Proc. A. Razmadze Math. Inst. 126(2001), 99–102.

I. Bukhnikashvili

On an Approximate Solution of the Linear Operator Equation by Richardson's Cyclic Method

(Reported on 20.09.2001)

In the Hilbert space, let there be given a linear equation

$$A\varphi = f \tag{1}$$

with a positive definite self-conjugate operator A, whose spectrum of eigen values $\lambda_i(i = 1, 2, \dots)$ is on the segment

$$[m, M], \quad M > m > 0. \tag{2}$$

An approximate solution of equation (1) can be obtained by the cyclic [1] iterative scheme

$$\varphi_s = \varphi_{s-1} - \frac{1}{\gamma_s}(A\varphi_{s-1} - f), \quad \gamma_{n+s} = \gamma_s, \quad s = 1, 2 \dots, \tag{3}$$

where as iterative parameters γ_s $(s = 1, \dots, n)$ are taken zeros of multiplicity k of Chebyshev [2] normed first kind polynomial of order n, raised to the power k and written as

$$P_n^k(x) = \prod_{i=1}^{n} \left(1 - \frac{x}{\gamma_i}\right)^k. \tag{4}$$

After k cycles, using scheme (3) for estimation of the error φ_{kn} upon the approximation to the exact solution φ, we obtain (see [1]) the following inequality:

$$\|\varphi_{kn} - \varphi\| \le \max \left|P_n^k(x)\right| \frac{1}{m} \left\|f\right\|. \tag{5}$$

From this estimation we can see that in order to accelerate the rate of convergence of approximations φ_{kn} to the exact solution φ the Chebyshev polynomial $P_n(x)$ must be taken with minimal module-maxima on the segment (2).

However, instead of the polynomial of type (4) with k-multiple zeros we can consider the polynomial of the same order kn, but with $k + d$ and $k - d$ multiple zeros, $(d < k)$. The polynomial of the above-mentioned type for even n can formally be written in terms of

$$R_{kn}(x) = \prod_{i=1}^{\frac{n}{2}} \left(1 - \frac{x}{a_i}\right)^{k-d} \left(1 - \frac{x}{b_i}\right)^{k+d}. \tag{6}$$

It turns out that if we choose zeros a_i and b_i $(i = 1, \dots, \frac{n}{2})$ of the polynomial $R_{kn}(x)$ such that all $n + 1$ module-maxima are equal between themselves, then under certain additional conditions the inequality

$$\max |R_{kn}(x)| < \max |P_n^k(x)|, \quad x \in [m, M] \tag{7}$$

will be fulfilled.

2000 *Mathematics Subject Classification:* 65J10, 65F15.

Key words and phrases. Linear operator equation, Approximate solution, eiqenvalues, Richardson's cyclic method.

If we need to perform the cyclic iterative process by using scheme (3) for a number of cycles, multiple to k, i.e., equal to lk, inequality (7) with regard for (5) gives every reason to prefer as iterated parameters the zeros of polynomial (6) to those of polynomial (4).

To construct the polynomial of type (6) with aligned module-maxima, it is convenient first to transform it by substituting

$$t = \frac{2x - M - m}{M - m} \tag{8}$$

and replacing (2) by the segment

$$[-1, 1] \tag{9}$$

and then to construct on the segment (9) the polynomial

$$\overline{R}_{2k}(t) = \frac{(t-g)^{k-d}(t-v)^{k+d}}{(t_1-u)^{k-d}(t_1-v)^{k+d}}, \tag{10}$$

which is a particular case of polynomial (6) for $n = 2$ with normalization at the point

$$t_1 = -\frac{p + \frac{1}{p}}{2} = -\frac{M+m}{M-m},$$

where the point with the abscissa t_1 corresponds to that of the normalization $x = 0$ with the transformation (8), and p denotes (see, e.g., [3]) the geometric characteristics of segment (2), which is given by the formula

$$p = \frac{\sqrt{M} + \sqrt{m}}{\sqrt{M} - \sqrt{m}}.$$

In the conditions mentioned above we choose zeros u and v of polynomial (10) so as its three module-maxima to be equal between themselves. If by t_0 we denote a point at which $R'_{2k}(t_0) = 0$, then for the determination of zeros u and v we obtain a system of equations

$$\begin{cases} |\overline{R}_{2k}(-1)| &= |\overline{R}_{2k}(t_0)| \\ |\overline{R}_{2k}(1)| &= |\overline{R}_{2k}(t_0)| \end{cases} \tag{11}$$

which, written in detail, looks as

$$\begin{cases} (1+v)^{k+d}(1+u)^{k-d} &= (\frac{1}{2} + \frac{d}{2k})^{k+d}(\frac{1}{2} - \frac{d}{2k})^{k-d}(v-u)^{2k} \\ (1-v)^{k+d}(1-u)^{k-d} &= (\frac{1}{2} + \frac{d}{2k})^{k+d}(\frac{1}{2} - \frac{d}{2k})^{k-d}(v-u)^{2k}. \end{cases} \tag{12}$$

