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ON THE EXACT ESTIMATION FOR THE EXPONENT OF SOLUTIONS OF ONE EQUATION WITH DISTRIBUTED DELAY

Dedicated to the blessed memory of Professor N. V. Azbelev

Abstract. We consider a linear autonomous differential equation with distributed delay and obtain the exact estimate for the exponent of solutions to this equation.

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1 Introduction

For autonomous functional differential equations with aftereffect, quite a lot of necessary and sufficient conditions of exponential stability have been obtained, and exponential stability regions have been constructed in the parameter space of the original equation. However, in all these criteria, only the exponent indicator sign is set. In this paper, for the equation with distributed delay, an exact estimate of the exponent is found.

Let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{C}$ be the set of complex numbers.

2 Preliminaries

Consider the differential equation with a distributed delay

$$\dot{x}(t) + b \int_{t-h}^{t} x(s)\, ds = 0, \quad t \in \mathbb{R}_+, \tag{2.1}$$

where $h \geqslant 0$, $b \in \mathbb{R}$.

We define the function x for negative values of the argument by the initial function φ. The requirements of "continuous docking" $x(0) = \varphi(0)$ are not considered mandatory in this article, although in the papers where the solution is understood as a continuous continuation of the initial function, this requirements are assumed to be fulfilled.

Definition 2.1. Equation (2.1) is called *exponentially stable* if there exist N, $\gamma > 0$ such that for all $\varphi \in L_1[0, \omega]$, $x(0) \in \mathbb{R}$. the following inequality holds:

$$|x(t)| \leqslant N e^{-\gamma t} \big(|x(0)| + \|\varphi\|_1 \big). \tag{2.2}$$

For studying the stability of equation (2.1) the following function turns out to be useful:

$$g(p) = p + b\, \frac{1 - e^{-ph}}{p}\,, \quad p \in \mathbb{C}.$$

We call this function a characteristic function, and the corresponding equation $g(p) = 0$ is a characteristic equation. The function $g(p)$ is analytical, and the characteristic equation has a denumerable set of roots on the complex plane. Further, the following property is especially important. The whole half-plane $\{p \in \mathbb{C} :\ \operatorname{Re} p \geqslant \alpha\}$ may contain only a finite number of roots of the characteristic equation.

The criterion of exponential stability for equation (2.1) is easily formulated in terms of the function $g(p)$.

Theorem 2.1. *Equation* (2.1) *is exponentially stable if and only if all roots of the characteristic equation lay to the left of the imaginary axis.*

Obviously, if estimation (2.2) holds for some $\gamma > 0$, then it holds for all $\gamma' < \gamma$. The number γ' can be chosen arbitrarily close to zero, making estimation (2.2) less accurate. Another problem is much more interesting and difficult: to find the largest of the exponents γ for which (2.2) is true. However, finding the exact exponent through ratio (2.2) is complicated by the fact that it is necessary to take into account a whole class of functions – solutions of equation (2.1) defined by all initial functions $\varphi \in L_1[0, \omega]$. It is much more convenient to study one solution of (2.1), the estimation of this solution allows us to estimate all other solutions. To define this solution, we use a different representation for (2.1).

Denote by S_h the following operator, which is acting in the space of continuous (piecewise continuous, summable) functions,

$$(S_h y)(t) = \begin{cases} y(t-h), & t \geqslant h, \\ 0, & t < h. \end{cases}$$

Along with (2.1), let us consider the nonhomogeneous equation

$$\dot{x}(t) + b \int_0^h (S_\xi x)(t)\, ds = f(t), \quad t \in \mathbb{R}_+, \tag{2.3}$$

where the function $f : \mathbb{R}_+ \to \mathbb{R}$ is locally integrable.

