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ON SHARP ESTIMATION OF THE EXPONENT OF SOLUTIONS TO SOME CLASSES OF FUNCTIONAL DIFFERENTIAL EQUATIONS

Abstract. A method for determining the sharp exponent of solutions to exponentially stable differential-difference equations is proposed. The equations with a single delay and with real and complex coefficients, as examples, are considered. For these equations, the dependence of such an exponent on the coefficient in both analytical and geometric form is found.

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Key words and phrases. Functional differential equation, fundamental solution, exponential stability, upper exponent.

რეზიუმე. მოცემულია მეთოდი ექსპონენციალურად სტაბილური დიფერენციალურ-სხვაობიანი განტოლებების ამონახსნების ზუსტი მაჩვენებლის დასადგენად. მაგალითის სახით განხილულია განტოლებები ერთჯერადი დაგვიანებით და ნამდვილი და კომპლექსური კოეფიციენტებით. ამ განტოლებისთვის ნაპოვნია ასეთი მაჩვენებლის დამოკიდებულება კოეფიციენტზე როგორც ანალიზური, ასევე გეომეტრიული ფორმით.

1 Introduction

As is known, exponential stability in its classical version is defined as follows: an ordinary differential equation $\dot{x} = Ax$ is called *exponentially stable* if there exist $N, \omega > 0$ such that for every solution supplemented with the initial condition, the estimate $|x(t)| \le Ne^{-\omega t}|x(0)|$ is valid. The largest ω for which this inequality is true is called the *upper exponent*. For autonomous ordinary differential equations (ODEs), computing or estimating the upper exponent is a well-known problem, and various methods have been developed to solve it. In particular, for the scalar equation $\dot{x} + ax = 0$, obviously, $\omega = a$; for systems, the exponent $(-\omega)$ coincides with the greatest real part of the eigenvalues of the matrix A.

The transition from ordinary differential to functional differential equations (FDEs) did not require fundamental changes in the definitions of stability; the initial function φ appears in the definition instead of the initial point, and the definition of exponential stability is transformed as follows: $|x(t)| \le Ne^{-\omega t}\|\varphi\|$. Just as for ODEs, the formulated above problem is reduced to studying location of the roots of the characteristic function on the complex plane. For a fairly wide class of FDEs, it was possible to find exponential stability criteria and even to construct stability regions in parameter space, that is, to give an effective answer to the question, when the roots of the characteristic function are situated to the left of the imaginary axis (and, moreover, are separated from it).But here it should be emphasized that all these criteria only assert the existence of the exponent ω and indicate its sign, but they do not provide a way to calculate or even estimate ω. The reason is that the characteristic function for FDE *is not a polynomial*, and therefore finding the real part of its nearest to the imaginary axis root is a qualitatively more difficult problem than the analogous problem for a polynomial. Nevertheless, it needs a solution, otherwise the question on the exponential stability of FDE cannot be considered as fully investigated.

2 Statement of the problem

Consider a FDE of retarded type

$$\dot{x}(t) + \sum_{j=0}^{J} b_j x(t - h_j) = 0, \quad t \in \mathbb{R}_+, \tag{2.1}$$

in the following assumptions: $b_k \in \mathbb{C}$, $0 \le h_0 < h_1 < \cdots < h_J$.

Define the function x for negative values of argument by an initial function φ. The requirement of "continuous junction" $x(0) = \varphi(0)$ is not considered mandatory within the framework of this article, although it is assumed to be satisfied in papers where the solution is understood as a continuous extension of the initial function [6, 7].

Definition 2.1. Equation (2.1) is called *exponentially stable* if there exist the constants $N, \gamma > 0$ such that for all $\varphi \in L_1[0, \omega]$ and $x(0) \in \mathbb{R}$ the corresponding solution of equation (2.1) is subjected to the estimate

$$|x(t)| \le Ne^{-\gamma t}\big(|x(0)| + \|\varphi\|_1\big). \tag{2.2}$$

If it is assumed that solutions of equation (2.1) are the continuous extensions of initial functions, then one can replace the sum $|x(0)| + \|\varphi\|_1$ in inequality (2.2) by $\max_{\theta \in [-h_J, 0]} |\varphi(\theta)|$.

The function

$$g(p) = p + \sum_{j=0}^{J} b_j e^{-ph_j}, \quad p \in \mathbb{C},$$

proves to be useful when investigating the stability of equation (2.1). We call this function the *characteristic polynomial*; the corresponding to it equation $g(p) = 0$ is called the *characteristic equation*. The function $g(p)$ is analytic on the whole complex plane. The characteristic equation has a countable set of roots. The following property is especially important for the further constructions: for each real α, the half-plane $\{p \in \mathbb{C} : \operatorname{Re} p \ge \alpha\}$ may contain only a finite set of roots of the characteristic equation.

In terms of the function $g(p)$, a criterion of the exponential stability for equation (2.1) is formulated easily.

