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UNIQUENESS OF POSITIVE SOLUTIONS OF THE SUSCEPTIBLE-INFECTIOUS-RECOVERED-DECEASED EPIDEMIC MODEL

Dedicated to Professor Kusano Takaŝi
on the occasion of his 90-th birthday

Abstract. The uniqueness of positive solutions of Susceptible-Infectious-Recovered-Deceased (SIRD) epidemic model is investigated and it is shown that there exists one and only one solution of an initial value problem for SIRD differential system in the class of positive solutions. Our approach is based on the uniqueness of positive solutions of an initial value problem for some nonlinear differential equation of first order which is satisfied by the number of recovered individuals $R(t)$.

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Key words and phrases. Uniqueness, SIRD epidemic model, positive solution, an initial value problem, SIRD differential system.

რეზიუმე. გამოკვლეულია მგრძნობიარე-ინფექციური-გამოჯანმრთელებული-დაღუპული (SIRD) ეპიდემიური მოდელის დადებითი ამონახსნების ერთადერთობა. ნაჩვენებია, რომ დადებით ამონახსნთა კლასში არსებობს SIRD დიფერენციალური სისტემისთვის დასმული საწყისი ამოცანის ერთადერთი ამონახსნი. ჩვენი მიდგომა ემყარება საწყისი ამოცანის დადებითი ამონახსნების ერთადერთობას პირველი რიგის გარკვეული არაწრფივი დიფერენციალური განტოლებისთვის, რომელსაც აკმაყოფილებს გამოჯანმრთელებული ინდივიდების რაოდენობა $R(t)$.

1 Introduction

In 1927, Kermack and McKendrick [4] proposed the Susceptible-Infectious-Recovered (SIR) epidemic model. Various epidemic models have been studied so far, and effort has been made to establish exact solutions of epidemic models in recent years. We refer the reader to [1,3,5,7] for exact solutions of SIR epidemic models and to [6,8] for exact solutions of Susceptible-Infectious-Recovered-Deceased (SIRD) epidemic model or Susceptible-Exposed-Infectious-Recovered (SEIR) epidemic model. It seems that little is known about the uniqueness of positive solutions of epidemic models. The purpose of this paper is to establish the uniqueness results of positive solutions of an initial value problem for SIRD epidemic model. Our approach is an adaptation of the standard arguments using a Lipschits condition. Since a positive exact solution of SIRD epidemic model is derived in [6], we conclude that there exists one and only one solution of an initial value problem for SIRD differential system in the class of positive solutions.

We are concerned with the initial value problem for the SIRD differential system

$$\frac{dS(t)}{dt} = -\beta S(t)I(t), \tag{1.1}$$

$$\frac{dI(t)}{dt} = \beta S(t)I(t) - \gamma I(t) - \mu I(t), \tag{1.2}$$

$$\frac{dR(t)}{dt} = \gamma I(t), \tag{1.3}$$

$$\frac{dD(t)}{dt} = \mu I(t) \tag{1.4}$$

for $t > 0$, subject to the initial conditions

$$S(0) = \widetilde{S}, \quad I(0) = \widetilde{I}, \quad R(0) = \widetilde{R}, \quad D(0) = \widetilde{D}, \tag{1.5}$$

where β, γ and μ are the positive constants, and $\widetilde{S}$, $\widetilde{I}$, $\widetilde{R}$, $\widetilde{D}$ are the constants satisfying the following hypotheses:

(A$_1$) $\widetilde{S} + \widetilde{I} + \widetilde{R} + \widetilde{D} = N$ (positive constant);

(A$_2$) $\widetilde{S} > \dfrac{\gamma + \mu}{\beta}$;

(A$_3$) $\widetilde{I} > 0$;

(A$_4$) $0 \le \widetilde{R} < \dfrac{\gamma}{\beta} \log\left(1 + \left(\dfrac{\widetilde{I}}{\widetilde{S}}\right)\right)$;

(A$_5$) $0 \le \widetilde{D} < \dfrac{\mu}{\beta} \log\left(1 + \left(\dfrac{\widetilde{I}}{\widetilde{S}}\right)\right)$.