Equation (2.1) we can be rewritten as (2.3) if the outer perturbation $f(t)$ is replaced by $\sigma(t)$, where

$$\sigma(t) = -b \int_{\min\{t-h,0\}}^{0} \varphi(s)\, ds.$$

Definition 2.2. We define a *solution* of equation (2.1) as a function $x : \mathbb{R}_+ \to \mathbb{R}$ that is absolutely continuous on every finite interval and satisfies equality (2.1) almost everywhere on $\mathbb{R}_+$.

As is known [1, Section 5.1, Theorem 1.1], for every given initial value $x(0) \in \mathbb{R}$, equation (2.3) has a unique solution in the class of locally absolutely continuous functions, and the solution can be written in the form

$$x(t) = X(t)x(0) + \int_0^t X(t-s)f(s)\, ds, \tag{2.4}$$

where $X : \mathbb{R}_+ \to \mathbb{R}$ is called *the fundamental solution* of equation (2.1). Formula (2.4) is said to be *the Cauchy formula.* The fundamental solution, as follows from the representation (2.4), does not depend either on the initial condition $x(0)$ or on the outer perturbation f, so it determines any solution of equation (2.1), too. It is convenient to define the fundamental solution on the negative semi-axis by zero.

3 Auxiliary results

For equations (2.1) and (2.3), the asymptotic behavior of the fundamental solution can be characterized in terms of the roots of the characteristic equation [2]. Let $p_1, p_2, \ldots, p_m$ be the roots of the characteristic equation with the largest real part α. Then

$$X(t) = \sum_{k=1}^{m} q_k(t)e^{p_k t} + r(t), \tag{3.1}$$

where $q_k(t)$ is a polynomial with degree determined by the multiplicity of the root p_k and

$$\lim_{t\to\infty} r(t)e^{-\alpha t} = 0.$$

It follows that for the fundamental solution of equation (2.1), there is a limit

$$\lim_{t\to\infty} \frac{\ln|X(t)|}{t} = \alpha < \infty. \tag{3.2}$$

According to the terminology adopted in the theory of ordinary differential equations [3], we call the number α the (strict) *upper exponent* of the function X. From formula (3.2), which defines α, it follows

Lemma 3.1. *For all $\gamma < -\alpha$, there is $N > 0$ such that for the fundamental solution of equation* (2.1)*, the estimate*

$$|X(t)| \leqslant Ne^{-\gamma t}, \quad t \geqslant 0, \tag{3.3}$$

holds.

From Lemma 3.1 and the definition of exponential stability follows

Lemma 3.2. *Equation* (2.1) *is exponentially stable if and only if* $\alpha < 0$.

When studying exponential stability, another characteristic of the rate of decrease of the solution turns out to be convenient.

Denote by Γ a set of numbers $\gamma \in \mathbb{R}$ where estimation (3.3) is valid. From (3.2), it is clear that $\Gamma \neq \varnothing$. Therefore, the following definition is correct.

Definition 3.1. We call the number $\omega \triangleq \sup\{\gamma : \ \gamma \in \Gamma\}$ *the exact exponent of the fundamental solution.*

Lemma 3.3. *If* $\gamma_0 \in \Gamma$, *then* $(-\infty, \gamma_0] \in \Gamma$.

Proof. If $\gamma_0 \in \Gamma$ and $\gamma < \gamma_0$, then $|X(t)| \leqslant Ne^{-\gamma_0 t} < Ne^{-\gamma t}$ for all $t \geqslant 0$, i.e., $\gamma \in \Gamma$. □

Now, it is easy to give a description of the structure of the set Γ. Let us consider two cases.

1. Let $\omega \in \Gamma$. From Lemma 3.3 it follows that $(-\infty, \omega] \subseteq \Gamma$. On the other hand, from Definition 3.1, we get $\Gamma \subseteq (-\infty, \omega]$. Hence $\Gamma = (-\infty, \omega]$.