Theorem 2.1. *For equation* (2.1) *to be exponentially stable, it is necessary and sufficient for all roots of the characteristic equation to be located to the left of the imaginary axis.*

It is obvious that if estimate (2.2) holds for some $\gamma > 0$, then it holds for all $\gamma' < \gamma$; one can choose the number γ' to be arbitrarily close to zero, making estimate (2.2) less exact. There is a problem more interesting and difficult: *find the greatest of exponents γ for which estimate* (2.2) *remains valid.* However, finding the exact exponent in terms of relation (2.2) is complicated by the fact that it is necessary to take into account a whole class of functions which is the class of solutions to equation (2.1) determined by all initial functions $\varphi \in L_1[0, \omega]$.

It would be much more convenient to deal with only one solution of equation (2.1) such that the estimation of this solution makes it possible to estimate all other solutions. To indicate such a solution, we use another notation for (2.1).

Denote by S_h the *shift operator* that is acting in the space of continuous (piecewise continuous, integrable) functions

$$(S_h y)(t) = \begin{cases} y(t-h), & t-h \geq 0, \\ 0, & t-h < 0. \end{cases}$$

Along with equation (2.1), consider the inhomogeneous equation

$$\dot{x}(t) + \sum_{j=0}^{J} b_j (S_{h_j} x)(t) = f(t), \quad t \in \mathbb{R}_+\,, \tag{2.3}$$

where the external perturbation function $f : \mathbb{R}_+ \to \mathbb{R}$ is assumed to be integrable on every finite segment. Equation (2.1) can be rewritten in the form (2.3) by replacing the external perturbation $f(t)$ by the function

$$\sigma(t) = -\sum_{j=1}^{J} b_j \varphi(t-h_j)\chi_j(t),$$

where $\chi_j(t)$ is the characteristic function of the set $(-\infty, h_j)$. We understand a *solution* of (2.3) as a function $x : \mathbb{R}_+ \to \mathbb{R}$ satisfying (2.3) almost everywhere on $\mathbb{R}_+$.

As is known (see [2, p. 84]), equation (2.3) with the given initial conditions is uniquely solvable, and its solution is represented in the form

$$x(t) = X(t)x(0) + \int_0^t X(t-s) f(s)\, ds, \tag{2.4}$$

where X is called the *fundamental solution.* It is convenient to consider it as defined on the entire axis $\mathbb{R}$, having added zero values on the negative semiaxis. The fundamental solution, as follows from representation (2.4), does not depend on either the initial condition $x(0)$, or the external perturbation f, and determines every solution to equation (2.1).

The asymptotic behavior of the fundamental solution of equations (2.1) and (2.3) can be described in terms of roots of the characteristic equation [10].

Suppose $p_1, p_2, \ldots, p_m$ are the roots of the characteristic equation with the greatest real part α. Then

$$X(t) = \sum_{k=1}^{m} q_k(t) e^{p_k t} + r(t), \tag{2.5}$$

where $q_k(t)$ is a polynomial, the degree of which coincides with the multiplicity of the root p_k, and $\lim_{t\to\infty} r(t)e^{-\alpha t} = 0$. It follows that for the fundamental solution of equation (2.1) there exists the limit

$$\lim_{t\to\infty} \frac{\ln|X(t)|}{t} = \alpha < \infty. \tag{2.6}$$

According to the terminology accepted in the theory of ODEs [5], we say that the number α is the (strict) *upper exponent* of the function X. Formula (2.6), which defines α, implies

Lemma 2.1. *For every $\gamma < -\alpha$, there exists $N > 0$ such that for the fundamental solution of equation* (2.1) *the estimate*

$$|X(t)| \leq Ne^{-\gamma t}, \quad t \in \mathbb{R}_+, \tag{2.7}$$

is satisfied.

It follows from Lemma 2.1 and the definition of exponential stability that equation (2.1) is exponentially stable if and only if $\alpha < 0$.

When studying exponential stability, another characteristic of the decay rate of the solution turns out to be convenient. Denote by Γ the set of numbers $\gamma \in \mathbb{R}$ for which estimate (2.7) is valid. It is clear from relation (2.6) that $\Gamma \neq \varnothing$. Therefore, the following definition is well-posed.

Definition 2.2. The number $\omega = \sup\{\gamma : \ \gamma \in \Gamma\}$ is called the *sharp exponent of the fundamental solution.*

Lemma 2.2. *If $\gamma_0 \in \Gamma$, then $(-\infty, \gamma_0] \in \Gamma$.*

Proof. If $\gamma_0 \in \Gamma$ and $\gamma < \gamma_0$, then $|X(t)| \leq Ne^{-\gamma_0 t} < Ne^{-\gamma t}$ for all $t \in \mathbb{R}_+$, that is, $\gamma \in \Gamma$. □

Now, it is easy to describe the structure of the set Γ. Consider two cases.

