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A NOTE ON ZYGMUND'S PROBLEM OF A STRONG DIFFERENTIABILITY

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Let $B=\{B(x):x\in\mathbf{R}^n\}$ be a differentiation basis in $\mathbf{R}^n(n\geqslant 2)$ (see [1], Chapter 2).

For $f\in L(\mathbf{R}^n)$ and $x\in\mathbf{R}^n$, the upper derivative and the derivative of the integral $\int f$ with resect to B at x are denoted by $\overline{D}_B\big(\int f\big)(x)$ and $D_B\big(\int f\big)(x)$, respectively (see [1], Chapter 3).

Let $2\leqslant k\leqslant n(n\geqslant 2)$. Denote by B_k the differentiation basis in $\mathbf{R}^n$ formed of all n-dimensional open intervals whose sides have no more than k different sizes (when $k=n$, we have the standard interval basis B_n).

Denote by P_n the differentiation basis in $\mathbf{R}$ formed of all n-dimensional open rectangles.

Let us recall some fundamental results from the integral differentiation theory.

(A). B_k differentiates $L(1+\log^+ L)^{k-1}(\mathbf{R}^n)$ [3].

(B). In each integral class $\Phi(L)(\mathbf{R}^n)(n\geqslant 2)$ wider than $L(1+\log^+ L)^{k-1}(\mathbf{R}^n)$ there exists a summable function f such that $\overline{D}_{B_k}\big(\int f\big)(x)=+\infty$ at every $x\in\mathbf{R}^n$ [3].

(C). P_n does not differentiate $L^\infty(\mathbf{R}^n)$ (see [1], Chapter 5).

Let γ be a union of n mutually orthogonal straight lines $\{\gamma_1,\dots,\gamma_n\}$ intersecting at the origin. The set of all such unions is denoted by $\Gamma(\mathbf{R}^n)$ and its elements are called directions.

For a fixed direction γ denote by $B_{k\gamma}$ the differentiation basis in $\mathbf{R}^n$ formed of all n-dimensional open rectangles from B_k whose sides have no more than k different sizes and are parallel to the straight lines from γ (when γ is the standard direction, we get B_n). Now, if $B_{n\gamma}$ differentiates $\int f$ at x, then the integral $\int f$ is said to be strongly differentiable at x along the direction γ (when γ is the standard direction, then the integral $\int f$ is said to be strongly differentiable at x).

As mentioned above, the basis P_n formed of all n-dimensional arbitrarily rotating rectangles has differentiation properties different from those of the basis B_n with fixed orientated rectangle sides. The dependence of differentiation properties on the orientation of the rectangles leads to Zygmund's problem [1], Chapter 4: Given a function $f\in L(\mathbf{R}^n)$, is it always possible to choose a direction $\gamma\in\Gamma(\mathbf{R}^n)$ such that $\int f$ be strongly differentiable along γ (almost averythere)?

Let $W(\mathbf{R}^n)$ denote a class of all locally summable functions on $\mathbf{R}^n$ whose strong upper derivatives $\overline{D}_{B_{n\gamma}}\big(\int f\big)(x)$ are equal to $+\infty$ almost everywhere along each fixed direction γ. In solving Zygmund's problem, Marstrand [4] showed that the class $W(\mathbf{R}^2)$ is not empty, and thus answered the above problem in the negative. Subsequently, L. Melero [5] and A. Stokolos [6] proved stronger results.

The following question naturally arises in connection with Zygmund's problem: Do there exist a function $f\in L(\mathbf{R}^n)$ and a direction γ such that $\int f$ be strongly differentiable only on the set of zero measure, but be strongly differentiable almost averywhere along γ?

A solution of this question for $n=2$ was given in [7]. A stronger result has been proved in [9] and the similar one later in [8].

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For $n \geqslant 3$ denote by $\Theta(\mathbf{R}^n)$ the set of all directions γ from $\Gamma(\mathbf{R}^n)$ for which the component straight line γ_j of the direction γ coincides with the $0x_i$-axis at least for one pair of numbers i, j $1 \leqslant i, j \leqslant n$.