2. Let $\omega \notin \Gamma$. We take the increasing sequence $\gamma_n \in \Gamma$, $n \in \mathbb{N}$, such that $\gamma_n \to \omega$. From Lemma 3.3 it follows that $(-\infty, \gamma_n] \subseteq \Gamma$. Therefore, $(-\infty, \omega) = \bigcup\limits_{n=1}^{\infty} (-\infty, \gamma_n] \subseteq \Gamma$. On the other hand, from Definition 3.1, we get $\Gamma \subseteq (-\infty, \omega)$. Then $\Gamma = (-\infty, \omega)$.

Thus, the set Γ is always a semi-axis of the form $(-\infty, \omega]$ or $(-\infty, \omega)$. Below, we will verify that both of these cases are realized for equation (2.1).

Theorem 3.1. *The upper and exact exponents of the fundamental solution of equation* (2.1) *are related by the equality* $\omega = -\alpha$.

Proof. From inequality (3.3) and the definition of α, it follows that $\omega \leqslant -\alpha$. On the other hand, from Lemma 3.2, it follows that $\alpha < 0$. Therefore, starting from some n, all numbers are $\alpha + \frac{1}{n} < 0$. From Definition 3.2, we get $-\alpha - \frac{1}{n} \in \Gamma$. Then $-\alpha - \frac{1}{n} \leqslant \omega$. Passing to the limit as $n \to \infty$, we have that $\omega \geqslant -\alpha$, that is, $\omega = -\alpha$. □

Theorem 3.2. *If the fundamental solution of equation* (2.1) *has estimate* (3.3) *with the exponent* $\gamma > 0$, *then any solution of equation* (2.1) *has estimate* (2.2) *with the same exponent* γ.

Proof. The proof follows from formula (2.4) and the definition of the function σ. □

For any other solution of equation (2.1), the upper exponent can be similarly determined by using formula (3.2). From Theorem 3.2, it follows that all solutions have the exponent no more than α. On the other hand, X is one of the solutions of equation (2.1), so the exponent α cannot be reduced if we consider it as a universal characteristic of all solutions of equation (2.1).

For exponentially stable equations, we give an equivalent reformulation of the definition of the upper exponent.

The following theorem gives the basic idea to solve the problem of the exact exponent of the fundamental solution.

Theorem 3.3. *The exact exponent of the fundamental solution of equation* (2.1) *is equal to* ω *if and only if the upper exponent of the fundamental solution of equation*

$$\dot{y}(t) - \omega y(t) + b \int\limits_{t-h}^{t} e^{\omega(t-s)} y(s)\, ds = 0, \quad t \in \mathbb{R}_+, \tag{3.4}$$

is equal to zero.

Proof. We add equation (2.1) with the initial condition $x(0) = 1$ and change variables $x(t) = e^{-\omega t} y(t)$ in equation (2.1). It is obvious that y is the solution of equation (3.4). The statement of the theorem follows from Theorem 3.1 and formula (3.2). □

Thus, Theorem 3.3 offers a scheme to find the exact exponent of equation (2.1). This exponent corresponds to those parameters of equation (2.1) where the upper exponent of equation (3.4) is equal to zero. Therefore, it is important to obtain effective signs (and especially areas) of stability for equations of form (3.4).

The following result, which follows from formulas (3.1) and (3.2), solves the question on the zero upper exponent in terms of zeros of the characteristic function.

Theorem 3.4. *The upper exponent of the fundamental solution of equation* (3.4) *is equal to zero if and only if the characteristic function of equation* (3.4) *has no zeros in the open right half-plane, but there are roots on the imaginary axis.*

Remark 3.1. Note that knowing the number ω does not yet give grounds to replace the exponent γ by ω in estimation (3.3). This is possible only if, in Definition 3.1, in the formula for ω, the exact upper bound is attainable for some $\gamma = \omega$. If this boundary is unattainable, then estimate (3.3) is valid only for all γ arbitrarily close to ω, but subordinate to the inequality $\gamma < \omega$.

Theorem 3.5. *Let ω be the exact exponent of equation* (2.1). *The fundamental solution of equation* (2.1) *has estimate* (3.3) *with the exponent $\gamma = \omega$ if and only if the fundamental solution of equation* (3.4) *is bounded.*

We present criterion for the boundedness of the fundamental solution in terms of the characteristic function. This result follows from formulas (3.1) and (3.2).