- Suppose $\omega \in \Gamma$. Then it follows from Lemma 2.2 that $(-\infty, \omega] \subseteq \Gamma$. On the other hand, we have from Definition 2.2 that $\Gamma \subseteq (-\infty, \omega]$. Hence $\Gamma = (-\infty, \omega]$.
- Suppose $\omega \notin \Gamma$. Take an increasing sequence $\gamma_n \in \Gamma$, $n \in \mathbb{N}$, such that $\gamma_n \to \omega$. It follows from Lemma 2.2 that $(-\infty, \gamma_n] \subseteq \Gamma$, hence $(-\infty, \omega) = \bigcup\limits_{n=1}^{\infty} (-\infty, \gamma_n] \subseteq \Gamma$. On the other hand, we have from Definition 2.2 that $\Gamma \subseteq (-\infty, \omega)$, hence $\Gamma = (-\infty, \omega)$.

Thus, the set Γ is always a semiaxis of the form $(-\infty, \omega]$ or $(-\infty, \omega)$. Below, we will verify that both of these cases are realized for equation (2.1).

Theorem 2.2. *The upper and the sharp exponents of the fundamental solution of equation* (2.1) *are connected by the equality $\omega = -\alpha$.*

Proof. It follows from inequality (2.7) and the definition of α that $\omega \leq -\alpha$. On the other hand, Lemma 2.1 implies that $-\alpha - 1/n \in \Gamma$ for every $n \in \mathbb{N}$, hence it follows from the definition of ω that $-\alpha - 1/n \leq \omega$. Passing to the limit as $n \to \infty$, we obtain $\omega \geq -\alpha$, that is, $\omega = -\alpha$. □

Theorem 2.3. *If the fundamental solution of equation* (2.1) *is subjected to estimate* (2.7) *with the exponent $\gamma > 0$, then every solution of equation* (2.1) *is subjected to estimate* (2.2) *with the same exponent γ.*

Proof. Follows from formula (2.4) and the definition of the function σ. □

Using a formula similar to (2.6), one can define the upper exponent for every solution of equation (2.1). It follows from Theorem 2.3 that for all solutions this exponent is not greater than α. On the other hand, X is one of the solutions of equation (2.1), so the exponent α cannot be reduced if we consider it as a universal characteristic of all solutions of equation (2.1). The following theorem gives the idea of solving the problem of the sharp exponent of the fundamental solution.

Theorem 2.4. *The sharp exponent of the fundamental solution of equation* (2.1) *is equal to ω if and only if the upper exponent of the fundamental solution of the equation*

$$\dot{y}(t) - \omega y(t) + \sum_{k=0}^{K} b_k e^{\omega h_k} y(t - h_k) = 0, \quad t \in \mathbb{R}_+\,, \tag{2.8}$$

is equal to zero.

Proof. Supplement equation (2.1) with the initial condition $x(0) = 1$ and make in it a change of variables $x(t) = e^{-\omega t}y(t)$. It is easily seen that y is a solution of equation (2.8). The assertion of the theorem follows from Theorem 2.2 and formula (2.6). □

Theorem 2.4 proposes a scheme for finding a sharp exponent of equation (2.1): it corresponds to those parameters of equation (2.1) for which the upper exponent of equation (2.8) is equal to zero. This also implies the importance of effective criteria (and especially regions) of stability for equations of form (2.8), that is, for those equations in which the term $y(t)$ is presented.

The following fact, which follows from formulas (2.5) and (2.6), solves the problem of conditions for the upper exponent to be equal to zero in terms of zeros of the characteristic function.

Theorem 2.5. *The upper exponent of the fundamental solution of equation* (2.8) *is equal to zero if and only if the characteristic quasipolynomial of equation* (2.8) *has no zeros in the open right half-plane, but has roots on the imaginary axis.*

Remark. Note that knowing the number ω does not yet give grounds to replace the exponent γ by ω in estimate (2.7). This is possible only if in Definition 2.2, in the formula for ω, the least upper bound is *attainable* for some $\gamma = \omega$, that is, $\omega \in \Gamma$. If this bound is unattainable, then estimate (2.7) is valid only for all γ (arbitrarily close to ω) subordinated to the inequality $\gamma < \omega$.

Theorem 2.6. *Suppose ω is the sharp exponent of equation* (2.1)*. The fundamental solution of equation* (2.1) *is subjected to estimate* (2.7) *with the exponent $\gamma = \omega$ if and only if the fundamental solution of equation* (2.8) *is bounded.*

Let us present a criterion of the boundedness of the fundamental solution in terms of the characteristic quasipolynomial. The criterion follows from formula (2.5).

Theorem 2.7. *The fundamental solution of equation* (2.8) *is bounded if and only if the characteristic quasipolynomial of equation* (2.8) *has no zeros in the open right half-plane, and all roots located on the imaginary axis are simple.*

We start the search for the dependence of the sharp exponent on parameters of the original equation with the simplest representatives of equation (2.1).

