Memoirs on Differential Equations and Mathematical Physics

Volume 87, 2022, 165–176

Igor N. Sergeev

FIRST APPROXIMATION STUDY OF LYAPUNOV, PERRON AND UPPER-LIMIT STABILITY OR INSTABILITY

Abstract. The natural concepts of Lyapunov, Perron and upper-limit stability of the zero solution of a differential system are defined, as well as their numerous varieties: from global to particular stability or, respectively, instability. A complete coincidence of possibilities of research on the first approximation of stability and asymptotic stability of all three types is found. A similar coincidence was established for partial and particular stability. The complete coincidence of the same possibilities in the case of one-dimensional systems is proved.

2020 Mathematics Subject Classification. 34D05, 34D20, 34D23.

Key words and phrases. Differential system, Lyapunov stability, Lyapunov and Perron exponents, first approximation stability, Perron stability, upper-limit stability.

რეზიუმე. დიფერენციალური სისტემის ნულოვანი ამოხსნისთვის განსაზღვრულია ლიაპუნოვის, პერონისა და ზედა ზღვრული სტაბილურობის ბუნებრივი ცნებები, ისევე როგორც მათი მრავალ-რიცხოვანი ვარიაციები: გლობალურიდან კერძო სტაბილურობამდე ან არასტაბილურობამდე, შესაბამისად. ნაპოვნია სამივე ტიპის სტაბილურობისა და ასიმპტოტური სტაბილურობის კვლევის შესაძლებლობების სრული დამთხვევა პირველი მიახლოებისას. მსგავსი დამთხვევა დადგინდა ნაწილობრივი და კერძო სტაბილურობისთვის. დამტკიცებულია ერთი და იგივე ტიპის შესაძლებლობების სრული დამთხვევა ერთგანზომილებიანი სისტემების შემთხვევაში.

1 Introduction

This paper is devoted to the development and study of Lyapunov stability [10, § 1] – a classical concept, which has been investigated in detail (see, for example, the monographs [6–9, 11]), but allowing new and meaningful variations.

The properties of Lyapunov stability of solutions of differential systems and their numerous varieties, including their opposite properties, are studied here in a close connection with similar concepts of Perron stability (ascending to Perron exponents [12]) and upper-limit stability. They were introduced quite recently [13, 24], but have already received a certain development (see [1–5, 14–23, 25–27]).

The studied properties are considered here from the point of view of the fundamental possibility of their study in the first approximation, i.e., in terms of only linear terms in the expansion of the right-hand side of the system at zero. This research was initiated and significantly promoted primarily by A. M. Lyapunov himself. A detailed description of the current state of this direction is contained, for example, in the monograph [9, § 11].

It turned out that a positive outcome in the study of stability or asymptotic stability of any of the three considered types is provided by the sets of the same first approximations. A similar coincidence is also observed for the sets of first approximations that provide partial and particular stability.

2 Definitions

For a given $n \in \mathbb{N}$, in the Euclidean space $\mathbb{R}^n$ with standard norm $|\cdot|$, we consider systems of the form

$$\dot{x} = f(t,x), \quad f(t,0) \equiv 0, \ \ (t,x) \in \mathbb{R}_+ \times G \tag{2.1}$$

(admitting a zero solution), where $\mathbb{R}_+ \equiv [0,+\infty)$ is the time semi-axis*, $G \subseteq \mathbb{R}^n$ is a phase domain (naturally containing the point 0), and the right-hand side $f(t,x)$ satisfies the conditions $f, f'_x \in C(\mathbb{R}_+ \times G)$ (ensuring the existence and uniqueness of solutions to the Cauchy problem).

Denote by $S_*(f)$ the set of all non-extendable non-zero solutions $x(\cdot) \neq 0$ of system (2.1), and by $S_\delta(f) \subseteq S_*(f)$ the subset of all those solutions with initial conditions $|x(0)| < \delta$.

First of all, we define three main types of stability and instability going back to the works [10, 12].

Definition 2.1. Let us say that the system (more precisely, its zero solution) (2.1) has the following property:

(a) *Lyapunov, Perron* or *upper-limit stability*, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that any solution $x \in S_\delta(f)$ satisfies the corresponding condition:

– the *Lyapunov* one

$$\sup_{t\in\mathbb{R}_+} |x(t)| < \varepsilon; \tag{2.2}$$

– the *Perron* (i.e., *lower-limit*) one

$$\varliminf_{t\to+\infty} |x(t)| < \varepsilon; \tag{2.3}$$

– the *upper-limit* one

$$\varlimsup_{t\to+\infty} |x(t)| < \varepsilon \tag{2.4}$$

(all conditions (2.2)–(2.4) tacitly assume that the solution x is defined on the entire semi-axis $\mathbb{R}_+$ (otherwise it reaches the boundary of the phase region G in a finite time);

(b) *Lyapunov, Perron* or *upper-limit instability*, if it does not have the stability of the corresponding type.

Now, we define numerous varieties of stability and instability properties that strengthen or weaken the primary concepts described in Definition 2.1.

