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Noureddine Azzouz, Bouharket Benaissa, Ahmed Bensedik

SOME GENERALIZATIONS OF INTEGRAL INEQUALITIES AND THEIR CONFORMABLE FRACTIONAL INTEGRAL VERSIONS

Abstract. The aim of this paper is to present new integral inequalities by using a power β and a weight function satisfying some hypothesis, in particular, in the case of monotone functions. On the other hand, we derive new versions of integral inequalities with conformable fractional calculus for $\beta = 1$.

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რეზიუმე. ნაშრომის მიზანია ახალი ინტეგრალური უტოლობის წარმოდგენა β სიმძლავრის ხარისხის მაჩვენებლისა და წონის ფუნქციის გამოყენებით, რომლებიც აკმაყოფილებს გარკვეულ ჰიპოთეზას, კერძოდ, მონოტონური ფუნქციების შემთხვევაში. მეორე მხრივ, $\beta = 1$-თვის ჩვენ გამოვიყვანთ ინტეგრალური უტოლობების ახალ ვერსიებს კონფორმული წილადური რიგის აღრიცხვის გამოყენებით.

1 Introduction and Preliminaries

A number of new definitions have been introduced to provide a new fractional calculation method, particularly a conformable derivative based on limits was introduced in [3], which were followed by several recent articles (for more details, we refer the reader to [6, 8, 9].

Definition 1.1 (Conformable fractional derivative)**.** Given a function $f : [0, +\infty) \to \mathbb{R}$, the "conformable fractional derivative" of order α of f is defined by

$$D_\alpha(f)(t) = \lim_{\epsilon\to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$$

for all $t > 0$, $\alpha \in (0, 1]$. If f is α-differentiable in some interval $(0, a)$, $a > 0$, and $\lim_{t\to 0^+} D_\alpha(f)(t)$ exists, then define

$$D_\alpha(f)(0) = \lim_{t\to 0^+} D_\alpha(f)(t).$$

In addition, if the conformable fractional derivative of order α of f exists, then we simply say f is α-differentiable.

Definition 1.2 (Conformable fractional integral)**.** Let $\alpha \in (0, 1]$ and $0 \le a < b$. A function $f : [0, +\infty) \to \mathbb{R}$ is α-fractional integrable on $[a, b]$ if the integral

$$\int_a^b f(t)\, d_\alpha t := \int_a^b f(t) t^{\alpha-1}\, dt$$

exists and is finite.

Definition 1.3 (Conformable fractional integral operator)**.** Let $\alpha \in (0, 1]$ and $f : [a, +\infty) \to \mathbb{R}$ for $a \ge 0$. The conformable fractional integral operator of order α of f is defined by

$$I_\alpha^a f(x) = \int_a^x f(t)\, d_\alpha t := \int_a^x f(t) t^{\alpha-1}\, dt$$

for all $x \ge a$, $\alpha \in (0, 1]$.

For $a = 0$, we denote $I_\alpha f := I_\alpha^0 f$.

Theorem 1.1. *Let $f : (a, b) \to \mathbb{R}$ be differentiable and $\alpha \in (0, 1]$. Then for all $x \ge a$ we have*

$$I_\alpha^a D_\alpha f(x) = f(x) - f(a),$$
$$D_\alpha I_\alpha^a f(x) = f(x).$$

In [7], the authors proved the following

Theorem 1.2. *Let $M > 0$, $0 < p < 1$ and $-1 < r < p-1$. If f is a non-negative measurable function on $(0, +\infty)$ satisfying for almost all $x > 0$ the inequality*

$$f(x) \le \frac{M}{x}\left(\int_0^x (f^p(t)t^{p-1})\, dt\right)^{\frac{1}{p}} \quad a.e.\ \ x > 0, \tag{1.1}$$

then

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\, dt\right)^p x^r\, dx \le C^p \int_0^\infty f^p(x) x^r\, dx,$$

where the $C^p := \frac{p^p M^{p(1-p)}}{p-r-1}$ is sharp.

We state the following theorem which is useful in proving the main results.

Theorem 1.3 (Minkowski's integral inequality, [1]). *Let $-\infty \le a < b \le +\infty$ and $-\infty \le c < d \le +\infty$. Suppose that f is measurable non-negative (non-positive) function on $(a,b)\times(c,d)$ and $f(\cdot,y) \in L_p(a,b)$ for almost all $y \in (c,d)$. Then*

1. *For $p \ge 1$,*
$$\left\| \int_c^d f(x,y)\,dy \right\|_{L_p(a,b)} \le \int_c^d \|f(x,y)\|_{L_p(a,b)}\,dy, \tag{1.2}$$
if the right-hand side is finite.