3 Equation with a real coefficient

Consider an equation

$$\dot{x}(t) + bx(t-1) = 0, \quad t \in \mathbb{R}_+\,, \tag{3.1}$$

in which $b \in \mathbb{R}$. The choose of delay $h = 1$ is not a restriction, since any equation with a single nonzero delay h may be reduced to the form (3.1) by replacing argument $t \mapsto th$.

The exponential stability criterion for equation (3.1) is known: *the equation is exponentially stable if and only if* $0 < b < \pi/2$. However, the sharp exponent for this equation has not been found so far – in the literature there are only indications that this exponent is determined by the real part of the root of the characteristic quasipolynomial nearest to the imaginary axis. This indication is not constructive, since only in exceptional cases it is possible to find the nearest to the imaginary axis root of the quasipolynomial exactly.

Put the following equation of form (2.8) in correspondence with equation (3.1):

$$\dot{y}(t) - \omega y(t) + be^{\omega}y(t-1) = 0, \quad t \in \mathbb{R}_+. \tag{3.2}$$

Equation (3.2) is a special case of an equation

$$\dot{y}(t) + \alpha y(t) + \beta y(t-1) = 0, \quad t \in \mathbb{R}_+, \tag{3.3}$$

for which the criterion of exponential stability is known and the region of stability is constructed [1], see Fig. 1. The boundaries of the infinite angle composing the stability region can be determined analytically: the rectilinear boundary has the form $u = -v$, and a parametric notation is convenient for the curvilinear boundary: $u = -\theta \cot\theta$, $v = \frac{\theta}{\sin\theta}$, $\theta \in [0, \pi/2]$ (for $\theta = 0$, the fraction $\frac{\theta}{\sin\theta}$ is defined by continuity). Denote the interior of the angle by D and its border by ∂D.

As the study of the characteristic equation shows [3], if $(\alpha, \beta) \in D$, then all zeros of the function $g(p) = p + \alpha + \beta e^{-p}$ lie in the open left half-plane; if $(\alpha, \beta) \in \partial D$, then the function $g(p)$ has no zeros to the right of the imaginary axis, but has roots on the imaginary axis; if $(\alpha, \beta) \notin (D \cup \partial D)$, then $g(p)$ has zeros in the open right half-plane. It follows from Theorem 2.5 that the upper exponent of equation (3.3) is equal to zero if and only if $(\alpha, \beta) \in \partial D$. Equation (3.2) shows that we are only interested in a part of the angle where $u < 0$.

From Theorem 2.4 we obtain that the sharp exponent of equation (3.1) is equal to $\omega > 0$ if and only if $(-\omega, be^{\omega}) \in \{(u, v): \ u < 0\} \cap \partial D$. To solve effectively the problem of the sharp exponent of equation (3.1), we pass to the coordinates $\zeta = \omega$, $\eta = b$. The first boundary is now $\eta = \zeta e^{-\zeta}$, and since we are interested only in the region $\{(u, v) : u \ < 0\}$, we have $\zeta \in (0, 1)$. The function $\eta = \zeta e^{-\zeta}$ has the inverse function on the segment $[0, 1]$, denote it by $\zeta = \varphi_1(\eta)$. Obviously, the function φ_1 is defined on the set $\eta \in [0, 1/e]$, is continuous on it and monotonically increases from 0 to 1. From the equation of the second boundary, we obtain the following parametrically defined curve $\zeta = \theta \cot\theta$, $\eta = \frac{\theta}{\sin\theta} e^{-\theta \cot\theta}$, where $\theta \in [0, \pi/2]$ (for $\theta = 0$, the fraction $\frac{\theta}{\sin\theta}$ is defined by continuity). It is easy to see that for $\eta \in [1/e, \pi/2]$, the above equalities define a single-valued, continuous, monotonically decreasing function $\zeta = \varphi_2(\eta)$. The graphs of the functions φ_1 and φ_2 that determine the dependence of the sharp exponent on the coefficient b for the fundamental solution of equation (3.1) are represented in Fig. 2. Thus, we have established the following result.

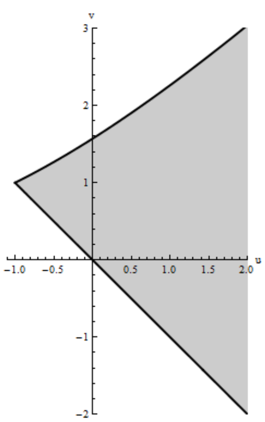

Figure 1

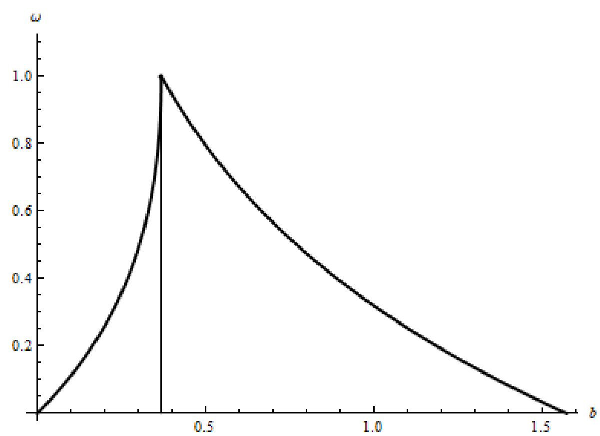

Figure 2

Theorem 3.1. *If equation* (3.1) *is exponentially stable, then its sharp exponent* ω *is defined by the following equalities:*

