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LIMIT CYCLES FOR PIECEWISE DIFFERENTIAL SYSTEMS FORMED BY AN ARBITRARY LINEAR SYSTEM AND A CUBIC ISOCHRONOUS CENTER

Abstract. In this paper, we study the existence and the maximum number of crossing limit cycles that can exhibit of some class of planar piecewise differential systems formed by two regions and separated by a straight line $x = 0$, where in the left region we define an arbitrary linear differential system and in the right region we define a cubic polynomial differential system with a homogeneous nonlinearity and an isochronous center at the origin. More precisely, we show that these systems may have at most zero or one or two explicit algebraic or non-algebraic limit cycles depending on the type of their linear differential system, i.e., if those systems have foci, center, saddle, node with different eigenvalues, non-diagonalizable node with equal eigenvalues or linear system without equilibrium points.

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Key words and phrases. Discontinuous piecewise differential systems, first integral, Poincaré map, crossing limit cycles.

რეზიუმე. ნაშრომში შესწავლილია გადამკვეთი ზღვრული ციკლების არსებობა და მაქსიმალური რაოდენობა, რომელიც შეიძლება ჰქონდეს ორი არით შექმნილ და $x = 0$ წრფით განცალკევებულ ბრტყელ ნაჭრობრივად დიფერენცირებად სისტემათა გარკვეულ კლასს, სადაც მარცხენა არეში განსაზღვრულია ნებისმიერი წრფივი დიფერენციალური სისტემა, ხოლო მარჯვენაში – კუბური პოლინომიალური დიფერენციალური სისტემა ერთგვაროვანი არაწრფივობით და იზოქრონული ცენტრით ნულში. უფრო ზუსტად, ნაჩვენებია, რომ ამ სისტემებს შეიძლება ჰქონდეთ მაქსიმუმ ნული ან ერთი ან ორი ცხადი ალგებრული ან არაალგებრული ზღვრული ციკლი იმისდა მიხედვით, თუ როგორია წრფივი დიფერენციალური სისტემის ტიპი, ე.ი. აქვს თუ არა ამ სისტემებს ფოკუსი, ცენტრი, უნაგირი, კვანძი განსხვავებული საკუთარი მნიშვნელობებით, არადიაგონალიზებადი კვანძი ტოლი საკუთარი მნიშვნელობებით, ან წრფივი სისტემა წონასწორობის წერტილების გარეშე.

1 Introduction and statement of the main result

The problem of the existence and the number of isolated periodic orbits, the so-called limit cycles, is one of the most challenging problems in the qualitative theory of planar ordinary differential equations. The search for a maximum number of limit cycles that polynomial differential systems of a given degree may have is a part of the 16th Hilbert's Problem (see [14]). This problem remains unsolved if $n \geq 2$. In the last few years, there has been an increasing interest in the study of the problem of bounding a number of limit cycles for planar piecewise differential systems (see [5, 7–9, 11, 15, 17, 23]). This interest has been mainly motivated by their wider range of applications in various fields of science (e.g., engineering, biology, control theory, design of electric circuits, mechanical systems, economics science, medicine, chemistry, physics, etc.).

There are many papers studying planar piecewise linear differential systems with two zones (see, e.g., [1–3, 18, 20, 21] and the references therein). For the discontinuous planar nonlinear differential systems there are several papers studying the number of limit cycles (see [5,8,15–17] and the references therein). Note that for the piecewise cubic polynomial differential system, there are two recent papers [12, 13] obtaining at least 18 and 24 small limit cycles, respectively.

Another interesting and natural problem is to express analytically the limit cycles. Nevertheless, in most of these papers explicit limit cycles do not appear. The present paper is a contribution in that direction, motivated by the recent publication of some research papers exhibiting planar polynomial systems with algebraic or non-algebraic limit cycles given analytically (see, e.g., [1, 3, 4, 19]).

The goal of this paper is to provide the exact maximum number of limit cycles of planar discontinuous piecewise differential systems separated by a straight line $x = 0$ and formed by an arbitrary linear differential system and cubic systems with homogeneous nonlinearity with an isochronous center at the origin.

More precisely, we consider planar discontinuous piecewise differential systems with two linearity regions separated by a straight line $\Sigma = \{(x, y) \in \mathbb{R}^2 : \ x = 0\}$. We assume that the two linearity regions in the phase plane are the left and right half-planes

$$\Sigma_L = \big\{(x, y) \in \mathbb{R}^2 : \ x < 0\big\}, \ \ \Sigma_R = \big\{(x, y) \in \mathbb{R}^2 : \ x > 0\big\},$$

formed by an arbitrary linear differential system and by cubic systems with homogeneous nonlinearity with an isochronous center at the origin. We can write such systems as

$$\begin{gathered}\dot{x} = -y + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \ \ \dot{y} = x + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3 \ \text{ in } \Sigma_R,\\ \dot{x} = \alpha x + \beta y + \gamma, \ \ \dot{y} = \eta x + \delta y + \xi \ \text{ in } \Sigma_L,\end{gathered} \tag{1.1}$$

where α, β, γ, η, δ, ξ and a_{ij}, b_{ij} for $i, j \in \{0, 1, 2, 3\}$, $i + j = 3$ are the real constants.

A center of a planar polynomial differential system is called an isochronous center if there exists a neighborhood such that all periodic orbits in this neighborhood have the same period. Due to Theorem 11.1 of [6], a cubic polynomial differential system with homogeneous nonlinearity and with an isochronous center at the origin has one of the following forms:

$$(S_1): \ \begin{cases}\dot{x} = -y - 3xy^2 + x^3,\\ \dot{y} = x + 3x^2y - y^3.\end{cases}$$

$$(S_2): \ \begin{cases}\dot{x} = -y + x^2y,\\ \dot{y} = x + 3xy^2.\end{cases}$$

$$(S_3): \ \begin{cases}\dot{x} = -y(1 + 3x^2),\\ \dot{y} = x(1 + 2x^2 - 9y^2).\end{cases}$$

$$(S_4): \ \begin{cases}\dot{x} = -y(1 - 3x^2),\\ \dot{y} = x(1 - 2x^2 + 9y^2).\end{cases}$$

It is known that system (S_1) has the first integral

$$H_1(x, y) = \frac{(x^2 + y^2)^2}{1 + 4xy}\,. \tag{1.2}$$

The period annulus of this system is given by $\{(x,y) \in \mathbb{R}^2 : \ H_1(x,y) = h_1, \ h_1 \in (0,+\infty)\}$.

System (S_2) has the first integral

$$H_2(x,y) = \frac{x^2+y^2}{1+x^2}\,. \tag{1.3}$$

The period annulus of this system is given by $\{(x,y) \in \mathbb{R}^2 : \ H_2(x,y) = h_2, \ h_2 \in (0,+\infty)\}$.

System (S_3) has the first integral

$$H_3(x,y) = \frac{(x+2x^3)^2+y^2}{(1+3x^2)^3}\,. \tag{1.4}$$

The period annulus of this system is given by $\{(x,y) \in \mathbb{R}^2 : \ H_3(x,y) = h_3, \ h_3 \in (0,\frac{4}{27})\}$.

System (S_4) has the first integral

$$H_4(x,y) = \frac{(x-2x^3)^2+y^2}{(1-3x^2)^3}\,. \tag{1.5}$$

The period annulus of this system is given by $\{(x,y) \in \mathbb{R}^2 : \ H_4(x,y) = h_4, \ h_4 \in (0,+\infty)\}$.

The linear differential system that we consider in the second half-plane Σ_L is either a focus (we include in this class the centers), or a saddle, or a node with different eigenvalues, or a node with equal eigenvalues whose linear part does not diagonalize, or linear without equilibrium points. Note that if piecewise differential systems with two pieces separated by a straight line has a star node (node with equal eigenvalues whose linear part diagonalize), this prevents the existence of periodic orbits.

Consider the piecewise differential systems (1.1). In order to state precisely our results, we introduce first some notations and definitions. In accordance with Filippov [10], we distinguish the following open regions in the discontinuity set Σ.

1. Crossing region:
$$\Sigma_c = \big\{(0,y) \in \Sigma : \ (a_{03}y^3 - y)(\beta y + \gamma) > 0\big\}. \tag{1.6}$$
2. Sliding region:
$$\Sigma_s = \big\{(0,y) \in \Sigma : \ (a_{03}y^3 - y)(\beta y + \gamma) \le 0\big\}. \tag{1.7}$$

As usual, isolated periodic orbits are called *limit cycles*. There are two types of limit cycles "crossing and sliding ones" in the planar discontinuous piecewise differential systems. The first type of the limit cycles contains some arc of discontinuity lines that separate the different differential systems (for more details see [22]), and the second type contains only isolated points of the lines of discontinuity. But we shall work only with crossing limit cycles. An equilibrium point is called a real (resp. *virtual*) singular point of the right system of (1.1) if this point locates in the region Σ_R (resp. Σ_L). A similar definition can be done for the left system of (1.1).

The main result of this paper is the following

Theorem 1.1. *The following statements hold for the discontinuous piecewise differential systems* (1.1)

(1) *if* (1.1) *is of the type linear focus and cubic isochronous center at the origin, then the piecewise differential systems* (1.1) *have at most two crossing limit cycles. Moreover, these limit cycles if there exists are non-algebraic and there are systems of this type with one or two limit cycles.*

(2) *if* (1.1) *is of the type linear center and cubic isochronous center at the origin, then the piecewise differential systems* (1.1) *have no crossing limit cycles.*

(3) *if* (1.1) *is of the type linear saddle and cubic isochronous center at the origin, then the piecewise differential systems* (1.1) *have at most one crossing limit cycle. Moreover, this limit cycle, if exists, is algebraic and here are systems of this type with one limit cycle.*

(4) *if* (1.1) *is of the type linear node with different eigenvalues and cubic isochronous center at the origin, then the piecewise differential systems* (1.1) *have at most one crossing limit cycle. Moreover, this limit cycle, if exists, is algebraic and here are systems of this type with one limit cycle.*

(5) *if* (1.1) *is of the type non-diagonalizable linear node with equal eigenvalues and cubic isochronous center at the origin, then the piecewise differential systems* (1.1) *have at most one crossing limit cycle. Moreover, this limit cycle, if exists, is non-algebraic and there are systems of this type with one limit cycle.*

(6) *if* (1.1) *of the type linear without equilibrium point and cubic isochronous center at the origin, then the piecewise differential systems* (1.1) *have at most one crossing limit cycle. Moreover, this limit cycle, if exists, is non-algebraic and there are systems of this type with one limit cycle.*

Theorem 1.1 will be proved in Section 2.

In the next proposition, we show that there are discontinuous piecewise differential systems (1.1) of the type linear real focus and cubic isochronous center at the origin, with two non-algebraic crossing limit cycles.

