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THE GENERAL BOUNDARY VALUE PROBLEMS FOR LINEAR SYSTEMS OF GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS, LINEAR IMPULSIVE DIFFERENTIAL AND ORDINARY DIFFERENTIAL SYSTEMS. NUMERICAL SOLVABILITY

Abstract. For the system of generalized linear ordinary differential equations, the boundary value problem

$$dx = dA(t) \cdot x + df(t) \quad (t \in I), \quad \ell(x) = c_0$$

is considered, where $I = [a, b]$ is a closed interval, $A : I \to \mathbb{R}^{n \times n}$ and $f : I \to \mathbb{R}^n$ are, respectively, the matrix- and vector-functions with components of bounded variation, ℓ is a linear bounded vector-functional, $c_0 \in \mathbb{R}^n$. Under a solution of the system is understood a vector-function $x : I \to \mathbb{R}^n$ with components of bounded variation satisfying the corresponding integral equality, where the integral is understood in the Kurzweil sense.

Along with a number of questions, such as solvability, construction of solutions, etc., we investigate the problem of the well-posedness. Effective sufficient conditions, as well as effective necessary and sufficient conditions, are established for each of these problems.

The obtained results are realized for the above boundary value problem for linear impulsive system

$$\frac{dx}{dt} = P(t)x + q(t), \quad x(\tau_l+) - x(\tau_l-) = G(\tau_l)x(\tau_l) + u(\tau_l) \quad (l = 1, 2, \dots),$$

where P and q are, respectively, the matrix- and vector-functions with Lebesgue integrable components, τ_l $(l = 1, 2, \dots)$ are the points of impulse actions, and $G(\tau_l)$ and $u(\tau_l)$ $(l = 1, 2, \dots)$ are the matrix- and vector-functions of discrete variables.

Using the well-posedness results, the effective sufficient conditions, as well as the effective necessary and sufficient conditions, are established for the convergence of difference schemes to the solution of linear boundary value problem for impulsive systems of differential equations, as well for ordinary differential equations. The analogous results are obtained for the stability of difference schemes.

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რეზიუმე. განზოგადებულ ჩვეულებრივ დიფერენციალურ განტოლებათა სისტემისთვის განხილულია სასაზღვრო ამოცანა

$$dx = dA(t) \cdot x + df(t) \quad (t \in I), \quad \ell(x) = c_0,$$

სადაც $I = [a, b]$ ნებისმიერი ჩაკეტილი ინტერვალია, $A : I \to \mathbb{R}^{n \times n}$ და $f : I \to \mathbb{R}^n$ არის შესაბამისად მატრიცული და ვექტორული ფუნციები, რომელთა კომპონენტები სასრული ვარიაციის ფუნქციებია, ℓ წრფივი შემოსაზღვრული ვექტორული ფუნქციონალია, $c_0 \in \mathbb{R}^n$. სისტემის ამონახსნის ქვეშ გაიგება ისეთი სასრული ვარიაციის ვექტორული ფუნქცია $x : I \to \mathbb{R}^n$, რომელიც აკმაყოფილებს შესაბამის ინტეგრალურ ტოლობას კურცვაილის აზრით.

ამოხსნადობისა და სხვა საკითხებთან ერთად, განხილულია ამ ამოცანის კორექტულობის საკითხი. განხილული ამოცანებისთვის დადგენილია როგორც ეფექტური საკმარისი პირობები, ასევე ეფექტური აუცილებელი და საკმარისი პირობები.

მიღებული შედეგები რეალიზებულია წრფივ იმპულსურ განტოლებათა

$$\frac{dx}{dt} = P(t)x + q(t), \quad x(\tau_l+) - x(\tau_l-) = G(\tau_l)x(\tau_l) + u(\tau_l) \quad (l = 1, 2, \dots)$$

სისტემისთვის აღნიშნული სასაზღვრო ამოცანისთვის, სადაც P და q შესაბამისად ლებეგის აზრით ინტეგრებადი მატრიცული და ვექტორული ფუნქციებია, τ_l $(l = 1, 2, \dots)$ იმპულსური ქმედების წერტილებია, ხოლო $G(\tau_l)$ და $u(\tau_l)$ $(l = 1, 2, \dots)$ კი დისკრეტული არგუმენტის მატრიცული და ვექტორული ფუნქციებია.

კორექტულობის შედეგების საფუძველზე დადგენილია როგორც ეფექტური საკმარისი პირობები, ასევე ეფექტური აუცილებელი და საკმარისი პირობები, რომლებიც უზრუნველყოფს სხვაობიანი სქემების კრებადობას წრფივი სასაზღვრო ამოცანის ამონახსნისკენ როგორც იმპულსურ დიფერენციალურ განტოლებათა სისტემებისთვის, ასევე ჩვეულებრივ დიფერენციალურ განტოლებათა სისტემებისთვის. გარდა ამისა, სხვაობიანი სქემების მდგრადობისთვის დადგენილია ანალოგიური შედეგები.

Introduction

In the present monograph, we consider the linear boundary value problems for systems of the so-called linear generalized ordinary differential equations in the Kurzweil sense. We propose the solvability and uniqueness conditions for the problems and consider the related questions on the well-posedness of the problem and the numerical solvability. The obtained results are realized for the the linear boundary value problem and its particular cases, that is, for the multi-point, Cauchy–Nicolletti, Cauchy–Nicolletti type and periodic problems, as well as for linear boundary value problems for linear systems of impulsive differential equations. The results on the well-posedness are used for the numerical solvability of the corresponding problems for systems of linear impulsive and ordinary differential equations and for the stability of difference schemes.

Since the middle of the past century, the question on the well-posedness of the initial problem for ordinary differential equations has become very topical among many mathematicians. In particular, such a question for the initial problem for linear systems was treated very thoroughly (see, e.g., [3, 15, 46, 47, 51, 63, 65, 75] and the references therein). The essence of the problem was to investigate under what conditions the small perturbations of the right-hand sides and the initial data of the given initial problem affect the nearness (in a uniform sense) of the solutions of the perturbed initial problem to the solutions of the given one. Note that unprovable sufficient conditions, as well as unprovable necessary and sufficient conditions were obtained in [3] both for the initial and for the linear boundary value problems.

The theory of generalized ordinary differential equations has been introduced by the Czech mathematician J. Kurzweil in 1957. In [52], he investigated the above problem and constructed an example of the problem which fails to have any solution in the classical sense, i.e., a "solution" has the points of discontinuity. The perturbation problems have a classical solution converging to the "solution" of the given problem only in a pointwise sense. So, in this case, the convergence may not occur in a uniform sense. In this connection, J. Kurzweil introduced an integral of certain type (see [52–54, 61, 71, 73, 74]) known in literature as the Kurzweil-Hanstock integral. He considered the solutions of differential equations defined as the functions satisfying the corresponding integral equations, where the integral is understood in the introduced sense. Such differential equations, called as generalized ordinary differential equations, may have solutions with the points of discontinuity. For such differential equations J. Kurzweil has proved the well-posed theorem. In such a case, the convergence takes place only in a pointwise sense. So, the above-constructed example was in conformity with the theorem.

To a considerable extent, the interest to the theory of generalized ordinary differential equations has also been stimulated by the fact that this theory enabled one to investigate ordinary differential, impulsive differential and difference equations from a unified point of view. In particular, all of them can be rewritten in the form of generalized ordinary differential equations

$$dx = dA(t) \cdot x + df(t),$$

where A and f are the matrix- and vector-functions of bounded variation, respectively, for the following systems : a) the impulsive system

$$\frac{dx}{dt} = P(t)x + q(t), \quad x(\tau_l+) - x(\tau_l-) = G(\tau_l)x(\tau_l) + u(\tau_l) \quad (l = 1, 2, \dots),$$

where P and q are, respectively, the matrix- and vector-functions with Lebesgue integrable components, τ_l $(l = 1, 2, \dots)$ are the points of impulse actions, and $G(\tau_l)$ $(l = 1, 2, \dots)$ and $u(\tau_l)$ $(l = 1, 2, \dots)$ are the matrix- and vector-functions of discrete variables;

b) the difference system

$$\Delta y(k-1) = G_1(k-1)y(k-1) + G_2(k)y(k) + g_0(k) \;\; (k = 1, \dots, m_0),$$

where m_0 is a fixed natural number, and G_1, G_2 and g_0 are, respectively, the matrix- and vector-functions of discrete variables; the differential-difference systems, etc.