Definition 2.2. Let us say that system (2.1) has the following *upper-limit* property:

(a) *asymptotic stability*, if for some $\delta > 0$, any solution $x \in S_\delta(f)$ satisfies the condition

$$\lim_{t\to+\infty} |x(t)| = 0; \tag{2.5}$$

(b) *global stability*, if any solution $x \in S_*(f)$ satisfies condition (2.5);

(c) *asymptotic* or *particular instability*, if it does not have asymptotic or global stability respectively.

(d) *complete instability*, if for some $\varepsilon, \delta > 0$ no solution $x \in S_\delta(f)$ satisfies condition (2.4);

(e) *global instability*, if for some $\varepsilon > 0$ no solution $x \in S_*(f)$ satisfies condition (2.4);

(f) *partial* or *particular stability*, if it does not have complete or global instability, respectively.

Definition 2.3. Let us say that system (2.1) has the *Perron asymptotic, global, partial, particular stability* or *asymptotic, particular, complete, global instability*, if it has, respectively, the upper-limit property of the same name with the upper limit at $t \to +\infty$, replaced by the lower one, and condition (2.5) replaced by the condition

$$\varliminf_{t\to+\infty} |x(t)| = 0. \tag{2.6}$$

Definition 2.4. Let us say that system (2.1) has the following *Lyapunov* properties:

(a) *asymptotic* or *global stability*, if it simultaneously possesses both the upper-limit stability of the same name and Lyapunov stability,

(b) *asymptotic* or *particular instability* – otherwise respectively;

(c) *complete* or *global instability*, if it has the same name of the upper-limit property at $t \to +\infty$, replaced by the least upper bound in $t \in \mathbb{R}_+$;

(d) *partial* or *particular stability* – otherwise, respectively.

The main objects studied in this paper are the relationships between classes of linear approximations that provide any of the considered properties from Definitions 2.1–2.4 for uniformly small (on the entire time semiaxis) nonlinear perturbations of the system.

Definition 2.5. If system (2.1) is represented as

$$\dot{x} = A(t)x + h(t,x) \equiv f(t,x), \quad A(t) \equiv f'_x(t,0) \in \operatorname{End}\mathbb{R}^n, \ \ (t,x) \in \mathbb{R}_+ \times G, \tag{2.7}$$

with uniform (and not only pointwise, which takes place automatically) smallness of the nonlinear additive

$$\sup_{t\in\mathbb{R}_+} |h(t,x)| = o(x), \ \ x \to 0, \tag{2.8}$$

then for system (2.1) we have the system of the *first* (or, that is the same, *linear*) *approximation*

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \ \ t \in \mathbb{R}_+, \ \ A \in C(\mathbb{R}_+). \tag{2.9}$$

Denote by $\mathcal{M}^n$ the set of all potential linear approximations (2.9) which we identify with the operator-functions A.

Definition 2.6. Let us say that the first approximation (2.9) *provides a given property* if it is possessed by any system (2.1) with this first approximation $A \in \mathcal{M}^n$ for some phase domain $G \subseteq \mathbb{R}^n$. The subset $\mathcal{K} \subseteq \mathcal{M}^n$ of those linear approximations that provide a given property is called the *class $\mathcal{K}$ of this property.* Let us introduce property classes:

(a) of stability at $\mathcal{K} = \mathcal{S}$ or instability at $\mathcal{K} = \mathcal{N}$ (non-stable);

(b) of different names denoted by superscripts $k = g, a, b, c, d$, which mean global, asymptotic, nameless (basic), complete or partial (chain), particular (dotty) properties, respectively;

(c) of different types denoted by subscripts $\varkappa = \lambda, \sigma, \pi$ corresponding to Lyapunov, upper-limit (super) or Perron type, respectively.

3 Theorems

The natural hierarchy between the various classes under consideration follows directly from the logical relationships between the respective properties.

Theorem 3.1. *For any $n \in \mathbb{N}$, the stability classes have the inclusions*

$$\varnothing \subseteq \mathcal{S}^g_\varkappa \subseteq \mathcal{S}^a_\varkappa \subseteq \mathcal{S}^b_\varkappa \subseteq \mathcal{S}^c_\varkappa \subseteq \mathcal{S}^d_\varkappa \subseteq \mathcal{M}^n, \quad \varkappa = \lambda, \sigma, \pi,$$
$$\mathcal{S}^k_\lambda \subseteq \mathcal{S}^k_\sigma \subseteq \mathcal{S}^k_\pi, \quad k = g, a, b, c, d.$$

Theorem 3.2. *For any $n \in \mathbb{N}$, the instability classes have the inclusions*

$$\varnothing \subseteq \mathcal{N}^g_\varkappa \subseteq \mathcal{N}^c_\varkappa \subseteq \mathcal{N}^b_\varkappa \subseteq \mathcal{N}^a_\varkappa \subseteq \mathcal{N}^d_\varkappa \subseteq \mathcal{M}^n, \quad \varkappa = \pi, \sigma, \lambda,$$
$$\mathcal{N}^k_\pi \subseteq \mathcal{N}^k_\sigma \subseteq \mathcal{N}^k_\lambda, \quad k = g, c, b, a, d.$$

As it turns out, the maximum total number of different non-empty stability classes is only two. The coincidence of classes asserted in the theorems below justifies the correctness of the corresponding unified notation for them.