Proposition 1.2. *The discontinuous piecewise differential systems* (1.1) *formed by one of the four cubic isochronous centers* (S_1), *or* (S_2), *or* (S_3), *or* (S_4), *and a family with one parameter of linear differential system of the form*

$$\dot{x} = -\frac{2}{5}\,x + \beta y - \frac{1}{8}\,, \quad \dot{y} = -\frac{1}{50\beta}\,(52x + 5)\,, \tag{1.8}$$

with $\beta \in (-\infty, -0.563\,86)$, *have exactly two nested non-algebraic crossing limit cycles. Moreover, these limit cycles are given by*

$$\begin{aligned}
\Gamma_1 &= \big\{(x,y) \in \Sigma_R : \ \ H_j(x,y) = h_j\big\} \cup \big\{(x,y) \in \Sigma_L : \ \ H_{f1}(x,y) = 5.974\,1 \times 10^{-2}\big\},\\
\Gamma_2 &= \big\{(x,y) \in \Sigma_R : \ \ H_j(x,y) = h'_j\big\} \cup \big\{(x,y) \in \Sigma_L : \ \ H_{f1}(x,y) = 0.101\,04\big\},
\end{aligned}$$

where $j \in \{1,2,3,4\}$,

$$\begin{aligned}
h_1 &= \Big(\frac{0.139\,44}{\beta}\Big)^4, \quad h_2 = h_3 = h_4 = \Big(\frac{0.139\,44}{\beta}\Big)^2,\\
h'_1 &= \Big(\frac{0.217\,03}{\beta}\Big)^4, \quad h'_2 = h'_3 = h'_4 = \Big(\frac{0.217\,03}{\beta}\Big)^2
\end{aligned}$$

and

$$H_{f1}(x,y) = \Big(\frac{26}{25}\,x^2 - \frac{2}{5}\,\beta xy + \frac{61}{260}\,x + \beta^2 y^2 - \frac{11}{52}\,\beta y + \frac{17}{832}\Big)e^{-\frac{2}{5}\,\arctan\frac{520x+50}{104x-520\beta y+55}}.$$

See Figure 1.

In the next proposition, we show that there are discontinuous piecewise differential systems (1.1) of the type linear virtual focus and cubic isochronous center at the origin, with one non-algebraic crossing limit cycle.

Proposition 1.3. *The discontinuous piecewise differential systems* (1.1) *formed by one of the four cubic isochronous centers* (S_1) *or* (S_2) *or* (S_3) *or* (S_4) *and a class with one parameter of linear differential system of the form*

$$\dot{x} = -x + \beta y + 1, \quad \dot{y} = \frac{1}{\beta}\,(3\beta y - 5x + 5), \tag{1.9}$$

with $\beta \in (-\infty, -5.466\,4)$, *have exactly one non-algebraic crossing limit cycle. Moreover, this limit cycle is given by*

$$\Gamma_1 = \big\{(x,y) \in \Sigma_R : \ \ H_j(x,y) = h_j\big\} \cup \big\{(x,y) \in \Sigma_L : \ \ H_{f2}(x,y) = 5.374\,3 \times 10^{-2}\big\},$$

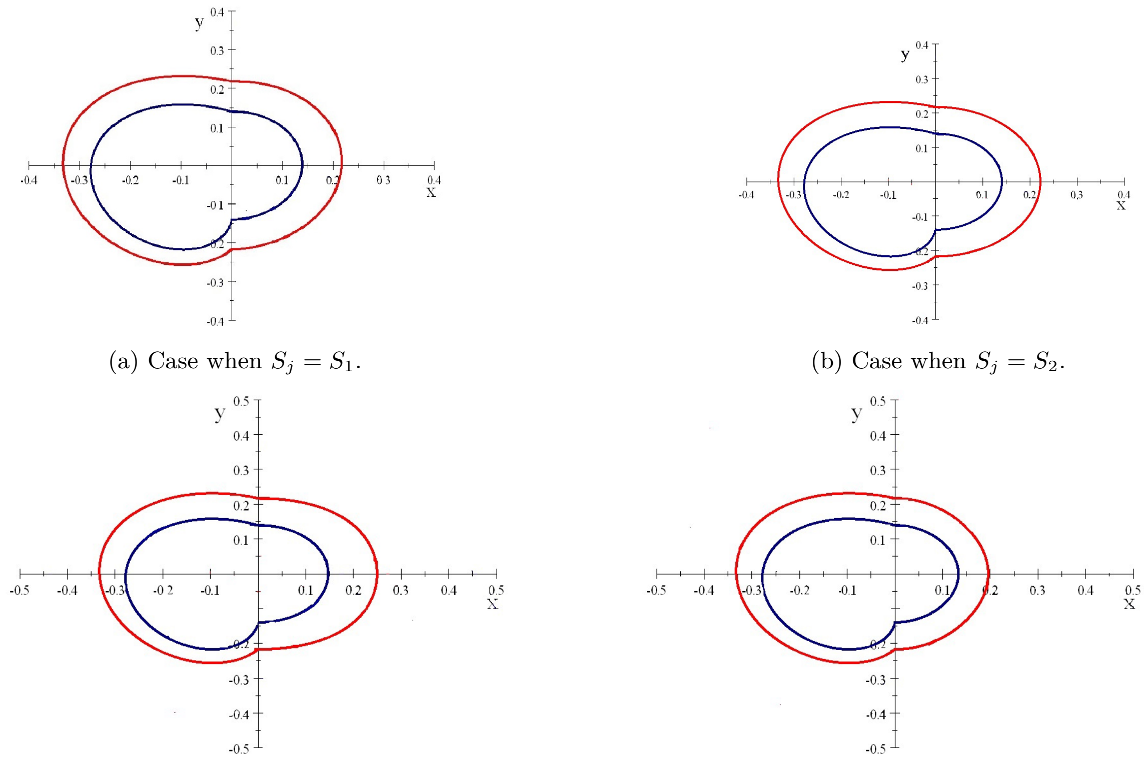
where $j \in \{1,2,3,4\}$,

$$h_1 = \Big(\frac{2.104\,0}{\beta}\Big)^4, \quad h_2 = h_3 = h_4 = \Big(\frac{2.104\,0}{\beta}\Big)^2$$

and

$$H_{f2}(x,y) = (5x^2 - 4\beta xy - 10x + \beta^2 y^2 + 4\beta y + 5)e^{-2\arctan\frac{x-1}{\beta y - 2x+2}}.$$

See Figure 2.



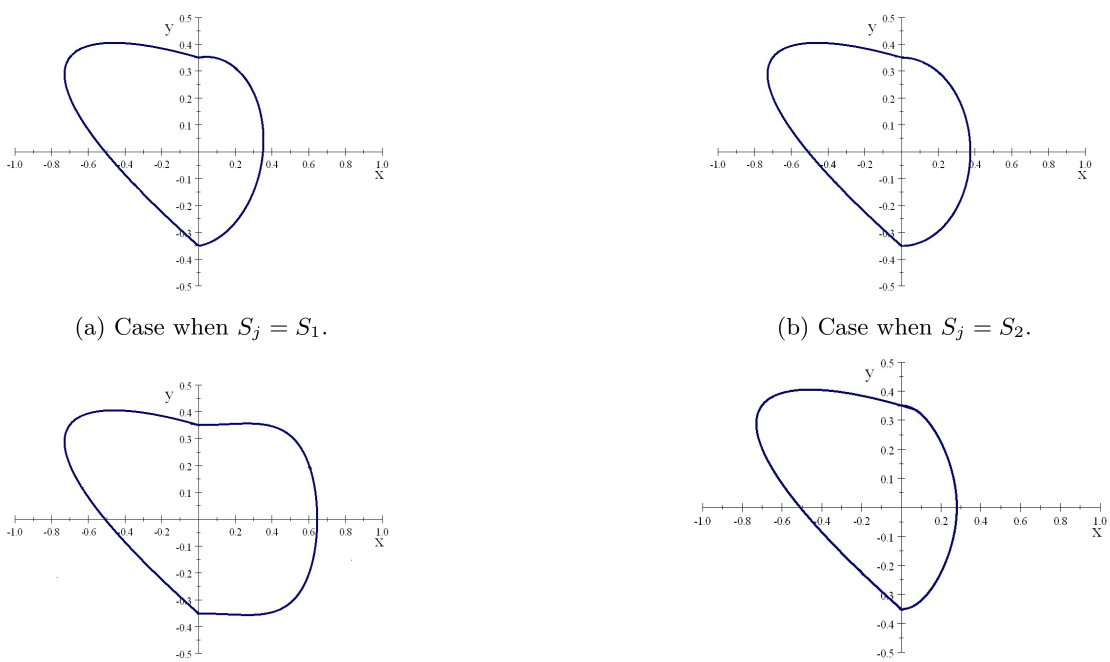
(a) Case when $S_j = S_1$.

(b) Case when $S_j = S_2$.

(c) Case when $S_j = S_3$.

(d) Case when $S_j = S_4$.

Figure 1: The two nested crossing limit cycles of systems (1.8) + (S_j) with $\beta = -1$.



(a) Case when $S_j = S_1$.

(b) Case when $S_j = S_2$.

(c) Case when $S_j = S_3$.

(d) Case when $S_j = S_4$.

Figure 2: The unique crossing limit cycle of systems (1.9) + (S_j) with $\beta = -6$.

In the next proposition, we show that there are discontinuous piecewise differential systems (1.1) of the type linear saddle and cubic isochronous center at the origin, with one algebraic crossing limit cycle.

Proposition 1.4. *The discontinuous piecewise differential systems* (1.1) *formed by one of the four cubic isochronous centers* (S_1) *or* (S_2) *or* (S_3) *or* (S_4) *and a class with one parameter of linear differential system of the form*

$$\dot{x} = x + \beta y + \frac{1}{10}, \quad \dot{y} = \frac{1}{4\beta}(3x + 4), \tag{1.10}$$

