Gamma_{(a,b)} \cap \Gamma_{(c,d)} = \Gamma_{(c,b)} \cup \Gamma_{(a,d)}$, $\Gamma_{(c,b)} \ne \varnothing$, $\Gamma_{(a,d)} \ne \varnothing$ then from $\rho \in W_p(\Gamma_{ab}) \cap W_p(\Gamma_{cd})$ follows $\rho \in W_p(\Gamma)$.*

Pr e points $e \in \Gamma_{(c,b)}$, $g \in \Gamma_{(a,d)}$ and introduce the notation



$$\delta_1 = \inf|z-t| \ \text{ for } \ z \in \Gamma_{[e,g]},\ \ t \in \Gamma_{[d,c]}$$
$$\delta_2 = \inf|z-t| \ \text{ for } \ z \in \Gamma_{[g,e]},\ \ t \in \Gamma_{[b,a]}$$
$$\delta = \frac{1}{2}\min(\delta_1,\ \delta_2).$$

It can be easily seen that if $z \in \Gamma_{[e,g]}$, then $\Gamma(z,\delta) \subset \Gamma_{[c,d]}$, and since $\rho \in W_p(\Gamma_{cd})$, we have

$$\sup_{\substack{0<z<\delta\\ z\in\Gamma_{[eg]}}} B\Big(\Gamma(z,r);\rho(z);p\Big) < \infty. \tag{9}$$

However, if $z \in \Gamma_{[g,e]}$, then $\Gamma(z;\delta) \in \Gamma_{[ab]}$, and as above

$$\sup_{\substack{0<z<\delta\\ z\in\Gamma_{[g,e]}}} B\Big(\Gamma(z,r);\rho(z);p\Big) < \infty. \tag{10}$$

Inequalities (9) and (10) together result in (7) for Γ such that mentioned in Lemma 2. Repeating concluding reasoning of Lemma 1, we obtain (4), which by virtue of Theorem 1^0 gives $\rho \in W_p(\Gamma)$. □

Proof of Theorem 1. By the condition of the theorem, to every inner point $t \in \Gamma$ there corresponds an open interval, while to the end points (if Γ is unclosed) there correspond half-open intervals which cover Γ. It is obvious that from that covering one can distinguish a finite covering of Γ by the arcs $\Gamma_{a_k b_k}$, $k = 1, 2 \dots, n$, for which $\rho \in W_p(\Gamma_{a_k b_k})$. We take from them two arbitrary arcs satisfying the conditions of Lemma 1. According to the lemma, a number of arcs will decrease by one. Moreover, if we get that for some i and j $\Gamma_{a_i b_i} \subset \Gamma_{a_j b_j}$, then we omit $\Gamma_{a_i b_i}$ from a system of arc intervals. Thus, after a finite number of steps we will arrive in the case of unclosed contour at the whole contour, while in the case of a closed contour we will arrive at the situation of Lemma 2, and this proves the theorem. □

References

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