By a *solution* of system (1.1)–(1.4) we mean a vector-valued function $(S(t), I(t), R(t), D(t))$ of class $C^1(0,\infty) \cap C[0,\infty)$ which satisfies (1.1)–(1.4). A solution $(S(t), I(t), R(t), D(t))$ of the SIRD differential system (1.1)–(1.4) is said to be *positive* if $S(t) > 0$, $I(t) > 0$, $R(t) > 0$ and $D(t) > 0$ for $t > 0$.

Associated with every continuous function $f(t)$ on $[0,\infty)$, we define

$$f(\infty) := \lim_{t\to\infty} f(t).$$

2 Uniqueness of positive solutions of SIRD differential system

In this section, we discuss the uniqueness of positive solutions of the SIRD differential system and, consequently, we deduce that there exists a unique solution of the initial value problem (1.1)–(1.5) in the class of positive solutions.

We need the following three theorems before discussing the uniqueness of positive solutions.

Theorem 2.1 ([6, Lemma 1])**.** *Let $(S(t), I(t), R(t), D(t))$ be a solution of the initial value problem (1.1)–(1.5) such that $S(t) > 0$ for $t > 0$. Then $R(t)$ satisfies the nonlinear differential equation of the first order*

$$R'(t) = \gamma\Big(N - \widetilde{D} + \frac{\mu}{\gamma}\,\widetilde{R} - \widetilde{S}e^{(\beta/\gamma)\widetilde{R}}e^{-(\beta/\gamma)R(t)} - \Big(1 + \frac{\mu}{\gamma}\Big)R(t)\Big), \quad t > 0, \tag{2.1}$$

and the initial condition

$$R(0) = \widetilde{R}. \tag{2.2}$$

Remark 2.1. If $I(t) > 0$ for $t > 0$, then $R(t)$ is increasing on $[0, \infty)$ because $R'(t) = \gamma I(t) > 0$ for $t > 0$. Since $R(0) = \widetilde{R} \geq 0$, it follows that $R(t) > 0$ for $t > 0$. Similarly, it can be shown that $D(t) > 0$ for $t > 0$ if $I(t) > 0$ for $t > 0$. If $S(t) > 0$ and $I(t) > 0$ for $t > 0$, we observe that $R(t)$ is increasing on $[0, \infty)$ and $R(t) = N - S(t) - I(t) - D(t) < N$. Therefore, there exists the limit $R(\infty)$.

Theorem 2.2 ([6, Theorem 1])**.** *Let $(S(t), I(t), R(t), D(t))$ be a solution of the initial value problem (1.1)–(1.5) such that $S(t) > 0$ and $I(t) > 0$ for $t > 0$. Then $(S(t), I(t), R(t), D(t))$ can be represented in the following parametric form:*

$$S(t) = S(\varphi(u)) = \widetilde{S}e^{(\beta/\gamma)\widetilde{R}}u, \tag{2.3}$$

$$I(t) = I(\varphi(u)) = N - \widetilde{D} + \frac{\mu}{\gamma}\,\widetilde{R} - \widetilde{S}e^{(\beta/\gamma)\widetilde{R}}u + \frac{\gamma + \mu}{\beta}\,\log u, \tag{2.4}$$

$$R(t) = R(\varphi(u)) = -\frac{\gamma}{\beta}\,\log u, \tag{2.5}$$

$$D(t) = D(\varphi(u)) = -\frac{\mu}{\beta}\,\log u + \widetilde{D} - \frac{\mu}{\gamma}\,\widetilde{R} \tag{2.6}$$

for $e^{-(\beta/\gamma)R(\infty)} < u \leq e^{-(\beta/\gamma)\widetilde{R}}$, where

$$t = \varphi(u) = \int_u^{e^{-(\beta/\gamma)\widetilde{R}}} \frac{d\xi}{\xi\psi(\xi)}$$

with $\psi(\xi)$ being

$$\psi(\xi) = \beta N - \beta\widetilde{D} + \frac{\beta\mu}{\gamma}\,\widetilde{R} - \beta\widetilde{S}e^{(\beta/\gamma)\widetilde{R}}\xi + (\gamma + \mu)\log\xi. \tag{2.7}$$