Theorem 3.6. *The fundamental solution of equation* (3.4) *is bounded if and only if the characteristic function of equation* (3.4) *has no zeros in the open right half-plane, and the roots lying on the imaginary axis are simple.*

4 Main result

Consider the equation

$$\dot{x}(t) + b\int_{t-1}^{t} x(s)\,ds = 0, \quad t \in \mathbb{R}_+, \tag{4.1}$$

where $b \in \mathbb{R}$. The choice of the delay $h = 1$ is not a limitation, because every equation with one nonzero delay h can be reduced to form (4.1) by replacing the argument $t \mapsto th$. For equation (4.1) the exponential stability criterion is known [4]: it is exponentially stable if and only if $0 < b < \pi^2/2$. However, so far no exact exponent has been found for this equation. In the literature, there are only indications that this exponent is determined by the real part of the root of the characteristic function nearest to the imaginary axis. Since it is possible to find exactly the root of the characteristic function nearest to the imaginary axis only in exceptional cases, this indication is unconstructive.

We correspond equation (4.1) to the equation of form (3.4):

$$\dot{y}(t) - \omega y(t) + b\int_{t-1}^{t} e^{\omega(t-s)} y(s)\,ds = 0, \quad t \in \mathbb{R}_+. \tag{4.2}$$

Equation (4.2) is a special case of the equation

$$\dot{y}(t) + \alpha y(t) + \beta\int_{t-1}^{t} e^{-\gamma(t-s)} y(s)\,ds = 0, \quad t \in \mathbb{R}_+. \tag{4.3}$$

For (4.3), the criterion of exponential stability is known and the stability region is constructed [5]. The boundaries of the region can be described analytically. One surface is given parametrically:

$$S_1 = \left\{ (u,v,w) : \begin{array}{l} u = \dfrac{\theta(-w\cos\theta + e^w w + \theta\sin\theta)}{-e^w\theta + w\sin\theta + \theta\cos\theta}, \\[2ex] v = \dfrac{e^w\theta(w^2+\theta^2)}{e^w\theta - w\sin\theta - \theta\cos\theta}, \end{array} \quad \theta \in \mathbb{R} \right\}.$$

The other surface has the form

$$S_2 = \Big\{(u,v,w): \ \ v = \frac{uw}{e^{-w}-1} \ \text{ for } \ w \neq 0, \ \ u+v=0 \ \text{ for } \ w=0\Big\}.$$

These surfaces have a rather complicated structure. They divide the parameter space $\{\alpha,\beta,\gamma\}$ into an infinite number of sets, among which we need to choose those that belong to the region of exponential stability. In addition, it is not known in advance a number of connectivity components. However, in paper [5], it is shown that the exponential stability region of equation (4.3), as well as its sections at fixed b, are linearly connected.

Theorem 4.1 ([5])**.** *Equation* (4.3) *is exponentially stable if and only if* $(\alpha,\beta,\gamma)\in D$.

In Fig. 1, the exponential stability region is depicted, it is located in the upper part closer to us, not including the boundaries, we denote it by D.

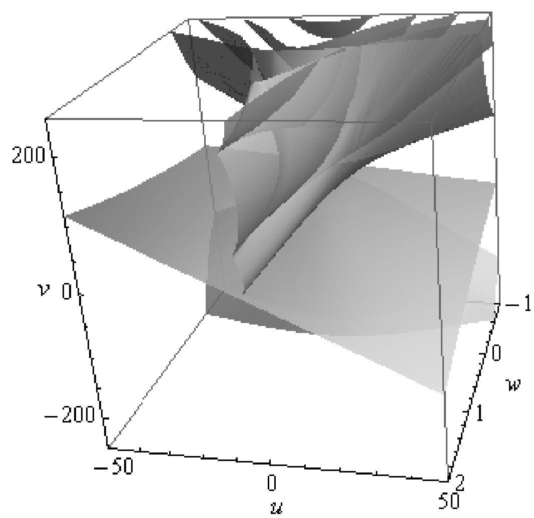


Figure 1. The exponential stability region of equation (4.3) (located closer to us).