We can show that the system of equations (12) is solvable by the method of iterations and has a unique solution in the corresponding interval, provided

$$u < v. \tag{13}$$

It should be noted that by introducing new variables $\frac{1+u}{1+v}$ and $\frac{1-v}{1-u}$ (which are positive, and by condition (13) less than unity), respectively, for the first and the second equations of system (12) we can iterate both equations separately and independently from each other, ensuring in both cases the convergence of iterations to the unique solution in the corresponding interval defined by the same condition (13).

Having found the unknowns u and v, with substitution (12) at hand allowing one to pass from polynomial (10) to polynomial (6), we can consider our problem to be fulfilled on segment (2) for the particular case $n = 2$. However, our writing of polynomial (10) makes it possible to solve the problem completely for an arbitrary even $n > 2$ as well. Bearing this in mind and using the already constructed polynomial (10), we take the superposition of the Chebyshev polynomial $\cos\frac{n}{2}\arccos t$ with normalization at the point

$$t_{\frac{n}{2}} = (-1)^{\frac{n}{2}} \frac{p^{\frac{n}{2}} + \frac{1}{p^{\frac{n}{2}}}}{2}$$

(see [3]). As a result, we obtain the polynomial of degree kn in the form

$$\overline{R}_{kn}(t) = \frac{(\cos\frac{n}{2}\arccos t - u)^{k-d}(\cos\frac{n}{2}\arccos t - v)^{k+d}}{(t_{\frac{n}{2}} - u)^{k-d}(t_{\frac{n}{2}} - v)^{k+d}},$$

or, using substitution (8) and getting back to segment (2), we find an unknown polynomial in the form

$$R_{kn}(x) = \frac{(\cos\frac{n}{2}\arccos\frac{2x-M-m}{M-m} - u)^{k-d}(\cos\frac{n}{2}\arccos\frac{2x-M-m}{M-m} - v)^{k+d}}{(t_{\frac{n}{2}} - u)^{k-d}(t_{\frac{n}{2}} - v)^{k+d}}. \quad (14)$$

The table below gives numerical examples for the solvability of equation (1) by means of iterative scheme (3) in case A is the matrix operator. In scheme (3), as iterative parameters are taken zeros of polynomials, respectively, (4) and (14). The first and the second vertical columns present error values of the lknth approximations to the exact solution which are obtained by scheme (3), when as γ_i $(i = 1, \ldots, n)$ are taken zeros, respectively, of polynomials (4) and (14). The third vertical column gives orders of matrices A; the Todd number $\frac{M}{m}$ of the matrix A is given in the fourth column. The fifth column provides us with the values $\frac{d}{k}$, and the sixth with the degree n of the "basis" polynomial. The orders of degrees of polynomials (4) and (14) can be found in the seventh vertical column.

$\Vert\varphi_{lkn} - \varphi\Vert$	$\Vert\varphi_{lkn} - \varphi\Vert$	N	$\frac{M}{m}$	$\frac{d}{k}$	n	lkn
$4 \cdot 10^{-4}$	$8 \cdot 10^{-8}$	80	10020	0,9	8	5600
10^{-5}	$7 \cdot 10^{-10}$	80	10020	0,9	8	7200
$2 \cdot 10^{-5}$	10^{-9}	250	10020	0,9	8	7200
10^{-7}	$2 \cdot 10^{-10}$	250	10020	0,6	8	1120
$3 \cdot 10^{-5}$	$5 \cdot 10^{-7}$	40	10002	0,6	24	2400
$6 \cdot 10^{-5}$	$7 \cdot 10^{-7}$	250	10002	0,6	24	2400
$2 \cdot 10^{-5}$	$5 \cdot 10^{-7}$	62	10002	0,6	8	7200
$2 \cdot 10^{-2}$	$2 \cdot 10^{-4}$	500	10002	0,9	4	720

References

1. V. Lebedev, S. Finogenov, On the order of choice of iteration parameters in Chebyshev cyclic iterative method. (Russian) *Zh. Vychisl. Mat. i Mat. Fiz.* **2**(1971), No. 2, 425–438.

2. I. Berezin, N. Zhitkov, Methods of calculations. (Russian) *Fizmatgiz, Moscow,* 1962.

3. V. Vazov, J. Forsyte, Difference methods of solution of partial differential equations. (Russian) *Izd. Inostr. Liter, Moscow,* 1963.

Author's address:

A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, Aleksidze St., Tbilisi, 380093
Georgia