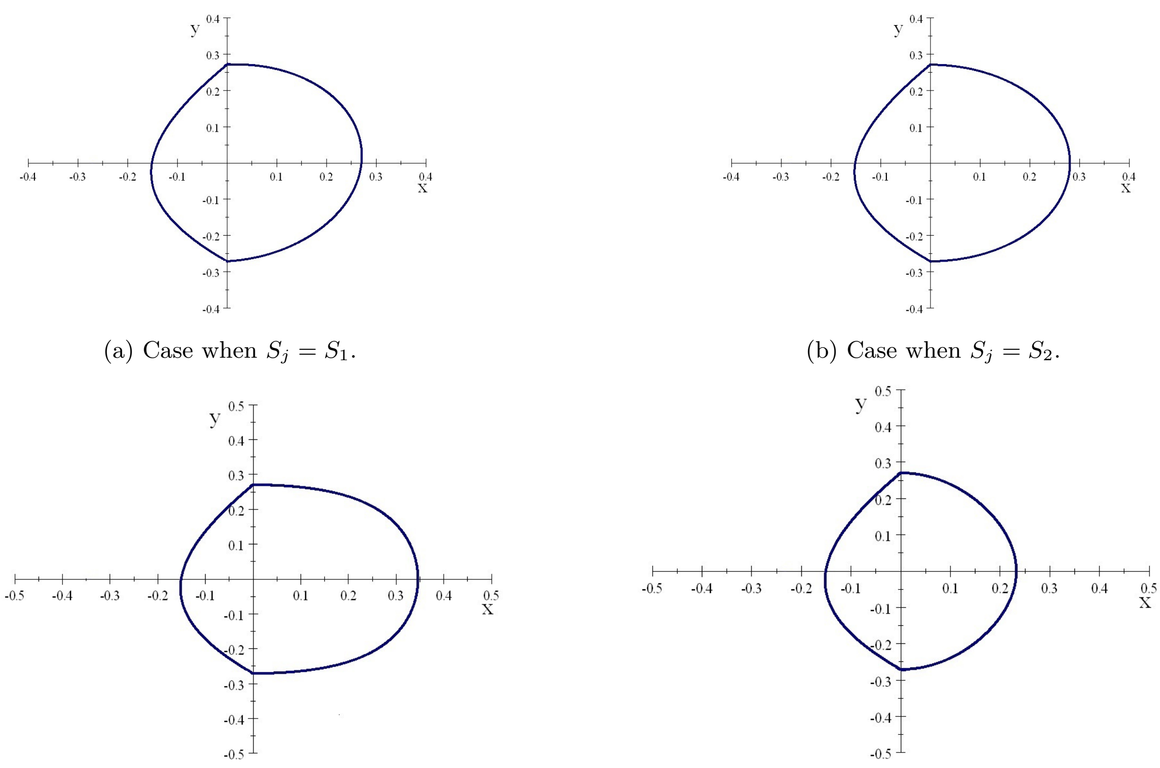
with $\beta \in (-\infty, -1.405\,6)$*, have exactly one algebraic crossing limit cycle. Moreover, this limit cycle is given by*

$$\Gamma = \big\{(x, y) \in \Sigma_R : \;\; H_j(x, y) = h_j\big\} \cup \big\{(x, y) \in \Sigma_L : \;\; H_s(x, y) = -3.281\,8\big\},$$

where $j \in \{1, 2, 3, 4\}$, $h_1 = (\frac{0.541\,03}{\beta})^4$ *and* $h_2 = h_3 = h_4 = (\frac{0.541\,03}{\beta})^2$,

$$H_s(x, y) = \Big(-\frac{1}{2}x + \beta y - \frac{19}{10}\Big)^3 \Big(\frac{3}{2}x + \beta y + \frac{23}{30}\Big).$$

See Figure 3.



(a) Case when $S_j = S_1$.

(b) Case when $S_j = S_2$.

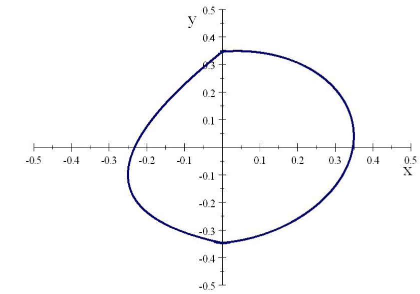
(c) Case when $S_j = S_3$.

(d) Case when $S_j = S_4$.

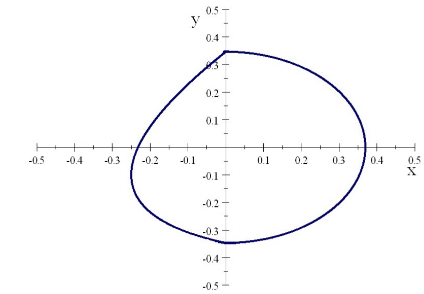
Figure 3: The unique crossing limit cycle of systems (1.10) + (S_j) with $\beta = -2$.

In the next proposition, we show that there are discontinuous piecewise differential systems (1.1) of the type linear node with different eigenvalues and cubic isochronous center at the origin, with one algebraic crossing limit cycle.

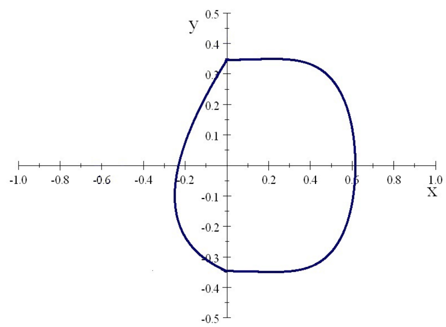
Proposition 1.5. *The discontinuous piecewise differential systems* (1.1) *formed by one of the four cubic isochronous centers* (S_1) *or* (S_2) *or* (S_3) *or* (S_4) *and a class with one parameter of linear*



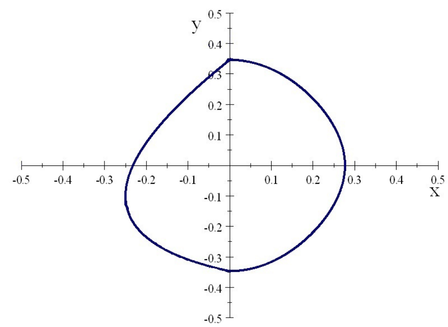
(a) Case when $S_j = S_1$.



(b) Case when $S_j = S_2$.



(c) Case when $S_j = S_3$.



(d) Case when $S_j = S_4$.

Figure 4: The unique crossing limit cycle of systems (1.11) + (S_j) with $\beta = -5$.

differential system of the form

$$\dot{x} = 6x + \beta y + 1, \quad \dot{y} = \frac{8}{\beta}\,x + \frac{4}{\beta}\,, \tag{1.11}$$

with $\beta \in (-\infty, -4.500\,1)$*, have exactly one algebraic crossing limit cycle. Moreover, this limit cycle is given by*

$$\Gamma = \big\{(x,y) \in \Sigma_R : \quad H_j(x,y) = h_j\big\} \cup \big\{(x,y) \in \Sigma_L : \quad H_n(x,y) = 36.008\big\},$$

where $j \in \{1,2,3,4\}$, $h_1 = (-\frac{1.732\,1}{\beta})^4$ *and* $h_2 = h_3 = h_4 = (-\frac{1.732\,1}{\beta})^2$

$$H_n(x,y) = \frac{(2x + \beta y + 3)^4}{(4x + \beta y + 2)^2}\,.$$

See Figure 4.

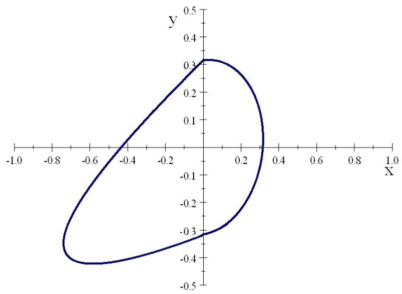
In the next proposition, we show that there are discontinuous piecewise differential systems (1.1) of the type linear non-diagonalizable node with equal eigenvalues and cubic isochronous center at the origin, with one non-algebraic crossing limit cycle.

Proposition 1.6. *The discontinuous piecewise differential systems* (1.1) *formed by one of the four cubic isochronous centers* (S_1) *or* (S_2) *or* (S_3) *or* (S_4) *and a family with one parameter of linear differential system of the form*

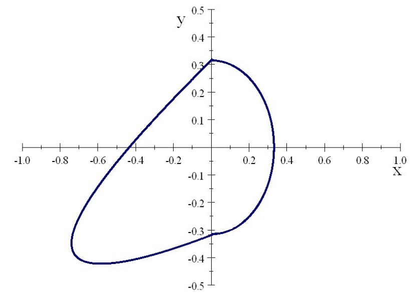
$$\dot{x} = x + \beta y - 1, \quad \dot{y} = -\frac{1}{\beta}\,(4x + 3\beta y - 4), \tag{1.12}$$

with $\beta \in (-\infty, -4.092\,5)$ *have exactly one non-algebraic crossing limit cycle. Moreover, this limit cycle is given explicitly by*

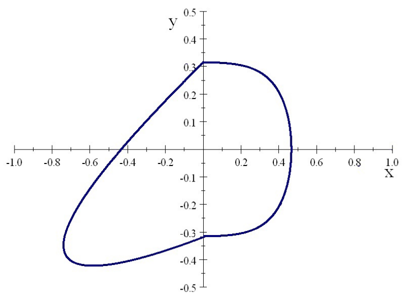
$$\Gamma = \big\{(x,y) \in \Sigma_R : \quad H_j(x,y) = h_j\big\} \cup \big\{(x,y) \in \Sigma_L : \quad H_{n'}(x,y) = 0.211\,46\big\},$$



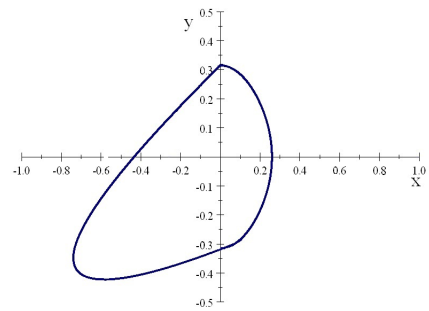
(a) Case when $S_j = S_1$.



(b) Case when $S_j = S_2$.



(c) Case when $S_j = S_3$.



(d) Case when $S_j = S_4$.

Figure 5: The unique crossing limit cycle of systems (1.12) + (S_j) with $\beta = -5$.

where $j \in \{1, 2, 3, 4\}$, $h_1 = (\frac{1.575\,2}{\beta})^4$ *and* $h_2 = h_3 = h_4 = (\frac{1.575\,2}{\beta})^2$

$$H_{n'}(x, y) = \frac{1}{2 - \beta y - 2x}\, e^{\frac{x-1}{2-\beta y-2x}}\,.$$

See Figure 5.

Remark 1.1. The assumption on the parameter β in Propositions 1.2–1.6 is a necessary condition such that the cubic polynomial differential system with homogeneous nonlinearity (S_3) has an unbounded period annulus surrounding the origin (i.e., is a necessary condition for $h_3 < \frac{4}{27}$) is also a necessary condition for the existence of crossing limit cycles of systems $(S_3) + (1.8) - (S_3) + (1.12)$.

For $(S_j) + (1.8) - (S_j) + (1.12)$, $j = 1$ or 2 or 4, the assumption $\beta < 0$ is a sufficient condition for the existence of crossing limit cycles because if $\beta < 0$, the two intersection points $(0, y_0)$ and $(0, y_1)$ of the orbit arc in Σ_R and the orbit arc in Σ_L, satisfy $(-y)(\beta y + \gamma) \leq 0$. This implies that the two intersection points $(0, y_0)$ and $(0, y_1)$ are sliding points and this prevents the existence of the crossing limit cycle.

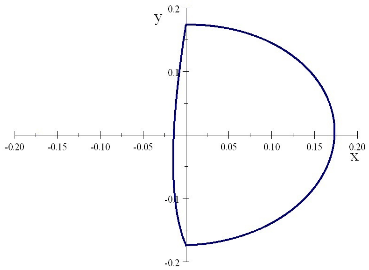
In the next proposition, we show that there are discontinuous piecewise differential systems (1.1) of the type linear without equilibria and cubic isochronous center at the origin, with one non-algebraic crossing limit cycle.