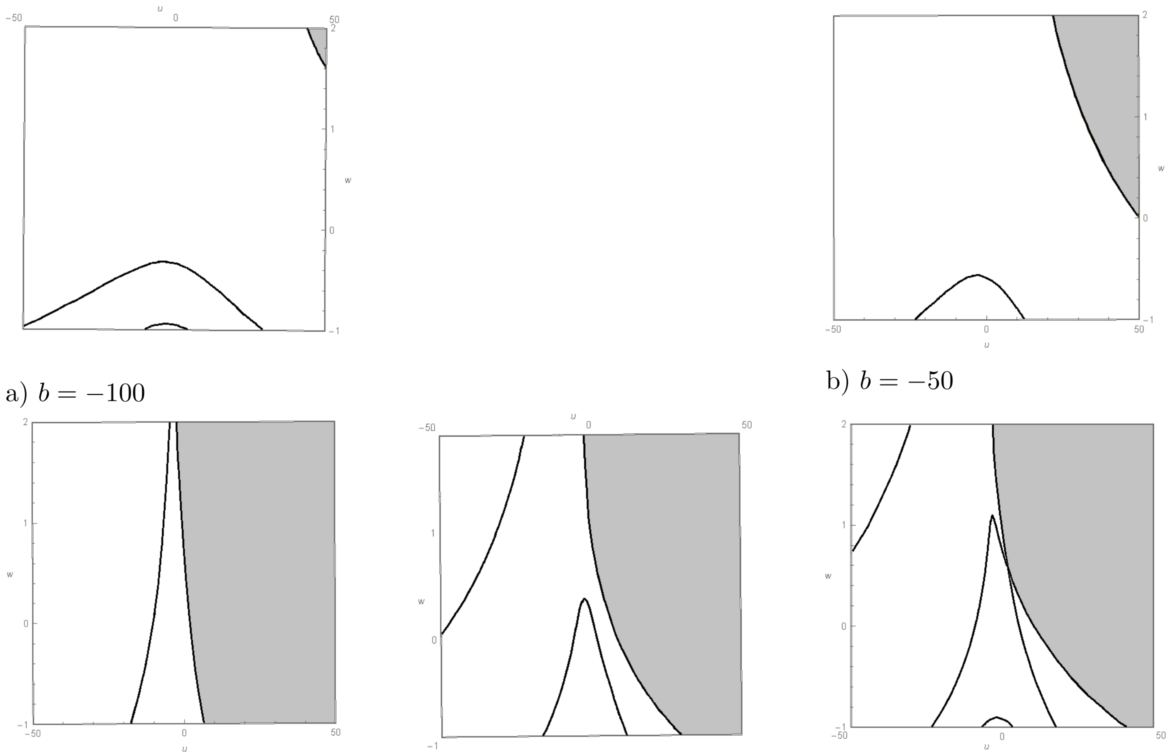
In Fig. 2, the sections of D-partition surfaces are depicted for some fixed $b\omega^2$, and the sections of the exponential stability region are also painted over.

Denote by ∂D the boundaries of the stability region. As the study of the characteristic equation shows [5], if $(\alpha,\beta,\gamma)\in D$, then all zeros of the function $g(p)=p+\alpha+\frac{\beta}{p+c}\,(1-e^{-p-\gamma})$ lie in the open left half-plane; if $(\alpha,\beta,\gamma)\in\partial D$, then the function $g(p)$ has no zeros to the right of the imaginary axis, but there are the roots on the imaginary axis; if $(\alpha,\beta,\gamma)\notin\partial D\cup D$, then $g(p)$ has zeros in the open right half-plane. It follows from Theorem 3.4 that the upper exponent of equation (4.3) is zero if and only if $(\alpha,\beta,\gamma)\in\partial D$. From equation (4.2), it is clear that we are interested only in that part of the region D where $u=w=-\omega$.