Therefore, we can consider the ordinary differential, impulsive differential and difference equations as such of the same type. In particular, when for the generalized ordinary differential equations we investigate the question of the well-posedness of the linear boundary value problems in the uniform sense, we obtain, as a particular case, the results dealing with the convergence of difference schemes to solutions of linear boundary value problems for impulsive differential and ordinary differential equations. Analogous concept has been used for investigation of the initial problem for linear systems of ordinary differential equations (see [15, 23]).

In the present work, we investigate a general question on the linear boundary value problems for linear generalized ordinary differential equations. Moreover, such a concept can be used for the initial and general boundary value problems for nonlinear cases.

Note that another conception of the investigation enabling one to study the continuous and discrete problems can be found in [30] (see also the references therein).

The initial and boundary value problems for generalized ordinary differential equations are investigated reasonably satisfactory for linear and nonlinear cases. The questions on the existence and well-posedness for linear problems are also considered. In particular, one of such questions for the initial problem for linear systems has been treated very thoroughly, e.g., in [2, 4, 10, 12, 15–25, 28, 29, 39, 40, 44, 61, 72–74] (see also the references therein). The same questions for the nonlinear case are studied in [5–9, 11, 13, 14, 52–54, 71] (see also the references therein).

The results obtained in the present monograph are new, they make more precise similar results given in our earlier works.

In particular, we investigate the question on the solvability of the linear boundary value problem satisfying the following particular cases of the boundary value problem: the general multi-point boundary problem, the Cauchy–Nicoletti problem, the two-point problem and periodic problem.

We present a short description of the results given in the paper.

The work consists of six chapters.

Chapter 1. Section 1.1 is devoted to the question of solvability of general linear boundary value problems for systems of linear generalized ordinary differential equations. It is well known that on a closed interval there does not exist a unified form of a linear functional given on the set of functions with bounded variations. In this connection, we consider two types of general linear boundary value problems. The first case considers the linear operator without any restriction to its form. As to the second case, the linear functional here is of specific form, in particular, of integral form. For each case, the Green type theorems are proved for the unique solvability of the problems, and the solutions are represented by the Green formula.

In the same section, we propose the spectral type theorems on the unique solvability of the problem.

Section 1.2 studies the question of the well-posedness of the general linear boundary value problems for linear systems of generalized differential equations. Here, we establish new effective sufficient conditions and an effective criterion for the well-posedness of the problem in the uniform sense on the closed interval. The results obtained in the paper are new for impulsive differential systems and some of them for ordinary differential case, as well.

Chapter 2. In Section 2.1, the results of Section 1.1 are realized for the general multi-point boundary value problem. We present here the spectral type theorems on the solvability of the problem under consideration. The special type existence theorems corresponding to the Cauchy–Nicoletti type and Cauchy–Nicoletti problems are established in Section 2.2. In Section 2.3, we established the conditions guaranteeing the existence of nonnegative solutions of the Cauchy–Nicoletti type and Cauchy–Nicoletti problems. A method for constructing solutions of the Cauchy–Nicoletti type and Cauchy–Nicoletti problems is established in Section 2.4.

Chapter 3 is devoted to the realization of the results obtained in Chapter 2 for the two-point boundary value problems for linear systems of generalized ordinary differential equations. In Sections 3.1–3.3, we suggest the results concerning the unique solvability, existence of nonnegative solutions and also a method for constructing solutions.

In Chapter 4, we consider the periodic problem for systems of generalized ordinary differential equations.

In Section 4.1 we formulate specific theorems on the existence and uniqueness of solutions. Section 4.2 deals with auxiliary propositions and proofs of the results.

Chapter 5 proposes investigation of linear boundary value problems for systems of linear impulsive differential equations. In the same chapter, we realize the results of Chapter 1 for the impulsive differential systems. Sections 5.1–5.3 consider the general linear boundary value problems, periodic problem and the numerical solvability of the general linear boundary value problem. Section 5.4 investigates the question on the stability of difference schemes. Some questions involving solvability, well-posedness, stability in the Lyapunov sense, etc., are studied in earlier works [2, 18, 21, 24, 26, 27, 32, 33, 57, 60, 64, 68, 69] (see also the references therein). Unlike another works, in this chapter we obtain somewhat different necessary and sufficient conditions for the well-posedness and stability of difference schemes.

The questions on the well-posedness and numerical solvability of the general linear boundary value problems for systems of ordinary differential equations are considered in Chapter 6. In the same chapter, we realize the results of Chapter 1 for ordinary differential systems. In particular, in Sections 6.1–6.3, we present the necessary and sufficient conditions guaranteeing the well-posedness of the problem, convergence of the difference schemes to the solution of the problem and also convergence of discontinuous vector-functions to the solution of the given problem, respectively. The results, obtained for this case, generalize our earlier results. Note that for the convergence of difference schemes we have used the concept that it is possible to consider both continuous and difference problems as generalized ones and, therefore, the convergence is a particular case of the well-posedness in the uniform sense for the latter problems.

The problem of numerical solvability of the initial and boundary value problems for the differential systems is classical one. The questions of solvability, stability and convergence of difference schemes were studied earlier in [1,2,17,18,23,30,34,35,37,38,42,56,58,66,70] for linear and nonlinear difference systems. In the above-cited papers, there take place only the sufficient conditions for the convergence of difference schemes, and it should be noted that neither necessary and, the more so, nor necessary and sufficient conditions were found therein. As we have noted above, unlike our earlier works, in the present work, we have obtained the necessary and sufficient conditions (i.e., the criterion) for the convergence and stability of the difference schemes.

The above-considered difference schemes are of the 1-order type. As to the 2-order type difference schemes, as in [17], the 2-order $n \times n$-difference linear problem can be reduced to some 1-order $2n \times 2n$-difference linear problem. Therefore, we can obtain the necessary and sufficient conditions for the convergence of the corresponding 2-order difference schemes. Analogously, we can consider the 3-order difference problem, etc.

The above-investigated problems are actual for the functional differential equations, as well. Some related problems are studied in works [31, 37, 41, 48–50] (see also the references therein). Obviously, the methods used in the present monograph can, likewise, be applied to the study of similar problems for functional differential equations.

Basic notation and definitions

In the present monograph, the use will be made of the following notation and definitions.

$\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{Z}$ is the set of all integers.

$\mathbb{R} =]-\infty, +\infty[$, $\mathbb{R}_+ = [0, +\infty[$; $[a, b]$ and $]a, b[$ $(a, b \in \mathbb{R})$ are, respectively, closed and open intervals.

I is an arbitrary finite or infinite interval from $\mathbb{R}$. We say that some properties are valid in I if they are valid on every closed interval from I.

$[t]$ is the integer part of $t \in \mathbb{R}$.

$\chi_{_M}$ is the characteristic function of the set $M \subset \mathbb{R}$, i.e., $\chi_{_M}(t) = 1$ for $t \in M$, and $\chi_{_M}(t) = 0$ for $t \notin M$; we use the designation $\chi_{_a}(t) \equiv \chi_{_M}(t)$ if $M = \{a\}$.

$\mathbb{R}^{n\times m}$ is the space of all real $n \times m$ matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\| = \max_{j=1,\dots,m} \sum_{i=1}^{n} |x_{ij}|.$$

$\mathbb{R}_+^{n\times m} = \{(x_{ij})_{i,j=1}^{n,m} :\ x_{ij} \geq 0\ (i = 1, \dots, n;\ j = 1, \dots, m)\}$.

$O_{n\times m}$ (or O) is the zero $n \times m$ matrix. We designate the zero n vector by 0_n (or 0), as well.

If $X = (x_{ij})_{i,j=1}^{n,m} \in \mathbb{R}^{n\times m}$, then

$$|X| = \big(|x_{ij}|\big)_{i,j=1}^{n,m}.$$

$X^\top$ is the matrix transposed to X, i.e., $X^\top = (x_{ji})_{i,j=1}^{m,n}$.

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$; $\mathbb{R}_+^n = \mathbb{R}_+^{n\times 1}$.

$x * y$ is the scalar product of the vectors $x, y \in \mathbb{R}^n$.

