Proc. A. Razmadze Math. Inst. 123(2000), 162–165

A. Meskhi

ON THE MEASURE OF NON-COMPACTNESS AND SINGULAR NUMBERS FOR THE VOLTERRA INTEGRAL OPERATORS

(Reported on May 4, 2000)

In the present note we deal with the measure of non-compactness and singular numbers of the Volterra integral operator

$$K_v(f)(x) = v(x) \int\limits_0^x k(x,y)f(y)dy, \quad k \geq 0, \quad x \in \mathbb{R}_+,$$

where $\mathbb{R}_+ \equiv (0,\infty)$ and v is a measurable function on $\mathbb{R}_+$. The following definitions are from [1] (see also [2]):

Definition 1. We say that a kernel $k : \{(x,y) : 0 < y < x < \infty\} \to \mathbb{R}_+$ belongs to V ($k \in V$) if there exists a positive constant b_1 such that for all x, y, z with $0 < y < z < x < \infty$ the inequality

$$k(x,y) \leq b_1 k(x,z)$$

holds.

Definition 2. Let $1 < \lambda < \infty$. A kernel k belongs to V_λ ($k \in V_\lambda$) if there exists a positive constant b_2 such that for all x, $x \in \mathbb{R}_+$, the inequality

$$\int\limits_{\frac{x}{2}}^x k^{\lambda'}(x,y)dy \leq b_2 x k^{\lambda'}(x,\frac{x}{2}),$$

is fulfilled, where $\lambda' = \lambda/(\lambda-1)$.

If $k(x,y) = k_1(x-y)$, where k_1 is a positive nonincreasing function on $\mathbb{R}_+$ satisfying the condition

$$\int\limits_0^{\frac{x}{2}} k_1^{\lambda'}(y)dy \leq b_3 x k_1^{\lambda'}(\frac{x}{2}), \quad 1 < \lambda < \infty,$$

with some positive constant b_3, then $k \in V \cap V_\lambda$. In particular, if $k_1(x) = x^{\alpha-1}$, where $\frac{1}{\lambda} < \alpha \leq 1$, then $k(x,y) = k_1(x-y)$ belongs to $V \cap V_\lambda$ (for other examples of kernel k see [1-2]).

Let H be a separable Hilbert space and let $\sigma_\infty(H)$ be the class of all compact operators $T : H \to H$, which forms an ideal in the normed algebra $\mathbb{B}$ of all bounded linear operators in H. To construct a Schatten-von Neumann ideal $\sigma_p(H)$ $(0 < p \leq \infty)$ in $\sigma_\infty(H)$, the sequence of singular numbers $s_j(T) \equiv \lambda_j(|T|)$ $(|T| \equiv (T^*T)^{1/2})$ is used, where the eigenvalues $\lambda_j(|T|)$, are non-negative and are repeated according to their multiplicity and

1991 *Mathematics Subject Classification:* 26A33, 42B15, 47B10, 47G10.

Key words and phrases. Integral operators with positive kernels, ball fractional integrals, boundedness, compactness, weight, measure of non-compactness, singular numbers, Schatten-von Neumann ideals.

arranged in decreasing order. A Schatten-von Neumann quasinorm (norm if $1 \le p \le \infty$) is defined as follows:

$$\|T\|_{\sigma_p(H)} \equiv \Big(\sum_j s_j^p(T) \Big)^{1/p}, \quad 0 < p < \infty,$$

with the usual modification if $p = \infty$. Thus we have that $\|T\|_{\sigma_\infty(H)} = \|T\|$ and $\|T\|_{\sigma_2(H)}$ is the Hilbert- Schmidt norm: $\|T\|_{\sigma_2(H)} = \Big(\int\int |T_1(x,y)|^2 dxdy \Big)^{1/2}$ for an integral operator $Tf(x) = \int T_1(x,y) f(y) dy$. We refer, for example, to [3] for more information concerning with Schatten-von Neumann ideals.

We denote by $L^p_w(\Omega)$, $1 < p < \infty$, the weighted Lebesgue space with respect to a measurable set $\Omega \subseteq \mathbb{R}^n$ and a weight w. If $w \equiv 1$, then we use the notation $L^p_w(\Omega) \equiv L^p(\Omega)$.

Let $v \in L^2_{\overline{k}}([2^n, 2^{n+1}))$ for all $n \in \mathbb{Z}$, where $\overline{k}(x) \equiv k(x, x/2)$. The following statement is true (see [1-2]):

Theorem A. *Let $1 < p \le q < \infty$ and let $k \in V \cap V_p$. Then K_v is bounded from $L^p(\mathbb{R}_+)$ into $L^q(\mathbb{R}_+)$ if and only if $B \equiv \sup_{n \in \mathbb{Z}} B_{pq}(n) < \infty$, where*

$$B_{pq}(n) \equiv \Big(\int_{2^n}^{2^{n+1}} |v(x)|^q x^{q/p'} k^q(x, x/2) dx \Big)^{1/q}.$$

Moreover, $\|K_v\| \approx B$.