Theorem 3.3. *For any $n \in \mathbb{N}$, there are the coincidences*

$$\mathcal{S}^g_\lambda = \mathcal{S}^g_\sigma = \mathcal{S}^g_\pi \equiv \mathcal{S}^g, \quad \mathcal{S}^a_\varkappa = \mathcal{S}^b_\varkappa \equiv \mathcal{S}^{ab}, \quad \mathcal{S}^c_\varkappa = \mathcal{S}^d_\varkappa \equiv \mathcal{S}^{cd}, \quad \varkappa = \lambda, \sigma, \pi. \tag{3.1}$$

Theorem 3.4. *For any $n \in \mathbb{N}$, the combined classes* (3.1) *satisfy the chain of relations*

$$\varnothing = \mathcal{S}^g \subsetneq \mathcal{S}^{ab} \subseteq \mathcal{S}^{cd} \subsetneq \mathcal{M}^n. \tag{3.2}$$

Theorem 3.5. *For $n > 1$, the middle of three inclusions* (3.2) *is strict*

$$\mathcal{S}^{ab} \subsetneq \mathcal{S}^{cd}.$$

So far, the results for instability classes are much more modest than those for stability classes.

Theorem 3.6. *For any $n \in \mathbb{N}$, the following strict inclusions are true:*

$$\varnothing \subsetneq \mathcal{N}^g_\varkappa, \quad \mathcal{N}^d_\varkappa \subsetneq \mathcal{M}^n, \quad \varkappa = \pi, \sigma, \lambda. \tag{3.3}$$

Theorem 3.7. *For any $n \in \mathbb{N}$, there are the strict inclusions*

$$\mathcal{N}^k_\pi \subsetneq \mathcal{N}^k_\sigma, \quad k = g, c, b, a, d,$$

guaranteed by a single non-inclusion

$$\mathcal{N}^d_\pi \not\supset \mathcal{N}^g_\sigma.$$

There are the relations between the stability and instability classes that supplement Theorem 3.1 and Theorem 3.2.

Theorem 3.8. *For any $n \in \mathbb{N}$, we have the relations*

$$\begin{gathered}\mathcal{S}^g_\varkappa \cap \mathcal{N}^d_\varkappa = \mathcal{S}^k_\varkappa \cap \mathcal{N}^k_\varkappa = \mathcal{S}^d_\varkappa \cap \mathcal{N}^g_\varkappa = \varnothing, \quad \varkappa = \pi, \sigma, \lambda, \quad k = a, b, c, \\ 0 \notin \mathcal{S}^d_\pi \cup \mathcal{N}^d_\lambda \subsetneq \mathcal{M}^n.\end{gathered} \tag{3.4}$$

Theorem 3.9. *For $n > 1$, there are the strict inclusions*

$$\mathcal{N}^c_\varkappa \subsetneq \mathcal{N}^b_\varkappa, \quad \varkappa = \sigma, \lambda,$$

guaranteed by one inequality

$$\mathcal{S}^c_\lambda \cap \mathcal{N}^b_\sigma \neq \varnothing.$$

In the one-dimensional case, the coincidences of the classes from Theorem 3.3 are supplemented by a whole series of new coincidences with the active participation now of the instability classes.

Theorem 3.10. *For $n = 1$, there is the coincidencies*

$$\mathcal{S}^{ab} = \mathcal{S}^{cd} \equiv \mathcal{S}^{abcd},$$
$$\mathcal{N}^g_\pi = \mathcal{N}^c_\pi = \mathcal{N}^b_\pi = \mathcal{N}^a_\pi = \mathcal{N}^d_\pi \equiv \mathcal{N}_\pi,$$
$$\mathcal{N}^g_\varkappa = \mathcal{N}^c_\varkappa = \mathcal{N}^b_\varkappa = \mathcal{N}^a_\varkappa = \mathcal{N}^d_\varkappa \equiv \mathcal{N}_{\sigma\lambda}, \quad \varkappa = \sigma, \lambda,$$

and the combined classes satisfy the strict inclusion

$$\mathcal{N}_\pi \subsetneq \mathcal{N}_{\sigma\lambda}.$$

4 Proofs

4.1 Theorems 3.1, 3.2 and 3.8

All assertions of the mentioned theorems are fully explained by the following reasonings.

I. First of all, for each type separately (Perron, Lyapunov or upper-limit), all ten properties from Definitions 2.1–2.4 are divided into five pairs of properties that serve as a logical negation of each other (see the first chain of equalities in Theorems 3.8), namely, of the same type:
- stability and instability;
- partial stability and complete instability;
- asymptotic stability and asymptotic instability;
- global stability and particular instability;
- particular stability and global instability.

II. Further, all the same properties from Definitions 2.1–2.4 can be covered in a somewhat different, but also uniform way of looking at them (dividing them into Lyapunov, Perron, or upper-limit ones):

(a) *stability properties* – the following five properties of the same type: global stability, asymptotic stability, stability, partial stability, and particular stability;

(b) *instability properties* – the remaining five properties of the same type: global instability, complete instability, instability, asymptotic instability, and particular instability.

Moreover, within each of these fives, the properties are logically ordered within each of these fives, i.e., each next property follows from the previous one. That proves the first chains of inclusions in Theorems 3.1 and 3.2.