If $X \in \mathbb{R}^{n\times n}$, then X^{-1}, $\det(X)$ and $r(X)$ are, respectively, the matrix inverse to X, the determinant of X and the spectral radius of X;

$\operatorname{diag}(X_1, \dots, X_m)$, where $X_i \in \mathbb{R}^{n_i\times n_i}$ $(i = 1, \dots, m)$, $n_1 + \dots + n_m = n$ is a quasidiagonal $n \times n$-matrix. In particular, if $X = (x_{ij})_{i,j=1}^n$, then $\operatorname{diag}(X) = \operatorname{diag}(x_{11}, \dots, x_{nn})$.

$\lambda_0(X)$ and $\lambda^0(X)$ are, respectively, the minimum and maximum eigenvalues of the symmetric matrix $X \in \mathbb{R}^{n\times m}$.

I_n is the identity $n \times n$-matrix; $\operatorname{diag}(\lambda_1, \dots, \lambda_n)$ is the diagonal matrix with diagonal elements $\lambda_1, \dots, \lambda_n$.

δ_{ij} is the Kroneker symbol, i.e., $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$ $(i, j = 1, \dots)$; $Z_n = (\delta_{i+1\,j})_{i,j=1}^n$.

The inequalities between the real matrices are understood componentwise.

$\bigvee\limits_a^b(X)$ is the sum of total variations of components x_{ij} $(i = 1, \dots, m;\ j = 1, \dots, m)$ of the matrix-function $X : [a, b] \to \mathbb{R}^{n\times m}$, $\bigvee\limits_b^a(X) = -\bigvee\limits_a^b(X)$; $V(X)(t) = (v(x_{ij})(t))_{i,j=1}^{n,m}$ for $t \in [a, b]$, where $v(x_{ij})(a) = 0$, $v(x_{ij})(t) \equiv \bigvee\limits_a^t(x_{ij})$.

$X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of X at the point t ($X(a-) = X(a)$ and $X(b+) = X(b)$).

$d_1X(t) = X(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$.

$\|X\|_\infty = \sup\{\|X(t)\| :\ t \in I\}$, $|X|_\infty = (|x_{ij}|_\infty)_{i,j=1}^{n,m}$.

$\mathrm{BV}([a, b]; \mathbb{R}^{n\times m})$ is the set of all matrix-functions $X : [a, b] \to \mathbb{R}^{n\times m}$ with bounded variation (i.e., such that $\bigvee\limits_a^b(X) < \infty$).

$\mathrm{BV}_\infty([a, b]; \mathbb{R}^{n\times m})$ is the normed space of all matrix-functions $X : [a, b] \to \mathbb{R}^{n\times m}$ with bounded variation with the norm $\|X\|_\infty$.

$\mathrm{BV}_{loc}(I; D)$, where $D \subset \mathbb{R}^{n\times m}$, is the set of all matrix-functions $X : I \to D$ for which $\bigvee\limits_a^b(X) < \infty$ for every closed interval $[a, b]$ from I.

$\mathrm{BV}_{loc}(I; \mathbb{R}_+^{n\times m}) = \{X \in \mathrm{BV}_{loc}(I; \mathbb{R}^{n\times m}) :\ X(t) \geq O_{n\times m} \text{ for } t \in I\}$.

$\mathrm{BV}_\omega(\mathbb{R};\mathbb{R}^{n\times m})$, where $\omega>0$, is the set of all matrix-functions $X:\mathbb{R}\to\mathbb{R}^{n\times m}$, whose restrictions on $[0,\omega]$ belong to $\mathrm{BV}([0,\omega],\mathbb{R}^{n\times m})$, and there exists a constant matrix $C\in\mathbb{R}^{n\times m}$ such that

$$X(t+\omega)=X(t)+C \quad \text{for} \quad t\in\mathbb{R}. \tag{0.0.1}$$

$C(I;\mathbb{R}^{n\times m})$ is the space of all continuous and bounded matrix-functions $X:I\to\mathbb{R}^{n\times m}$ with the norm $\|X\|_{\infty,I}=\sup\{\|X(t)\|:\ t\in I\}$.

$C(I;D)$, where $D\subset\mathbb{R}^{n\times m}$, is the set of all continuous and bounded matrix-functions $X:I\to D$.

If $X\in C([a,b];\mathbb{R}^{n\times m})$, then $\|X\|_c=\max\big\{\|X(t)\|:\ t\in[a,b]\big\}$.

$\mathrm{AC}([a,b];D)$ is the set of all absolutely continuous matrix-functions $X:[a,b]\to D$.

$\mathrm{AC}_{loc}(I;D)$ is the set of all matrix-functions $X:I\to D$, whose restrictions to an arbitrary closed interval $[a,b]$ from I belong to $\mathrm{AC}([a,b];D)$.

$\mathrm{AC}_{loc}(I\setminus T;D)$, where $T=\{\tau_1,\tau_2,\dots\}$, $\tau_l\in I$ $(l=1,2,\dots)$, $\tau_l\neq\tau_k$ $(i\neq k)$, is the set of all matrix-functions $X:I\to D$, whose restrictions to an arbitrary closed interval $[a,b]$ from $I\setminus T$ belong to $\mathrm{AC}([a,b],D)$.

$\mathrm{BVAC}_{loc}(I,T;D)=BV(I;D)\cap\mathrm{AC}_{loc}(I\setminus T;D)$.

$B(T;\mathbb{R}^{n\times m})$ is the set of all matrix-functions $G:T\to\mathbb{R}^{n\times m}$ such that

$$\sum_{l=1}^{+\infty}\|G(\tau_l)\|<+\infty.$$

$B_{loc}(T,\mathbb{R}^{n\times m})$ is the set of all matrix-functions $G:T\to\mathbb{R}^{n\times m}$ such that

$$\sum_{\tau_l\in T_{[a,b]}}\|G(\tau_l)\|<+\infty \quad \text{for every} \quad [a,b]\subset I.$$

Let $\omega>0$. If the set $T=\{\tau_1,\tau_2,\dots\}$, where $\tau_l\in\mathbb{R}$ $(l=1,2,\dots)$, $\tau_l\neq\tau_k$ $(i\neq k)$, is such that $\tau_l+\omega\in T$ $(l=1,2,\dots)$, then by $B_\omega(T,\mathbb{R}^{n\times m})$ we denote the set of all ω-periodic matrix-functions $G:T\to\mathbb{R}^{n\times m}$ such that

$$\sum_{\tau_l\in[0,\omega]}\|G(\tau_l)\|<+\infty.$$

$|||\ell|||$ is the usual norm of the linear bounded operator ℓ.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its components is such.

A matrix-function $X=(x_{ij})_{i,j=1}^n:[a,b]\to\mathbb{R}^{n\times n}$ is quasi-nondecreasing if the functions x_{ij} $(i\neq j;\ i,j=1,\dots,n)$ are nondecreasing on $[a,b]$.

We say that a matrix-function $X:I\to\mathbb{R}^{n\times n}$ is nonsingular if $\det(X(t))\neq0$ for every $t\in I$.

If B_1 and B_2 are normed spaces, then an operator $g:B_1\to B_2$ (nonlinear, in general) is positive homogeneous if

$$g(\lambda x)=\lambda g(x)$$

for every $\lambda\in\mathbb{R}_+$ and $x\in B_1$.

An operator $\varphi:\mathrm{BV}([a,b],\mathbb{R}^n)\to\mathbb{R}^n$ is called nondecreasing if for every $x,y\in\mathrm{BV}([a,b],\mathbb{R}^n)$ such that $x(t)\le y(t)$ for $t\in[a,b]$, the inequality $\varphi(x)(t)\le\varphi(y)(t)$ holds for $t\in[a,b]$.

If $\alpha\in\mathrm{BV}([a,b],\mathbb{R})$ has no more than a finite number of discontinuity points and $m\in\{1,2\}$, then $D_{\alpha m}=\{t_{\alpha m1},\dots,t_{\alpha mn_{\alpha m}}\}$ $(t_{\alpha m1}<\dots<t_{\alpha mn_{\alpha m}})$ is the set of all points from $[a,b]$ for which $d_m\alpha(t)\neq0$.

$\mu_{\alpha m}=\max\{d_m\alpha(t):\ t\in D_{\alpha m}\}$ $(m=1,2)$.