Let γ be a fixed direction from $\gamma \in \Gamma(\mathbf{R}^n) \setminus \Theta(\mathbf{R}^n)$. According to [7], p. 633, in any class of functions $L\Phi(L)(\mathbf{R}^n)$ which is wider than $L\log^{n-1}(\mathbf{R}^n)$, there exists a nonnegative summable function f for which the relations $D_{B_n}(\int f)(x) = +\infty$ and $D_{B_{n\gamma}}(\int f)(x) = f(x)$ hold almost everywhere on $\mathbf{R}^n$.

A stronger result has been announced by G.Oniani in [10]. Similar results later have been anounced in [11].

Theorem [10]. *For any nonnegative function $f \in L(\mathbf{R}^n) \backslash L(1+\log^+ L)^{n-1}(\mathbf{R}^n) (n \geqslant 3)$ there exists its equimeasurable function g such that:*

1) *for $\gamma \in \Gamma(\mathbf{R}^n) \setminus \Theta(\mathbf{R}^n)$*

$$D_{B_{n\gamma}}\Big(\int f\Big)(x) = f(x), \quad a.e. x \ \in \mathbf{R}^n;$$

2) *for $\gamma \in \Theta(\mathbf{R}^n)$*

$$\overline{D}_{B_{n\gamma}}\Big(\int f\Big)(x) = +\infty, \quad a.e.\ x \in \mathbf{R}^n.$$

It is known (see [1], Chapter 3) that if $\int |f|$ is strongly differentiable almost everywhere on $\mathbf{R}^n$, then the same statement is true for $\int f$. A. Papoulis [12] established that the inverse statement does not hold in general. More precisely, he gave an example of the function for which $\int f$ is strongly differentiable almost everywhere, while $\int |f|$ on the nul-set only.

A stronger result was later proved by T. Zerekidze [13]. In particular, he established that for any summable function $f \in L(\mathbf{R}^n)$ there exists a function g, $|g| = |f|$, such that $\int g$ is strongly differentiable almost everythere along each fixed direction $\gamma \in \Gamma(\mathbf{R}^n)$. In other words, by changing the sign of the function on the set of positive measure the differentiability properties of its integral can be improved along any fixed direction.

By our next theorem we observe that that the change of the sign of the function (on the set of positive measure) can be used to improve the differentiation properties of its integral only for a certain set of directions in general. The Theorem strengthens the results of A. Papoulis [12] and J. Marstrand [4].

Theorem 1. *Let $n \geqslant 3$ and $0 < \beta < \frac{1}{n-1}$.*
There exists such a function $f \in L(1+\log^+ L)^\beta(\mathbf{R}^n)$ that:

1) *for $\gamma \in \Gamma(\mathbf{R}^n) \setminus \Theta(\mathbf{R}^n)$*

$$D_{B_{n\gamma}}\Big(\int f\Big)(x) = f(x), \quad a.e.\ x \in \mathbf{R}^n;$$

2) *for $\gamma \in \Theta(\mathbf{R}^n)$*

$$\overline{D}_{B_{2\gamma}}\Big(\int f\Big)(x) = +\infty, \quad a.e.\ x \in \mathbf{R}^n;$$

3) *for $\gamma \in \Gamma(\mathbf{R}^n)$*

$$\overline{D}_{B_{2\gamma}}\Big(\int |f|\Big)(x) = +\infty, \quad a.e.\ x \in \mathbf{R}^n.$$

Remark. The method used in proving this result enables one to prove the following generalization of the theorem [14].

Theorem 2. *Let $0 < \beta < 1$, and let the sequence of directions $(\gamma_n)_{n\geqslant 1} \subset \Gamma(\mathbf{R}^2)$ be given. Then there exists a summable function $f \in L(\log^+ L)^\beta(\mathbf{R}^2)$ such that:*

1) *for almost every direction $\gamma \in \Gamma(\mathbf{R}^2)$, $\gamma \neq \gamma_n$*

$$D_{B_{2\gamma}}\Big(\int f\Big)(x) = f(x), \quad a.e. \ x \in \mathbf{R}^2;$$

2) *for every direction γ_n, $n \in \mathbf{N}$,*

$$\overline{D}_{B_{2\gamma}}\Big(\int f\Big)(x) = +\infty, \quad a.e. \ x \in \mathbf{R}^2;$$

3) *for every direction $\gamma \in \Gamma(\mathbf{R}^2)$*

$$\overline{D}_{B_{2\gamma}}\Big(\int |f|\Big)(x) = +\infty, \quad a.e. \ x \in \mathbf{R}^n.$$

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