III. Finally, all properties from Definitions 2.1–2.4 are logically ordered within each of the ten triplets of the same name Perron, Lyapunov or upper-limit properties (that proves the second chains of inclusions in Theorems 3.1 and 3.2), namely:

(1) any upper-limit stability property follows from the Lyapunov one, but entails the Perron one of the same name;

(2) any upper-limit instability property follows from the Perron one, but entails the Lyapunov one of the same name.

IV. As a linear approximation that does not provide any of the considered properties at all, the first (zero system; see the last relations (3.4) in Theorem 3.8) of the following three systems is suitable:

$$\dot{x} = 0 \cdot x, \quad \dot{x} = -|x| \cdot x, \quad \dot{x} = |x| \cdot x, \ \ x \in \mathbb{R}^n.$$

It serves as a linear approximation for the other two systems, whose right-hand sides are sufficiently smooth and satisfy condition (2.8). Moreover, the second of them is Lyapunov globally stable, and the third is Perron globally unstable. Therefore, the first system provides neither the logically weakest instability, the Lyapunov particular one, nor the logically weakest stability, the Perron particular one.

4.2 Theorems 3.3, 3.4 and 3.6

The second equality (3.1) in Theorem 3.3 was first proved in [19], and technically it was quite difficult. Subsequently, its proof was significantly simplified [21] due to the ideas of [20].

V. First, we prove the first equalities (3.1) and (3.2) in Theorems 3.3 and 3.4. Let system (2.7) with the right-hand side f, with some phase domain G and with the given linear approximation, have some kind of global stability: Perron, Lyapunov or upper limit. Then we choose an r-neighbourhood $U_r \Subset G$ and define a new right-hand side by the equality

$$g(t,x) \equiv \varphi(x) f(t,x), \quad \varphi(x) \equiv \begin{cases} 1, & |x| \le \dfrac{r}{2}, \\ 0, & |x| \ge r, \end{cases} \quad \varphi \in C^1(\mathbb{R}^n).$$

A system of the same form (2.7) with the right-hand side g will have the same phase region and the same linear approximation. However, it will not have any global stability, since all its solutions starting outside the neighborhood U_r will turn out to be stationary, which means that they will not satisfy any of requirements (2.5) and (2.6).

VI. It suffices to verify that if the linear approximation ensures the logically weakest of the six properties involved in the formulation of the second equality (3.1) in Theorem 3.3 – Perron stability, then it also ensures the strongest of them – Lyapunov asymptotic stability.

So, let, on the contrary, some *initial* system (2.1) not have Lyapunov asymptotic stability. Then we indicate a *new* system of the same form, with some *inner* phase subdomain $U_r \Subset G$ (which is an r-neighbourhood of zero), with the right-hand side g and with the same linear approximation, but already Perron unstable and, moreover, admitting for each $\delta > 0$ a solution $y \in S_\delta(g)$, not defined on the entire semi-axis $\mathbb{R}_+$.

If the initial system also does not have Lyapunov stability, then for some fixed $\varepsilon > 0$ and each $\delta > 0$ there is a solution $y \in S_\delta(g)$ that does not satisfy requirement (2.2). Therefore, the restriction of the original system to some inner subdomain U_r for $r \in (0, \varepsilon)$ is suitable as a new system.

Now, let the initial system be Lyapunov stable. Then we fix an arbitrary inner subdomain U_r and, setting $t_0 \equiv 0$ and $\varepsilon \equiv 2\beta_0 \equiv r$, construct a new system in accordance with items (1)–(4) below.

1. We choose an arbitrary number $\delta_1 \in (0, \beta_0)$, and then choose a solution $x_1 \in S_{\delta_1}(f)$, satisfying for some $t_0 > 0$ the relations

$$\sup_{0 \le t \le t_0} |x_1(t)| \le \beta_0 < \varepsilon, \quad 0 < \varlimsup_{t \to +\infty} |x_1(t)| \le \sup_{t \in \mathbb{R}_+} |x_1(t)| < \varepsilon.$$

2. For the function

$$y_1(t) \equiv \begin{cases} x_1(t) e^{\delta_1 \varphi(t - t_0)}, & t \ge t_0, \\ x_1(t), & 0 \le t \le t_0, \end{cases} \quad \varlimsup_{t \to +\infty} |y_1(t)| = +\infty,$$

where

$$\varphi(\tau) \equiv \begin{cases} \tau - \dfrac{2}{3}, & \tau \ge 1, \\ \dfrac{\tau^3}{3} \ge \tau - \dfrac{2}{3}, & 0 \le \tau \le 1, \\ 0, & \tau \le 0, \end{cases} \quad \varphi \in C^2(\mathbb{R}), \tag{4.1}$$

we choose the smallest root $s = s_1 > t_0$ of the equation $|y_1(s)| = \varepsilon$ and set

$$t_1 \equiv s_1 + 1, \quad 0 < 2\beta_1 \equiv \min_{0 \le t \le s_1} |y_1(t)| < \delta_1 < r.$$

