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EXISTENCE OF RAPIDLY DECAYING POSITIVE SOLUTIONS OF QUASILINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH ARBITRARY NONLINEARITIES

Dedicated to Professor Takaŝi Kusano on his 90th birthday

Abstract. Quasilinear ordinary differential equations are considered without assuming monotonicity conditions of nonlinear terms. New existence results of rapidly decaying positive solutions are established.

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1 Introduction and statement of the results

Let us consider the ordinary differential equation

$$\big(p(t)|y'|^{\alpha-1}y'\big)' + q(t)f(y) = 0, \quad t \ge t_0 \ (> 0), \tag{1.1}$$

without assuming monotonicity conditions on $f(y)$. The following conditions (A1)–(A4) are assumed throughout the paper without further mention:

(A1) $\alpha > 0$ is a positive constant;

(A2) $f : (0,\infty) \to (0,\infty)$ is a continuous function satisfying

$$\int_0 f(y)\,dy < \infty;$$

(A3) $p : [t_0,\infty) \to (0,\infty)$ is a continuous function satisfying

$$\int^\infty \frac{dt}{p(t)^{1/\alpha}} < \infty;$$

(A4) $q : [t_0,\infty) \to \mathbf{R}$ is a continuous function such that $p(t)^{1/\alpha}q(t)$ is of class C^1.

A C^1 positive-valued function $y = y(t)$ defined for sufficiently large t is called a positive solution of equation (1.1) if $p(t)|y'|^{\alpha-1}y'$ is also of class C^1 and satisfies (1.1) for sufficiently large t.

By the assumptions (A2) and (A3), we can introduce the auxiliary functions $\pi(t)$ and $F(y)$ by

$$\pi(t) \equiv \int_t^\infty \frac{ds}{p(s)^{1/\alpha}} \quad \text{and} \quad F(y) \equiv \int_0^y f(z)\,dz,$$

respectively. Note that $F : (0,\infty) \to (0,\infty)$ becomes automatically an increasing function. This fact is essentially employed in this paper.

Let $q(t) \ge 0$ and $y(t)$ be an arbitrary positive solution of (1.1). Then $(p(t)|y'|^{\alpha-1}y')' \le 0$, which shows that $y(t)$ satisfies the estimates

$$c_1\pi(t) \le y(t) \le c_2 \quad \text{for sufficiently large } t$$

for some positive constants c_1 and c_2 [2]. So, to investigate those positive solutions which behave like positive constant multiples of $\pi(t)$ is of some theoretical interest. In the present paper, we call such positive solutions as rapidly decaying solutions.

Definition. A positive solution y of equation (1.1) is called a rapidly decaying positive solution if

$$0 < \liminf_{t\to\infty} \frac{y(t)}{\pi(t)} \le \limsup_{t\to\infty} \frac{y(t)}{\pi(t)} < \infty. \tag{1.2}$$

Remark.

(i) Even though $q(t)$ changes the sign near ∞, we call positive solutions $y(t)$ satisfying (1.2) rapidly decaying positive solutions.

(ii) As will be seen in the sequel, some rapidly decaying positive solutions y may satisfy the property

$$\lim_{t\to\infty} \frac{y(t)}{\pi(t)} = const > 0, \tag{1.3}$$

which shows more precise behavior than (1.2).

(iii) When $q(t) \geq 0$, equation (1.1) may have positive solutions $y(t)$ which decay slower than rapidly decaying positive solutions (see [2] for the details).

The main object of this article is to present a new existence criterion of rapidly decaying positive solutions of (1.1) without assuming monotonicity conditions on $f(y)$. Such a problem was discussed in [5] under the conditions that $p(t) = t^\beta$, $\beta > \alpha$ and $q(t) \geq 0$ without the integrability assumption of f in (A2). In the present paper, we intend to consider this problem based on the other calculation. Note that related results are found in [1].