Theorem 2.3. *Let $(S(t), I(t), R(t), D(t))$ be a solution of the initial value problem (1.1)–(1.5) such that $S(t) > 0$ and $I(t) > 0$ for $t > 0$. Then we find that*

$$S(t) = \widetilde{S}e^{(\beta/\gamma)\widetilde{R}}e^{-(\beta/\gamma)R(t)}, \tag{2.8}$$

$$I(t) = N - \widetilde{D} + \frac{\mu}{\gamma}\,\widetilde{R} - \widetilde{S}e^{(\beta/\gamma)\widetilde{R}}e^{-(\beta/\gamma)R(t)} - \Big(1 + \frac{\mu}{\gamma}\Big)R(t), \tag{2.9}$$

$$D(t) = \widetilde{D} - \frac{\mu}{\gamma}\,\widetilde{R} + \frac{\mu}{\gamma}\,R(t) \tag{2.10}$$

for $t \geq 0$.

Proof. It follows from (2.5) that

$$u = e^{-(\beta/\gamma)R(t)} \tag{2.11}$$

holds. Substituting (2.11) into (2.3), (2.4) and (2.6), we obtain (2.8), (2.9) and (2.10), respectively. □

We define the function $f(r)$ by

$$f(r) := \gamma\Big(N - \widetilde{D} + \frac{\mu}{\gamma}\,\widetilde{R} - \widetilde{S}e^{(\beta/\gamma)\widetilde{R}}e^{-(\beta/\gamma)r} - \Big(1 + \frac{\mu}{\gamma}\Big)r\Big), \quad r > 0.$$

Since

$$f'(r) = \beta\widetilde{S}e^{(\beta/\gamma)\widetilde{R}}e^{-(\beta/\gamma)r} - (\gamma + \mu),$$

we have

$$|f'(r)| \le \beta\widetilde{S}e^{(\beta/\gamma)\widetilde{R}} + (\gamma+\mu)\ (\equiv K)$$

and hence $f(r)$ satisfies the Lipschitz condition on $(0,\infty)$ with Lipschitz constant K, i.e.,

$$|f(r_1) - f(r_2)| \le K|r_1 - r_2| \tag{2.12}$$

holds for all $r_1, r_2 \in (0,\infty)$ (cf. Coddington [2, p. 208]).

Theorem 2.4. *Let $R_i(t)$ $(i=1,2)$ be solutions of the initial value problem (2.1), (2.2) such that $R_i(t) > 0$ for $t > 0$ $(i=1,2)$. Then we observe that*

$$R_1(t) \equiv R_2(t) \ \text{ on } [0,\infty).$$

Proof. Integrating (2.1) with $R(t) = R_i(t)$ $(i=1,2)$ over $[\varepsilon, t]$ $(\varepsilon > 0)$, taking the limit as $\varepsilon \to +0$ and using (2.2), we obtain

$$R_i(t) = \widetilde{R} + \int_0^t f(R_i(s))\,ds \ \ (i=1,2)$$

(see, e.g., Coddington [2, p. 200, Theorem 4]). We easily see that

$$R_1(t) - R_2(t) = \int_0^t \big(f(R_1(s)) - f(R_2(s))\big)\,ds$$

and therefore

$$|R_1(t) - R_2(t)| \le \int_0^t \big|f(R_1(s)) - f(R_2(s))\big|\,ds \le K\int_0^t |R_1(s) - R_2(s)|\,ds, \ \ t > 0 \tag{2.13}$$

by taking into account (2.12). If we define

$$W(t) := \int_0^t |R_1(s) - R_2(s)|\,ds,$$

we observe, using (2.13), that

$$W'(t) - KW(t) \le 0, \ \ t > 0,$$

or

$$\big(e^{-Kt}W(t)\big)' \le 0, \ \ t > 0.$$

Hence $e^{-Kt}W(t)$ is nonincreasing on $[0,\infty)$ and we see that

$$e^{-Kt}W(t) \le W(0) = 0, \ \ t \ge 0,$$

i.e.,

$$W(t) \le 0, \ \ t \ge 0. \tag{2.14}$$

Combining (2.13) with (2.14) gives

$$|R_1(t) - R_2(t)| \le KW(t) \le 0, \ \ t > 0,$$

and therefore we see that

$$R_1(t) \equiv R_2(t) \ \text{ on } (0,\infty).$$

Since $R_1(0) = R_2(0) = \widetilde{R}$, we conclude that

$$R_1(t) \equiv R_2(t) \ \text{ on } [0,\infty). \qquad \square$$

Theorem 2.5. *Let* $(S_i(t), I_i(t), R_i(t), D_i(t))$ $(i = 1, 2)$ *be solutions of the initial value problem* (1.1)–(1.5) *such that* $S_i(t) > 0$ *and* $I_i(t) > 0$ *for* $t > 0$. *Then we see that*