3. We perturb the original system so that it admits the solution $y_1(t)$ for $0 \le t \le s_1$, namely: add a term to its right-side that vanishes (together with its first derivatives) for $t = t_0, t_1$ and for $|x| \le \beta_1$ and having the form

$$\Delta(t,x) \equiv g(t,x) - f(t,x) = \theta_{t_1}^{s_1}(t) \theta_{\beta_1}^{2\beta_1}(|x|) \cdot \psi(t,x), \quad t \in [t_0, t_1], \tag{4.2}$$

where for $a < b$ the functions $\theta_a^b, \theta_b^a \in C^1(\mathbb{R})$ are given by the formulas

$$\theta_a^b(\xi) \equiv \begin{cases} 0, & \xi \le a, \\ 1, & \xi \ge b, \end{cases} \quad \theta_b^a(\xi) \equiv 1 - \theta_a^b(\xi)$$

and

$$\psi(t,x) \equiv \delta_1 \dot{\varphi}(t-t_0)x - h(t,x) + e^{\delta_1\varphi(t-t_0)} h(t, xe^{-\delta_1\delta(t-t_0)}). \tag{4.3}$$

4. We repeat the arguments from items 1–3 above, increasing the indices of all parameters by 1 on each next circle. As a result, we have built the sequences of functions x_i, y_i ($i \in \mathbb{N}$), the sequences decreasing to zero $\beta_0 > \delta_1 > \beta_1 > \delta_2 > \cdots$ and the unbounded sequence $0 \equiv t_0 < s_1 < t_1 < s_2 < \cdots$, and hence the new system with the right-hand side g and with the phase region U_r – Perron by virtue of the construction itself.

5. The new system has the same linear approximation as the initial one, because for each $\alpha > 0$, for some $N(\alpha) \in \mathbb{N}$ and $\rho(\alpha) \in (0,r)$, the estimates

$$\delta_i < \alpha, \quad i > N(\alpha), \quad \eta(\rho) \equiv \sup_{t\in\mathbb{R}_+,\quad 0<|x|\le\rho} \frac{|h(t,x)|}{|x|} < \alpha, \quad 0 < \rho \le \rho(\alpha),$$

are satisfied (see representation (2.8)), whence for $|x| \le \min\{\beta_{N(\alpha)}, \rho(\alpha)\}$, we derive:

(a) if $t \le t_{N(\alpha)}$, then $\Delta(t,x) = 0$;

(b) if $t > t_{N(\alpha)}$, then for some $i > N(\alpha)$, we have $t \in [t_{i-1}, t_i]$, and due to equalities (4.2) and (4.3), we obtain the estimates

$$|\Delta(t,x)| \le \delta_i|x| + \eta(\rho)|x| + e^{\delta_i\varphi(t-t_{i-1})}\eta(\rho)|x|e^{-\delta_i\varphi(t-t_{i-1})} \equiv (\delta_i + 2\eta(\rho))|x| \le 3\alpha|x|.$$

VII. Let the given linear approximation (2.9) not provide Lyapunov partial stability (logically the strongest of the six properties involved in the formulation of the third equality (3.1)). Then some system (2.7) with this linear approximation and with some phase domain $G \subset \mathbb{R}^n$ is Lyapunov completely unstable, i.e., for some $\varepsilon, \delta > 0$, no solution $x \in S_\delta(f)$ satisfies the requirement (2.2) and, without loss of generality, we can assume that $U_\varepsilon \subset G$ and $\delta < \varepsilon/2$. But then, after the restriction of this system to the subdomain $U_{\varepsilon/2}$, none of the same solutions will be defined on the entire semiaxis $\mathbb{R}_+$. Therefore, the resulting restriction will be Perron completely unstable and even globally unstable. Thus the linear approximation (2.9) does not provide Perron particular stability (logically the weakest of the six properties mentioned), that completes the proof of Theorem 3.3 (see V and VI above).

VIII. Finally, there are systems that provide (see [6, Theorem 15.2.1]) the next properties of any type:

(a) asymptotic stability – for example, the system $\dot{x} = -x$;

(b) global instability – for example, the system $\dot{x} = x$.

These facts justify the left strict inclusions (3.2), (3.3) in Theorem 3.4, 3.6, and remaining right strict ones follow from the relations (3.4) of Theorem 3.8.

4.3 Theorems 3.5, 3.7 and 3.9

It is curious (although not essential now) that the linear system considered in IX below serves as a first approximation for some autonomous system (2.1), that is, Perron globally stable.

IX. For $n > 1$, consider the linear autonomous system (2.9) defined in the coordinate space $\mathbb{R}^n$ by the following equations:

$$\dot{x}_1 = x_1, \quad \dot{x}_i = -x_i, \ \ i = 2, \dots, n.$$

This is a linear approximation that has all of the following properties at once:

(a) does not provide Perron stability (because it is itself Perron unstable);

(b) provides partial stability of any type. Really, for any system (2.7) with the specified linear approximation, one can choose a sufficiently small value $r > 0$ for which, for all $|x| < r$, the value $|h(t,x)|/|x|$ from condition (2.8) turns out to be so small (uniformly by $t \in \mathbb{R}_+$) that the conclusion of Theorem 15.2.1 [6] will hold, according to which for any $\delta < r$, there exists a solution $x \in S_\delta(f)$, exponentially decreasing as $t \in \mathbb{R}_+$, so the system certainly has Lyapunov, and even more Perron and upper-limit partial stability;

(c) provides Lyapunov instability, because for any system (2.7) from item (b) above, there exists a solution $x \in S_\delta(f)$ growing exponentially at least until its phase curve leaves the r-neighborhood of zero U_r;

(d) provides even more upper-limit instability, because for any system (2.7), the property from item (b) above can be possessed only with those solutions whose initial values lie on the image of the hyperplane $x_1 = 0$ under some diffeomorphism (that is, a manifold of dimension $n-1$, possibly only a part of it), and all other solutions have the property from item (c) above, i.e., every time after the next hit in the neighborhood of U_r it is inevitably left over time.