2. *For $0 < p < 1$,*
$$\left\| \int_c^d f(x,y)\,dy \right\|_{L_p(a,b)} \ge \int_c^d \|f(x,y)\|_{L_p(a,b)}\,dy, \tag{1.3}$$
if the left-hand side is finite.

Hardy-type inequalities have a great diversity in different branches of analysis and integrative equations. The aim of this paper is to present some new weighted Hardy-type inequalities by using Minkowski's integral inequality, and to derive new conformal fractional integral inequalities.

2 Main results

Theorem 2.1. *Let $\alpha \in (0,1]$, $\beta \ge 1$, $p > 1$ and f be a non-negative measurable function on $(0,+\infty)$. Then the inequality*

$$\int_0^\infty \left(\frac{1}{x^\beta} \int_0^x f(t)t^{\alpha-1}\,dt \right)^p dx \le \left(\frac{p}{\beta p - 1} \right)^p \int_0^\infty (f(x)x^{\alpha-\beta})^p\,dx \tag{2.1}$$

holds if the right-hand side is finite.

Proof. For $x > 0$, we have

$$\frac{1}{x^\beta} \int_0^x f(t)t^{\alpha-1}\,dt = \int_0^1 f(\tau x)(\tau x)^{\alpha-1} x^{1-\beta}\,d\tau.$$

We denote by Lhs the left-hand side of inequality (2.1) Using the Minkowski inequality (1.2), we get

$$\begin{aligned}
(Lhs)^{\frac{1}{p}} &= \left(\int_0^\infty \left(\int_0^1 f(tx)(tx)^{\alpha-1}x^{1-\beta}\,dt \right)^p dx \right)^{\frac{1}{p}} \le \int_0^1 \left(\int_0^\infty x^{p(1-\beta)}(f(tx)(tx)^{\alpha-1})^p\,dx \right)^{\frac{1}{p}} dt \\
&= \int_0^1 \left(\int_0^\infty \frac{(tx)^{p(1-\beta)}}{t^{p(1-\beta)}} (f(tx)(tx)^{\alpha-1})^p\,dx \right)^{\frac{1}{p}} dt = \int_0^1 \left(\int_0^\infty (f(\mu)\mu^{\alpha-\beta})^p \frac{d\mu}{t} \right)^{\frac{1}{p}} \frac{1}{t^{(1-\beta)}}\,dt \\
&= \int_0^1 \frac{1}{t^{\frac{1}{p}+1-\beta}}\,dt \left(\int_0^\infty (f(\mu)\mu^{\alpha-\beta})^p\,d\mu \right)^{\frac{1}{p}} = \left(\frac{p}{\beta p - 1} \right) \left(\int_0^\infty (f(\mu)\mu^{\alpha-\beta})^p\,d\mu \right)^{\frac{1}{p}}.
\end{aligned}$$
□

From the equality

$$\frac{1}{x^\beta} \int_0^x f(t)t^{\alpha-1}\,dt = \frac{1}{x^\beta} \int_0^x f(t)\,d_\alpha t$$

and for $\beta = 1$, we obtain the following Corollary.

Corollary 2.1. *Let $\alpha \in (0,1]$, f be a non-negative measurable function on $(0,+\infty)$ and $\int\limits_0^\infty \frac{f^p(x)}{x^{p(1-\alpha)}}\,dx < \infty$, then for $p > 1$ we have*

$$\int\limits_0^\infty \Big(\frac{1}{x} I_\alpha(x)\Big)^p\,dx \le \Big(\frac{p}{p-1}\Big)^p \int\limits_0^\infty \Big(\frac{1}{x^{1-\alpha}} f(x)\Big)^p\,dx.$$

Theorem 2.2. *Let $\alpha \in (0,1]$, $\beta, p \ge 1$, $r < 0$ and let f, w be non-negative measurable functions on $(0,+\infty)$, where the weight function w satisfies the following hypothesis:*

$$\text{for all } t \in (0,1),\quad w(tx) \le t\,w(x). \tag{2.2}$$

Then the inequality

$$\int\limits_0^\infty \Big(\frac{1}{x^\beta}\int\limits_0^x f(t)t^{\alpha-1}\,dt\Big)^p w^r(x)\,dx \le \Big(\frac{p}{\beta p - r - 1}\Big)^p \int\limits_0^\infty (f(x)x^{\alpha-\beta})^p w^r(x)\,dx \tag{2.3}$$

holds if the right-hand side is finite.