As an initial result of this problem, we can introduce the following [2]

Theorem 1.1. *Suppose that $q(t) \geq 0$ and there is a nondecreasing continuous function $f^* : (0,\infty) \to (0,\infty)$ satisfying $f(y) \leq f^*(y)$ and*

$$\int^\infty q(t) f^*(k\pi(t))\, dt < \infty \ \text{ for some constant } \ k > 0.$$

Then equation (1.1) has a rapidly decaying positive solution y satisfying (1.3).

Though Theorem 1.1 itself is not given explicitly in [2], the close look at the proof of [2, Theorem 1.2] enables us to establish Theorem 1.1.

Our main results are as follows:

Theorem 1.2. *Suppose that there is a constant $k > 0$ satisfying*

$$\limsup_{t\to\infty} p(t)^{1/\alpha} |q(t)| F(k\pi(t)) < \frac{\alpha}{2(\alpha+1)}\, k^{\alpha+1}$$

and

$$\int^\infty \left|(p(t)^{1/\alpha} q(t))'\right| F(k\pi(t))\, dt < \infty.$$

Then, equation (1.1) has a rapidly decaying positive solution.

Corollary 1.1. *Suppose that there is a constant $k > 0$ satisfying*

$$\lim_{t\to\infty} p(t)^{1/\alpha} q(t) F(k\pi(t)) = 0$$

and

$$\int^\infty \left|(p(t)^{1/\alpha} q(t))'\right| F(k\pi(t))\, dt < \infty.$$

Then equation (1.1) has a rapidly decaying positive solution y satisfying (1.3).

Corollary 1.2. *Suppose that*

$$q(t) \geq 0 \ \text{ and } \ \left[p(t)^{1/\alpha} q(t)\right]' \leq 0.$$

Then equation (1.1) has a rapidly decaying positive solution y satisfying (1.3).

This paper is organized as follows. In Section 2, the proofs of our results are given. Section 3 provides illustrative examples.

2 Proof of the results

Proof of Theorem 1.2. A rapidly decaying positive solution $y(t)$ will be obtained as a positive solution of the following integral equation:

$$y(t) = \left(\frac{\alpha+1}{\alpha}\right)^{\frac{1}{\alpha+1}} \int_t^\infty p(s)^{1/\alpha} \Bigg[C_0 - p(s)^{1/\alpha} q(s) F(y(s)) + \int_s^\infty \left[-p(r)^{1/\alpha} q(r)\right]' F(y(r))dr\Bigg]^{\frac{1}{\alpha+1}} ds, \quad t \geq T_0,$$

with some constants $C_0 > 0$ and $T_0 \geq t_0$. We employ the fixed point theorem to solve this equation.

For $t \geq t_0$, we put

$$I(t) = \int_t^\infty \left|(p(s)^{1/\alpha} q(s))'\right| F(k\pi(s))\,ds.$$

Let $m_2 > 0$ be a constant satisfying

$$p(t)^{1/\alpha}|q(t)|F(k\pi(t)) < m_2 < \frac{\alpha}{2(\alpha+1)}\,k^{\alpha+1},\ \ t \geq T_1,$$

where $T_1 \geq t_0$ is a sufficiently large number. For this m_2, we can choose a constant $m_1 > 0$ satisfying

$$m_1 + m_2 < \frac{\alpha}{\alpha+1}\,k^{\alpha+1} \ \text{ and } \ m_1 - m_2 > 0.$$

Then there is a sufficiently large $T \geq T_1$ satisfying

$$m_1 + m_2 + I(T) \leq \frac{\alpha}{\alpha+1}\,k^{\alpha+1} \tag{2.1}$$

and

$$m_1 - m_2 - I(T) > 0.$$

We put

$$m_1 - m_2 - I(T) = \frac{\alpha}{\alpha+1}\,k_1^{\alpha+1}. \tag{2.2}$$

(Note that automatically $0 < k_1 < k$.)