Proposition 1.7. *The discontinuous piecewise differential systems (1.1) formed by one of the four cubic isochronous centers (S_1) or (S_2) or (S_3) or (S_4) and a class with one parameter of linear differential system of the form*

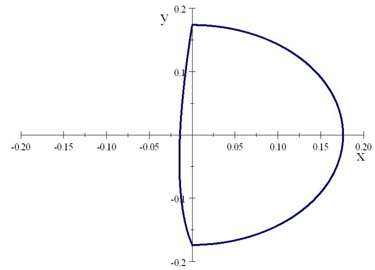
$$\dot{x} = (\mu - 1)x - y - \frac{1}{100}\,, \quad \dot{y} = \mu(\mu - 1)x - \mu y - \frac{\mu + 100}{100}\,, \tag{1.13}$$

when $\mu \neq 0$, have one explicit non-algebraic crossing limit cycle given by

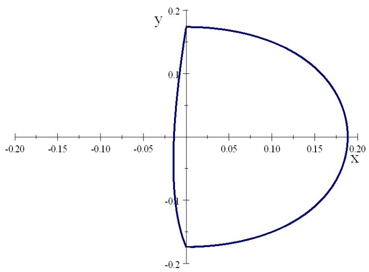
$$\Gamma = \big\{(x, y) \in \Sigma_R : \; H_j(x, y) = h_j\big\} \cup \big\{(x, y) \in \Sigma_L : \; H_w(x, y) = 0.995\,01\big\},$$



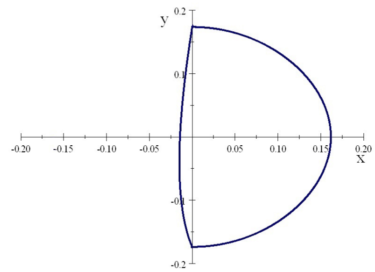
(a) Case when $S_j = S_1$.



(b) Case when $S_j = S_2$.



(c) Case when $S_j = S_3$.



(d) Case when $S_j = S_4$.

Figure 6: The unique crossing limit cycle of systems (1.13) + (S_j) with $\mu = 3$.

where $j = j \in \{1, 2, 3, 4\}$, $h_1 = (0.173\,38)^4$ *and* $h_2 = h_3 = h_4 = (0.173\,38)^2$ *and*

$$H_w(x, y) = \Big(\frac{101}{100} + (1-\mu)x + y\Big)e^{\mu x - y}.$$

See Figure 2.3.

Remark 1.2. The assumption $b < 0$ in Proposition 1.7 is a necessary condition for the existence of a crossing limit cycle of thesystem because the crossing region of these systems is given by $-by(y+\frac{1}{100}) > 0$, hence this last inequality implies that the crossing region is an open interval $(-\frac{1}{100}, 0)$ of the line Σ if $b > 0$ and is an open interval $(0, +\infty) \cup (-\infty, -\frac{1}{100})$ of the line Σ if $b < 0$. Since the intersection points $(0, y_1)$ and $(0, y_2)$, where $y_1 = -0.173\,38$ and $y_2 = 0.173\,38$, are located in $(-\infty, -\frac{1}{100}) \cup (0, +\infty)$, we have to choose $b < 0$.

2 Proof of Theorem 1.1

To prove our main result, we need the following lemmas.

Lemma 2.1 ([2]). *A linear differential system without equilibrium points can be written as*

$$\dot{x} = ax + by + c, \quad \dot{y} = \mu ax + \mu by + d, \tag{2.1}$$

where a, b, c, μ *and* d *are real constants such that* $d \neq \mu c$ *and* $\mu \neq 0$. *Moreover, this system has the first integral*

$$H_w(x, y) = \begin{cases} b\mu^2x^2 - 2b\mu xy - 2dx + by^2 + 2cy & \text{if } a + b\mu = 0, \\ \big((a+b\mu)(ax+by) + ac + bd\big)e^{\frac{a+b\mu}{d-c\mu}(\mu x - y)} & \text{if } a + b\mu \neq 0. \end{cases} \tag{2.2}$$

The following lemma provides a normal form for an arbitrary linear differential system having a real focus (resp. a center), saddle, node with different eigenvalues and non-diagonalizable node with equal eigenvalues, respectively

Lemma 2.2.

(i) *A linear differential system having a focus (resp. a center) can be written as*

$$\dot{x} = \alpha x + \beta y + \gamma, \quad \dot{y} = -\frac{1}{\beta}\left((\alpha-\lambda)^2 + \omega^2\right)x + (2\lambda - \alpha)y + \xi, \tag{2.3}$$

with $\omega > 0$, $\beta \neq 0$, $\lambda \neq 0$ (resp. $\omega > 0$, $\beta \neq 0$ and $\lambda = 0$). Moreover, when $\lambda \neq 0$, this system has the first integral

$$\begin{aligned} H_f(x,y) = \Bigg(&(\omega^2 + (\alpha-\lambda)^2)x^2 + 2\beta(\alpha-\lambda)xy + \beta^2 y^2 \\ &+ 2\,\frac{\lambda\gamma((\alpha-\lambda)^2+\omega^2) + \beta\xi(\alpha\lambda - \lambda^2 - \omega^2)}{\lambda^2+\omega^2}\,x \\ &+ 2\beta\Big(\gamma + \frac{\lambda(\beta\xi - \gamma(2\lambda-\alpha))}{\lambda^2+\omega^2}\Big)y + \frac{\gamma^2((\alpha-\lambda)^2+\omega^2) + \beta\xi(2\gamma(\alpha-\lambda)+\beta\xi)}{\lambda^2+\omega^2}\Bigg) \\ &\times e^{-\frac{2\lambda}{\omega}\arctan\frac{\omega((\lambda^2+\omega^2)x + 2\lambda\gamma - \alpha\gamma - \beta\xi)}{(\lambda^2+\omega^2)(\alpha-\lambda)x + \beta(\lambda^2+\omega^2)y + (\omega^2-\lambda^2+\alpha\lambda)\gamma + \beta\lambda\xi}}, \end{aligned} \tag{2.4}$$

and if $\lambda = 0$, the first integral of (2.3) *is*

$$H_c(x,y) = (\omega^2 + \alpha^2)x^2 + 2\beta\alpha xy + \beta^2y^2 - 2\beta\xi x + 2\beta\gamma y + \frac{\gamma^2(\alpha^2+\omega^2) + \beta\xi(2\gamma\alpha + \beta\xi)}{\omega^2}.$$

(ii) *A linear differential system having a saddle (resp. a node with different eigenvalues) (resp. a non-diagonalizable node with equal eigenvalues) can be written as*

$$\dot{x} = \alpha x + \beta y + \gamma, \quad \dot{y} = \frac{1}{\beta}\left(\rho^2 - (\alpha - r)^2\right)x + (2r-\alpha)y + \xi, \tag{2.5}$$

with $\beta \neq 0$ and $\rho^2 > r^2 > 0$ (resp. $\beta \neq 0$ and $r^2 > \rho^2 > 0$) (resp. $\beta \neq 0$, $r \neq 0$ and $\rho = 0$). Moreover, when $\rho \neq 0$, this system has the first integral

$$\begin{aligned} &H_{s,n}(x,y) \\ &= \Big((\alpha-r-\rho)x + \beta y + \gamma + \frac{\beta\xi - \gamma(2r-\alpha)}{r-\rho}\Big)\Big((\alpha-r+\rho)x + \beta y + \gamma + \frac{\beta\xi-\gamma(2r-\alpha)}{r+\rho}\Big)^{\frac{\rho-r}{r+\rho}}, \end{aligned} \tag{2.6}$$

when $\rho = 0$, the first integral of (2.5) *is given by*

$$H_{n'}(x,y) = \frac{1}{r(r-\delta)x + \beta r y + r\gamma + \beta\xi - \gamma\delta}\, e^{\frac{r^2x - \beta\xi + \gamma\delta}{r(r-\delta)x + \beta r y + r\gamma + \beta\xi - \gamma\delta}}. \tag{2.7}$$

Proof. Consider the general linear differential system

$$\dot{x} = \alpha x + \beta y + \gamma, \quad \dot{y} = \eta x + \delta y + \xi. \tag{2.8}$$

Its eigenvalues are given by $\lambda_{1,2} = \frac{1}{2}\left(\alpha + \delta \pm \sqrt{(\alpha-\delta)^2 + 4\beta\eta}\right)$.

(i) We know that system (2.8) has a focus if $\frac{1}{2}(\alpha+\delta) = \lambda$ and $(\alpha-\delta)^2 + 4\beta\eta = -4\omega^2$ for some $\omega > 0$, $\beta\eta < 0$ and $\lambda \in \mathbb{R}$, then $\delta = 2\lambda - \alpha$ and $\eta = -\frac{1}{\beta}((\alpha-\lambda)^2 + \omega^2)$. Therefore, we obtain system (2.3).

(ii) The linear differential system (2.8) has a saddle if $\frac{1}{2}(\alpha+\delta) = r$ and $(\alpha-\delta)^2 + 4\beta\eta = 4\rho^2$ for some $r^2 < \rho^2$, then $\alpha = 2r - \delta$ and $\eta = -\frac{1}{\beta}((r-\delta)^2 - \rho^2)$. Therefore, we obtain system (2.5).

Analogously to the previous case, the linear differential system (2.8) has a node with different eigenvalues if $\frac{1}{2}(\alpha+\delta) = r$ and $(\alpha-\delta)^2 + 4\beta\eta = 4\rho^2$ for some $r^2 > \rho^2$, then $\alpha = 2r - \delta$, $\eta = -\frac{1}{\beta}\left((r-\delta)^2 - \rho^2\right)$.

We know that system (2.8) has a non-diagonalizable node with equal eigenvalues if $(\alpha-\delta)^2+4\beta\eta = 0$ and $\frac{1}{2}(\alpha+\delta) = r \neq 0$, then $\delta = 2r-\alpha$ and $\eta = -\frac{1}{\beta}(r-\delta)^2$. Therefore, we obtain system (2.5) with $\rho = 0$.

It is clear that H_f, H_c, $H_{s,n}$, and $H_{n'}$ are the first integrals of systems (2.3), (2.3) with $\lambda = 0$, (2.5) with $\rho \neq 0$ and (2.5) with $\rho = 0$, respectively. In fact, all the following equations are satisfied:

$$\frac{dH_i}{dt} = \dot{x}\,\frac{\partial H_i}{\partial x} + \dot{y}\,\frac{\partial H_i}{\partial y} \equiv 0, \quad i = f, c, s, n, n'. \qquad \square$$

Remark 2.1. According to Lemma 2.1 and Lemma 2.2, it seems clear that limit cycles (if exist) of discontinuous planar piecewise differential systems (1.1) are algebraic, when the left subsystem of (1.1) is one of the following types:
- a linear center;
- a linear saddle;
- a linear node with different eigenvalues;
- a linear system without equilibria (2.1) with $a + b\mu = 0$.

While these limit cycles (if exist) of discontinuous planar piecewise differential systems (1.1) are non-algebraic when the left subsystem of (1.1) is one of the following types:
- a linear focus;
- a non-diagonalizable node with equal eigenvalues;
- a linear system without equilibria (2.1) with $a + b\mu \neq 0$.

Proof of Theorem 1.1. We consider the discontinuous piecewise differential systems (1.1). If there exists a limit cycle of the discontinuous piecewise differential systems (1.1), it must intersect the discontinuity line Σ at two different points $(0, y_0)$ and $(0, y_1)$. In order to investigate the limit cycles of these systems, we use the Poincaré map of (1.1).