$$(S_1(t), I_1(t), R_1(t), D_1(t)) \equiv (S_2(t), I_2(t), R_2(t), D_2(t)) \ \text{ on } [0, \infty).$$

Proof. First, we note that $R_i(t) > 0$ and $D_i(t) > 0$ for $t > 0$ $(i = 1, 2)$ by Remark 2.1. It follows from Theorem 2.3 that

$$\begin{aligned} S_i(t) &= \widetilde{S} e^{(\beta/\gamma)\widetilde{R}} e^{-(\beta/\gamma)R_i(t)}, \\ I_i(t) &= N - \widetilde{D} + \frac{\mu}{\gamma}\widetilde{R} - \widetilde{S} e^{(\beta/\gamma)\widetilde{R}} e^{-(\beta/\gamma)R_i(t)} - \Big(1 + \frac{\mu}{\gamma}\Big) R_i(t), \\ D_i(t) &= \widetilde{D} - \frac{\mu}{\gamma}\widetilde{R} + \frac{\mu}{\gamma} R_i(t) \end{aligned}$$

for $t \geq 0$ $(i = 1, 2)$. We observe, using Theorems 2.1 and 2.4, that

$$R_1(t) = R_2(t) \ \text{ for } \ t \geq 0,$$

and therefore we find that

$$S_1(t) = S_2(t), \ \ I_1(t) = I_2(t) \ \text{ and } \ D_1(t) = D_2(t) \ \text{ for } \ t \geq 0. \qquad \square$$

Let α be the unique solution of the transcendental equation

$$x = F(N, \widetilde{D}, \widetilde{R}, \gamma, \mu) - \frac{\gamma}{\gamma + \mu} \widetilde{S} e^{(\beta/\gamma)\widetilde{R}} e^{-(\beta/\gamma)x} \tag{2.15}$$

such that $\widetilde{R} < \alpha < F(N, \widetilde{D}, \widetilde{R}, \gamma, \mu) < N$, where

$$F(N, \widetilde{D}, \widetilde{R}, \gamma, \mu) := \frac{\gamma}{\gamma + \mu} N - \frac{\gamma}{\gamma + \mu} \widetilde{D} + \frac{\mu}{\gamma + \mu} \widetilde{R}$$

(cf. [6, Lemma 2.4]).

We note that the hypothesis (A_4) is equivalent to

(A_4') $\widetilde{R} \geq 0$ and $N - \widetilde{D} > \widetilde{S} e^{(\beta/\gamma)\widetilde{R}} + \widetilde{R}$,

since $N - \widetilde{D} = \widetilde{S} + \widetilde{I} + \widetilde{R}$ and $\widetilde{S} + \widetilde{I} > \widetilde{S} e^{(\beta/\gamma)\widetilde{R}}$.

We assume that the hypothesis

(A_6) $\widetilde{S} < \dfrac{\gamma + \mu}{\beta} e^{(\beta/\gamma)(\alpha - \widetilde{R})}$

holds. We easily see that (A_6) is equivalent to

(A_6') $\dfrac{\gamma + \mu}{\beta} > N - \widetilde{D} + \dfrac{\mu}{\gamma}\widetilde{R} - \dfrac{\gamma + \mu}{\gamma}\alpha$

in light of

$$\widetilde{S} e^{(\beta/\gamma)\widetilde{R}} e^{-(\beta/\gamma)\alpha} = N - \widetilde{D} + \frac{\mu}{\gamma}\widetilde{R} - \frac{\gamma + \mu}{\gamma}\alpha.$$

The following theorem is due to Yoshida [6, Theorem 2.5 and Remark 3.15].