Let $C[T,\infty)$ be the Frechét space with the topology of uniform convergence of functions on every compact subinterval of $[T,\infty)$. We define the closed convex subset $Y \subset C[T,\infty)$ as

$$Y = \big\{y \in C[T,\infty) \mid \ k_1\pi(t) \leq y(t) \leq k\pi(t) \ \text{ for } \ t \geq T\big\}.$$

For $y \in Y$, we put

$$\Phi y(t) = m_1 - p(t)^{1/\alpha} q(t) F(y(t)) - \int_t^\infty \big[p(s)^{1/\alpha} q(s)\big]' F(y(s))\,ds,\ \ t \geq T,$$

and

$$\mathcal{F}y(t) = \Big(\frac{\alpha+1}{\alpha}\Big)^{\frac{1}{\alpha+1}} \int_t^\infty p(s)^{-1/\alpha} [\Phi y(s)]^{\frac{1}{\alpha+1}}\,ds,\ \ t \geq T.$$

Below, we will show that the Schauder–Tychonoff fixed point theorem [4, Theorems 2.3.8 and 4.5.1] is applicable to $\mathcal{F}$ and Y.

(i) *We show that* $\mathcal{F}(Y) \subset Y$. Let $y \in Y$. By (2.1), we have

$$\begin{aligned}\Phi y(t) &\leq m_1 + p(t)^{1/\alpha}|q(t)|F(k\pi(t)) \\ &\quad + \int_T^\infty \left|[p(s)^{1/\alpha} q(s)]'\right| F(k\pi(s))\,ds \leq m_1 + m_2 + I(T) \leq \frac{\alpha}{\alpha+1}\,k^{\alpha+1},\ \ t \geq T.\end{aligned}$$

Similarly, we find from (2.2) that

$$\Phi y(t) \geq m_1 - m_2 - I(T) = \frac{\alpha}{\alpha+1}\,k_1^{\alpha+1},\ \ t \geq T.$$

Therefore, we have

$$\mathcal{F}y(t) \leq \Big(\frac{\alpha+1}{\alpha}\Big)^{\frac{1}{\alpha+1}} k \Big(\frac{\alpha}{\alpha+1}\Big)^{\frac{1}{\alpha+1}} \int_t^\infty p(s)^{-1/\alpha}\,ds = k\pi(t),\ \ t \geq T,$$

and

$$\mathcal{F}y(t) \geq \Big(\frac{\alpha+1}{\alpha}\Big)^{\frac{1}{\alpha+1}} k_1 \Big(\frac{\alpha}{\alpha+1}\Big)^{\frac{1}{\alpha+1}} \int_t^\infty p(s)^{-1/\alpha}\, ds = k_1\pi(t), \quad t \geq T.$$

Consequently, $\mathcal{F}y \in Y$, and hence $\mathcal{F}(Y) \subset Y$.

(ii) *We show that $\mathcal{F}$ is a continuous mapping.* Let $\{y_n\} \subset Y$ and $y \in Y$ be, respectively, a sequence and an element which satisfy $\lim_{n\to\infty} y_n(t) = y(t)$ uniformly on every finite interval of $[T,\infty)$. Let $T' > T$ be an arbitrary constant. We show that $\lim_{n\to\infty} \mathcal{F}y_n(t) = \mathcal{F}y(t)$ uniformly on $[T,T']$.

As a first step, we show that

$$\lim_{n\to\infty} \int_T^\infty \big|[p(s)^{1/\alpha}q(s)]'\big|\,\big|F(y_n(s)) - F(y(s))\big|\, ds = 0. \tag{2.3}$$

In fact, since

$$\big|[p(s)^{1/\alpha}q(s)]'\big|\,\big|F(y_n(s)) - F(y(s))\big| \leq 2\big|[p(s)^{1/\alpha}q(s)]'\big|F(k\pi(s)), \quad s \geq T,$$

and

$$\int_T^\infty \big|[p(s)^{1/\alpha}q(s)]'\big|F(k\pi(s))\, ds < \infty,$$

the Lebesgue dominated convergence theorem implies (2.3). Therefore,

$$\lim_{n\to\infty} \Phi y_n(t) = \Phi y(t) \ \text{ uniformly on } [T,T'].$$

Next, we notice that

$$|\mathcal{F}y_n(t) - \mathcal{F}y(t)| \leq \Big(\frac{\alpha+1}{\alpha}\Big)^{\frac{1}{\alpha+1}} \int_T^\infty p(s)^{-1/\alpha}\Big|[\Phi y_n(s)]^{\frac{1}{\alpha+1}} - [\Phi y(s)]^{\frac{1}{\alpha+1}}\Big|\, ds, \quad t \geq T.$$