(1) $\omega = \varphi_1(b)$ *for* $b \in (0, 1/e)$*;*

(2) $\omega = \varphi_2(b)$ *for* $b \in (1/e, \pi/2)$*;*

(3) $\omega = 1$ *for* $b = 1/e$.

It remains to find out what estimates the fundamental solution has for each of the three cases given in Theorem 3.1.

Theorem 3.2. *Suppose equation* (3.1) *is exponentially stable and* $b \neq 1/e$. *Then for the fundamental solution of equation* (3.1) *the estimate* $|X(t)| \leq Ne^{-\omega t}$ *is valid, where* ω *is defined in items* (1) *and* (2) *of Theorem 3.1.*

Proof. Suppose $b \in (0, 1/e)$ and $\omega = \varphi_1(b)$. Then $be^{\omega} = \omega$ in equation (3.2), the point $(-\omega, \omega)$ belongs to the rectilinear boundary of the angle D and does not coincide with the point $(-1, 1)$. It is shown in [3] that in this case the fundamental solution of equation (3.2) is bounded.

Suppose now $b \in (1/e, \pi/2)$ and $\omega = \varphi_2(b)$. Then the coefficients of equation (3.2) form the point $(-\omega, be^{\omega})$ which belongs to the curvilinear boundary of the angle D and does not coincide with the point $(-1, 1)$. It was shown in [3] that in this case the fundamental solution of (3.2) is also bounded. Thus, for $b \neq 1/e$, the sharp exponent of equation (3.1) turns out to be attainable: it follows from Theorem 2.6 that the fundamental solution of equation (3.1) has an exponential estimate with the sharp exponent ω given in Theorem 3.1. □

Theorem 3.2 does not include the case $b = 1/e$; we consider it separately. It follows from Theorem 3.1 that $\omega = 1$, but in this case the sharp exponent of equation (3.1) is not attainable. The coefficients of equation (3.2) form the point $(-1, 1)$, and the fundamental solution of this equation, as is shown in [3], has a linear growth. Therefore, the estimate $|X(t)| \leq Ne^{(-1+\varepsilon)t}$ holds for all positive ε, but for $\varepsilon = 0$ it is not true any more. Let us show that the asymptotic behavior of the fundamental solution of equation (3.1) for $b = 1/e$ can be characterized more precisely.

Consider an equation

$$\dot{y}(t) - y(t) + y(t-1) = 0, \quad t \in \mathbb{R}_+. \tag{3.4}$$

For this equation, prove the following assertion.

Lemma 3.1. *For some $M, \nu > 0$, for the fundamental solution $Y(t)$ of equation* (3.4) *the following estimate is valid:*

$$\left| Y(t) - 2\Big(t + \frac{1}{3}\Big) \right| \leq Me^{-\nu t}, \quad t \in \mathbb{R}_+.$$

Proof. Equation (3.4) is of form (3.3) for $\alpha = -1$, $\beta = 1$. For these parameters, the point (α, β) is located on the boundary of the stability region of equation (3.3) (see Fig. 1), and the quasipolynomial $g(p) = p - 1 + e^{-p}$ has no roots to the right of the imaginary axis, but has a single root of multiplicity 2 on the imaginary axis $p_0 = 0$. Construct a rectangular contour with sides parallel to the real and imaginary axes, containing one root of the function g, which is the point $(0, 0)$, inside it. Using the inverse Laplace transform formula and the Cauchy theorem, we have

$$\left| Y(t) - \operatorname*{res}_{p=0} \frac{e^{pt}}{p - 1 + e^{-p}} \right| \leq Ne^{-\nu t}.$$

Since

$$\operatorname*{res}_{p=0} \frac{e^{pt}}{p - 1 + e^{-p-1}} = \lim_{p \to 0} \frac{d}{dp} \left(\frac{p^2 e^{pt}}{p - 1 + e^{-p-1}} \right) = 2t + \frac{2}{3}\,,$$

we obtain the desired inequality. □

Theorem 3.3. *Suppose $b = 1/e$ in equation* (3.1). *Then for some $M, \nu > 0$, for the fundamental solution of* (3.1) *the following estimate holds:*

$$\left| X(t)e^{t} - 2\Big(t + \frac{1}{3}\Big) \right| \leq Me^{-\nu t}. \tag{3.5}$$

Proof. Since the equation

$$\dot{x}(t) + \frac{1}{e}\, x(t-1) = 0$$

is transformed into equation (3.4) by the change of variables $x(t) = e^{-t} y(t)$, the assertion of the theorem follows from Lemma 3.1. □

Note the following properties of the sharp exponent of an exponentially stable equation (3.1), which follow from the properties of the functions φ_1, φ_2 (they are geometrically obvious in Fig. 2).