If $\beta\in\mathrm{BV}([a,b],\mathbb{R})$, then

$$\nu_{\alpha m\beta j}=\max\left\{d_j\beta(t_{\alpha ml})+\sum_{t_{\alpha m\,l+1-m}<\tau<t_{\alpha m\,l+2-m}}d_j\beta(\tau):\ l=1,\dots,n_{\alpha m}\right\}\ \ (j,m=1,2);$$

here, $t_{\alpha20}=a-1$, $t_{\alpha1n_{\alpha1}+1}=b+1$.

$s_1,\ s_2,\ s_c : \mathrm{BV}_{loc}(I;\mathbb{R}) \to \mathrm{BV}_{loc}(I;\mathbb{R})$ are the operators defined, respectively, by

$$s_1(x)(a) = s_2(x)(a) = 0, \quad s_c(x)(a) = x(a)$$

(somewhere we use the designations s_0 instead of the operator s_c);

$$s_1(x)(t) = s_1(x)(s) + \sum_{s<\tau\le t} d_1x(\tau), \ \ s_2(x)(t) = s_2(x)(s) + \sum_{s\le\tau<t} d_2x(\tau)$$

and

$$s_c(x)(t) = s_c(x)(s) + x(t) - x(s) - \sum_{j=1}^{2}(s_j(x)(t) - s_j(x)(s)) \ \text{ for } \ s<t, \ \ s,t\in I,$$

where $a\in I$ is an arbitrary fixed point.

If $g\in \mathrm{BV}([a,b];\mathbb{R})$, $f:[a,b]\to\mathbb{R}$ and $a\le s<t\le b$, then we assume

$$\int_s^t x(\tau)\,dg(\tau) = (L-S)\int_{]s,t[} x(\tau)\,dg(\tau) + f(t)\,d_1g(t) + f(s)\,d_2g(s),$$

where $(L-S)\int_{]s,t[} f(\tau)\,dg(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$. It is known (see [52, Theorem 1.2.1] and [67, Chapter VI, (8.1) Theorem]) that if the integral exists, then the Kurzweil–Stieltjes integral $(K-S)\int_s^t f(\tau)\,dg(\tau)$ exists and the right-hand side of the last integral equality is equal to the Kurzweil–Stieltjes integral and, therefore, $\int_s^t f(\tau)\,dg(\tau) = (K-S)\int_s^t f(\tau)\,dg(\tau)$.

If $a=b$, then we assume

$$\int_a^b x(t)\,dg(t) = 0.$$

Moreover, we put

$$\int_s^{t+} x(\tau)\,dg(\tau) = \lim_{\varepsilon\to0,\,\varepsilon>0}\int_s^{t+\varepsilon} x(\tau)\,dg(\tau), \quad \int_s^{t-} x(\tau)\,dg(\tau) = \lim_{\varepsilon\to0,\,\varepsilon>0}\int_s^{t-\varepsilon} x(\tau)\,dg(\tau).$$

$L^{+\infty}([a,b],\mathbb{R};g)$ is the space of all $\mu(g)$-measurable and $\mu(g)$-essentially bounded functions $x:[a,b]\to\mathbb{R}$ with the norm

$$\|x\|_{\infty,g} = \operatorname*{ess\,sup}_{g}\big\{|x(t)|\big\} \equiv \inf\big\{r>0:\ |x(t)|\le r \text{ for } \mu(g) \text{ almost all } t\in[a,b]\big\}.$$

$L([a,b],\mathbb{R};g)$, where $g(t)\equiv g_1(t)-g_2(t)$ and g_i $(i=1,2)$ are nondecreasing functions, is the set of all functions $x:[a,b]\to\mathbb{R}$, measurable and integrable with respect to the measures $\mu(g_i)$ $(i=1,2)$, i.e., such that

$$\int_a^b |x(t)|\,dg_i(t) < +\infty \ \ (i=1,2).$$

If $G=(g_{ik})_{i,k=1}^{l,n}\in \mathrm{BV}([a,b];\mathbb{R}^{l\times n})$ and $X=(x_{kj})_{k,j=1}^{n,m}:[a,b];\to\mathbb{R}^{n\times m}$, then

$$S_c(G)(t) \equiv \big(s_c(g_{ik})(t)\big)_{i,k=1}^{l,n}, \quad S_j(G)(t) \equiv \big(s_j(g_{ik})(t)\big)_{i,k=1}^{l,n} \ \ (j=1,2)$$

and

$$\int_a^b dG(\tau)\cdot X(\tau) = \bigg(\sum_{k=1}^{n}\int_a^b x_{kj}(\tau)\,dg_{ik}(\tau)\bigg)_{i,j=1}^{l,m}.$$

Sometimes we use the designation $\int\limits_a^{\cdot} dG(s)\cdot X(s)$ for the integral $\int\limits_a^t dG_s\cdot X(s)$ as the vector-function to the variable t.

$L^p([a,b],\mathbb{R}^{n\times m};G)$ is the space of all matrix-functions $X=(x_{kj})_{k,j=1}^{n,m}:[a,b]\to\mathbb{R}^{n\times m}$ satisfying $x_{kj}\in L^p([a,b],\mathbb{R};g_{ik})$ with the norm

$$\|X\|_{p,G}=\sum_{i,k,j=1}^{n}\|x_{kj}\|_{p,g_{ik}}.$$

If $G(t)\equiv\operatorname{diag}(t,\dots,t)$, then we assume $\|X\|_{L^p}=\|X\|_{p,G}$ and omit G in the notation containing G.

$L^p([a,b],D;G)$, where $D\subset\mathbb{R}^{n\times m}$, is the set of all matrix-functions $X\in L^P([a,b],\mathbb{R}^{n\times m};G)$ such that $X(t)\in D$ for $t\in[a,b]$.

$L^p_\omega(\mathbb{R},\mathbb{R}^{n\times m};G)$ is the set of all matrix-functions $X:\mathbb{R}\to\mathbb{R}^{n\times m}$ satisfying condition (0.0.1), whose restrictions on $[0,\omega]$ belong to $L^p([0,\omega],\mathbb{R}^{n\times m};G)$.

$L_\omega(\mathbb{R},\mathbb{R}^{n\times m})$ is the set of all ω-periodic matrix-functions $X:\mathbb{R}\to\mathbb{R}^{n\times m}$ which are integrable on $[0,\omega]$.

We introduce the operators $\mathcal{A}(X,Y)$, $\mathcal{B}(X,Y)$ and $\mathcal{I}(X,Y)$ in the following way:

(a) if $X\in\mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$, $\det(I_n+(-1)^j d_jX(t))\neq 0$ for $t\in I$ $(j=1,2)$, and $Y\in\mathrm{BV}_{loc}(I;\mathbb{R}^{n\times m})$, then

$$\begin{aligned}\mathcal{A}(X,Y)(a)&=O_{n\times m},\\ \mathcal{A}(X,Y)(t)-\mathcal{A}(X,Y)(s)&=Y(t)-Y(s)+\sum_{s<\tau\le t}d_1X(\tau)\,(I_n-d_1X(\tau))^{-1}\,d_1Y(\tau)\\ &\quad-\sum_{s\le\tau<t}d_2X(\tau)\,(I_n+d_2X(\tau))^{-1}\,d_2Y(\tau)\ \text{ for }\ s<t,\ \ s,t\in I,\end{aligned}\tag{0.0.2}$$

(b) if $X\in\mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ and $Y:I\to\mathbb{R}^{n\times n}$, then

$$\mathcal{B}(X,Y)(a)=O_{n\times m},$$
$$\mathcal{B}(X,Y)(t)=X(t)Y(t)-X(a)Y(a)-\int\limits_a^t dX(\tau)\cdot Y(\tau)\ \text{ for }\ t\in I,\tag{0.0.3}$$

(c) if $X\in\mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$, $\det(X(t))\neq 0$, and $Y:I\to\mathbb{R}^{n\times n}$, then

$$\mathcal{I}(X,Y)(a)=O_{n\times m},$$
$$\mathcal{I}(X,Y)(t)=\int\limits_a^t d\big(X(\tau)+\mathcal{B}(X,Y)(\tau)\big)\cdot X^{-1}(\tau)\ \text{ for }\ t\in I,\tag{0.0.4}$$

where $a\in I$ is a fixed point.

In addition, we use the following notation and definitions:

$\widetilde{\mathbb{N}}=\{0,1,\dots\}$.

For $l\in\mathbb{N}$, we denote $\mathbb{N}_l=\{1,\dots,l\}$ and $\widetilde{\mathbb{N}}_l=\{0,1,\dots,l\}$.