Since $0 \leq \Phi y_n(t), \Phi y(t) \leq m_1 + m_2 + I(T)$, we find that

$$p(s)^{-1/\alpha}\Big|[\Phi y_n(s)]^{\frac{1}{\alpha+1}} - [\Phi y(s)]^{\frac{1}{\alpha+1}}\Big| \leq 2(m_1+m_2+I(T))^{\frac{1}{\alpha+1}} p(s)^{-1/\alpha}, \quad s \geq T.$$

By assumption (A3), the Lebesgue dominated convergence theorem implies that

$$\lim_{n\to\infty} \sup_{[T,\infty)} |\mathcal{F}y_n(t) - \mathcal{F}y(t)| = 0.$$

Therefore, $\{\mathcal{F}y_n\}$ converges to $\mathcal{F}y$ uniformly on $[T,T']$.

(iii) *We show that $\mathcal{F}Y$ is relatively compact.* Since $\mathcal{F}(Y) \subset Y$, the set $\mathcal{F}(Y)$ is bounded on every compact subinterval of $[T,\infty)$. Next, let $y \in Y$. Then we obtain

$$\begin{aligned} |(\mathcal{F}y)'(t)| &= \Big(\frac{\alpha+1}{\alpha}\Big)^{\frac{1}{\alpha+1}} p(t)^{-1/\alpha}[\Phi y(t)]^{\frac{1}{\alpha+1}} \\ &\leq \Big(\frac{\alpha+1}{\alpha}\Big)^{\frac{1}{\alpha+1}} p(t)^{-1/\alpha}\Big(\frac{\alpha}{\alpha+1}\,k^{\alpha+1}\Big)^{\frac{1}{\alpha+1}} = kp(t)^{-1/\alpha}, \quad t \geq T. \end{aligned}$$

So, the set $\{(\mathcal{F}y)' \mid y \in Y\}$ is bounded on every compact subinterval of $[T,\infty)$. By the Ascoli–Arzelà theorem, we find that $\mathcal{F}Y$ is relatively compact.

By the above consideration, the Schauder-Tychonoff fixed point theorem shows that there is a fixed element $y \in Y : \mathcal{F}y = y$. The element y satisfies

$$y(t) = \Big(\frac{\alpha+1}{\alpha}\Big)^{\frac{1}{\alpha+1}} \int_t^\infty p(s)^{-1/\alpha}[\Phi y(s)]^{\frac{1}{\alpha+1}}\, ds, \quad t \geq T.$$

We show that $y(t)$ is a solution of (1.1). From this formula, we know that

$$p(t)(-y'(t))^{\alpha} = \Big(\frac{\alpha+1}{\alpha}\Big)^{\frac{\alpha}{\alpha+1}} [\Phi y(t)]^{\frac{\alpha}{\alpha+1}}, \ \ t \ge T.$$

So,

$$\big[p(t)(-y')^{\alpha}\big]^{\frac{\alpha+1}{\alpha}} = \frac{\alpha+1}{\alpha}\Bigg[m_1 - p(t)^{1/\alpha}q(t)F(y) - \int_t^{\infty} \big[p(s)^{1/\alpha}q(s)\big]' F(y(s))\,ds\Bigg], \ \ t \ge T. \tag{2.4}$$

Differentiating both sides, we obtain

$$\frac{\alpha+1}{\alpha}\big[p(t)(-y')^{\alpha}\big]^{1/\alpha} \cdot \big(p(t)(-y')^{\alpha}\big)' = \frac{\alpha+1}{\alpha} p(t)^{1/\alpha} q(t) f(y)(-y'), \ \ t \ge T.$$

Since $y'(t) < 0$, we get

$$\big(p(t)(-y')^{\alpha}\big)' = q(t)f(y), \ \ t \ge T,$$

which is equivalent to equation (1.1).