Theorem 2.6. *Under the hypotheses* (A_1)–(A_6), *the function* $(S(t), I(t), R(t), D(t))$ *given by*

$$\begin{aligned} S(t) &= \widetilde{S} e^{(\beta/\gamma)\widetilde{R}} \varphi^{-1}(t), \\ I(t) &= N - \widetilde{D} + \frac{\mu}{\gamma}\widetilde{R} - \widetilde{S} e^{(\beta/\gamma)\widetilde{R}} \varphi^{-1}(t) + \frac{\gamma + \mu}{\beta} \log \varphi^{-1}(t), \\ R(t) &= -\frac{\gamma}{\beta} \log \varphi^{-1}(t), \\ D(t) &= -\frac{\mu}{\beta} \log \varphi^{-1}(t) + \widetilde{D} - \frac{\mu}{\gamma}\widetilde{R} \end{aligned}$$

is a positive solution of the initial value problem (1.1)–(1.5)*, where* $\varphi^{-1}(t)$ *denotes the inverse function of* $\varphi : (e^{-(\beta/\gamma)\alpha}, e^{-(\beta/\gamma)\widetilde{R}}] \to [0, \infty)$ *such that*

$$t = \varphi(u) = \int_{u}^{e^{-(\beta/\gamma)\widetilde{R}}} \frac{d\xi}{\xi\psi(\xi)}$$

with $\psi(\xi)$ *given by* (2.7).

Remark 2.2. Since $\alpha = R(\infty)$ and $I(\infty) = 0$, it follows that

$$\begin{aligned} N - \widetilde{D} + \frac{\mu}{\gamma}\widetilde{R} - \frac{\gamma+\mu}{\gamma}\,\alpha \\ &= N - R(\infty) - \Big(\frac{\mu}{\gamma}R(\infty) + \widetilde{D} - \frac{\mu}{\gamma}\widetilde{R}\Big) = N - R(\infty) - D(\infty) - I(\infty) = S(\infty) \end{aligned}$$

(cf. [6, Theorems 3.3 and 3.5]). Therefore, (A_6') reduces to

(A_6'') $\dfrac{\gamma+\mu}{\beta} > S(\infty)$.

It is easy to see that the hypothesis (A_5) is equivalent to

(A_5') $\widetilde{D} \geq 0$ and $N - \widetilde{R} > \widetilde{S}e^{(\beta/\mu)\widetilde{D}} + \widetilde{D}$.

Under the hypothesis (A_5), the transcendental equation

$$y = G(N, \widetilde{R}, \widetilde{D}, \mu, \gamma) - \frac{\mu}{\mu+\gamma}\,\widetilde{S}e^{(\beta/\mu)\widetilde{D}}e^{-(\beta/\mu)y} \tag{2.16}$$

has a unique solution α_* such that

$$\widetilde{D} < \alpha_* < N,$$

where

$$G(N, \widetilde{R}, \widetilde{D}, \mu, \gamma) := \frac{\mu}{\mu+\gamma}\,N - \frac{\mu}{\mu+\gamma}\,\widetilde{R} + \frac{\gamma}{\mu+\gamma}\,\widetilde{D}.$$

We see that (2.15) is equivalent to (2.16) under the transformation

$$x = \widetilde{R} + \frac{\gamma}{\mu}\,(y - \widetilde{D})$$

and therefore we obtain

$$\alpha = \widetilde{R} + \frac{\gamma}{\mu}\,(\alpha_* - \widetilde{D}).$$

Since

$$e^{(\beta/\gamma)(\alpha-\widetilde{R})} = e^{(\beta/\gamma)(\gamma/\mu)(\alpha_*-\widetilde{D})} = e^{(\beta/\mu)(\alpha_*-\widetilde{D})},$$

we find that (A_6) is equivalent to

(A_7) $\widetilde{S} < \dfrac{\mu+\gamma}{\beta}e^{(\beta/\mu)(\alpha_*-\widetilde{D})}$

which reduces to

(A_7') $\dfrac{\mu+\gamma}{\beta} > N - \widetilde{R} + \dfrac{\gamma}{\mu}\,\widetilde{D} - \dfrac{\mu+\gamma}{\mu}\,\alpha_*$

in view of

$$\widetilde{S}e^{(\beta/\mu)\widetilde{D}}e^{-(\beta/\mu)\alpha_*} = N - \widetilde{R} + \frac{\gamma}{\mu}\,\widetilde{D} - \frac{\mu+\gamma}{\mu}\,\alpha_*.$$

