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ON THE BOUNDARY VALUE PROBLEM OF LINEAR CONJUGATION IN THE CASE OF NON-SMOOTH LINES AND SOME MEASURABLE COEFFICIENTS

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1. Let Γ be the rectifiable Jordan curve dividing the plane into domains D_Γ^+ and D_Γ^- (it is assumed that $\infty \in D_\Gamma^-$). In the sequel we will need classes of analytic functions $E_p(D^\pm)$ which are usually called the Hausdorff-Smirnov classes. For the boundary value problems we will use the following definition of these classes.

We will say that $\phi(z) \in E_p(D_\Gamma)$, where $p > 0$ and D_γ denotes either D_Γ^+ or D_Γ^- if:

(a) $\phi(z)$ is analytic in D_Γ;

(b) $\phi(\infty) = 0$ if $D_\Gamma = D_\Gamma^-$;

(c) there exists a sequence of curves $\{\Gamma_n\}_{n=1}^\infty \in D_\Gamma$ such that $\infty \overline{\in} \Gamma_n$, $\Gamma_n \to \Gamma$ as $n \to \infty$, and

$$\sup_n \int_\Gamma |\phi(z)|^p ds < \infty.$$

(under the convergence $\Gamma_n \to \Gamma$ is understood the same as in [1], p. 203).

As usual, we denote by $(S_\Gamma \varphi(t))(\tau)$ and $(K_\Gamma \varphi(t))(z)$ the integrals

$$\Big(S_\Gamma \varphi(t)\Big)(z) = \frac{1}{\pi i} \int_\Gamma \varphi(t)(t-\tau)^{-1} dt, \quad \tau \in \Gamma,$$

and

$$\Big(K_\Gamma \varphi(t)\Big)(z) = \frac{1}{2\pi i} \int_\Gamma \varphi(t)(t-z)^{-1} dt, \quad z \overline{\in} \Gamma,$$

where $\varphi \in L(\Gamma)$. The first integral is understood in the sense of the Cauchy principal value (see, e.g., [2]). Sometimes for these integrals we will use the notation $S_\Gamma \varphi$ and $K_\Gamma \varphi$ or $S\varphi$ and $K\varphi$.

We will say that $\Gamma \in R$ if

$$\|S_\Gamma \varphi\|_{L_p(\Gamma)} \le M_p \|\varphi\|_{L_p(\Gamma)}, \quad \forall \varphi \in L_p(\Gamma)$$

and that the positive measurable function $\rho(t)$ is the weight, and we write $\rho \in W_p(\Gamma)$ if

$$\|\rho S_\Gamma \, \rho^{-1} \varphi\|_{L_p(\Gamma)} \le M_p \|\varphi\|_{L_p(\Gamma)}, \quad \forall \varphi \in L_p(\Gamma).$$

2. The boundary value problem of linear conjugation is formulated as follows: Find the function $\phi(z) \in E_p(D_\Gamma^\pm)$ whose angular values satisfy the condition

$$\phi^+(t) = G(t)\phi^-(t) + f(t), \quad t \in \Gamma, \tag{1}$$

where $f \in L_p(\Gamma)$, $G \in L_\infty(\Gamma)$ are the given functions, $p > 1$, $\Gamma \in R$.

The main results dealing with this problem can be found in [2].

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We say that assertion (A) for problem (1) holds if all solutions of that problem can be written in conventional form

$$\phi(z) = X(z)\Big(K_\Gamma(f/X^+)\Big)(z) + X(z)P_{\varkappa-1}(z),$$

where $P_{\varkappa-1}(z)$ is the polynomial of degree $\varkappa - 1$. Here we assume that $P_k(z) = 0$ if $k < 0$ and

$$X(z) = \begin{cases} X_0(z), & \text{for } z \in D_\Gamma^+ \\ (z-a)^{-\varkappa}X_0(z), & \text{for } z \in D_\Gamma^- \end{cases}$$

$$X_0(z) = \exp\Big(K_\Gamma \ln(t-a)^{-\varkappa}G(t)\Big)(z);$$

if $\varkappa \geq 0$, then the problem is, undoubtedly, solvable, while if $\varkappa < 0$, then for its solvability it is necessary and sufficient that

$$\int_\Gamma t^k f(t)\Big(X^+(t)\Big)^{-1}dt = 0, \quad k = 0, 1, \dots, \varkappa - 1.$$

In the case, where Γ is the Ljapunov curve and $G(t)$ is measurable, Simonenko [3] cited the conditions under which assertion (A) holds. These conditions can be written as follows:

(a) $0 < \operatorname*{vrai\,sup}_{t\in\Gamma} |G(t)| < \infty;$ (2)

(b) the argument of the function $G(t)$ can be chosen so as $\varphi = \varphi_1 + \varphi_2$ to hold for $\varphi(t) = \arg G(t)$, where $\varphi_1(t)$ is continuous on Γ, except possibly one point, where it has discontinuity equal to $2\pi\varkappa$; $\varkappa$ is an integer and φ_2 satisfies the condition

$$\operatorname*{vrai\,sup}_{t\in\Gamma} |\varphi_2(t)| < \frac{\pi}{\max(p,q)}, \quad q = p(p-1)^{-1}. \tag{3}$$

The function whose argument satisfies condition (2) is called sectorial. Conditions (a) and (b) can be fulfilled both on closed and on open lines.

Simonenko's result was transferred by various authors to cases of many sufficiently general boundary lines, however, the existence of cusps on them is excluded.

Our aim is to consider the cases when cusps do take place.

To formulate the main result it is necessary to quote one more definition.

Denote by Γ_{ab} an arc with the ends a and b directed from a to b.

We say that the arc $\Gamma_{ab} \in R_A$, if it can be supplemented to the closed Jordan contour Γ_0 so that for any function $G(t) = \exp i\varphi_2(t)$, where φ_2 satisfies inequality (3) and $\varphi_2(t) = 0$ for $t \in \Gamma_0 - \Gamma_{ab}$, assertion (A) holds.

The Basic Result. *In problem* (1) *if a simple closed Jordan curve* $\Gamma \in R$, $\Gamma = \bigcup_{k=1}^{n-1} \Gamma_{a_k a_{k+1}}$, $\Gamma_{a_k a_{k+1}} \in R_A$ $(k = 1, \dots, n-1)$ *and* $G(t)$ *is the measurable function on* Γ *satisfying conditions* (*a*) *and* (*b*), *and moreover, for the points* $a_k (k = 1, \dots, n)$ *there exist on* Γ *angular neighborhoods* $\Gamma_{b_k a_k}$ *and* $\Gamma_{a_k c_k}$ *in which for some* $\alpha \in [0; 1]$ *the conditions*

$$\operatorname*{vrai\,sup}_{t\in\Gamma_{b_k a_k}} |\varphi_2(t)| < \alpha\frac{\pi}{\max(p,q)} \tag{4}$$

and

$$\operatorname*{vrai\,sup}_{t\in\Gamma_{a_k c_k}} |\varphi_2(t)| < (1-\alpha)\frac{\pi}{\max(p,q)}, \quad q = p(p-1)^{-1},$$

are fulfilled, then assertion (A) *for problem* (1) *is valid.*

Obviously, in these conditions the cusps and the other cases are quite possible at the points a_k.

The proof of that assertion is based on two lemmas which are given below.

Denote by $\chi_k(t)$ the characteristic function of the set $\{t : t \in \Gamma_k\}$. Below, Γ will be assumed to be the same as in the statement of the basic result.

Lemma 1. *If condition* (3) *is fulfilled, then*

$$\rho(t) = \exp \frac{i}{2} S_\Gamma \chi_k \varphi_2 \in W_p(\Gamma).$$

The proof is based on the consideration of the boundary value problem

$$\phi^+(t) = (\exp i\chi_k\varphi_2)\phi^-(t) + f(t), \quad t \in \Gamma,$$

which by means of substitution $\phi(z) = \phi_1(z) + \phi_2(z)$, $\phi_1(z) = \Big(K_\Gamma(\chi_k\psi)\Big)(z)$, $\phi_2(z) = \Big(K_\Gamma(\chi_{\Gamma-\Gamma_k} f)\Big)(z)$ is reduced to the boundary value problem on the arc Γ_k.

Lemma 2. *If condition* (3) *for the function* φ_2 *is sutisfied and, in addition, condition* (4) *at the points* a_k *is fulfilled, then*

$$\rho(t) = \exp \frac{i}{2} S_\Gamma \varphi_2 \in W_p(\Gamma).$$

To prove the lemma we apply Stein's theorem and Theorem 4.2 from [4].

Using somehow modified methods of solving the boundary value problems, from Lemmas 1 and 2 we obtain the above-formulated basic assertion.

Corollary. *In Simonenko's conditions assertion* (A) *together with condition* (4) *is valid both for piecewise smooth lines and for lines with bounded rotation not excluding the cases for the existence of cusps.*

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