Since $y \in Y$, $y(t)$ satisfies (1.2) by the definition of Y. This completes the proof. □

Proof of Corollary 1.1. Since the assumptions imply those of Theorem 1.2, we can find a rapidly decaying positive solution $y(t)$ of (1.1) satisfying (2.4). We show that actually (1.3) holds. Since

$$\lim_{t\to\infty} p(t)^{1/\alpha} q(t) F(y(t)) = 0,$$

we find from (2.4) that

$$\lim_{t\to\infty} p(t)[-y'(t)]^{\alpha} = \Big(\frac{\alpha+1}{\alpha} m_1\Big)^{\frac{\alpha}{\alpha+1}}.$$

By L'Hôspital's rule, we find that

$$\lim_{t\to\infty} \frac{y(t)}{\pi(t)} = \lim_{t\to\infty} p(t)^{1/\alpha}(-y'(t)) = \lim_{t\to\infty} \big[p(t)(-y'(t))^{\alpha}\big]^{1/\alpha} = \Big(\frac{\alpha+1}{\alpha} m_1\Big)^{\frac{1}{\alpha+1}}.$$

This completes the proof. □

Proof of Corollary 1.2. Recall that $\lim_{t\to\infty} F(k\pi(t)) = 0$ for any constant $k > 0$. The assumptions imply that there is a limit $\lim_{t\to\infty} p(t)^{1/\alpha} q(t) \in [0,\infty)$. So, the assumptions of Corollary 1.1 hold.

This completes the proof. □

3 Examples

Example 3.1. Let $\beta > 0$, $\delta > 2$ and $r > 1$ be the constants. Let us define the sequence of closed intervals $\{I_n\}$ by

$$I_n = \Big[\frac{1}{n} - \frac{1}{n^{\delta}}, \frac{1}{n} + \frac{1}{n^{\delta}}\Big]$$

for sufficiently large $n \in \mathbf{N}$. There is a sufficiently large $n_0 \in \mathbf{N}$ such that

$$I_n \cap I_{n+1} = \varnothing, \ \text{ and } \ r^{-n} < \frac{1}{(n+1)^{\beta}} \ \text{ for } \ n \ge n_0.$$

Define the function $f_1(y)$ on $(0, (1/n_0) + (1/n_0^{\delta}))$ by

$$f_1(y) = \begin{cases} n^{\delta-\beta}\Big(y - \dfrac{1}{n}\Big) + \dfrac{1}{n^{\beta}} & \text{if } \dfrac{1}{n} - \dfrac{1}{n^{\delta}} \le y \le \dfrac{1}{n}, \ \ n \ge n_0, \\ -n^{\delta-\beta}\Big(y - \dfrac{1}{n}\Big) + \dfrac{1}{n^{\beta}} & \text{if } \dfrac{1}{n} \le y \le \dfrac{1}{n} + \dfrac{1}{n^{\delta}}, \ \ n \ge n_0, \\ 0 & \text{if } y \notin \displaystyle\bigcup_{n=n_0}^{\infty} I_n. \end{cases}$$

Further, define the function $f_2(y)$ by

$$f_2(y) = r^{-n} \text{ if } \frac{1}{n+1} < y \le \frac{1}{n}, \quad n \ge n_0.$$

Put

$$f(y) = \max\left\{f_1(y), f_2(y)\right\} \text{ for } y \in \left(0, \frac{1}{n_0}\right],$$

and for $y \in [1/n_0, \infty)$ we define $f(y)$ in such a way that $f(y)$ is a continuous positive function. Then it is found that $f : (0,\infty) \to (0,\infty)$, $f(+0) = 0$, f is continuous and $f(y) \le y^\beta$ near $+0$. Further, we find that for some constants $C_1, C_2 > 0$,

$$C_1 y^{\delta+\beta-1} \le F(y) \equiv \int_0^y f(z)\,dz \le C_2 y^{\delta+\beta-1} \text{ for } y \text{ near } +0. \tag{3.1}$$

Note that $f(y)$ is not a monotone function near $+0$.