Analogous result for the Riemann-Liouville operator

$$R_{\alpha,v} f(x) = v(x) \int_0^x \frac{f(y)dy}{(x-y)^{1-\alpha}},$$

where $\alpha > 1/p$, was derived in [4] (see [5] for $\alpha > 1/2$).

Theorem 1. *Let $1 < p \le q < \infty$ and let $k \in V \cap V_p$. Then K_v acts compactly from $L^p(\mathbb{R}_+)$ into $L^q(\mathbb{R}_+)$ if and only if $B < \infty$ and $\lim_{n \to +\infty} B_{pq}(n) = \lim_{n \to -\infty} B_{pq}(n) = 0$.*

This follows in the same way as Theorem 5 in [1].

Similar theorem for integral operators with positive kernels defined on (a,b) $(-\infty < a < b < \infty)$ was proved in [1]. For the compactness criteria of the operator $R_{\alpha,v}$ defined on $\mathbb{R}_+$, when $\alpha > 1/p$, see [4] (see [5] for $p = 2$).

The necessary and sufficient conditions for weight functions ensuring the boundedness of $R_{\alpha,v}$ from $L^p_w(\mathbb{R}_+)$ into $L^q(\mathbb{R}_+)$ $(1 < p < q < \infty$, $\alpha \in (0,1))$ were found in [6], Chapter 3. Analogous problem for $\alpha > 1$ and $1 < p \le q < \infty$ was solved in [7-8]. The boundedness and compactness two - weight criteria for integral operators with Oinarov (see [9]) kernels (involving the operator $R_{\alpha,v}$ only for $\alpha \ge 1$) was derived in [10].

Theorem 2. *Let $1 < p \le q < \infty$ and let $k \in V \cap V_p$. Suppose that K_v is bounded from $L^p(\mathbb{R}_+)$ to $L^q(\mathbb{R}_+)$. Then there exist positive constants d_1 and d_2 depending only on p, q, b_1 and b_2 (see Definitions 1 and 2) such that*

$$d_1 J \le \operatorname{dist}(K, \mathbb{K}(L^p, L^q)) \le d_2 J,$$

where $\mathbb{K}(L^p, L^q)$ is a space of all compact operators from $L^p(\mathbb{R}_+)$ into $L^q(\mathbb{R}_+)$ and $J = \overline{\lim_{n \to -\infty}} B_{pq}(n) + \overline{\lim_{n \to +\infty}} B_{pq}(n)$.

This statement is proved in the same manner as Theorem 9 from [1].

Analogous problem in the case of Volterra type integral operator defined on (a, b) $(-\infty < a < b < +\infty)$ and for $R_{\alpha,v}$ on $\mathbb{R}_+$ (where $\alpha > 1/p$) was studied in [1]. Two-sided estimates of the distance of the Hardy operator $H_{v,w}f(x) = v(x) \int_0^x fw$ from compact oparators were established in [11] (for the Riemann-Liouville operator of order one and greater and for integral operators with kernels of Oinarov type see [12-13]).

Theorem 3. *Let $2 \le p < \infty$ and let $k \in V \cap V_2$. Then $K_v \in \sigma_p(L^2(\mathbb{R}_+))$ if and only if $\{B_n\} \in l_p(\mathbb{Z})$, where $B_n \equiv B_{22}(n)$. Moreover, there exist positive constants c_1 and c_2 such that $c_1 \|B_n\|_{l_p(\mathbb{Z})} \le \|K_v\|_{\sigma_p(L^2(\mathbb{R}_+))} \le c_2 \|B_n\|_{l_p(\mathbb{Z})}$.*

Two-sided estimates of the Schatten-von Neumann norms for the Riemann-Liouville operator $R_{\alpha,v}$, when $\alpha > 1/2$ and $p > 1/\alpha$, were established in [5] (for $\alpha = 1$ and $p > 1$ see [14]). The similar problems for $\alpha \in \mathbb{N}$ in the case of two weights, when $p \ge 1$, was solved in [15].

Let us now consider the multidimensional case. Let for measurable $f : \mathbb{R}^n \to \mathbb{R}^1$

$$B_u^\alpha(f)(x) = u(x) \int\limits_{|y|<|x|} \frac{(|x|^2 - |y|^2)^\alpha}{|x-y|^n} f(y)dy, \quad \alpha > 0,$$

be its ball fractional integral (see e.g. [16], Section 29 and [17], Chapter 7), where u is a measurable function on $\mathbb{R}^n$ with $u \in L^2(\{2^k < |x| < 2^{k+1}\})$ for all $k \in \mathbb{Z}$. The necessary and sufficient conditions for u guaranteeing the boundedness (compactness) of B_u^α from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ were found in [18]. Here we give two-sided estimates of the measure of non-compactness and Schatten-von Neumann norms for B_u^α.