From Theorem 3.3, we get that the exact exponent of equation (4.1) is $\omega>0$ if and only if $(-\omega,\beta,-\omega)\in\{(u,v,w): \ u=w<0\}\cap\partial D$. To solve effectively the problem of the exact exponent of equation (4.1), we turn to the coordinates $\zeta=\omega$, $\eta=b$. The second boundary now has the form $\eta=\frac{\zeta^2}{e^{\zeta}-1}$, and since we are interested only in the domain $(u,v,w): u=w<0$, then $\zeta\in(0,\zeta_0)$, where ζ_0 is the positive root of the equation

$$e^{-\xi}=1-\frac{\xi}{2}\,.$$

It is easy to calculate that $\zeta_0\approx 1.593624260$. On the segment $[0,\zeta_0]$, the function $\eta=\frac{\zeta^2}{e^{\zeta}-1}$ has an inverse, denote it by $\zeta=\varphi_1(\eta)$. Obviously, the function φ_1 is defined on the set $\zeta\in(0,\zeta_0)$, is continuous on this set and monotonically increases from 0 to η_0, $\eta_0\approx 0.647610$.



a) $b = -100$ b) $b = -50$

c) $b = 10$ d) $b = 50$ e) $b = 70$

Figure 2. The exponential stability region of equation (4.3).

From the equation of the first boundary, we obtain a curve given as follows:

$$\eta = -\frac{\zeta e^{-\zeta}(\theta^2 + \zeta^2)}{\zeta e^{-\zeta} - \zeta\cos\theta - \theta\sin\theta}\,, \tag{4.4}$$

$$2\theta\zeta(e^{-\zeta} - \cos\theta) + (\zeta^2 - \theta^2)\sin\theta = 0, \quad \theta \in [0, \pi]. \tag{4.5}$$

By rewriting equation (4.5) as

$$e^{-\zeta} - \cos\theta = -\frac{\sin\theta}{2\theta}\Big(\zeta - \frac{\theta^2}{\zeta}\Big),$$

we see that this equation for every fixed $\theta \in [0, \pi]$ has a unique solution ζ.

We see that on the segment $\eta \in [\eta_0, \pi^2/2]$ the above equalities define the univalent, continuous function $\zeta = \varphi_2(\eta)$. In Fig. 3, the graphs of the functions φ_1 and φ_2 are given. These functions determine the dependence of the general exponent on the coefficient b for the fundamental solution of equation (4.1). Thus, we have established the following result.

Theorem 4.2. *If equation* (4.1) *is exponentially stable, then its exact exponent* ω *is determined by the following equalities:*

(1) $\omega = \varphi_1(b)$ *for* $b \in (0, \eta_0)$*;*

(2) $\omega = \varphi_2(b)$ *for* $b \in (\eta_0, \pi^2/2)$*;*

(3) $\omega = \zeta_0$ *for* $b = \eta_0$*.*

We obtain the estimates for the fundamental solution in each of the three cases given in Theorem 4.2.

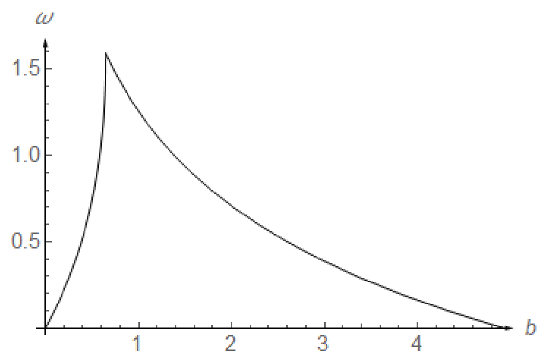


Figure 3.