If $J\subset\mathbb{Z}$, then $\mathrm{E}(J;\mathbb{R}^{n\times m})$ is the space of all bounded matrix-functions $Y:J\to\mathbb{R}^{n\times m}$ with the norm

$$\|Y\|_J=\max\big\{\|Y(k)\|:\ k\in J\big\}.$$

Δ is the difference operator of the first order, i.e.,

$$\Delta Y(k-1)=Y(k)-Y(k-1)\ \text{ for }\ Y\in E(\widetilde{\mathbb{N}}_l,\mathbb{R}^{n\times m}),\ \ k\in\mathbb{N}_l.$$

If a function Y is defined on $\mathbb{N}_l$ or $\widetilde{\mathbb{N}}_{l-1}$, then we assume $Y(0)=O_{n\times m}$, or $Y(l)=O_{n\times m}$, respectively, if necessary.

For $m\in\mathbb{N}$, $Y\in\mathrm{E}(\widetilde{\mathbb{N}}_m;\mathbb{R}^{n\times m})$ and $i\in\mathbb{N}_m$, $\tau_m=(b-a)/m$, $\tau_{0m}=a$, $\tau_{km}=a+k\tau_m$ and $I_{km}=]\tau_{k-1\,m},\tau_{km}[$ for $m\in\mathbb{N}$ and $k\in\mathbb{N}_m$. Moreover, for $m\in\mathbb{N}$, we define the function ν_m by

$$\nu_m(t)=\Big[\frac{t-a}{b-a}\,m\Big]\ \text{ for }\ t\in[a,b].$$

Obviously, $\nu_m(\tau_{km})=k$ for all $m\in\mathbb{N}_m$ and $k\in\widetilde{\mathbb{N}}_m$.

If $\alpha\in E(J,\mathbb{R}_+)$, then

$$\|Y\|_{\nu,\alpha}=\Big(\sum_{k\in J}\alpha(k)\|Y(k)\|^\nu\Big)^{\frac{1}{\nu}}\ \text{ if }\ 1\le\nu<+\infty,\ \text{ and }\ \|Y\|_{+\infty,\alpha}=\|Y\|_J$$

(if $\alpha(k)\equiv1$, then we omit α in this notation).

For all $m\in\mathbb{N}$, define the operators $p_m:\mathrm{BV}([a,b];\mathbb{R}^n)\to\mathrm{E}(\widetilde{N}_m;\mathbb{R}^n)$ and $q_m:\mathrm{E}(\widetilde{N}_m;\mathbb{R}^n)\to\mathrm{BV}([a,b];\mathbb{R}^n)$, respectively, by

$$p_m(x)(k)=x(\tau_{km})\ \text{ for }\ x\in\mathrm{BV}([a,b];\mathbb{R}^n)\ \text{ and }\ k\in\widetilde{\mathbb{N}}_m$$

and

$$q_m(y)(t)=\begin{cases}y(k) & \text{if }\ t=\tau_{km}\ \text{ for some }\ k\in\widetilde{\mathbb{N}}_m,\\ y(k)-\dfrac{1}{m}\,G_{1m}(k)y(k)-\dfrac{1}{m}\,g_{1m}(k) & \text{if }\ t\in]\tau_{k-1\,m},\tau_{k\,m}[\ \text{ for some }\ k\in\widetilde{\mathbb{N}}_m\end{cases}$$
$$\text{for }\ y\in\mathrm{E}(\widetilde{N}_m;\mathbb{R}^n),\ \ t\in[a,b].$$

We say that the matrix-function $X\in\mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ satisfies the Lappo–Danilevskiĭ condition at the point a if the matrices $S_c(X)(t)-S_c(X)(a)$, $S_1(X)(t)-S_1(X)(a)$ and $S_2(X)(t)-S_2(X)(a)$ are pairwise permutable and

$$\int\limits_a^t S_c(X)(\tau)\,dS_c(X)(\tau)=\int\limits_a^t dS_c(X)(\tau)\cdot S_c(X)(\tau)\ \text{ for }\ \ t\in[a,b]. \tag{0.0.5}$$

Here, the use is made of the following formulas:

$$\int\limits_a^b f(t)\,dg(t)=\int\limits_a^b f(t)\,dg(t-)+f(b)\,d_1g(b), \tag{0.0.6}$$

$$\int\limits_a^b f(t)\,dg(t)=\int\limits_a^b f(t)\,dg(t+)+f(a)\,d_2g(a), \tag{0.0.7}$$

$$\int\limits_a^{t-} x(\tau)\,dg(\tau)=\int\limits_a^t x(\tau)\,dg(\tau)-x(t)\,d_1g(t), \tag{0.0.8}$$

$$\int\limits_a^{t+} x(\tau)\,dg(\tau)=\int\limits_a^t x(\tau)\,dg(\tau)+x(t)\,d_2g(t). \tag{0.0.9}$$

$$\int\limits_a^b f(t)\,dg(t)+\int\limits_a^b g(t)\,df(t)=f(b)g(b)-f(a)g(a)+\sum_{a<t\le b}d_1f(t)\cdot d_1g(t$$
$$-\sum_{a\le t<b}d_2f(t)\cdot d_2g(t)\ \text{ (integration-by-parts formula)}, \tag{0.0.10}$$

$$\int_a^b h(t)\,d(f(t)g(t)) = \int_a^b h(t)f(t)\,dg(t) + \int_a^b h(t)g(t)\,df(t) - \sum_{a<t\le b} h(t)\,d_1 f(t)\cdot d_1 g(t \\ + \sum_{a\le t<b} h(t)\,d_2 f(t)\cdot d_2 g(t) \ \ \text{(general integration-by-parts formula)}, \quad (0.0.11)$$

$$\int_a^b f(t)\,ds_1(g)(t) = \sum_{a<t\le b} f(t)\,d_1 g(t), \quad \int_a^b f(t)\,ds_2(g)(t) = \sum_{a<t\le b} f(t)\,d_2 g(t), \quad (0.0.12)$$

$$\int_a^b f(t)\,d\bigg(\int_a^s g(s)\,dh(s)\bigg) = \int_a^b f(t)g(t)\,dh(t) \ \ \text{for} \ \ t\in I, \quad (0.0.13)$$

$$d_j\bigg(\int_a^t f(s)\,dg(s)\bigg) = f(t)\,d_j g(t) \ \ \text{for} \ \ t\in I \ \ (j=1,2), \quad (0.0.14)$$

$$\int_a^b f^k(t)\,df(t) = \frac{1}{k+1}\bigg[f^{k+1}(b) - f^{k+1}(a) \\ + \sum_{m=0}^{k-1}\bigg(\sum_{a<t\le b} f^m(t)\,d_1 f(t)\cdot d_1 f^{k-m}(t) - \sum_{a\le t<b} f^m(t)\,d_2 f(t)\cdot d_2 f^{k-m}(t)\bigg)\bigg] \ \ (k=1,2,\dots) \quad (0.0.15)$$

and

$$\int_a^b \operatorname{sgn} g(t)\,dg(t) = |g(b)| - |g(a)| \\ + \sum_{a<t\le b}\big(|g(t-)| - g(t-)\operatorname{sgn} g(t)\big) - \sum_{a\le t<b}\big(|g(t+)| - g(t+)\operatorname{sgn} g(t)\big) \quad (0.0.16)$$

for $f,g\in \mathrm{BV}([a,b];\mathbb{R})$.

The proof of formulas (0.0.6), (0.0.10), (0.0.12) and (0.0.13) one can find e.g. in [73, Theorems I.4.25, I.4.33 and Lemma I.4.23]. As to formulas (0.0.11), (0.0.15) and (0.0.16), they are proved in [23, Lemmas 1.1.1 and 1.1.2].

Chapter 1

Linear boundary value problems for systems of generalized ordinary differential equations

1.1 General linear boundary value problems. Unique solvability

1.1.1 Statement of the problem and formulation of the results

Let $A \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ and $f \in \mathrm{BV}([a,b];\mathbb{R}^n)$, i.e., $A : [a,b] \to \mathbb{R}^{n\times n}$ and $f : [a,b] \to \mathbb{R}^n$ be, respectively, matrix- and vector-functions with bounded total variation components on the closed interval $[a,b]$.