Let $\mathbb{K}(L^p(\mathbb{R}^n), L^q(\mathbb{R}^n))$ be a set of all compact operators from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. The following statements are valid:

Theorem 4. *Let $1 < p \le q < \infty$ and let $\alpha > n/p$. Assume that B_u^α is bounded from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$. Then there exist positive constants c_1 and c_2 depending only on p, q, n and α such that*

$$c_1 I \le \operatorname{dist}\{\, B_u^\alpha,\ \mathbb{K}(L^p(\mathbb{R}^n), L^q(\mathbb{R}^n)) \,\} \le c_2 I,$$

where $I = \overline{\lim_{k\to-\infty}} a_{pq}(k) + \overline{\lim_{k\to+\infty}} a_{pq}(k)$,

$$a_{pq}(n) \equiv \Bigg(\int\limits_{\{x:2^k<|x|<2^{k+1}\}} |u(x)|^q |x|^{(2\alpha - n/p)q} dx \Bigg)^{1/q}.$$

Theorem 5. *Let $2 \le p < \infty$ and let $\alpha > n/2$. Then $B_u^\alpha \in \sigma_p(L^2(\mathbb{R}^n))$ if and only if $\{a_k\} \in l_p(\mathbb{Z})$, where $a_k \equiv a_{22}(k)$. Moreover, there exist positive constants d_1 and d_2 such that $d_1 \|a_k\|_{l_p(\mathbb{Z})} \le \|B_u^\alpha\|_{\sigma_p(L^2(\mathbb{R}^n))} \le d_2 \|a_k\|_{l_p(\mathbb{Z})}$.*

References

1. A. Meskhi, Criteria for the boundedness and compactness of integral transforms with positive kernels. *Proc. Edinburgh Math. Soc.* (accepted for publication).

2. V. Kokilashvili and A. Meskhi, Boundedness and compactness criteria for some classical integral operators. *In: Lecture Notes in Pure and Appl. Math.* **213**, *"Function Spaces V", Proceedings of the Conference, Poznań, Poland, August* 28-*September* 3, 1998 (*Ed. H. Hudzik and L. Skrzypczak*), 279–296, *New York, Bazel, Marcel Dekker,* 2000.

3. I. C. Gohberg and M. G. Krein, Introduction to the theory of linear non-selfadjoint operators. *Amer. Math. Soc.* Transl. **128**, *Amer. Math. Society, Providence, RI*, 1969.

4. A. Meskhi, Solution of some weight problems for the Riemann-Liouville and Weyl operators. *Georgian Math. J.* **5**(1998), No. 6, 565–574.

5. J. Newman and M. Solomyak, Two-sided estimates on singular values for a class of integral operators on the semi-axis. *Integral Equations Operator Theory* **20**(1994), 335–349.

6. I. Genebashvili, A. Gogatishvili, V. Kokilashvili, and M. Krbec, Weight theory for integral transforms on spaces of homogeneous type. *Pitman Monographs and Surveys in Pure and Applied Mathematics* **92**, *Longman, Harlow,* 1998.

7. J. F. Martin-Reyes and E. Sawyer, Weighted inequalities for Riemann- Liouville fractional integrals of order one and greater. *Proc. Amer. Math. Soc.* **106** (1989), 727–733.

8. V. Stepanov, Two-weight estimates for the Riemann-Liouville operators. (Russian) *Izv. Akad. Nauk SSSR,* **54** (1990), N0. 3, 645–656.

9. R. Oinarov, Two-sided estimates of certain classes of integral operators. (Russian) *Proc. steklov Inst. Math.* **204**(1993), 240–250.

10. V. Stepanov, Weighted norm inequalities for integral operators and related topics. *Nonlinear Analysis, Function spaces and Applications,* **5**, 139–176, *Olimpa Press, Prague,*1994.

11. D. E. Edmunds, W. D. Evans, and D. J. Harris, Approximation numbers of certain Volterra integral operators, *J. London Math. Soc.* **37**(1988), No. 2, 471–489.

12. D. E. Edmunds and V. D. Stepanov, The measure of non-compactness and approximation numbers of certain Volterra integral operators. *Math. Ann.* **298**(1994), 41–66.

13. B. Opic, On the distance of the Riemann- Liouville operators from compact operators. *Proc. Amer. Math. Soc.* **122**(1994), No. 2, 495–501.

14. K. Nowak, Schatten ideal behavior of a generalized Hardy operator, *Proc. Amer. Math. Soc.* **118**(1993), 479–483.

15. D. E. Edmunds and V. D. Stepanov, On the singular numbers of certain Volterra integral operators. *J. Funct. Anal.* **134**(1995), No. 1, 222–246.

16. S. G. Samko, A. A. Kilbas, and O. I. Marichev, Integrals and Derivatives. Theory and Applications. *Gordon and Breach Science Publishers, London-New York*, 1993.

17. B. S. Rubin, Fractional integrals and potentials. *Addison Wesley Longman, Essex, UK*, 1996.

18. A. Meskhi, On the boundedness and compactness of ball fractional integral operators. *Fractional Calculus, Applied Analysis* **3**(2000), No. 1, 13–30.

Author's address:
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 380093
Georgia