Theorem 4.3. *Suppose equation* (4.1) *is exponentially stable and* $b \neq \eta_0$. *Then for the fundamental solution of equation* (4.1), *the estimate* $|X(t)| \leqslant Ne^{-\omega t}$, *where* ω *is defined in items* (1) *and* (2) *of Theorem 4.2, holds.*

Proof. Let $b \in (0, \eta_0)$ and $\omega = \varphi_1(b)$. Then in equation (4.2), $b = \frac{\omega^2}{e^{\omega}-1}$, the point $(-\omega, \beta, -\omega)$ belongs to the surface S_2 and does not coincide with the point $(-\zeta_0, \eta_0, -\zeta_0)$. Taking into account [5], we get that the fundamental solution of equation (4.2) is bounded. Now, let $b \in (\eta_0, \pi^2/2)$, and $\omega = \varphi_2(b)$. Then the coefficients of equation (4.2) form the point $(-\omega, \beta, -\omega)$ belonging to the surface S_1 that does not coincide with the point $(-\zeta_0, \eta_0, -\zeta_0)$. Taking into account [5], we get that in this case, the fundamental solution of equation (4.2) is bounded, as well. Thus, for $b \neq \eta_0$, the exact exponent of equation (4.1) turns out to be achievable: from Theorem 3.5 it follows that the fundamental solution of equation (4.1) has an exponential estimate with the exact exponent ω specified in Theorem 4.2. □

Theorem 4.3 does not include the case $b = \eta_0$; consider it. It follows from Theorem 4.2 that $\omega = \zeta_0$, but in this case, the exact exponent of equation (4.1) is not attainable. The coefficients of equation (4.2) form the point $(-\zeta_0, \eta_0, -\zeta_0)$, and the fundamental solution of this equation, as follows from [2], has linear growth. Therefore, the fundamental solution has the estimate $|X(t)| \leqslant Ne^{(-\zeta_0+\varepsilon)t}$ for all positive ε, but for $\varepsilon = 0$, it ceases to be true.

We show that the asymptotic behavior of the fundamental solution of (4.1) for $b = \eta_0$ can be characterized more precisely.

Consider the equation

$$\dot{y}(t) - \zeta_0 y(t) + \eta_0 \int\limits_{t-1}^{t} e^{\zeta_0(t-s)} y(s)\, ds = 0, \quad t \in \mathbb{R}_+, \tag{4.6}$$

and prove the following result.

Lemma 4.1. *For some* M, $\nu > 0$, *for the fundamental solution* $Y(t)$ *of equation* (4.6), *the estimate*

$$\left| Y(t) - \frac{\zeta_0}{\zeta_0 - 1}\, t - \frac{\zeta_0^2 - 3\zeta_0 + 3}{(\zeta_0 - 1)^2} \right| \leqslant Me^{-\nu t}, \quad t \in \mathbb{R}_+,$$

holds.

Proof. Equation (4.6) is of form (4.3) for $\alpha = \gamma = -\zeta_0$, $\beta = \eta_0$. These parameters correspond to hitting the boundary of the stability region of equation (4.3) (see Fig. 3), and the function

$$g(p) = p - \zeta_0 + \eta_0\, \frac{1 - e^{-p+\zeta_0}}{p - \zeta_0}$$

has no roots to the right of the imaginary axis, but has a single root on imaginary axis $p_0 = 0$ of multiplicity 2. Construct a rectangular contour with sides parallel to the real and imaginary axes and containing within itself one root of the function g – the point $(0,0)$. Using the Laplace inverse transformation formula and Cauchy's theorem, we have

$$\left|Y(t) - \underset{p=0}{res}\,\frac{e^{pt}}{p-\zeta_0+\eta_0\,\frac{1-e^{-p+\zeta_0}}{p-\zeta_0}}\right| \leqslant Ne^{-\nu t}.$$