- For $b \neq 1/e$, the sharp exponent ω of equation (3.1) belongs to the interval $(0, 1)$.
- There are two and only two values of b, namely $\varphi_1^{-1}(\omega)$ and $\varphi_2^{-1}(\omega)$, that correspond to each $\omega \in (0, 1)$; for all $b \in [\varphi_1^{-1}(\omega),\, \varphi_2^{-1}(\omega)]$, the sharp exponent of equation (3.1) is not less than ω.
- The maximum sharp exponent for equation (3.1) is equal to 1 and is attained if and only if $b = 1/e$; in this case, estimate (2.7) should be replaced by inequality (3.5).

Let us illustrate the work of Theorem 3.1 by several examples.

Example 3.1. Let us find the sharp exponent for the equation

$$\dot{x}(t) + \frac{1}{3}\, x(t-1) = 0, \quad t \in \mathbb{R}_+.$$

Since $b = 1/3 < 1/e$, one should apply the first item of Theorem 3.1. Solving the equation $\omega e^{-\omega} = 1/3$, find $\omega \approx 0.619$. By virtue of Theorem 3.2, we get that the fundamental solution of the considered equation has an exponential estimate with the obtained exponent $\omega \approx 0.619$.

Example 3.2. Let us find the sharp exponent for the equation

$$\dot{x}(t) + x(t-1) = 0, \quad t \in \mathbb{R}_+.$$

Since $b = 1 > 1/e$, one should apply the second item of Theorem 3.1. Solving the equation $\frac{\theta}{\sin\theta} e^{-\theta\cot\theta} = 1$, find $\theta \approx 1.337$, it follows that $\omega = \theta\cot\theta \approx 0.318$. By virtue of Theorem 3.2, we get that the fundamental solution of the considered equation has an exponential estimate with the obtained exponent $\omega \approx 0.318$.

Example 3.3. Let us find conditions on parameters of equation (3.1) under which the equation has the sharp exponent $\omega = 0.5$. In accordance with the above-noted properties of the functions φ_1 and φ_2, we find that there are exactly two values of b for which equation (3.1) has the desired sharp exponent. The first value is $b = \varphi_1^{-1}(0.5) = 0.5e^{-0.5} \approx 0.303$. The second value is determined in two steps: first, from equation $\theta\cot\theta = 0.5$ we find $\theta_0 \approx 1.166$, and then $b = \varphi_2^{-1}(0.5) = \frac{\theta_0}{\sin\theta_0} e^{-0.5} \approx 0.769$. For all $b \in [\varphi_1^{-1}(0.5), \varphi_2^{-1}(0.5)]$, the sharp exponent of equation (3.1) is no less than 0.5.

4 Equation with a complex coefficient

Consider an equation

$$\dot{x}(t) + (b+ic)x(t-1) = 0, \quad t \in \mathbb{R}_+\,, \tag{4.1}$$

where $b, c \in \mathbb{R}$.

Denote by G the domain in a plane bounded by the curve (the curve itself is not included) $u = \theta\sin\theta$, $v = \theta\cos\theta$, $\theta \in [-\pi/2, \pi/2]$. The domain G is shown in Fig. 3. As is known [9], *equation* (4.1) *is exponentially stable if and only if* $(b, c) \in G$. Further, we assume that equation (4.1) is exponentially stable and find its sharp exponent.

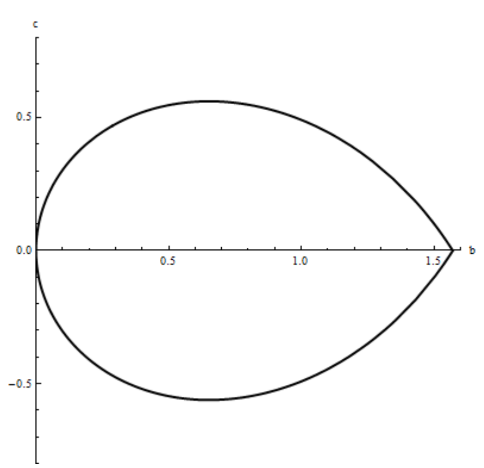

Figure 3

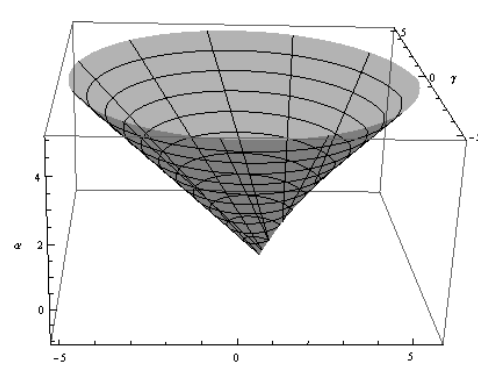

Figure 4

Put the following equation of form (2.8) in correspondence with equation (4.1):

$$\dot{y}(t) - \omega y(t) + (b + ic)e^{\omega} y(t-1) = 0, \quad t \in \mathbb{R}_+ . \tag{4.2}$$

