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A NOTE ON THE STRONG DIFFERENTIABILITY OF INTEGRALS ALONG DIFFERENT DIRECTIONS IN $\mathbf{R}^n$ $(n \geqslant 3)$

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In this note we present some results concerning Zygmunds problem on strong differentiability of integrals along different directions in $\mathbf{R}^n$ $(n \geqslant 3)$.

Let $B = \{\cup B(x) : x \in \mathbf{R}^n\}$ be a differentiation basis in $\mathbf{R}^n$.

For $f \in L(\mathbf{R}^n)$ and $x \in \mathbf{R}^n$ let us denote respectively by $\bar{D}_B(f)(x)$ and $D_B(f)(x)$ the upper derivative and derivative of the integral $\int f$ with respect to B at x (see [1]).

Let σ be the union of n mutually orthogonal straight lines $\sigma_1, \dots, \sigma_n$ in $\mathbf{R}^n$ $(n \geqslant 2)$ which intersects at the origin. The set of such unions will be denoted by $\Gamma(\mathbf{R}^n)$. Elements of this set will be called directions.

For a fixed direction σ we denote by $B_{2\sigma}$ the differentiation basis in $\mathbf{R}^n$ formed of all n-dimensional open rectangles with the sides parallel to the straight lines from σ. If $B_{2\sigma}$ differentiates $\int f$ at x, then the integral $\int f$ is said to be strongly differtiable along σ at x (in the case σ- standard direction, then the integral $\int f$ is said to be strongly differentiable at x).

The following problem was proposed by Zygmund (see [1], ch. 4); given a function $f \in L(\mathbf{R}^n)$, is it possible to choose a direction σ such that integral $\int f$ would be strongly differentiable along σ?

Let $W(\mathbf{R}^n)$, $n \geqslant 2$ denote a class of locally integrable functions on $\mathbf{R}^n$ whose strong upper derivatives $\bar{D}_{B_{2\sigma}}(f)(x)$ are equal to $+\infty$ almost everywhere along each fixed direction σ. When solving Zygmunds problem, Marstrand [2] showed that the class $W(\mathbf{R})^2$ is not empty, and thus his answer to above stated problem was negative. Stronger results later have been proofed by Lopes Melero [3] and Stokolos [4].

In connection with Zygmunds problem the following natural question arises poised by Zerekidze: does there exists a function $f \in L(\mathbf{R}^n)$ $(n \geqslant 2)$ and a direction σ such that $\int f$ is strongly differentiable on a set of zero measure only, but along the direction σ is strongly differentiable almost everywhere?

A positive answer to this question for $n = 2$ was given in [5]. In particular it is established, that for a given sequence of directions $(\sigma_m)_{m=1}^{\infty} \subset \Gamma(\mathbf{R}^2)$ there exists a function $f \in L(\mathbf{R}^2)$, $f \geqslant 0$ such that for almost every direction $\sigma \in \Gamma(R^2) \backslash (\sigma_m)_{m=1}^{\infty}$ the integral $\int f$ is strongly differentiable along σ, but for every σ_m, $m = 1, 2, \dots$ the strong upper derivative $\bar{D}_{B_{2\sigma}}(f)(x)$ equals to $+\infty$ almost everywhere. Stronger results later have been proofed in papers [6]-[7].

In the present paper we give a positive answer to this question for all $n \geqslant 3$.

Theorem 1. *Let $\varphi :]0, +\infty[\to]0, +\infty[$, $\varphi(t) = o(t \log(t))$ and $\varphi(t) \uparrow +\infty$, $t \uparrow +\infty$. Let σ is an arbitrary (nonstandart) direction from $\Gamma(\mathbf{R})^n$ $(n \geqslant 3)$. There exists a nonnegative summable function $f \in \varphi(L)(\mathbf{R}^n)$ such that*

$$\bar{D}_{B_2}(f)(x) = +\infty, \quad a.e.\ x \in \mathbf{R}^n$$

and

$$D_{B_{2\sigma}}(f)(x) = f(x), \quad a.e.\ x \in \mathbf{R}^n.$$

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Denote by $\Theta(\mathbf{R}^n)$ $(n \geqslant 3)$ the set of all such directions σ from $\Gamma(\mathbf{R}^n)$ that for at last one pair of numbers i, j, $1 \leqslant i$, $j \leqslant n$ the angle between the axes Ox_i and component straight line σ_j of the direction σ equals to zero.

Theorem 2. *Let $\varphi :]0,+\infty[\to]0,+\infty[$, $\varphi(t) = o\Big(t(\log(t))^{n-1}\Big)$ and $\varphi(t) \uparrow +\infty$, $t \uparrow +\infty$. There exists a nonnegative summable function $f \in \varphi(L)(\mathbf{R}^n)$ such that*

1) *For $\sigma \in \Theta(\mathbf{R}^n)$*

$$\bar{D}_{B_{2\sigma}}(f)(x) = +\infty, \quad a.e\ x \in \mathbf{R}^n;$$

2) *For $\sigma \in \Gamma(\mathbf{R}^n)\backslash\Theta(\mathbf{R}^n)$*

$$D_{B_{2\sigma}}(f)(x) = f(x), \quad a.e\ x \in \mathbf{R}^n.$$

References

1. M. de Guzman, Differentiation of integrals in R^n. *Lecture Notes in Math. Springer, New York* **481**(1975).
2. J. M. Marstrand, A counter-examle in the theory of strong differentiation. *Bull. London Math. Soc.* **9**(1977), 209–211.
3. B. Lopez Melero, A negatve result in differentiation theory. *Studia Math.* **72**(1982), 173–182.
4. A. M. Stokolos, An anequality for equimeasurable rearrangements and its application in the theory of differentiation of integrals. *Anal. Math.* **9**(1983), 133–146.
5. G. Lepsveridze, On strong differentiability of integrals along different directions. *Georgian Math. J.* **2**(1995), No. 6, 617–635.
6. G. Oniani, On the differentiability of integrals with respect to the bases $B_2(\Omega)$. *East J. Approx.* **3**(1997), No. 3, 275–301.
7. G. Lepsveridze, To the problem of a strong differentiability of integrals along different directions. *Georgian Math. J.* **5**(1998), No. 2, 157–176.
8. A. Ctokolos, On Zugmynd's problem (Russian). *Mat. Zametki* **5**(1998), No. 64, 749–762.

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