All this together proves the validity of Theorems 3.5 and 3.9.

X. To prove Theorem 3.7, consider the linear system (2.9) of scalar type defined by an operator-function $A \equiv aI$, where the only continuous coefficient $a : \mathbb{R}_+ \to \mathbb{R}$ for some special sequence $0 \equiv t_0 < t_1 < t_2 < \cdots$ is chosen so that the function a is, for example, linear on each interval $[t_k + 1, t_{k+1}]$ and $[t_k, t_k + 1]$ for $k = 0, 1, \ldots$, and also satisfies the following conditions:

$$a(0) = 1, \quad a(t) = \begin{cases} -2, & t \in [t_{k-1}+1, t_k], \quad k = 1, 3, \ldots, \\ 2, & t \in [t_{k-1}+1, t_k], \quad k = 2, 4, \ldots, \end{cases}$$

$$\int_{t_k}^{t_k+1} a(\tau)\,d\tau = 0, \quad k = 0, 1, \ldots, \qquad \int_0^{t_k} a(\tau)\,d\tau = \begin{cases} -t_k, & k = 1, 3, \ldots, \\ t_k, & k = 2, 4, \ldots . \end{cases}$$

Any solution x of this system satisfies the conditions

$$x(t) = x(0)e^{\int_0^t a(\tau)\,d\tau}, \quad t \in \mathbb{R}_+, \quad x(t_k) = \begin{cases} x(0)e^{-t_k}, & k = 1, 3, \ldots, \\ x(0)e^{t_k}, & k = 2, 4, \ldots . \end{cases}$$

Therefore, we have:

(a) its Perron global stability;

(b) both Lyapunov and upper-limit global instability of all systems with the given linear approximation. Indeed, for any such system (2.7), a sufficiently small $r > 0$ can be chosen so that for all $|x| < r$, the quantity $|h(t,x)|/|x|$ is uniformly small enough to that the conclusion of Theorem 15.2.1 [6] is true, according to which any solution $x(t)$ grows exponentially over the sequence of time points $t = t_2, t_4, \ldots$, starting from any moment (in particular, from $t_0 = 0$) when its phase curve turns out to be an r-neighborhood of zero, and until it leaves this neighborhood, that, therefore, will definitely happen.

4.4 Theorem 3.10

Below, only the case $n = 1$ is considered everywhere.

XI. Let the linear approximation (2.9) not ensure the stability of at least one (and then, according to Theorem 3.3, any) of the three types, let us say, Perron's. Then for some nonlinear additive h, satisfying condition (2.8), and some fixed $\varepsilon > 0$, each number $\delta > 0$ can be associated with the solution $x \in S_\delta(f)$ of system (2.7) that does not satisfy requirement (2.3).

1. Take an arbitrary sequence $\delta_i \to +0$ $(i \to +\infty)$ and use it to construct a sequence of corresponding solutions x_i. We assume, without loss of generality, that all these solutions (if necessary, passing to their subsequences) take values on the phase line only on one particular side of the point $x = 0$, and their initial values converge to zero strictly monotonically.

2. Redefine the function $h(t, x)$ on the other side of zero for the reasons of oddness in x and get a *new* system that has:

(a) the former linear approximation (2.9) and the smoothness of the new nonlinear addition is provided by the equalities $h(t, 0) = h'_x(t, 0) = 0$, $t \in \mathbb{R}_+$, arising from condition (2.8);

(b) a sequence of pairs of solutions $\pm x_i$ $(i \in \mathbb{N})$ with positive initial values $x_i(0) \to +0$ $(i \to +\infty)$.

3. The new system already has Perron complete instability, since any of its non-zero solutions x do not satisfy requirement (2.3), since one of the solutions $\pm x_i$ (which no longer satisfies this requirement) for some $i \in \mathbb{N}$ starts on the phase line between the points $x(0)$ and 0, which means that for all $t > 0$, it will also be located between the solution x and the zero solution.

Thus, system (2.9) does not provide partial Perron, and hence no other partial or even partial stability, which completes the proof of the first line of the equalities of Theorem 3.10.

XII. If the one-dimensional system (2.1) has:

(a) complete instability of some type, then it also has a global instability of the same type, since if some non-zero solution does not satisfy any of requirements (2.3), (2.4), then any other solution, starting on the same ray farther from zero, does not satisfy this requirement all the more;

(b) Lyapunov complete instability, then it also has an upper limit complete instability, because if some nonzero solution satisfies requirement (2.4), then by the theorem on the continuous dependence of solutions on initial values [8, § 7], any other solution starting on the same ray, close enough to zerom, also satisfies requirement (2.2).