Equation (4.2) is a special case of an equation

$$\dot{y}(t) + \alpha y(t) + (\beta + i\gamma) y(t-1) = 0, \quad t \in \mathbb{R}_+, \tag{4.3}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$. The criterion of exponential stability for this equation is also known, and the stability region is constructed [8].

Consider a surface defined in the space $Ouvw$ parametrically: $u = \theta \sin\theta - w\cos\theta$, $v = \theta\cos\theta + w\sin\theta$, $\theta \in [-\theta_0, \theta_0]$, where $\theta_0 \in (0, \pi)$ is a root of the equation $w = -\theta\cot\theta$. This surface bounds an infinite curvilinear cone with the apex at the point $M(1, 0, -1)$ (see Fig. 4). Denote the interior of the cone by K and its boundary by ∂K. As the study of the characteristic equation shows [8], if $(\beta, \gamma, \alpha) \in K$, then all zeros of the function $g(p) = p + \alpha + (\beta + i\gamma)e^{-p}$ lie to the left of the imaginary axis; if $(\beta, \gamma, \alpha) \in \partial K$, then the function $g(p)$ has no zeros to the right of the imaginary axis, but has roots on the imaginary axis; if $(\beta, \gamma, \alpha) \notin (K \cup \partial K)$, then $g(p)$ has zeros in the open right half-plane. It follows from Theorem 2.5 that the upper exponent of equation (4.3) is equal to zero if and only if $(\beta, \gamma, \alpha) \in \partial K$. It can be seen from equation (4.2) that we are interested only in that part of the cone K, where $w < 0$. We obtain from Theorem 2.6 that the sharp exponent of equation (4.1) is equal to $\omega > 0$ if and only if $(be^{\omega}, ce^{\omega}, -\omega) \in \{(u, v, w) : \ w < 0\} \cap \partial K$.

In order to solve effectively the problem of estimating the exponent, we pass to the coordinates $\xi = b$, $\eta = c$, $\zeta = \omega$, in which the equation of the surface ∂K has the form $\xi = (\theta\sin\theta + \zeta\cos\theta)e^{-\zeta}$, $\eta = (\theta\cos\theta - \zeta\sin\theta)e^{-\zeta}$, $\theta \in [-\theta_1, \theta_1]$, where $\theta_1 \in (0, \pi/2)$ is a root of the equation $\zeta = \theta\cot\theta$. In the space $O\xi\eta\zeta$, these equalities uniquely define the surface $\zeta = \zeta(\xi, \eta)$; for $\zeta = 0$, it coincides with the boundary of G. Since $\omega > 0$, we consider this surface only in the half-space $\zeta > 0$. In Fig. 5, the plot of the surface $\zeta = \zeta(\xi, \eta)$ is shown for $\zeta > 0$. The surface has the form of a curvilinear cone with apex at the point $(1/e, 0, 1)$. It is obvious that the constructed surface solves in geometric language the problem of finding the sharp exponent of equation (4.1). Thus, we have established the following result.

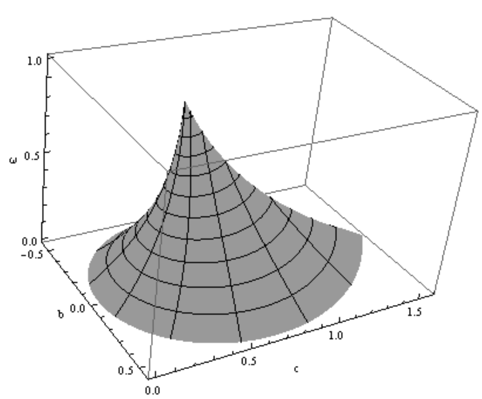

Figure 5

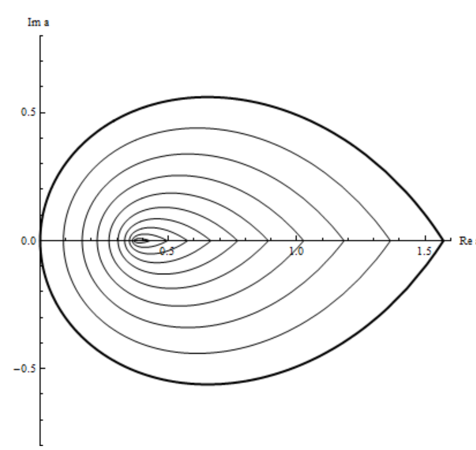

Figure 6

Theorem 4.1. *If equation* (4.1) *is exponentially stable, then its sharp exponent* ω *is determined by the equality* $\omega = \zeta(b, c)$.