We can define a right return map P_R as $y_1 = P_R(y_0)$ and a left return map P_L as $y_2 = P_L(y_1)$. Composing the right return map P_R with the left return map P_L, the Poincaré map P of (1.1) can be constructed by P_L and P_R as follows:

$$y_2 = P(y_0) = P_L \circ P_R(y_0).$$

It is obvious that the zeros of $F(y_0) = y_0 - P(y_0)$ correspond to the limit cycles of the discontinuous piecewise differential systems (see Figure 7).

In what follows, we give the detailed calculations for the right and left return maps. To determine the right return map P_R, we use the first integrals for the right side systems of (1.1). Assume that the orbits starting at the point $(0, y_0)$ go into the right zone Σ_R under the flow of the right differential systems. If these orbits can reach Σ again at some point $(0, y_1)$, then $(0, y_0)$ and $(0, y_1)$ must satisfy the following equation:

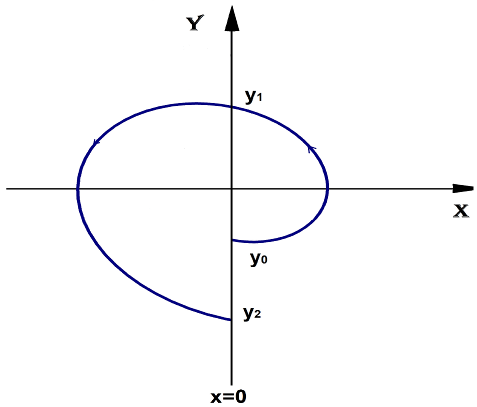
$$e_j = H_j(0, y_0) - H_j(0, y_1) = 0,$$

where $j \in \{1, 2, 3, 4\}$ and H_j are given by (1.2), (1.3), (1.4) and (1.5), respectively. The equations $e_j = 0$ for $j \in \{1, 2, 3, 4\}$ are equivalent to $(y_0 - y_1)(y_1 + y_0) = 0$. From this equation, the unique solution satisfying $y_0 < y_1$ is $y_1 = -y_0$. Then we can define a right Poincaré map as

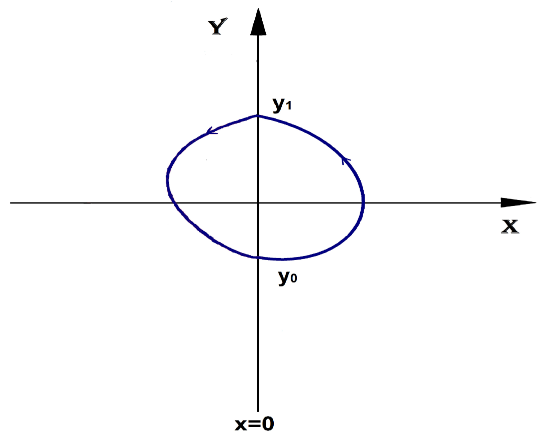
$$P_R(y_0) = -y_0. \tag{2.9}$$

Proof of statement (1) of Theorem 1.1. First, we consider the case where the left subsystem of (1.1) is a linear focus type satisfying (2.3) with $\lambda \neq 0$. To determine the left return map P_L, we use the parametric representation of the solution of the linear differential system (2.3) in Σ_L. Thus the solution of this system with $\lambda \neq 0$ starting at the point $(0, y_1)$ is given by

$$\begin{aligned} x_L(t) &= \frac{e^{\lambda t}(\lambda(\beta\xi - \gamma(2\lambda-\alpha)) + (\lambda^2+\omega^2)(\gamma+\beta y_1))\sin\omega t}{\omega(\lambda^2+\omega^2)} - \frac{(\beta\xi - \gamma(2\lambda-\alpha))(e^{t\lambda}\cos\omega t - 1)}{\lambda^2+\omega^2}, \\ y_L(t) &= \frac{(\gamma((\alpha-\lambda)^2+\omega^2) + \beta\xi\alpha)(e^{t\lambda}\cos\omega t - 1)}{\beta(\lambda^2+\omega^2)} + e^{\lambda t}\Big(\cos\omega t - \frac{\alpha-\lambda}{\omega}\,\sin\omega t\Big)y_1 \\ &\quad - \frac{(\lambda(\alpha-\lambda)(\gamma(\alpha-\lambda)+\beta\xi) + (\lambda\gamma-\beta\xi)\omega^2)e^{\lambda t}\sin\omega t}{\omega\beta(\lambda^2+\omega^2)}. \end{aligned}$$



(a) The poincaré map of (1.1)



(b) The periodic solution of (1.1)

Figure 7: The Poincaré map and the periodic solution of (1.1)

Then from the equation $x_L(t) = 0$, we obtain

$$y_1(t) = -\frac{\omega(\beta\xi - \gamma(2\lambda - \alpha))(e^{-t\lambda} - \cos\omega t) + (\gamma(\lambda^2 + \omega^2) + \lambda(\beta\xi - \gamma(2\lambda - \alpha)))\sin\omega t}{\beta(\lambda^2 + \omega^2)\,\sin\omega t}. \tag{2.10}$$

For this case, the parametric representation of the left return map P_L is

$$P_L(y_1) = \frac{(\gamma((\alpha - \lambda)^2 + \omega^2) + \beta\xi\alpha)(e^{t\lambda}\cos\omega t - 1)}{\beta(\lambda^2 + \omega^2)} + e^{\lambda t}\Big(\cos\omega t - \frac{(\alpha - \lambda)}{\omega}\,\sin\omega t\Big)y_1 - \frac{(\lambda(\alpha - \lambda)(\gamma(\alpha - \lambda) + \beta\xi) + (\lambda\gamma - \beta\xi)\omega^2)e^{\lambda t}\sin\omega t}{\omega\beta(\lambda^2 + \omega^2)}.$$

Since $y_1 = -y_0$, the zeros of the function F are the zeros of the function G given by $G(t) = -y_1(t) - P_L(y_1(t))$. When substituting the previous expressions of $y_1(t)$ and $P_L(y_1(t))$ into the equation $G(t) = 0$, we obtain the equation

$$\frac{1}{\beta(\lambda^2 + \omega^2 -)}\Big((\gamma(\omega^2 - \lambda^2) + \lambda(\beta\xi + \gamma\alpha)) - \omega(\beta\xi - \gamma(2\lambda - \alpha))\,\frac{\sinh\lambda t}{\sin\omega t}\Big) = 0. \tag{2.11}$$

Now, it is easy to see that the existence of a crossing limit cycle is equivalent to the existence of a positive t satisfying (2.11). For convenience, we use the notation

$$f_1(t) = \big(\gamma(\omega^2 - \lambda^2) + \lambda(\beta\xi + \gamma\alpha)\big) - \omega\big(\beta\xi - \gamma(2\lambda - \alpha)\big)\,\frac{\sinh\lambda t}{\sin\omega t}. \tag{2.12}$$

So, the way of solving equation (2.11) is the same as that of the equation $f_1(t) = 0$. In order to investigate a number of solutions of $f_1(t) = 0$, and since f_1 is a C^1-function in $\mathbb{R}\setminus\{0\}$, we use the first derivative of the function f_1 with respect to the variable t. Simple calculations yield

$$f_1'(t) = -\frac{\omega(\alpha\gamma - 2\lambda\gamma + \beta\xi)}{\sin^2\omega t}\,(\lambda\sin\omega t\cosh\lambda t - \omega\cos\omega t\sinh\lambda t).$$

Note that the zeros of $f_1'(t)$ are the zeros of $K_1(t)$, where

$$K_1(t) = \lambda\sin\omega t\cosh\lambda t - \omega\cos\omega t\sinh\lambda t.$$

Note that the left linear differential system (2.3) has the eigenvalues $\lambda \pm i\sqrt{\omega}$, $\omega > 0$, at its singularity

$$(x_0, y_0) = \Big(\frac{\beta\xi - \gamma(2\lambda - \alpha)}{\lambda^2 + \omega^2}, -\frac{\gamma((\alpha - \lambda)^2 + \omega^2) + \beta\xi\alpha}{\beta(\lambda^2 + \omega^2)}\Big).$$

So, it follows that the frequency is ω and, consequently, if (x_0, y_0) is a virtual focus, we have $t \in (0, \frac{\pi}{\omega})$ and $t \in (\frac{\pi}{\omega}, \frac{2\pi}{\omega})$ for (x_0, y_0) is a real focus.

(i) If (x_0, y_0) is a virtual focus, i.e., if $\beta\xi - \gamma(2\lambda - \alpha) > 0$, we have $K_1(0) = 0$ and $K_1(t) \neq 0$ for $t \neq 0$, since $K_1'(t) = (\lambda^2 + \omega^2)\sin\omega t \sinh\lambda t$ cannot vanish in $(0, \frac{\pi}{\omega})$, so, $f_1'(t) \neq 0$ for $t \in (0, \frac{\pi}{\omega})$. Therefore, the equation $f_1(t) = 0$ with $\beta\xi - \gamma(2\lambda - \alpha) > 0$ may have at most one solution in $(0, \frac{\pi}{\omega})$. Hence systems (1.1) have at most one crossing limit cycle.

(ii) If (x_0, y_0) is a real focus, i.e., if $\beta\xi - \gamma(2\lambda - \alpha) < 0$, and since $K_1(\frac{\pi}{\omega}) = \omega\sinh(\lambda\frac{\pi}{\omega})$ and $K_1(\frac{2\pi}{\omega}) = -\omega\sinh(\lambda\frac{2\pi}{\omega})$ and sign $(K_1(\frac{\pi}{\omega})K_1(\frac{2\pi}{\omega})) < 0$, while $K_1'(t) \neq 0$ in $(\frac{\pi}{\omega}, \frac{2\pi}{\omega})$, then K_1 is a strictly monotone function in $(\frac{\pi}{\omega}, \frac{2\pi}{\omega})$. Thus $f_1'(t) = 0$ has exactly one solution in $(\frac{\pi}{\omega}, \frac{2\pi}{\omega})$ and, consequently, equation (2.12) has at most two zeros in $(\frac{\pi}{\omega}, \frac{2\pi}{\omega})$. From the above analysis, we conclude that systems (1.1) have at most two crossing limit cycles when $\beta\xi - \gamma(2\lambda - \alpha) < 0$. Using the first integrals of both differential systems of (1.1) and knowing that the non-algebraic crossing periodic orbits pass through the points $(0, y_{1i})$ and through the points $(0, y_{0i})$, $i = 1, 2$, where y_{1i} is defined by (2.10) and $y_{0i} = -y_{1i}$, we obtain the following expressions:

$$\Gamma_i = \{(x,y) \in \Sigma_R : \ H_j(0,y) = H_j(0,y_{0i})\} \cup \{(x,y) \in \Sigma_L : \ H(x,y) = H(0,y_{0i})\}, \ \ i = 1,2,$$

where $j \in \{1,2,3,4\}$ and H_j are given by (1.2), (1.3), (1.4), (1.5), respectively.