Remark 2.1. Note that inequality (2.2) is satisfied, for example, by polynomial functions $w(x) = x^n$ for any integer $n \ge 1$, by constant functions $w(x) = c$ where c is a strictly negative constant.

Proof. We denote by Lhs the left-hand side of inequality (2.3). Using the Minkowski inequality (1.2) and hypothesis (2.2), we conclude that

$$\begin{aligned}
(Lhs)^{\frac{1}{p}} &= \Big(\int\limits_0^\infty \Big(\frac{1}{x^\beta}\int\limits_0^x f(t)t^{\alpha-1}\,dt\Big)^p w^r(x)\,dx\Big)^{\frac{1}{p}} = \Big(\int\limits_0^\infty \Big(\int\limits_0^1 f(tx)(tx)^{\alpha-1}x^{1-\beta}w^{\frac{r}{p}}(x)\,dt\Big)^p dx\Big)^{\frac{1}{p}} \\
&\le \int\limits_0^1 \Big(\int\limits_0^\infty x^{p(1-\beta)}(f(tx)(tx)^{\alpha-1})^p w^r(x)\,dx\Big)^{\frac{1}{p}} dt = \int\limits_0^1 \Big(\int\limits_0^\infty (f(tx)(tx)^{\alpha-\beta})^p w^r(x)\,dx\Big)^{\frac{1}{p}} \frac{1}{t^{1-\beta}}\,dt \\
&\le \int\limits_0^1 \Big(\int\limits_0^\infty (f(tx)(tx)^{\alpha-\beta})^p\,\frac{w^r(tx)}{t^r}\,dx\Big)^{\frac{1}{p}} \frac{1}{t^{1-\beta}}\,dt = \int\limits_0^1 \Big(\int\limits_0^\infty (f(\mu)\mu^{\alpha-\beta})^p\,\frac{w^r(\mu)}{t^r}\,\frac{d\mu}{t}\Big)^{\frac{1}{p}} \frac{1}{t^{1-\beta}}\,dt \\
&= \int\limits_0^1 \frac{1}{t^{\frac{r+1}{p}+1-\beta}}\,dt\Big(\int\limits_0^\infty (f(\mu)\mu^{\alpha-\beta})^p w^r(\mu)\,d\mu\Big)^{\frac{1}{p}} = \Big(\frac{p}{\beta p - r - 1}\Big)\Big(\int\limits_0^\infty (f(\mu)\mu^{\alpha-\beta})^p w^r(\mu)\,d\mu\Big)^{\frac{1}{p}}. \quad \square
\end{aligned}$$

Taking $\beta = 1$ and $w(x) = x$, we obtain the following

Corollary 2.2. *Let $\alpha \in (0,1]$, $r < 0$, and let f be a non-negative measurable function on $(0,+\infty)$ and $\int\limits_0^\infty \frac{f^p(x)}{x^{p(1-\alpha)}}\,dx < \infty$, then for $p \ge 1$ we have*

$$\int\limits_0^\infty \Big(\frac{1}{x} I_\alpha(x)\Big)^p x^r\,dx \le \Big(\frac{p}{p-r-1}\Big)^p \int\limits_0^\infty \Big(\frac{1}{x^{1-\alpha}} f(x)\Big)^p x^r\,dx.$$

Now we present some new inequalities related to the monotone functions.

Proposition. *Let $\alpha \in (0,1]$, $p > 0$, $1 \le \beta < 1 + p\alpha$ and let f be a non-negative measurable and decreasing function on $(0,+\infty)$, then*

$$f(x) \le \frac{K}{x^\lambda}\Big(\int\limits_0^x (f^p(t)t^{p\alpha-\beta})\,dt\Big)^{\frac{1}{p}}, \tag{2.4}$$

where $K > 0$ and $0 < \lambda \le \alpha$.