Since

$$\underset{p=0}{res}\,\frac{e^{pt}}{p-\zeta_0+\eta_0\,\frac{1-e^{-p+\zeta_0}}{p-\zeta_0}} = \lim_{p\to 0}\frac{d}{dp}\left(\frac{p^2e^{pt}}{p-\zeta_0+\eta_0\,\frac{1-e^{-p+\zeta_0}}{p-\zeta_0}}\right) = \frac{\zeta_0}{\zeta_0-1}\,t + \frac{\zeta_0^2-3\zeta_0+3}{(\zeta_0-1)^2}\,,$$

we see that the desired inequality follows from here. □

Theorem 4.4. *Suppose in equation* (4.1) $b=\eta_0$. *Then for some* M, $\nu>0$, *the following estimation for the fundamental solution holds:*

$$\left|X(t)e^{\zeta_0 t} - \frac{\zeta_0}{\zeta_0-1}t - \frac{\zeta_0^2-3\zeta_0+3}{(\zeta_0-1)^2}\right| \leqslant Me^{-\nu t}, \quad t\in\mathbb{R}_+.$$

Proof. Since the equation

$$\dot{x}(t) + \eta_0\int_{t-1}^{t} x(s)\,ds = 0, \quad t\in\mathbb{R}_+,$$

by change of variables $x(t)=e^{-\zeta_0 t}y(t)$, is transformed into equation (4.6), we see that the statement of the theorem follows from Lemma 4.1. □

Note that

$$\frac{\zeta_0}{\zeta_0-1}\approx 2.68457, \qquad \frac{\zeta_0^2-3\zeta_0+3}{(\zeta_0-1)^2}\approx 0.717733.$$

We illustrate Theorem 4.2 with a few examples.

Example 4.1. Find the exact exponent for the equation

$$\dot{x}(t) + \frac{1}{2}\int_{t-1}^{t} x(s)\,ds = 0, \quad t\in\mathbb{R}_+.$$

Since $b=1/2<\eta_0$, we apply the first part of Theorem 4.2. Solve the equation

$$\frac{\omega^2}{e^{\omega}-1} = \frac{1}{2}$$

and find $\omega\approx 0.74085$. By virtue of Theorem 4.3, we obtain that the fundamental solution of the considered equation has an exponential estimate with the found exponent $\omega\approx 0.74085$.

Example 4.2. Find the exact exponent for the equation

$$\dot{x}(t) + \int_{t-1}^{t} x(s)\,ds = 0, \quad t\in\mathbb{R}_+.$$

Since $b=1>\eta_0$, we apply the second part of Theorem 4.2. Solve the system of equations

$$-\frac{\omega e^{-\omega}(\theta^2+\omega^2)}{\omega e^{-\omega}-\omega\cos\theta-\theta\sin\theta} = 1,$$
$$e^{-\omega}-\cos\theta = -\frac{\sin\theta}{2\theta}\left(\omega-\frac{\theta^2}{\omega}\right), \quad \theta\in[0,\pi],$$

and find $\theta\approx 1.369636$, $\omega\approx 1.255976$. By virtue of Theorem 4.3, we obtain that the fundamental solution of the considered equation has an exponential estimate with the found exponent $\omega\approx 1.255976$.

Example 4.3. Find the conditions for the parameters of equation (4.1) under which the equation has an exact exponent $\omega = 0.5$. In accordance with the properties of the functions φ_1, φ_2 noted above, we have that there are two values of b for which equation (4.1) has the desired exact exponent. The first value is

$$b = \varphi_1^{-1}(0.5) = \frac{0.5^2}{e^{0.5} - 1} \approx 0.385374.$$

The second value is determined in two steps: first, from the equation

$$e^{-0.5} - \cos\theta = -\frac{\sin\theta}{2\theta}\left(0.5 - \frac{\theta^2}{0.5}\right)$$

we find $\theta_0 \approx 2.514957$, and then

$$b = -\frac{0.5e^{-0.5}(\theta_0^2 + 0.5^2)}{-0.5e^{-0.5} - 0.5\cos\theta_0 - \theta_0\sin\theta_0} \approx 2.601190.$$

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