Proof of statement (2) of Theorem 1.1. Using the notation introduced in the proof of statements (1), we consider that the left subsystem of (1.1) is a linear center type satisfying (2.3) with $\lambda = 0$. The solution of system (2.3) with $\lambda = 0$ starting at the point $(0, y_1)$ is

$$\begin{aligned} x_L(t) &= \frac{1}{\omega^2}\big((\alpha\gamma + \beta\xi) + \omega(\gamma + \beta y_1)\sin\omega t - \alpha\gamma\cos\omega t - \beta\xi\cos\omega t\big), \\ y_L(t) &= \frac{(\gamma(\alpha^2+\omega^2) + \beta\xi\alpha)(\cos\omega t - 1)}{\beta\omega^2} + \Big(\cos\omega t - \frac{\alpha}{\omega}\sin\omega t\Big)y_1 + \frac{\xi\sin\omega t}{\omega}. \end{aligned}$$

Then, from the equation $x_L(t) = 0$, we obtain

$$y_1(t) = -\frac{(\beta\xi + \gamma\alpha) + \gamma\omega\sin\omega t + (-\gamma\alpha - \beta\xi)\cos\omega)}{\beta\omega\sin\omega t}.$$

For this case, the parametric representation of the left return map P_L is

$$P_L(y_1) = \frac{(\gamma(\alpha^2+\omega^2) + \beta\xi\alpha)(\cos\omega t - 1)}{\beta\omega^2} + \Big(\cos\omega t - \frac{\alpha}{\omega}\sin\omega t\Big)y_1 + \frac{\xi\sin\omega t}{\omega}.$$

Since $y_1 = -y_0$, substituting the previous two expressions into the equation $G(t) = -y_1(t) - P_L(y_1(t)) = 0$, we obtain $\frac{2}{\beta}\gamma = 0$. Hence, if $\gamma \neq 0$, this last equality does not hold, the equation $F(y_0) = 0$ has no solutions and, consequently, the discontinuous piecewise differential systems (1.1) have no periodic solutions. If $\gamma = 0$, then $G(t) = 0$ for all $t > 0$, i.e., $F(y_0) = 0$ has a continuum of solutions. So, the discontinuous piecewise differential systems (1.1) either does not have periodic solutions, or it has a continuum of periodic orbits and, consequently, these differential systems have no limit cycles.

Proof of statements (3) and (4) of Theorem 1.1. Now, we assume that the left subsystem of (1.1) is a linear system satisfying (2.5) with $\rho \neq 0$. We recall that if $r^2 > \rho^2 > 0$, then system (2.5) has a real or a virtual node with two different eigenvalues, while if $r^2 > \rho^2 > 0$, the system has a real or a virtual saddle. We have to study these cases simultaneously. To determine the left return map P_L of (1.1), we use the parametric representation of the solution of the linear differential system (2.5) with $\rho \neq 0$ in Σ_L starting at the point $(0, y_1)$, this solution is

$$\begin{aligned} x_L(t) = \frac{e^{(r+\rho)t}(\gamma(\alpha - r + \rho) + \beta\xi + \beta(r+\rho)y_1)}{2\rho(\rho + r)} \\ - \frac{\beta\xi - \gamma(2r - \alpha)}{\rho^2 - r^2} + \frac{e^{(r-\rho)t}(\gamma(\alpha - r - \rho) + \beta\xi + \beta(r-\rho)y_1)}{2\rho(\rho - r)}, \end{aligned}$$

$$\begin{aligned} y_L(t) = & \frac{e^{(r+\rho)t}(\gamma\rho^2 - \gamma(\alpha-r)^2 + (\beta\xi + \beta(r+\rho)y_1)(\rho + r - \alpha))}{2\beta\rho(r+\rho)} \\ & - \frac{(\gamma(\alpha-r)^2 + 2\beta\xi r - \beta\xi(2r-\alpha) - \gamma\rho^2)}{\beta(r^2-\rho^2)} \\ & + \frac{e^{(r-\rho)t}(\gamma(\alpha-r)^2 - \gamma\rho^2 + (\beta\xi + \beta(r-\rho)y_1)(\alpha - r + \rho))}{2\beta\rho(r-\rho)}. \end{aligned}$$

Hence, the left Poincaré map is written as follows:

$$\begin{aligned} P_L(y_1) = & \frac{\gamma\rho^2 - \gamma(\alpha-r)^2 + (\beta\xi + \beta(r+\rho)y_1)(\rho + r - \alpha)}{2\beta\rho(r+\rho)} e^{(r+\rho)t} \\ & + \frac{\gamma(\alpha-r)^2 - \gamma\rho^2 + (\beta\xi + \beta(r-\rho)y_1)(\alpha - r + \rho)}{2\beta\rho(r-\rho)} e^{(r-\rho)t} - \frac{\gamma(\alpha-r)^2 - \gamma\rho^2 + \beta\xi(\alpha-r)}{\beta(r^2-\rho^2)}, \end{aligned}$$

and from the equation $x_L(t) = 0$, we obtain

$$\begin{aligned} y_1(t) = & \frac{e^{(\rho+r)t}(\rho - r)(\gamma(\alpha - r + \rho) + \beta\xi) - 2\rho(\beta\xi - \gamma(2r-\alpha))}{\beta(\rho^2 - r^2)(e^{(r-\rho)t} - e^{(r+\rho)t})} \\ & + \frac{e^{(r-\rho)t}(r+\rho)(\gamma(\alpha - r - \rho) + \beta\xi)}{\beta(\rho^2 - r^2)(e^{(r-\rho)t} - e^{(r+\rho)t})}. \end{aligned} \tag{2.13}$$

But $y_1 = -y_0$, so this reduces the equation $y_0 - P(y_0) = 0$ to the form

$$\frac{1}{\beta(\rho^2 - r^2)} \left((\rho^2 - r^2)(-2\gamma) + 2r(\beta\xi - \gamma(2r-\alpha))\right) \frac{\sinh \rho t}{\sinh rt} - 2\rho(\beta\xi - \gamma(2r-\alpha)) = 0. \tag{2.14}$$

For convenience, we use the notation

$$f_2(t) = \left(2r(\beta\xi - \gamma(2r-\alpha)) - 2\gamma(\rho^2 - r^2)\right) \frac{\sinh \rho t}{\sinh rt} - 2\rho(\beta\xi - \gamma(2r-\alpha)). \tag{2.15}$$

Now, the way of solving (2.14) is equivalent to that of finding the solutions t of the equation $f_2(t) = 0$. In order to investigate a number of solutions of $f_2(t) = 0$, and since f_2 is a C^1-function in $\mathbb{R} \setminus \{0\}$, we use the first derivative of the function f with respect to the variable t. Simple calculations yield

$$f_2'(t) = -\frac{2r(\beta\xi - \gamma(2r-\alpha)) - 2\gamma(\rho^2 - r^2)(r\cosh rt \sinh \rho t - \rho \sinh rt \cosh \rho t)}{\sinh^2 rt}.$$

Note that the zero of f_2' is the zero of K_2, where

$$K_2(t) = r\cosh rt \sinh t\rho - \rho \sinh rt \cosh t\rho.$$

Since $K_2(0) = 0$ and $K_2'(t) = (r^2 - \rho^2)\sinh rt \sinh \rho t \neq 0$ for any $t > 0$ (because $r \neq 0$ and $\rho \neq 0$), we can conclude that equation (2.14) has at most one real solution, and there are the values of r, γ, ρ, β, ξ and α for which this solution exists. Hence systems (1.1) have at most one crossing limit cycle. Using the first integrals of both differential systems of (1.1) and knowing that the algebraic crossing periodic orbit passes through the points $(0, y_0)$ and $(0, y_1)$, where y_1 is defined by (2.13) and $y_0 = -y_1$, we get the following expressions:

$$\Gamma = \left\{(x,y) \in \Sigma_R : \;\; H_j(0,y) = H_j(0,y_0)\right\} \cup \left\{(x,y) \in \Sigma_L : \;\; H(x,y) = H(0,y_0)\right\},$$

where $j \in \{1,2,3,4\}$ and H_j are given by (1.2), (1.3), (1.4), (1.5), respectively. This completes the proof of statements (3) and (4) of Theorem 1.1.

Proof of statement (5) of Theorem 1.1. Now, we consider the case, where the left subsystem of (1.1) is a linear system satisfying (2.5) with $\rho = 0$. By an analogous analysis of previous statements, the solution $(x_L(t), y_L(t))$ of system (2.5) with $\rho = 0$ which passes through the point $(0, y_1)$ is

$$x_L(t) = \frac{e^{rt}(\gamma(2r-\alpha) - \beta\xi) - (\gamma(2r-\alpha) - \beta\xi)}{r^2} + \frac{te^{rt}(\gamma(\alpha - r) + \beta\xi + \beta r y_1)}{(r)},$$

$$\begin{aligned} y_L(t) = &\frac{e^{rt}(\gamma(\alpha-r)^2+\beta(\alpha\xi+r^2y_1))-(\gamma(\alpha-r)^2+\beta\xi(\alpha-r))}{\beta r^2} \\ &- \frac{te^{rt}(\gamma(\alpha-r)^2+\beta(\alpha-r)(\xi+ry_1))}{\beta r}, \end{aligned}$$

whence for the left Poincaré map we get the following parametric representation:

$$\begin{aligned} P_L(y_1) = &\frac{e^{rt}(\gamma(\alpha-r)^2+\beta(\alpha\xi+r^2y_1))-(\gamma(\alpha-r)^2+\beta\xi(\alpha-r))}{\beta r^2} \\ &- \frac{te^{rt}(\gamma(\alpha-r)^2+\beta(\alpha-r)(\xi+ry_1))}{\beta r}. \end{aligned} \tag{2.16}$$

From the equation $x_L(t)=0$, we obtain

$$y_1(t) = -\frac{1}{t\beta e^{rt}}\Big(-\frac{1}{r^2}(\gamma(2r-\alpha)-\beta\xi)+\frac{1}{r^2}e^{rt}(\gamma(2r-\alpha)-\beta\xi)+\frac{t}{r}e^{rt}(\beta\xi-\gamma(r-\alpha))\Big). \tag{2.17}$$

Using (2.16), (2.17), and taking into account that $y_0=-y_1(t)$ and $P(y_0)=P_L(y_1(t))$, the equation $y_0-P(y_0)=0$ becomes

$$\frac{1}{\beta r}\Big(\frac{(\gamma(2r-\alpha)-\beta\xi)\sinh rt}{rt}+\gamma(\alpha-r)+\beta\xi\Big)=0, \tag{2.18}$$

the previous equation is equivalent to $f_3(t)=0$, where

$$f_3(t)=\frac{\sinh rt}{rt}+\frac{\beta\xi-\gamma(r-\alpha)}{\gamma(2r-\alpha)-\beta\xi}. \tag{2.19}$$

Now, the way of solving (2.18) is equivalent to that of finding the solutions t of the equation $f_3(t)=0$. The study of the maximum number of zeros of $f_3(t)=0$ is equivalent to finding of the maximum number of intersection points z_i of the curve $\mathcal{F}: y=\frac{\sinh z}{z}$ with the horizontal line $\mathcal{L}: y=-\frac{\beta\xi-\gamma(r-\alpha)}{\gamma(2r-\alpha)-\beta\xi}$.

It is easy to check that $K_3(z)=\frac{\sinh z}{z}$ is an even function and $K_3(z)$ for $z>0$ is strictly increasing and strictly decreasing for $z<0$, and $\lim\limits_{z\to 0}K_3(z)=1$.