Theorem 2.7. *Under the hypotheses* (A_1)–(A_6)*, the function* $(S_*(t), I_*(t), R_*(t), D_*(t))$ *given by*

$$S_*(t) = \widetilde{S}e^{(\beta/\mu)\widetilde{D}}\varphi_*^{-1}(t), \tag{2.17}$$

$$I_*(t) = N - \widetilde{R} + \frac{\gamma}{\mu}\,\widetilde{D} - \widetilde{S}e^{(\beta/\mu)\widetilde{D}}\varphi_*^{-1}(t) + \frac{\mu+\gamma}{\beta}\,\log\varphi_*^{-1}(t), \tag{2.18}$$

$$R_*(t) = -\frac{\gamma}{\beta}\,\log\varphi_*^{-1}(t) + \widetilde{R} - \frac{\gamma}{\mu}\,\widetilde{D}, \tag{2.19}$$

$$D_*(t) = -\frac{\mu}{\beta}\,\log\varphi_*^{-1}(t) \tag{2.20}$$

is a positive solution of the initial value problem (1.1)–(1.5)*, where* $\varphi_*^{-1}(t)$ *denotes the inverse function of* $\varphi_* : (e^{-(\beta/\mu)\alpha_*}, e^{-(\beta/\mu)\widetilde{D}}] \to [0,\infty)$ *such that*

$$t = \varphi_*(u) = \int_u^{e^{-(\beta/\mu)\widetilde{D}}} \frac{d\xi}{\xi\psi_*(\xi)}$$

with $\psi_*(\xi)$ *being*

$$\psi_*(\xi) = \beta N - \beta\widetilde{R} + \frac{\beta\gamma}{\mu}\,\widetilde{D} - \beta\widetilde{S}e^{(\beta/\mu)\widetilde{D}}\xi + (\mu+\gamma)\log\xi,$$

and we find that

$$\big(S_*(t), I_*(t), R_*(t), D_*(t)\big) \equiv \big(S(t), I(t), R(t), D(t)\big) \quad \text{on } [0,\infty). \tag{2.21}$$

Proof. By starting our arguments utilizing (1.4) instead of (1.3) in [6], we see that (2.17)–(2.20) is a positive solution of the initial value problem (1.1)–(1.5) (cf. [6, Remark 3.16]). The identity (2.21) follows from a result of Yoshida [6, Remark 3.16]. □

We are now ready to state our main theorem about the existence and uniqueness of positive solutions to SIRD differential system.

Theorem 2.8. *Assume that the hypotheses* (A_1)–(A_6) *hold. The function* $(S(t), I(t), R(t), D(t))$ *given by*

$$\begin{aligned}
S(t) &= \widetilde{S}e^{(\beta/\gamma)\widetilde{R}}\varphi^{-1}(t) = \widetilde{S}e^{(\beta/\mu)\widetilde{D}}\varphi_*^{-1}(t),\\
I(t) &= N - \widetilde{D} + \frac{\mu}{\gamma}\,\widetilde{R} - \widetilde{S}e^{(\beta/\gamma)\widetilde{R}}\varphi^{-1}(t) + \frac{\gamma+\mu}{\beta}\,\log\varphi^{-1}(t)\\
&= N - \widetilde{R} + \frac{\gamma}{\mu}\,\widetilde{D} - \widetilde{S}e^{(\beta/\mu)\widetilde{D}}\varphi_*^{-1}(t) + \frac{\mu+\gamma}{\beta}\,\log\varphi_*^{-1}(t),\\
R(t) &= -\frac{\gamma}{\beta}\,\log\varphi^{-1}(t) = -\frac{\gamma}{\beta}\,\log\varphi_*^{-1}(t) + \widetilde{R} - \frac{\gamma}{\mu}\,\widetilde{D},\\
D(t) &= -\frac{\mu}{\beta}\,\log\varphi^{-1}(t) + \widetilde{D} - \frac{\mu}{\gamma}\,\widetilde{R} = -\frac{\mu}{\beta}\log\varphi_*^{-1}(t)
\end{aligned}$$

is a positive solution of the initial value problem (1.1)–(1.5) *and is unique in the class of positive solutions.*