Proof. f is assumed decreasing on $(0,x)$, then

$$\Big(\int_0^x (f^p(t)t^{p\alpha-\beta})\,dt\Big)^{\frac{1}{p}} \geq f(x)\Big(\int_0^x t^{p\alpha-\beta}\,dt\Big)^{\frac{1}{p}} = \frac{x^{\alpha+\frac{1-\beta}{p}}}{(p\alpha+1-\beta)^{\frac{1}{p}}}\,f(x) = \frac{x^{\lambda}}{K}\,f(x),$$

and since $1 \leq \beta < 1+p\alpha$, we get $-p\alpha < 1-\beta \leq 0$, so $0 < \alpha + \frac{1-\beta}{p} \leq \alpha$. □

Condition (2.4) is a more general condition of monotonicity and is a generalization of (1.1).

Lemma. *Let $\alpha \in (0,1]$, $M > 0$, $\beta \geq 1$ and $0 < p < 1$, let f be a non-negative measurable function on $(0,+\infty)$ satisfying the following condition:*

$$f(x) \leq \frac{M}{x^{\alpha}}\Big(\int_0^x (f^p(t)t^{p\alpha-\beta})\,dt\Big)^{\frac{1}{p}} \quad a.e.\ \ x > 0. \tag{2.5}$$

Then the inequality

$$\Big(\int_0^x f(t)t^{\alpha-\beta}\,dt\Big)^{p} \leq p^p M^{p(1-p)} \int_0^x (f^p(t)t^{p\alpha-\beta})\,dt$$

holds if the right-hand side is finite.

Proof. Let $x > 0$ and f satisfy inequality (2.5) almost everywhere in $(0,x)$. Since

$$f(t) = (f(t)t)^{1-p}(f^p(t)t^{p-1}),$$

we have

$$\begin{aligned} f(t) &\leq \left[\frac{M}{t^{\alpha}}\Big(\int_0^t f^p(\mu)\mu^{p\alpha-\beta}\,d\mu\Big)^{\frac{1}{p}} t\right]^{1-p}(f^p(t)t^{p-1}) \\ &= M^{1-p}\Big(\int_0^t f^p(\mu)\mu^{p\alpha-\beta}\,d\mu\Big)^{\frac{1}{p}-1}(f^p(t)t^{p\alpha-\alpha}). \end{aligned}$$

Hence we obtain

$$f(t)t^{\alpha-\beta} \leq M^{1-p}\Big(\int_0^t f^p(\mu)\mu^{p\alpha-\beta}\,d\mu\Big)^{\frac{1}{p}-1}(f^p(t)t^{p\alpha-\beta}).$$

Integrating the above inequality on $(0,x)$ and taking $\psi(t) = \int_0^t f^p(\mu)\mu^{p\alpha-\beta}\,d\mu$, we obtain

$$\begin{aligned} \int_0^x f(t)t^{\alpha-\beta}\,dt &\leq M^{1-p}\int_0^x \left[\Big(\int_0^t f^p(\mu)\mu^{p\alpha-\beta}\,d\mu\Big)^{\frac{1}{p}-1}(f^p(t)t^{p\alpha-\beta})\right]dt \\ &= M^{1-p}\int_0^x (\psi(t))^{\frac{1}{p}-1}\psi'(t)\,dt = pM^{1-p}(\psi(x))^{\frac{1}{p}} = pM^{1-p}\Big(\int_0^x f^p(\mu)\mu^{p\alpha-\beta}\,d\mu\Big)^{\frac{1}{p}}, \end{aligned}$$

which completes the proof. □

Remark 2.2. Lemma 2 is a new generalization of Lemma 2.1 [7].

Taking $\beta = 1$ in Lemma 2, we obtain the following

Corollary 2.3. *Let $\alpha \in (0,1]$, $M > 0$ and $0 < p < 1$, let f be a non-negative measurable function on $(0,+\infty)$ satisfying the following condition:*

$$f(x) \leq \frac{M}{x^\alpha}\Big(\int_0^x (f^p(t)t^{p\alpha-1})\,dt\Big)^{\frac{1}{p}} \quad a.e. \;\; x > 0,$$

Then the inequality

$$\Big(\int_0^x f(t)t^{\alpha-1}\,dt\Big)^p \leq p^p M^{p(1-p)} \int_0^x (f^p(t)t^{p\alpha-1})\,dt$$

holds if the right-hand side is finite.