Consider a linear system of generalized ordinary differential equations of the form

$$dx = dA(t) \cdot x + df(t) \ \text{ for } \ t \in [a,b]. \tag{1.1.1}$$

We investigate the problem on the existence and uniqueness of solutions of system (1.1.1) satisfying the linear boundary condition

$$\ell(x) = c_0, \tag{1.1.2}$$

where $\ell : \mathrm{BV}_\infty([a,b];\mathbb{R}^n) \to \mathbb{R}^n$ is a linear vector-functional bounded with respect to the norm $\|\cdot\|_\infty$, and $c_0 \in \mathbb{R}^n$.

In particular, we establish Green's type theorem on the unique solvability of problem (1.1.1), (1.1.2) and give the representation of the solution.

Also, we consider the same problem for the generalized system of the form

$$dx = dA(t) \cdot x + dB(t) \cdot q(t) \ \text{ for } \ t \in [a,b], \tag{1.1.3}$$

where $B \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ and $q : [a,b] \to \mathbb{R}^n$ is a vector-function with integrable components on the closed interval $[a,b]$ with respect to B in the Kurzweil–Stieltes sense.

The boundary value problem (1.1.1), (1.1.2) is considered without restriction on the form of the linear functional ℓ. Note that there are no normal forms of presentation of the linear functionals on $\mathrm{BV}([a,b];\mathbb{R}^n)$. In this connection, we consider the generalized linear differential system of form (1.1.3). The same situation for ordinary differential case is considered in [46, 47].

In particular, we investigate system (1.1.1) for the following particular cases of the boundary condition (1.1.2):

(a)

$$\int_a^b d\mathcal{L}(t) \cdot x(t) = c_0, \tag{1.1.4}$$

where $c_0 = (c_{0i})_{i=1}^n \in \mathbb{R}^n$ and $\mathcal{L} \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$;

(b) the general multi-point boundary condition

$$\sum_{j=1}^{n_0} L_j x(t_j) = c_0, \tag{1.1.5}$$

where $t_j \in [a,b]$ $(j = 1,\dots,n_0)$, $L_j \in \mathbb{R}^{n\times n}$ $(j = 1,\dots,n_0)$ are constant matrixes, and n_0 is a fixed natural number;

(c) the Cauchy–Nicoletti type problem

$$x_i(t_i) = \ell_i(x_1,\dots,x_n) + c_{0i} \;\; (i = 1,\dots,n),$$

where $\ell_i : \mathrm{BV}_\infty([a,b];\mathbb{R}^n) \to \mathbb{R}$ $(i = 1,\dots,n)$ are linear bounded functionals;

(d) the Cauchy–Nicoletti problem

$$x_i(t_i) = c_{0i} \;\; (i = 1,\dots,n), \tag{1.1.6}$$

where $c_{0i} \in \mathbb{R}$, and x_i is the i-th component of the vector-function x for every $i \in \{1,\dots,n\}$;

(e) the periodic problem

$$x(a) = x(b).$$

Note that condition (1.1.4) is the particular case of (1.1.2), where

$$\ell(x) \equiv \int_a^b d\mathcal{L}(t)\cdot x(t). \tag{1.1.7}$$

In the present paper, we establish the effective necessary and sufficient conditions for a unique solvability of the general problem (1.1.3), (1.1.2) (of problem (1.1.1), (1.1.2)). The obtained results differ from those given in [16, 18, 72].

The boundary value problems with condition (1.1.6) have been first considered by O. Nicoletti [62] for systems of ordinary differential equations. The optimal conditions for the solvability and unique solvability of the problem with the boundary condition (1.1.4) for the linear and nonlinear cases are established in [46, 47, 59] (see also the references therein).

The multi-point boundary value problems for functional differential, impulsive differential and difference equations are investigated in [18, 27, 37] (see also the references therein).

The multi-point problems for systems of generalized ordinary differential equations were studied in [6, 8, 14, 18].

The results presented in the present monograph generalize the results obtained for the linear case and presented in our last papers.

A vector-function $x \in \mathrm{BV}([a,b];\mathbb{R}^n)$ is said to be a solution of system (1.1.1) if

$$x(t) - x(s) = \int_s^t dA(\tau)\cdot x(\tau) + f(t) - f(s) \;\text{ for }\; a \le s < t \le b.$$

Under a solution of problem (1.1.1), (1.1.2) we understand a solution $x \in \mathrm{BV}([a,b];\mathbb{R}^n)$ of system (1.1.1) satisfying condition (1.1.2).

By a solution of the system of generalized ordinary differential inequalities

$$dx(t) \le dA(t)\cdot x(t) + f(t) \;\; (\ge)$$

we mean a vector-function $x \in \mathrm{BV}([a,b];\mathbb{R}^n)$ such that

$$x(t) - x(s) \le \int_s^t dA(\tau)\cdot x(\tau) + f(t) - f(s) \;\; (\ge) \;\text{ for }\; a \le s \le t \le b.$$

We assume that

$$\det\big(I_n + (-1)^j\, d_j A(t)\big) \neq 0 \ \text{ for } \ t \in [a,b] \ \ (j=1,2). \tag{1.1.8}$$

Moreover, without loss of generality, we assume $A(a) = O_{n\times n}$ and $f(a) = 0_n$ for every system of type (1.1.1).

The above inequalities guarantee the unique solvability of the Cauchy problem for the corresponding systems (see, e.g., [73, Theorem III.1.4]).

If $s \in \mathbb{R}$ and $\beta \in \mathrm{BV}[a,b],\mathbb{R})$ are such that

$$1 + (-1)^j\, d_j\beta(t) \neq 0 \ \text{ for } \ (-1)^j(t-s) < 0 \ \ (j=1,2),$$

then by $\gamma_\beta(\,\cdot\,,s)$ we denote the unique solution of the Cauchy problem

$$d\xi(t) = \xi(t)\, d\beta(t), \quad \xi(s) = 1.$$

It is known (see [39, 44, 71]) that

$$\gamma_\beta(t,s) = \begin{cases} \exp(s_c(\beta)(t) - s_c(\beta)(s)) \prod\limits_{s<\tau\le t} (1 - d_1\beta(\tau))^{-1} \prod\limits_{s\le\tau<t} (1 + d_2\beta(\tau)) & \text{for } \ t > s, \\ \exp(s_c(\beta)(t) - s_c(\beta)(s)) \prod\limits_{t<\tau\le s} (1 - d_1\beta(\tau)) \prod\limits_{t\le\tau<s} (1 + d_2\beta(\tau))^{-1} & \text{for } \ t < s, \\ 1 & \text{for } \ t = s. \end{cases} \tag{1.1.9}$$

Alongside with system (1.1.1) and the boundary condition (1.1.2), we consider the corresponding homogeneous system

$$dx = dA(t)\cdot x \tag{1.1.1$_0$}$$

and the homogeneous boundary condition

$$\ell(x) = 0_n. \tag{1.1.2$_0$}$$

Definition 1.1.1. Let condition (1.1.8) hold and let $B \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$, $B(a) = O_{n\times n}$. A matrix-function $\mathcal{G}_B : [a,b]\times[a,b] \to \mathbb{R}^{n\times n}$ is said to be the Green matrix of problem $(1.1.1_0), (1.1.2_0)$ with respect to the matrix-function B if:

(a) there exist numbers $\alpha_1 \in \mathbb{R}$ and $\alpha_2 \in \mathbb{R}$, $\alpha_1 + \alpha_2 = 1$, such that the restrictions of the matrix-function $\mathcal{G}_B(\,\cdot\,,s)$ on $[a,s[$ and $]s,b]$ satisfy, respectively, the matrix equations

$$dX = dA(t)\cdot X + \alpha_1 dB(t)$$

and

$$dX = dA(t)\cdot X - \alpha_2 dB(t)$$

for every $s \in \,]a,b[\,$;

(b)

$$\big(I_n - d_1A(t)\big)^{-1}\big(\mathcal{G}_B(t-,t) + \alpha_1 d_1B(t)\big) = (I_n + d_2A(t))^{-1}\big(\mathcal{G}_B(t+,t) + \alpha_2 d_2B(t)\big) \ \text{ for } \ t \in \,]a,b[\,;$$

(c) $\mathcal{G}_B(t,\,\cdot\,) \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ for every $t \in [a,b]$;

(d) the vector-function $x(t) = \int\limits_a^b d_s\mathcal{G}_B(t,s)\cdot q(s)$ satisfies condition $(1.1.2_0)$ for every $q \in L([a,b],\mathbb{R}^n;B)$.