Let us consider the equation

$$\left(t^\rho |y'|^{\alpha-1} y'\right)' + t^{-\lambda} f(y) = 0, \quad t \ge t_0 (> 0), \tag{3.2}$$

where $\rho > \alpha > 0$ and $\lambda \in \mathbf{R}$. This equation satisfies conditions (A1)–(A4). We find that for equation (3.2), $\pi(t)$ is given by

$$\pi(t) = \frac{\alpha}{\rho - \alpha}\, t^{-\frac{\rho-\alpha}{\alpha}}.$$

Since $f(y) \le y^\beta$ near $+0$ and y^β is an increasing function, Theorem 1.1 asserts that equation (3.2) has a rapidly decaying positive solution if

$$\lambda > 1 - \frac{\beta(\rho - \alpha)}{\alpha}.$$

On the other hand, in view of (3.1), Corollary 1.1 asserts that equation (3.2) has a rapidly decaying positive solution if

$$\lambda > 1 - \frac{(\beta + \delta - 2)(\rho - \alpha)}{\alpha}.$$

Since $\delta > 2$, the latter condition is weaker than the former.

Example 3.2. This example gives an application of our results to the semilinear Laplace equations via the supersolution-subsolution method in [3]. (See [3] for the definitions of supersolutions and subsolutions of elliptic equations under consideration.)

Suppose that $N > 2$ is an integer, and put $\Omega_R = \{x \in \mathbf{R}^N \mid |x| > R\}$ for large $R > 0$. Let us consider the following semilinear Laplace equation near the ∞ of $\mathbf{R}^N$:

$$\Delta u + b(x) f(u) = 0, \tag{3.3}$$

where $x = (x_i) \in \mathbf{R}^N$ and

$$\Delta u = \Delta u(x) = \sum_{i=1}^N \frac{\partial^2 u}{\partial x_i^2}.$$

We assume that $b(x)$ is a nonnegative and locally Hölder continuous function (with exponent $\theta \in (0,1)$), and $f : (0,\infty) \to (0,\infty)$ is a locally Lipschitz continuous function satisfying

$$\int_0 f(u) du < \infty. \tag{3.4}$$

Let $b^* : [R_0, \infty) \to [0, \infty)$ be a C^1-function such that

$$0 \le b(x) \le b^*(|x|), \quad x \in \Omega_{R_0},$$

where $R_0 > 0$ is a sufficiently large number, and

$$\left(r^{2(N-1)}b^*(r)\right)' \leq 0.$$

Then we can show that equation (3.3) has a positive solution $u \in C^{2+\theta}_{loc}(\Omega_R)$, $R \geq R_0$, satisfying

$$0 < \liminf_{|x|\to\infty} |x|^{N-2}u(x) \leq \limsup_{|x|\to\infty} |x|^{N-2}u(x) < \infty. \tag{3.5}$$

To see this, we employ the supersolution-subsolution method in [3]. It is easily seen that a radial positive function $v(r), r = |x|$ satisfying

$$(r^{N-1}v')' + r^{N-1}b^*(r)f(v) = 0, \ \text{ near } \infty, \tag{3.6}$$

is a supersolution of equation (3.3). By assumption (3.4), we find that assumptions (A1)–(A4) hold for equation (3.6). Employing Corollary 1.2, we find that equation (3.6) has a rapidly decaying positive solution $v(r)$ satisfying

$$\lim_{r\to\infty} r^{N-2}v(r) = c \in (0,\infty).$$

On the other hand, the function

$$w(x) \equiv c_1|x|^{-(N-2)}, \ \ 0 < c_1 < c,$$

is a subsolution of equation (3.3) satisfying

$$w(x) \leq v(|x|), \ \text{ near } \infty.$$

Therefore, [3, Theorem 3.3] implies that there is a solution $u(x)$ of equation (3.3) of the class $C^{2+\theta}_{\text{loc}}$ satisfying

$$w(x) \leq u(x) \leq v(|x|), \ \text{ near } \infty.$$

Consequently, u satisfies (3.5).

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