Theorem 2.3. *Let $\alpha \in (0,1]$, $\beta \geq 1$, $r < \beta p - 1$, $0 < p < 1$ and let v be a weight function on $(0,+\infty)$. If $\frac{v(x)}{x}$ is non-decreasing and f is a non-negative measurable function on $(0,+\infty)$ satisfying condition (2.5), then the inequality*

$$\Big(\int_0^\infty \Big(\frac{1}{x^\beta}\int_0^x f(t)t^{\alpha-1}\,dt\Big)^p v^r(x)\,dx\Big)^{\frac{1}{p}} \leq C^p\Big(\int_0^\infty (f(x)x^{\alpha-\beta})^p x^{1-\beta} v^r(x)\,dx\Big)^{\frac{1}{p}} \tag{2.6}$$

holds if the right-hand side is finite, where $C^p := \frac{p^p M^{p(1-p)}}{\beta p - r - 1}$.

Proof. Let $x > 0$ and f satisfy inequality (2.5) almost everywhere in $(0,x)$. Denote by Lhs the integral in the left-hand side of inequality (2.6). By applying Lemma 2 and Fubini's Theorem, we get

$$\begin{aligned} Lhs &= \int_0^\infty \Big(\frac{1}{x^\beta}\int_0^x f(t)t^{\alpha-1}\,dt\Big)^p v^r(x)\,dx = \int_0^\infty \Big(\int_0^x f(t)t^{\alpha-1}\,dt\Big)^p \frac{v^r(x)}{x^{\beta p}}\,dx \\ &\leq \int_0^\infty p^p M^{p(1-p)} \int_0^x (f^p(t)t^{p\alpha-\beta})\,dt \frac{v^r(x)}{x^{\beta p}}\,dx = p^p M^{p(1-p)} \int_0^\infty \Big(\int_t^\infty \frac{v^r(x)}{x^{\beta p}}\,dx\Big) f^p(t)t^{p\alpha-\beta}\,dt. \end{aligned}$$

Since the function $\frac{v(x)}{x}$ is non-decreasing on $[t,\infty[$, we get

$$\forall\, x \in [t,\infty[\,, \quad \frac{v^r(x)}{x^r} \leq \frac{v^r(t)}{t^r}\,.$$

Consequently, we deduce that

$$\begin{aligned} Lhs &\leq p^p M^{p(1-p)} \int_0^\infty \Big(\frac{v^r(t)}{t^r}\int_t^\infty \frac{1}{x^{\beta p - r}}\,dx\Big) f^p(t)t^{p\alpha-\beta}\,dt \\ &= \frac{p^p M^{p(1-p)}}{\beta p - r - 1}\int_0^\infty \Big(\frac{v^r(t)}{t^r} t^{-\beta p + r + 1} f^p(t)t^{p\alpha-\beta}\Big)\,dt \\ &= \frac{p^p M^{p(1-p)}}{\beta p - r - 1}\int_0^\infty (f(t)t^{\alpha-\beta})^p v^r(t)t^{1-\beta}\,dt. \end{aligned}$$

□

Setting $\beta = 1$ and $v(x) = x$, we obtain the following

Corollary 2.4. *Let $\alpha \in (0,1]$, $r < p - 1$, $0 < p < 1$. If f is a non-negative measurable function on $(0,+\infty)$ and satisfies condition (2.5), then the inequality*

$$\Big(\int_0^\infty \Big(\frac{1}{x}\int_0^x f(t)\,d_\alpha t\Big)^p x^r\,dx\Big)^{\frac{1}{p}} \leq C^p\Big(\int_0^\infty (f(x)x^{\alpha-1})^p x^r\,dx\Big)^{\frac{1}{p}}$$

holds if the right-hand side is finite, where $C^p := \frac{p^p M^{p(1-p)}}{p - r - 1}$.

Remark 2.3. By taking $\alpha = 1$ in the above corollary, we get Theorem 1.2.

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Authors' addresses:

Noureddine Azzouz
1. Faculty of Science, University of Elbayadh, Algeria.
2. Laboratory of Informatics and Mathematics, University of Tiaret, Algeria.
E-mail: n.azzouz@cu-elbayadh.dz

Bouharket Benaissa
1. Faculty of Material Sciences, University of Tiaret, Algeria.
2. Laboratory of Informatics and Mathematics, University of Tiaret, Algeria.
E-mail: bouharket.benaissa@univ-tiaret.dz

Ahmed Bensedik
Laboratory of Dynamic Systems and Application, University of Tlemcen, Algeria.
E-mail: ahmed.bensedik@univ-tlemcen.dz