Below, we prove the existence of a matrix-function $H \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ such that the matrix-function $\mathcal{G}_B$ defined by the equalities

$$\mathcal{G}_B(t,s) = \begin{cases} Y(t)\big(H(s) + Q_B(s) - \alpha_2 Q_B(t)\big) & \text{for } \ a \le s < t \le b, \\ Y(t)\big(H(s) + \alpha_1 Q_B(t)\big) & \text{for } \ a \le t < s \le b, \\ \text{arbitrary} & \text{for } \ t = s \end{cases} \tag{1.1.10}$$

is the Green matrix of problem $(1.1.1_0), (1.1.2_0)$ with respect to B for every numbers $\alpha_1 \in \mathbb{R}$ and $\alpha_2 \in \mathbb{R}$ such that $\alpha_1 + \alpha_2 = 1$, where

$$Q_B(t) \equiv Y^{-1}(t)B(t) - \int_a^t dY^{-1}(\tau) \cdot B(\tau), \tag{1.1.11}$$

and Y is the fundamental matrix of the homogeneous system $(1.1.1_0)$ under the condition

$$Y(a) = I_n$$

(see the proof of Theorem 1.1.1).

The Green matrix is unique in the following sense. If $\mathcal{G}_B(t,s)$ and $\mathcal{G}_{1\,B}(t,s)$ are two matrix-functions corresponding to the common constants α_1 and α_2, satisfying conditions (a)–(c) of Definition 1.1.1, then

$$\mathcal{G}_B(t,s) - \mathcal{G}_{1\,B}(t,s) \equiv Y(t)H_*(s),$$

where $H_* \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ is a matrix-function such that

$$H_*(s+) = H_*(s-) = C = const \ \text{ for } \ s \in [a,b],$$

and $C \in \mathbb{R}^{n\times n}$ is a constant matrix.

We use the following propositions.

Proposition 1.1.1 (Variation-of-constants formula)**.** *Let the matrix-function $A \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ be such that condition* (1.1.8) *hold and Y be a fundamental matrix of the homogeneous system* $(1.1.1_0)$*. Then each solution of system* (1.1.1) *admits the representation*

$$\begin{aligned} x(t) &= f(t) - f(s) + Y(t)\bigg\{Y^{-1}(s)x(s) - \int_s^t dY^{-1}(\tau)\cdot(f(\tau)-f(s))\bigg\} \\ &= Y(t)Y^{-1}(s) + \int_s^t Y(t)Y^{-1}(\tau)\, d\mathcal{A}(A,f)(\tau) \ \text{ for } \ t,s \in [a,b]. \end{aligned} \tag{1.1.12}$$

Proposition 1.1.2. *Let the matrix-function $A \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ be such that condition* (1.1.8) *hold and Y be a fundamental matrix of the homogeneous system* $(1.1.1_0)$*. Then*

$$\begin{aligned} Y^{-1}(t) &= Y^{-1}(s) - Y^{-1}(t)A(t) + Y^{-1}(s)A(s) + \int_s^t dY^{-1}(\tau)\cdot A(\tau) \\ &= Y^{-1}(s) - \mathcal{B}(Y^{-1},A)(t) + \mathcal{B}(Y^{-1},A)(s) \ \text{ for } \ s,t \in [a,b] \end{aligned} \tag{1.1.13}$$

and

$$d_jY^{-1}(t) = -Y^{-1}(t)\, d_jA(t)\cdot\big(I_n + (-1)^j\, d_jA(t)\big)^{-1} \ \text{ for } \ t \in [a,b] \ \ (j=1,2), \tag{1.1.14}$$

$$d_1Y^{-1}(t) = -d_1A(t)\cdot Y^{-1}(t-), \quad d_2Y^{-1}(t) = -d_2A(t)\cdot Y^{-1}(t+) \ \text{ for } \ t \in [a,b]. \tag{1.1.15}$$

In addition,

$$dY^{-1}(t) = -Y^{-1}(t)\, d\mathcal{A}(A,A)(t) \ \text{ for } \ t \in [a,b], \tag{1.1.16}$$

where $\mathcal{A}$ is the operator defined by (0.0.2).

Theorem 1.1.1. *Let condition* (1.1.8) *hold. Then the boundary value problem* (1.1.3), (1.1.2) *is uniquely solvable if and only if the corresponding homogeneous problem* $(1.1.1_0), (1.1.2_0)$ *has only a trivial solution. If the last condition holds, then the solution x of problem* (1.1.3), (1.1.2) *admits the representation*

$$x(t) = x_0(t) + \int_a^b d_s\mathcal{G}_B(t,s)\cdot q(s) \ \text{ for } \ t \in [a,b], \tag{1.1.17}$$

where x_0 is a solution of problem $(1.1.1_0), (1.1.2)$*, and $\mathcal{G}_B$ is the Green matrix of problem* $(1.1.1_0)$, $(1.1.2_0)$ *with respect to the matrix-function B.*

Corollary 1.1.1. *Let condition* (1.1.8) *hold. Then the boundary value problem* (1.1.1), (1.1.2) *is uniquely solvable if and only if the corresponding homogeneous problem* $(1.1.1_0), (1.1.2_0)$ *has only a trivial solution. If the last condition holds, then the solution x of problem* (1.1.1), (1.1.2) *admits the representation*

$$x(t) = z_0(t) + f(t) + \int_a^b d_s\mathcal{G}_A(t,s)\cdot f(s) \ \text{ for } \ t\in[a,b],$$

where z_0 is the solution of homogeneous system $(1.1.2_0)$ *satisfying the condition*

$$\ell(x) = c_0 - \ell(f),$$

and $\mathcal{G}_A(t,s)$ is the Green matrix of problem $(1.1.1_0), (1.1.2_0)$ *with respect to the matrix-function A.*

Below, in the proof of Theorem 1.1.1, we will show that the homogeneous problem $(1.1.1_0), (1.1.2_0)$ has only a trivial solution if and only if

$$\det(D) \neq 0, \tag{1.1.18}$$

where $D = \ell(Y)$, and Y is a fundamental matrix of system $(1.1.1_0)$.

The following proposition is a simple modification of Lemma 3.3 from [72] related to problem (1.1.3), (1.1.2).

Proposition 1.1.3. *Let the matrix-function $A \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ be such that condition* (1.1.8) *hold. Then the boundary value problem* (1.1.3), (1.1.2) *is solvable if and only if the condition*

$$(c_0 - \ell(F))^\top \gamma = 0 \tag{1.1.19}$$

holds for every $\gamma\in\mathbb{R}^n$ such that

$$(\ell(Y))^\top\gamma = 0_n,$$

where

$$F(t) \equiv Y(t)\int_a^t Y^{-1}(\tau)\,d\mathcal{A}(A,f)(\tau), \ \text{ and } \ f(t) \equiv \int_a^t dB(\tau)\cdot q(\tau).$$

So, if condition (1.1.18) holds, then only the vector $\gamma = 0_n$ satisfies the homogeneous system appearing in Proposition 1.1.3 and, evidently, condition (1.1.19) hold. If condition (1.1.18) is violated, then problem (1.1.3), (1.1.2) is solvable only for c_0, that satisfies the conditions of the proposition.

In connection with problem (1.1.1), (1.1.4), we give the following

Definition 1.1.2. Let condition (1.1.8) hold and let ℓ be an integral operator given by (1.1.7), where $\mathcal{L}\in\mathrm{BV}([a,b];\mathbb{R}^{n\times n})$. A matrix-function $\mathcal{G} : [a,b]\times[a,b]\to\mathbb{R}^{n\times n}$ is said to be the Green matrix of system $(1.1.1_0)$ under the condition

$$\int_a^b d\mathcal{L}(t)\cdot x(t) = 0_n \tag{1.1.4$_0$}$$

if:

(a) for every $s\in[a,b[$, the matrix-function $\mathcal{G}(\,\cdot\,,s)$ satisfies the matrix equation $(1.1.1_0)$ both on $[a,s[$ and $]s,b]$;

(b)
$$\mathcal{G}(t,t+) - \mathcal{G}(t,t-) = Y(t)D^{-1}\bigg\{\int_a^t d\mathcal{L}(\tau)\cdot Y(\tau)Y^{-1}(t)(I_n - d_1A(t))^{-1}$$
$$+\int_t^b d\mathcal{L}(\tau)\cdot Y(\tau)Y^{-1}(t)(I_n+d_2A(t))^{-1} - d_1\mathcal{L}(t)\cdot(I_n-d_1A(t))^{-1} - d_2\mathcal{L}(t)\cdot(I_n+d_2A(t))^{-1}\bigg\}$$
$$\text{for } \ t\in\,]a,b[\,,$$

where

$$D = \int_a^b d\mathcal{L}(s) \cdot Y(s) \tag{1.1.20}$$

and Y is, as above, the fundamental matrix of system $(1.1.1_0)$ satisfying the condition $Y(a) = I_n$;

(c) $\mathcal{G}(t, \cdot\,) \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ for every $t \in [a,b]$;

(d) the vector-function $x(t) = \int_a^b d_s\mathcal{G}(t,s) \cdot f(s)$ satisfies condition $(1.1.4_0)$ for every $f \in \mathrm{BV}([a,b];\mathbb{R}^n)$.