XIII. Let us *assume* that the linear approximation (2.9) does not provide global, and hence complete instability – Perron or, respectively, upper limit, and hence Lyapunov (see XII). Then for some nonlinear additive h, satisfying condition (2.8), each pair of numbers $\varepsilon = \delta > 0$ corresponds to a solution $x \in S_\delta(f)$ of system (2.7) satisfying requirement (2.3) or, respectively, (2.2).

4. By analogy with items 1 and 2 above, we construct a sequence of numbers $\varepsilon_i = \delta_i \to +0$ $(i \to +\infty)$ and the corresponding pairs of solutions $\pm x_i$ of the new system with the same linear approximation. This system will already turn out to be Perron or, respectively, Lyapunov stable, since for each $i \in \mathbb{N}$, all its nonzero solutions x with initial conditions $|x(0)| \le x_i(0) < \delta_i$ will satisfy requirement (2.3) or, respectively, (2.2) for $\varepsilon = \varepsilon_i$.

5. If at least one solution x_i is *special*, i.e., also satisfies requirement (2.6) or, respectively, (2.5), then the system is Perron or Lyapunov asymptotically stable.

XIV. Let each of the solutions $x_i > 0$ $(i \in \mathbb{N})$ constructed in item 4 satisfy the requirement (2.2) but not (2.5), i.e., is not *special* in the Lyapunov sense. Then we construct a system of the same type (with an odd nonlinearity), but already having a special solution for which we set $t_0 = 0$ and

$$\varepsilon_i \ge \sup_{t \in \mathbb{R}_+} x_i(t) \ge \varlimsup_{t \to +\infty} x_i(t) \equiv 2\gamma_i \to +0, \quad i \to +\infty.$$

6. For the function (see formula (4.1))

$$y_1(t) \equiv x_1(t) e^{-\delta_1 \varphi(t - t_0)}, \quad t \ge t_0, \quad \varlimsup_{t \to +\infty} y_1(t) = 0, \tag{4.4}$$

choose a number $s_1 > t_0 + 1$ that satisfies the conditions

$$0 < \min_{t_0 \le t \le s_1} y_1(t) \equiv 2\beta_1 \le y_1(s_1) < \gamma_1 < x_2(s_1).$$

Then we choose the value of $t_1 > s_1$ so close to s_1 that the perturbed system (4.2), which admits a solution $y_1(t)$ for $t_0 \le t \le s_1$, provides z_1 for its continuation to the right from the point s_1, the condition $z_1(t_1) < x_2(t_1)$ is also fulfilled.

7. We repeat the arguments from item (6), each time increasing the indices of all parameters by 1. As a result, we obtain the sequences of functions x_i, y_i, z_i ($\mathbb{N} \ni i \to +\infty$), numbers $\varepsilon_i, \delta_i, \gamma_i, \beta_i \to +0$ and $0 \equiv t_0 < s_1 < t_1 < s_2 < \cdots \to +\infty$, and with them a new system with the same linear approximation (see 5 of VI above).

8. The resulting system, by virtue of the construction itself, has a singular solution u (for example, with the initial value $u(0) = x_1(0)$), and hence has Lyapunov asymptotic stability. To make this stability also global, we choose some value of $u_0 \in \overline{G}$ that majorizes the function u on the entire semi-axis $\mathbb{R}_+$ (which is possible due to its boundedness). Let us change the system (without changing it in the small tube of the zero solution and preserving the standard smoothness and oddness in the phase variable for its right-hand side) on the interval $t \in [0,1]$ so that some of its solutions v satisfies the conditions

$$v(1) = u(1) \le u_0 = \lim_{t\to+0} v(t) = \sup_{t\in\mathbb{R}_+} v(t).$$

Let us replace the original phase domain by the interval $G' \equiv (-u_0, u_0) \subseteq G$, as a result, the values of absolutely all solutions of the new system for each $t > 0$ will lie strictly between the numbers $\pm v(t)$.

XV. Let each of the solutions $x_i > 0$ ($i \in \mathbb{N}$) constructed in item 4 satisfy requirement (2.3), but not (2.6). Then, as in XIV, we construct a system that already has a singular solution in the Perron sense, and hence possesses Perron asymptotic stability. To do this, putting $t_0 = 0$ and

$$\varepsilon_i \ge \varliminf_{t\to+\infty} x_i(t) \equiv 2\gamma_i \to +0, \quad i \to +\infty,$$

let us repeat the constructions from items 6, 7 with one amendment: the upper limit in formula (4.4) is replaced by the lower limit. To obtain Perron global stability, we change the system in accordance with item 8 above in which, if the solution u is unbounded, we have to calculate $u_0 = +\infty$.

Thus system (2.9), under assumption XIII, does not provide Perron or, accordingly, Lyapunov even partial instability. This completes the proof of Theorem 3.10 (the final assertion of which is already contained in Theorem 3.7).

References

[1] A. A. Bondarev, One example of an unstable system. (Russian) *Differ. Uravn.* **55** (2019), no. 6, 899.

[2] A. A. Bondarev, The existence of a completely unstable in the sense of Lyapunov differential system with Perron and upper limit massive partial stability. (Russian) *Differ. Uravn.* **57** (2021), no. 6, 858–859.