Clearly, we can choose the values of the parameters of system (2.5) with $\rho=0$ such that the straight line $\mathcal{L}$ intersects the curve $\mathcal{F}$ at either zero point or at one or two points.

If $\mathcal{L}$ does not intersect $\mathcal{F}$, then $f_3(t)=0$ has no solution, and systems (1.1) has no limit cycles.

If $\mathcal{L}$ intersects $\mathcal{F}$ at a unique point, then the intersection point is multiple to two, this point should be $y=1$ and $z=0$. This implies that $t=0$; again, systems (1.1) have no limit cycles.

If the intersection points are two, we denote them by (z_1,y_1') and (z_2,y_2'). Taking into account the evenness of the function $K_3(z)=\frac{\sinh z}{z}$, it follows that $z_1=-z_2$ and $y_1'=y_2'$. So, the equation $f_3(t)=0$ has at most one solution in $t\in(0,+\infty)$ for $z=rt$ and, consequently, a unique solution for y_1 and y_0. Hence systems (1.1) has at most one crossing limit cycle. Using the first integrals of both differential systems of (1.1) and knowing that the algebraic crossing periodic orbit passes through the points $(0,y_0)$ and $(0,y_1)$, where y_1 is defined by (2.17) and $y_{0i}=-y_{1i}$, we obtain the following expressions:

$$\Gamma=\big\{(x,y)\in\Sigma_R:\ \ H_j(0,y)=H_j(0,y_0)\big\}\cup\big\{(x,y)\in\Sigma_L:\ \ H(x,y)=H(0,y_0)\big\},$$

where $j\in\{1,2,3,4\}$ and H_j are given by (1.2), (1.3), (1.4), (1.5), respectively. This completes the proof of statement (5) of Theorem 1.1.

Proof of statement (6) of Theorem 1.1. Finally, we consider the case where the left subsystem of (1.1) is a linear system having no equilibria, neither real nor virtual, satisfying (2.1). In a similar way as in the previous cases, the solution of system (2.1) with $a+\mu b\neq 0$, starting at the point $(0,y_1)$, is

$$x_L(t)=\frac{b(c\mu-d)(a+b\mu)t+(e^{at+bt\mu}-1)(ac+bd+b(a+b\mu)y_1)}{(a+b\mu)^2},$$

$$y_L(t) = \frac{a(d-c\mu)t + ay_1}{(a+b\mu)} - \frac{\mu(ac+bd)}{(a+b\mu)^2} + \frac{e^{at+bt\mu}\mu(ac+bd+b(a+b\mu)y_1)}{(a+b\mu)^2}.$$

Then the left Poincaré map is

$$P_L(y_1) = \frac{a(d-c\mu)t + ay_1}{(a+b\mu)} - \frac{\mu(ac+bd)}{(a+b\mu)^2} + \frac{e^{at+bt\mu}\mu(ac+bd+b(a+b\mu)y_1)}{(a+b\mu)^2},$$

and from equation $x_L(t) = 0$, we obtain

$$y_1(t) = -\frac{(e^{at+bt\mu}-1)(ac+bd) - b(a+b\mu)(d-c\mu)t}{b(e^{at+bt\mu}-1)(b\mu+a)}. \tag{2.20}$$

Since $y_1 = -y_0$, the equation $y_0 - P(y_0) = 0$ is equivalent to $-y_1(t) - P_L(y_1(t)) = 0$. Substituting the previous two expressions into $-y_1(t) - P_L(y_1(t)) = 0$, we obtain

$$-(d-c\mu)t\coth\left(\frac{1}{2}(a+b\mu)t\right) - \frac{2}{b(a+b\mu)}(ac+bd) = 0 \tag{2.21}$$

or, equivalently, $f_4(t) = 0$, where

$$f_4(t) = \frac{1}{2}(a+b\mu)t\coth\left(\frac{1}{2}(a+b\mu)t\right) + \frac{ac+bd}{b(d-c\mu)}. \tag{2.22}$$

By an analogous analysis of the previous case, in order to investigate the number of solutions of $f_4(t) = 0$, we find a number of intersection points z_i of the curve $\mathcal{F}' : y = z\coth(z)$ with the straight line $\mathcal{L}' : y = \frac{-1}{b(d-c\mu)}(ac+bd)$.

The function $K(z) = z\coth z$ is even and for $z > 0$ is strictly increasing and strictly decreasing for $z < 0$, and $K(0) = 0$.

Clearly, the straight line $\mathcal{L}'$ may intersect the curve $\mathcal{F}'$ at either zero point or at one or two points.

If $\mathcal{L}'$ does not intersect $\mathcal{F}'$, then $f_4(t) = 0$ has no solution, and systems (1.1) have no limit cycles.

If $\mathcal{L}'$ intersects $\mathcal{F}'$ at a unique point, then the intersection point is multiple of two, this point should be $y = 0$ and $z = 0$. This implies that $t = 0$ (because $a + b\mu \neq 0$); again, systems (1.1) have no limit cycles.

If the intersection points are two, we denote them by (z_1, y_1') and (z_2, y_2'). Since $K_3(z) = z\coth z$ is an even function and the straight line $\mathcal{L}'$ is horizontal, it follows that $z_1 = -z_2$ and $y_1' = y_2'$. So, equation (2.21) has at most one solution $t > 0$ for $z = \frac{1}{2}t(a+b\mu)$ and, consequently, a unique solution for y_1 and y_0 follows from (2.20) and (2.9), respectively. To obtain in this way at most one limit cycle for the discontinuous piecewise differential systems, we use the first integrals of both differential systems knowing that the non-algebraic periodic orbit passes through the points $(0, y_0)$ and $(0, -y_0)$, where y_1 is defined by (2.20) and $y_{0i} = -y_{1i}$, we get the following expression:

$$\Gamma = \big\{(x,y) \in \Sigma_R : \ \ H_j(0,y) = H_j(0,y_0)\big\} \cup \big\{(x,y) \in \Sigma_L : \ \ H(x,y) = H(0,y_0)\big\},$$

where $j \in \{1,2,3,4\}$, and H_j are given by (1.2), (1.3), (1.4), (1.5), respectively.

Now, we consider system (2.1) with $a + \mu b = 0$. In this case, the solution of system (2.1), starting at the point $(0, y_1)$, is

$$x_L(t) = (c + by_1)t + \frac{1}{2}(bd - bc\mu)t^2,$$
$$y_L(t) = y_1 + \frac{1}{2}(bd\mu - bc\mu^2)t^2 + (by\mu + d)t.$$

If $x_L(t) = 0$, we get

$$y_1(t) = -\frac{1}{2b}(2c + b(d-c\mu)t),$$

and the parametric representation of the left Poincaré map is

$$P_L(y_1) = y_1 + \frac{1}{2}(bd\mu - bc\mu^2)t^2 + (by_1\mu + d)t.$$

Since $y_1 = -y_0$, the equation $F(y_0) = y_0 - P(y_0) = 0$ is equivalent to $G(t) = -y_1(t) - P_L(y_1(t)) = 0$. Substituting the previous two expressions of $y_1(t)$ and $P_L(y_1(t))$ into $G(t) = 0$, we obtain $\frac{2}{b}c = 0$. Hence, if $c \neq 0$, this last equality does not hold, and $F(y_0) = 0$ has no solutions and, consequently, the discontinuous piecewise differential systems (1.1) have no periodic solutions. If $c = 0$, then $G(t) = 0$ for all $t > 0$, i.e., $F(y_0) = 0$ has a continuum of solutions. So, in this case, the discontinuous piecewise differential systems (1.1) either do not have periodic solutions, or have a continuum of periodic orbits and, consequently, these differential systems have no limit cycles. So, statement (6) of theorem 1.1 is proved. □

3 Proof of propositions

Proof of Proposition 1.2. We consider that we have the piecewise differential systems $(S_j)+(1.8)$ with $j = 1$ or 2 or 3 or 4, we remark that the equilibrium point $(-\frac{5}{52}, \frac{9}{104\beta})$ of system (1.8) has eigenvalues $-\frac{1}{5} \pm i$. So, it is a real focus. Those piecewise differential systems have the first integral

$$H_{f1}(x,y) = \Big(\frac{26}{25}x^2 - \frac{2}{5}\beta xy + \frac{61}{260}x + \beta^2 y^2 - \frac{11}{52}\beta y + \frac{17}{832}\Big)e^{-\frac{2}{5}\arctan\frac{520x+50}{104x-520y\beta+55}}$$

if $x \in \Sigma_L$, and the first integral H_j with $j = 1$ or 2 or 3 or 4, and H_j are given by (1.2), (1.3), (1.4) and (1.5), respectively if $x \in \Sigma_R$.

For the piecewise differential systems $(S_j) + (1.8)$ with $j \in \{1, 2, 3, 4\}$, the function (2.12) becomes

$$f_1(t) = \frac{1}{100\sinh\frac{1}{5}t}\Big(10\sinh\frac{1}{5}t + 11\sin t\Big).$$

The equation $f_1(t) = 0$ has exactly two positive zeros $t_1 = 4.1438$ and $t_2 = 4.7492$. From these values of $t_i, i = 1, 2$, and using (2.10), we get the values of $y_{11} = -\frac{0.13944}{\beta}$ and $y_{12} = -\frac{0.21703}{\beta}$, so, from $y_0 = -y_1$, we have $y_{01} = \frac{0.13944}{\beta}$ and $y_{02} = \frac{0.21703}{\beta}$. Thus these two solutions will correspond to the isolated periodic orbits Γ_1 and Γ_1 of systems $(S_j) + (1.8)$, i.e.,to two limit cycles of those systems. The smallest one Γ_1 intersects the switching line Σ at two points $(0, y_{01})$ and $(0, y_{11})$ and the biggest limit cycle Γ_2 intersects the switching line Σ at two points $(0, y_{02})$ and $(0, y_{12})$. Straightforward computations show that the solution of $(S_j) + (1.8)$ with $j = 1$ or 2 or 3, or 4, passing through the crossing points $(0, y_{01})$ and $(0, y_{11})$, corresponds to

$$\Gamma_1 = \big\{(x,y) \in \Sigma_R : \quad H_j(x,y) = h_j\big\} \cup \big\{(x,y) \in \Sigma_L : \quad H_{f1}(x,y) = 5.9741 \times 10^{-2}\big\},$$

where $h_1 = (\frac{0.13944}{\beta})^4$, $h_2 = h_3 = h_4 = (\frac{0.13944}{\beta})^2$, and the solution of $(S_j) + (1.8)$, passing through the crossing points $(0, y_{02})$ and $(0, y_{12})$, corresponds to

$$\Gamma_2 = \big\{(x,y) \in \Sigma_R : \quad H_j(x,y) = h'_j\big\} \cup \big\{(x,y) \in \Sigma_L : \quad H_{f1}(x,y) = 0.10104\big\},$$

where $h'_1 = (\frac{0.21703}{\beta})^4$, $h'_2 = h'_3 = h'_4 = (\frac{0.21703}{\beta})^2$. Moreover, Γ_1 and Γ_2 are non-algebraic and travel in a counterclockwise sense around the sliding segment $\Sigma_s = \{(0,y) \in \Sigma : \quad 0 \leq y \leq -\frac{1}{8\beta}\}$. □

Proof of proposition 1.3. We consider that we have the piecewise differential systems $(S_j)+(1.9)$ with $j = 1$ or 2 or 3 or 4. The equilibrium point $(1, 0)$ of system (1.9) has eigenvalues $1 \pm i$, so, it is a virtual focus, these piecewise differential systems have the first integral

$$H_{f2}(x,y) = (5x^2 - 4\beta xy - 10x + \beta^2 y^2 + 4\beta y + 5)e^{-2\arctan\frac{x-1}{y\beta-2x+2}}$$

if $x \in \Sigma_L$, and the first integral H_j with $j = 1$ or 2 or 3 or 4, where H_j are given by (1.2), (1.3), (1.4) and (1.5), respectively, if $x \in \Sigma_R$.