Proof. The conclusion follows by combining Theorems 2.5, 2.6 and 2.7. □

Remark 2.3. The function $R(t) = -(\gamma/\beta)\log\varphi^{-1}(t) = -(\gamma/\beta)\log\varphi_*^{-1}(t) + \widetilde{R} - (\gamma/\mu)\widetilde{D}$ is a unique solution of the initial value problem (2.1), (2.2) in the class of positive solutions. In fact, we obtain

$$\begin{aligned}
R'(t) &= -\frac{\gamma}{\beta}\,\frac{(\varphi^{-1}(t))'}{\varphi^{-1}(t)} = -\frac{\gamma}{\beta}\,\frac{1}{\varphi'(\varphi^{-1}(t))}\,\frac{1}{\varphi^{-1}(t)} = -\frac{\gamma}{\beta}\big(-\psi(\varphi^{-1}(t))\big) = \frac{\gamma}{\beta}\,\psi(\varphi^{-1}(t))\\
&= \frac{\gamma}{\beta}\Big(\beta N - \beta\widetilde{D} + \frac{\beta\mu}{\gamma}\,\widetilde{R} - \beta\widetilde{S}e^{(\beta/\gamma)\widetilde{R}}\varphi^{-1}(t) + (\gamma+\mu)\log\varphi^{-1}(t)\Big)\\
&= \gamma\Big(N - \widetilde{D} + \frac{\mu}{\gamma}\,\widetilde{R} - \widetilde{S}e^{(\beta/\gamma)\widetilde{R}}e^{-(\beta/\gamma)R(t)} - \frac{\gamma+\mu}{\gamma}\,R(t)\Big)
\end{aligned}$$

in view of $\varphi^{-1}(t) = e^{-(\beta/\gamma)R(t)}$, and therefore $R(t)$ is a solution of (2.1). Since

$$R(0) = -\frac{\gamma}{\beta}\log\varphi^{-1}(0) = -\frac{\gamma}{\beta}\log e^{-(\beta/\gamma)\widetilde{R}} = \widetilde{R},$$

we see that $R(t)$ satisfies (2.2). The uniqueness and the positivity of $R(t)$ follow from Theorems 2.4 and 2.6, respectively. Analogously, we have

$$\begin{aligned} R'(t) &= -\frac{\gamma}{\beta}\frac{(\varphi_*^{-1}(t))'}{\varphi_*^{-1}(t)} = \frac{\gamma}{\beta}\psi_*(\varphi_*^{-1}(t)) \\ &= \gamma\Big(N - \widetilde{R} + \frac{\gamma}{\mu}\widetilde{D} - \widetilde{S}e^{(\beta/\mu)\widetilde{D}}\varphi_*^{-1}(t) + \frac{\mu+\gamma}{\beta}\log\varphi_*^{-1}(t)\Big) \\ &= \gamma\Big(N - \widetilde{D} + \frac{\mu}{\gamma}\widetilde{R} - \widetilde{S}e^{(\beta/\gamma)\widetilde{R}}e^{-(\beta/\gamma)R(t)} - \frac{\mu+\gamma}{\gamma}R(t)\Big) \end{aligned}$$

and

$$R(0) = -\frac{\gamma}{\beta}\log\varphi_*^{-1}(0) + \widetilde{R} - \frac{\gamma}{\mu}\widetilde{D} = -\frac{\gamma}{\beta}\log e^{-(\beta/\mu)\widetilde{D}} + \widetilde{R} - \frac{\gamma}{\mu}\widetilde{D} = \frac{\gamma}{\mu}\widetilde{D} + \widetilde{R} - \frac{\gamma}{\mu}\widetilde{D} = \widetilde{R}.$$

We note that

$$\varphi_*^{-1}(t) = e^{-(\beta/\mu)\widetilde{D}}e^{(\beta/\gamma)\widetilde{R}}\varphi^{-1}(t) \text{ and } \psi(\varphi^{-1}(t)) = \psi_*(\varphi_*^{-1}(t)).$$

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