[3] A. A. Bondarev, An example of complete but not global Perron instability. (Russian) *Vestnik Moskov. Univ. Ser. I Mat. Mekh.* **2021**, no. 2, 43–47; translation in *Moscow Univ. Math. Bull.* **76** (2021), no. 2, 78–82.

[4] A. A. Bondarev, Example of a differential system with complete Perron and upper-limit instability but massive partial stability. (Russian) *Differ. Uravn.* **58** (2022), no. 2, 147–152; translation in *Differ. Equ.* **58** (2022), no. 2, 147–153.

[5] A. A. Bondarev, An example of a globally unstable system in the sense of Lyapunov with Perron and upper limit global stability. (Russian) *Differ. Uravn.* **58** (2022), no. 6, 860–861.

[6] B. F. Bylov, R. È. Vinograd, D. M. Grobman and V. V. Nemyckiĭ, *Theory of Ljapunov Exponents and its Application to Problems of Stability.* (Russian) Izdat. "Nauka", Moscow, 1966.

[7] B. P. Demidovich, *Lectures on the Mathematical Theory of Stability.* (Russian) Izdat. "Nauka", Moscow, 1967.

[8] A. F. Filippov, *Introduction to Theory of Differential Equations.* (Russian) Editorial URSS, Moscow, 2004.

[9] N. A. Izobov, *Lyapunov Exponents and Stability.* Stability, Oscillations and Optimization of Systems, 6. Cambridge Scientific Publishers, Cambridge, 2012.

[10] A. M. Lyapunov, *General Problem of the Stability of Motion.* (Russian) Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow–Leningrad, 1950.

[11] V. V. Nemytskii and V. V. Stepanov, *Qualitative Theory of Differential Equations.* (Russian) Gostekhizdat, Moscow–Leningrad, 1949.

[12] O. Perron, Die Ordnungszahlen linearer Differentialgleichungssysteme. (German) *Math. Z.* **31** (1930), no. 1, 748–766.

[13] I. N. Sergeev, Determination of Perron stability and its relationship with Lyapunov stability. (Russian) *Differ. Uravn.* **54** (2018), no. 6, 855–856.

[14] I. N. Sergeev, On the study of the Perron stability of one-dimensional and autonomous differential systems. (Russian) *Differ. Uravn.* **54** (2018), no. 11, 1561–1562.

[15] I. N. Sergeev, Investigation of Perron stability of linear differential systems. (Russian) *Differ. Uravn.* **54** (2018), no. 11, 1571–1572.

[16] I. N. Sergeev, On investigation of the Perron and Lyapunov stability properties in the first approximation. (Russian) *Differ. Uravn.* **55** (2019), no. 6, 897–899.

[17] I. N. Sergeev, On the dependence and independence of the Perron and Lyapunov stability properties on the phase region of the system. (Russian) *Differ. Uravn.* **55** (2019), no. 11, 1572–1573.

[18] I. N. Sergeev, Perron stability and its study at the first approximation. *Dokl. Math.* **99** (2019), no. 3, 252–254.

[19] I. N. Sergeev, Definition and some properties of Perron stability. (Russian) *Differ. Uravn.* **55** (2019), no. 5, 636–646; translation in *Differ. Equ.* **55** (2019), no. 5, 620–630.

[20] I. N. Sergeev, Dependence and independence of the Perron and Lyapunov stability properties on the system phase domain. (Russian) *Differ. Uravn.* **55** (2019), no. 10, 1338–1346; translation in *Differ. Equ.* **55** (2019), no. 10, 1294–1303.

[21] I. N. Sergeev, Studying the Perron and Lyapunov stability properties by the first approximation. (Russian) *Differ. Uravn.* **56** (2020), no. 1, 84–93; translation in *Differ. Equ.* **56** (2020), no. 1, 83–92.

[22] I. N. Sergeev, Lyapunov, Perron and upper-limit stability properties of autonomous differential systems. (Russian) *Izv. Inst. Mat. Inform.* **56** (2020), 63–78.

[23] I. N. Sergeev, Some features of the Perron and Lyapunov stability properties of autonomous systems. (Russian) *Differ. Uravn.* **56** (2020), no. 6, 830–831.

[24] I. N. Sergeev, Determination of upper-limit stability and its relationship with Lyapunov and Perron stability. (Russian) *Differ. Uravn.* **56** (2020), no. 11, 1556–1557.

[25] I. N. Sergeev, On the study of Lyapunov, Perron and upper limit properties of stability in the first approximation. (Russian) *Differ. Uravn.* **57** (2021), no. 6, 857–858.

[26] I. N. Sergeev, Massive and almost massive properties of stability and instability of differential systems. (Russian) *Differ. Uravn.* **57** (2021), no. 11, 1576–1578.

[27] I. N. Sergeev, Singularities of almost massive properties of stability and instability of one-dimensional and autonomous differential systems. (Russian) *Differ. Uravn.* **58** (2022), no. 6, 852–853.

(Received 16.07.2022; accepted 01.09.2022)

Author's address:

Igor N. Sergeev

Faculty of Mathematics and Mechanics, Lomonosov Moscow State University, 1 Leninskiye Gory, Moscow 119991, Russia.

E-mail: igniserg@gmail.com