For the piecewise differential systems $(S_j) + (1.9)$ with $j \in \{1, 2, 3, 4\}$, the function (2.12) becomes

$$f_1(t) = -\frac{1}{\sinh t}(2\sinh t - 4\sin t).$$

From the equation $f_1(t) = 0$, we obtain the unique solution $t = 1.435\,4$. From this value of t and using (2.10), we get the values of $y_1 = -\frac{2.104\,0}{\beta}$, because $y_1 = -y_0$, thus $y_0 = \frac{2.104\,0}{\beta}$. So, the discontinuous piecewise differential systems (S_j) + (1.9) have exactly one crossing limit cycle. Straightforward computations show that the solution of (S_j) + (1.9) passing through the crossing points $(0, y_0)$ and $(0, y_1)$ corresponds to

$$\Gamma_1 = \big\{(x,y) \in \Sigma_R : \quad H_j(x,y) = h_j\big\} \cup \big\{(x,y) \in \Sigma_L : \quad H_{f2}(x,y) = 5.374\,3 \times 10^{-2}\big\},$$

where $j \in \{1,2,3,4\}$, $h_1 = (\frac{2.104\,0}{\beta})^4$ and $h_2 = h_3 = h_4 = (\frac{2.104\,0}{\beta})^2$. We note that this limit cycle is non-algebraic and travels in a counterclockwise sense, around the sliding segment $\Sigma_s = \{(0,y) \in \Sigma : 0 \leq y \leq -\frac{1}{\beta}\}$. □

Proof of proposition 1.4. We consider the piecewise differential systems (S_j)+(1.10) with $j \in \{1,2,3,4\}$. Since the eigenvalues of matrices of the linear differential system (1.10) are $\frac{3}{2}$, $-\frac{1}{2}$, this system has a real saddle at the equilibrium point $(-\frac{4}{3}, \frac{37}{30\beta})$. The piecewise differential systems (S_j) + (1.10) with $j = 1$ or 2 or 3, or 4, have the first integral

$$H_s(x,y) = \Big(-\frac{1}{2}\,x + \beta y - \frac{19}{10}\Big)^3\Big(\frac{3}{2}\,x + \beta y + \frac{23}{30}\Big)$$

if $x \in \Sigma_L$, and the first integral H_j with $j = 1$ or 2 or 3, or 4, where H_j are given by (1.2), (1.3), (1.4) and (1.5), respectively, if $x \in \Sigma_R$.

For the piecewise differential systems (S_j)+(1.10) with $j \in \{1,2,3,4\}$, the function (2.15) becomes

$$f_2(t) = \frac{17}{20}\,\frac{\sinh t}{\sinh \frac{1}{2}\,t} - 2.$$

The unique solution of $f_1(t) = 0$ is $t = 1.171\,4$. From this value of t and using (2.10), we get the values of $y_1 = -\frac{0.541\,03}{\beta}$. Since $y_1 = -y_0$, we have $y_0 = \frac{0.541\,03}{\beta}$. So, the discontinuous piecewise differential systems (S_j) + (1.10) have exactly one crossing limit cycle. Straightforward computations show that the crossing limit cycle passing through the crossing points $(0, y_1)$ and $(0, y_2)$ corresponds to

$$\Gamma = \big\{(x,y) \in \Sigma_R : \quad H_j(x,y) = h_j\big\} \cup \big\{(x,y) \in \Sigma_L : \quad H_s(x,y) = -3.\,281\,8\big\},$$

where $j = 1$ or 2 or 3, or 4, $h_1 = (\frac{0.541\,03}{\beta})^4$ and $h_2 = h_3 = h_4 = (\frac{0.541\,03}{\beta})^2$. Moreover, this limit cycle is algebraic and the sliding region of systems (S_j)+(1.11) is defined by $\Sigma_s = \{(0,y) \in \Sigma : \ 0 \leq y \leq -\frac{1}{10\beta}\}$, which is inside the periodic orbit. Drawing the orbit Γ, we obtain the limit cycle in Figure 3, which travels in a counterclockwise sense. □

Proof of proposition 1.5. We consider the piecewise differential systems (S_j) + (1.11) with $j = 1$ or 2 or 3 or 4. The equilibrium point $(\frac{1}{2}, -\frac{4}{\beta})$ of system (1.11) has eigenvalues 4, 2, so, it is a virtual node. On the other hand, the piecewise differential systems (S_j) + (1.11) with $j \in \{1,2,3,4\}$ have the first integral

$$H_n(x,y) = \frac{(2x + \beta y + 3)^4}{(4x + \beta y + 2)^2}$$

if $x \in \Sigma_L$, and the first integral H_j with $j = 1$ or 2 or 3, or 4, where H_j are given by (1.2), (1.3), (1.4) and (1.5), respectively, if $x \in \Sigma_R$.

For the piecewise differential systems (S_j)+(1.11) with $j \in \{1,2,3,4\}$, the function (2.15) becomes

$$f_2(t) = 20\,\frac{\sinh t}{\sinh 3t} - 4.$$

Now, solving the equation $f_2(t) = 0$ with respect to the variable t, we get $t = 0.658\,48$. Using the expression of y_1 given by (2.13) and taking into account that $y_1 = -y_0$, we get $y_1 = -\frac{1.732\,1}{\beta}$ and $y_0 = \frac{1.732\,1}{\beta}$. So, the discontinuous piecewise differential systems (S_j) + (1.11) have exactly one crossing limit cycle. Straightforward computations show that the crossing limit cycle passing through the crossing points $(0, y_1)$ and $(0, y_2)$ corresponds to

$$\Gamma = \big\{(x,y) \in \Sigma_R : \quad H_j(x,y) = h_j\big\} \cup \big\{(x,y) \in \Sigma_L : \quad H_n(x,y) = 36.008\big\},$$

where $j \in \{1,2,3,4\}$, $h_1 = (-\frac{1.7321}{\beta})^4$ and $h_2 = h_3 = h_4 = (-\frac{1.7321}{\beta})^2$. Moreover, Γ is non-algebraic and travels in a counterclockwise sense, around the sliding set $\Sigma_s = \{(0,y) \in \Sigma : \ 0 \le y \le -\frac{1}{\beta}\}$. $\square$

Proof of proposition 1.6. We consider the piecewise differential systems $(S_j) + (1.12)$ with $j = 1$ or 2 or 3 or 4. Since the eigenvalues of the matrices of the linear differential system (1.12) is -1, this system has a virtual node with eigenvalue of multiplicity 2 whose linear part does not diagonalize at the equilibrium point $(1, 0)$. The piecewise differential systems $(S_j) + (1.12)$ with $j = 1$ or 2 or 3, or 4, have the first integral

$$H_{n'}(x,y) = \frac{1}{-2x - \beta y + 2}\, e^{\frac{x-1}{-2x-\beta y+2}}$$

if $x \in \Sigma_L$, and the first integral H_j with $j = 1$ or 2 or 3 or 4, where H_j are given by (1.2), (1.3), (1.4) and (1.5), respectively, if $x \in \Sigma_R$.

For the piecewise differential systems $(S_j)+(1.12)$ with $j \in \{1,2,3,4\}$, the function (2.19) becomes

$$f_3(t) = \frac{1}{t}\,(\sinh t - 2t).$$

Now, solving $f_3(t) = 0$, we get $t = 2.1773$, substituting this value of t into the expression of y_1 given by (2.17) and taking into account that $y_1 = -y_0$, we get $y_1 = -\frac{1.5752}{\beta}$ and $y_0 = \frac{1.5752}{\beta}$. So, the discontinuous piecewise differential systems $(S_j) + (1.12)$ have exactly one crossing limit cycle. Straightforward computations show that the crossing limit cycle passing through the crossing points $(0, y_1)$ and $(0, y_2)$ corresponds to

$$\Gamma = \big\{(x,y) \in \Sigma_R : \quad H_j(x,y) = h_j\big\} \cup \big\{(x,y) \in \Sigma_L : \quad H_{n'}(x,y) = 0.21146\big\},$$

where $j \in \{1,2,3,4\}$, $h_1 = (\frac{1.5752}{\beta})^4$ and $h_2 = h_3 = h_4 = (\frac{1.5752}{\beta})^2$. Moreover, this limit cycle is non-algebraic and surrounds the sliding segment $\Sigma_s = \{(0,y) \in \Sigma : \ \frac{1}{\beta} \le y \le 0\}$ counterclockwise. $\square$

Proof of proposition 1.7. We consider the piecewise differential systems $(S_j)+(1.13)$ with $j \in \{1,2,3,4\}$. The planar linear differential system (1.13) has the first integral

$$H_w(x,y) = \Big(\frac{101}{100} + (1-\mu)x + y\Big)e^{\mu x - y}$$

in Σ_L and the cubic polynomial differential systems (S_j) with $j \in \{1,2,3,4\}$ have the first integral H_j, where H_j are given by (1.2), (1.3), (1.4) and (1.5) with $j = 1$ or 2 or 3 or 4, respectively, in Σ_R. It is easy to see that (1.13) has no equilibria, neither real nor virtual.

Then for the discontinuous piecewise differential systems $(S_j)+(1.13)$, the function (2.22) becomes

$$f_4(t) = t \coth \frac{1}{2}\, t - \frac{101}{50}\,.$$

This function $f_4(t)$ has exactly a unique positive root $t = 0.34676$. From this value of t and using (2.20), we get the values of $y_1 = 0.17338$. Since $y_1 = -y_0$, we have $y_0 = -0.17338$. So, the discontinuous piecewise differential systems $(S_j)+(1.13)$ have exactly one non-algebraic crossing limit cycle. Straightforward computations show that the crossing limit cycle passing through the crossing points $(0, y_1)$ and $(0, y_2)$ corresponds to

$$\Gamma = \big\{(x,y) \in \Sigma_R : \quad H_j = h_j\big\} \cup \big\{(x,y) \in \Sigma_L : \quad H_w(x,y) = 0.99501\big\},$$

where $j \in \{1,2,3,4\}$, $h_1 = (0.17338)^4$ and $h_2 = h_3 = h_4 = (0.17338)^2$. This limit cycle surrounds the sliding set $\Sigma_s = \{(0,y) \in \Sigma : \ \frac{-1}{100} \le y \le 0\}$. $\square$

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