Let us indicate effective conditions for the fundamental solution to have an exponential estimate with the sharp exponent ω.

Theorem 4.2. *Suppose equation* (4.1) *is exponentially stable and* $b + ic \neq 1/e$. *Then for the fundamental solution of* (4.1) *the estimate* $|X(t)| \leq Ne^{-\omega t}$ *is valid, where* $\omega = \zeta(b, c)$.

Proof. Suppose ω is the sharp exponent of equation (4.1). Then $(be^{\omega}, ce^{\omega}, -\omega) \in \partial K$. It follows from Theorem 2.7 that the fundamental solution of equation (4.2) is bounded if the quasipolynomial corresponding to this point has no multiple roots. It is easy to see that for equation (4.3) the functions $g(p) = p + \alpha + (\beta + i\gamma)e^{-p}$ and $g'(p) = 1 - (\beta + i\gamma)e^{-p}$ have a common root $(p = 0)$ on the imaginary axis if and only if $\alpha = -1$, $\beta = 1$, $\gamma = 0$. In all other cases there are no common roots, hence the roots of $g(p)$ lying on the imaginary axis are simple. Note that for equation (4.2), the set of coefficients $\omega = 1$, $b = 1/e$, $c = 0$ corresponds to the unique multiple root. Therefore, under the conditions of Theorem 4.2, the fundamental solution of equation (4.2) is bounded. It remains to refer to Theorem 2.6 to complete the proof. □

Theorem 4.2 does not include the case $b + ic = 1/e$, but its separate study is not required, since it has already been considered in Theorem 3.3. Note the following properties of the sharp exponent of an exponentially stable equation (4.1) which follow from the properties of the surface $\zeta = \zeta(\xi, \eta)$ and are geometrically obvious from Fig. 4.

- For $b + ic \neq 1/e$, the sharp exponent ω of equation (4.1) belongs to the interval $(0, 1)$.
- For each $\omega \in (0, 1)$, there are infinitely many corresponding points (b, c) liying on the curve $\xi = (\theta \sin\theta + \omega\cos\theta)e^{-\omega}$, $\eta = (\theta\cos\theta - \omega\sin\theta)e^{-\omega}$, $\theta \in [-\theta_1, \theta_1]$, where $\theta_1 \in (0, \pi/2)$ is a root of the equation $\omega = \theta\cot\theta$.
- These curves are the sections of the surface $\zeta = \zeta(\xi, \eta)$ by the planes $\zeta = \omega$. In Fig. 6, several sections for different ω are shown. They are symmetrical with respect to the axis $O\xi$ and lie entirely in the half-plane $\eta \geq 0$.
- The maximum sharp exponent for equation (4.1) is equal to 1 and is attained if and only if $b + ic = 1/e$; in this case, estimate (2.7) should be replaced by inequality (3.5).

Theorem 4.1 describes general exponents for equations of form (4.1) in geometric language. However, along with such a description, it is useful to have a way to calculate the general exponent analytically. In order to do this, it is more convenient to write the coefficient of equation (4.1) in the exponential form: $b + ci = re^{i\delta}$. Without loss of generality, we can suppose that $\delta \in [0, \pi/2]$. Theorem 4.1 implies that if ω is the sharp exponent of equation (4.1), then $b = (\theta\sin\theta + \omega\cos\theta)e^{-\omega}$, $c = (\theta\cos\theta - \omega\sin\theta)e^{-\omega}$, which implies

$$\tan\delta = \frac{\theta\cos\theta - \omega\sin\theta}{\theta\sin\theta + \omega\cos\theta},$$

therefore, $\omega = \theta\cot(\theta + \delta)$. Further, $r = e^{-\omega}\sqrt{\theta^2 + \omega^2}$, hence

$$r = \frac{\theta}{\sin(\theta + \delta)}\, e^{-\theta\cot(\theta+\delta)}.$$

Corollary 4.1. *Suppose equation* (4.1) *is exponentially stable. Then if* $b + ic = re^{i\delta}$*, then the sharp exponent of this equation is determined by the equality* $\omega = \theta_0\cot(\theta_0 + \delta)$*, where* θ_0 *is a root of the equation* $r = \frac{\theta}{\sin(\theta+\delta)}\, e^{-\theta\cot(\theta+\delta)}$ *lying in the interval* $(-\pi/2, \pi/2)$.

Note another important property of the sharp exponent: if it is found, then we can give an unimprovable estimate for real roots of the characteristic function of equation (4.1).

Corollary 4.2. *Suppose* ω *is the sharp exponent of equation* (4.1)*. Then the characteristic function* $g(p)$ *has no roots in the open complex half-plane* $\{p \in \mathbb{C} : \operatorname{Re} p > -\omega\}$*, and has at least one root on the line* $\operatorname{Re} p = -\omega$.

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