Without loss of generality, we will assume that $\mathcal{L}(b) = O_{n\times n}$.

The Green matrix of problem $(1.1.1_0), (1.1.4_0)$ exists and is unique in the sense given above. In particular,

$$\mathcal{G}(t,s) = \begin{cases} -Y(t)D^{-1} \int_a^s d\mathcal{L}(\tau) \cdot Y(\tau)Y^{-1}(s) & \text{for } a \le s < t \le b, \\ Y(t)D^{-1} \int_s^b d\mathcal{L}(\tau) \cdot Y(\tau)Y^{-1}(s) & \text{for } a \le t < s \le b, \\ \text{an arbitrary} & \text{for } t = s, \end{cases} \tag{1.1.21}$$

where D is the constant matrix defined by (1.1.20) (cf. [72]).

Remark 1.1.1. If ℓ is an integral operator defined by (1.1.7), where $\mathcal{L} \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$, then for the matrix-function H appearing in (1.1.10), we have

$$H(s) \equiv -D^{-1}\bigg(\int_a^s d\mathcal{L}(\tau) \cdot Y(\tau)Q_B(\tau) + \int_s^b d\mathcal{L}(\tau) \cdot Y(\tau)Q_B(s) \bigg). \tag{1.1.22}$$

In addition, from (1.1.12), due to (1.1.13), we get $Q_A(t) \equiv I_n - Y^{-1}(t)$ and, therefore, from (1.1.22) we have

$$H(s) \equiv -I_n + D^{-1}\bigg(\mathcal{L}(s) - \mathcal{L}(a) + \int_s^b d\mathcal{L}(\tau) \cdot Y(\tau)Y^{-1}(s) \bigg) \tag{1.1.23}$$

for $B(t) \equiv A(t)$. Moreover, in this case, (1.1.10) has the form

$$\mathcal{G}_A(t,s) = \begin{cases} Y(t)\big(H(s) + Y^{-1}(t) - Y^{-1}(s)\big) & \text{for } a \le s < t \le b, \\ Y(t)H(s) & \text{for } a \le t < s \le b, \\ \text{arbitrary} & \text{for } t = s \end{cases} \tag{1.1.24}$$

if $\alpha_1 = 0$ and $\alpha_2 = 1$.

Corollary 1.1.2. *Let condition* (1.1.8) *hold and let ℓ be an integral operator given by* (1.1.7)*, where $\mathcal{L} \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$. Then the boundary value problem* (1.1.1), (1.1.4) *is uniquely solvable if and only if the corresponding homogeneous problem* $(1.1.1_0), (1.1.4_0)$ *has only the trivial solution. If the last condition holds, then the solution x of problem* (1.1.1), (1.1.4) *admits the representation*

$$x(t) = x_0(t) + \int_a^b d_s\mathcal{G}(t,s) \cdot f(s) \ \text{ for } t \in [a,b], \tag{1.1.25}$$

where x_0 is a solution of problem $(1.1.1_0), (1.1.4)$ *and $\mathcal{G} : [a,b] \times [a,b] \to \mathbb{R}^{n\times n}$ is the Green matrix $\mathcal{G}$ of problem* $(1.1.1_0), (1.1.4_0)$ (*see* (1.1.21)).

Remark 1.1.2. If problem $(1.1.1_0), (1.1.2_0)$ has a nontrivial solution, then for every $f \in \mathrm{BV}([a,b];\mathbb{R}^n)$ there exists a vector $c_0 \in \mathbb{R}^n$ such that problem (1.1.1), (1.1.2) has no solution.

Remark 1.1.3. If problem $(1.1.1_0), (1.1.2_0)$ has a nontrivial solution, and $\ell : \mathrm{BV}_\infty([a,b];\mathbb{R}^n) \to \mathbb{R}^n$ is a surjective mapping, then for every $c_0 \in \mathbb{R}^n$ there exists $f \in \mathrm{BV}([a,b];\mathbb{R}^n)$ such that problem (1.1.1), (1.1.2) has no solution.

Remark 1.1.4. Let the matrix-function A satisfy the Lappo–Danilevskiĭ condition at the point a. Then problem (1.1.1), (1.1.4) is uniquely solvable if and only if

$$\det\bigg(\int_a^b d\mathcal{L}(t)\cdot\exp(S_0(A)(t))\prod_{a\le\tau<t}(I_n+d_2A(\tau))\prod_{a<\tau\le t}(I_n-d_1A(\tau))^{-1}\bigg)\neq 0.$$

We give here another form of the Green theorem for the following one-dimensional problem:

$$du(t) = u(t)\,d\alpha(t) + d\varphi(t), \tag{1.1.26}$$

$$u(a) - u(b) = c_0, \tag{1.1.27}$$

where $\alpha \in \mathrm{BV}([a,b];\mathbb{R})$, $\varphi \in \mathrm{BV}([a,b];\mathbb{R})$, $\alpha(a) = \varphi(a) = 0$ and $c_0 \in \mathbb{R}$.

Alongside with (1.1.26), (1.1.27), consider the corresponding homogeneous boundary value problem

$$du(t) = u(t)\,d\alpha(t), \tag{$1.1.26_0$}$$

$$u(a) - u(b) = 0. \tag{$1.1.27_0$}$$

If

$$1 + (-1)^j\,d_j\alpha(t) \neq 0 \ \text{ for } \ t \in [a,b] \ \ (j = 1,2) \tag{1.1.28}$$

and

$$\lambda(\alpha)(b) \neq 1, \tag{1.1.29}$$

where

$$\lambda(\alpha)(t) = \exp(s_0(\alpha)(t))\prod_{a\le\tau<t}(1+d_2\alpha(\tau))\Big/\prod_{a<\tau\le t}(1-d_1\alpha(\tau)) \ \text{ for } \ t\in[a,b], \tag{1.1.30}$$

then we assume

$$g_0(\alpha)(t,\tau) = \begin{cases} (1-\lambda(\alpha)(b))^{-1}\lambda(\alpha)(b)\,\lambda(\alpha)(t)\lambda^{-1}(\alpha)(\tau) & \text{for } \ a \le t \le \tau \le b, \\ (1-\lambda(\alpha)(b))^{-1}\lambda(\alpha)(t)\lambda^{-1}(\alpha)(\tau) & \text{for } \ a \le \tau < t \le b; \end{cases} \tag{1.1.31}$$

$$g_j(\alpha)(t,\tau) = (1+(-1)^j\,d_j\alpha(\tau))^{-1}g_0(\alpha)(t,\tau) \ \text{ for } \ t\neq\tau, \ \ t,\tau\in[a,b] \ \ (j=1,2) \tag{1.1.32}$$

and

$$g_j(\alpha)(t,t) = (1+(-1)^j\,d_j\alpha(t))^{-1}\lambda^{j-2}(\alpha)(b)\cdot g_0(\alpha)(t,t) \ \text{ for } \ t\in[a,b] \ \ (j=1,2). \tag{1.1.33}$$

Notice that $\lambda(\alpha)$ is the unique solution of system $(1.1.26_0)$ under the condition $u(a) = 1$ (see [39,44]).

Theorem 1.1.1$_1$. *Let* (1.1.28) *hold. Then problem* (1.1.26), (1.1.27) *is uniquely solvable if and only if the corresponding homogeneous problem* $(1.1.26_0), (1.1.27_0)$ *has only a trivial solution. If the last condition holds, then the solution u of problem* (1.1.26), (1.1.27) *admits the representation*

$$u(t) = u_0(t) + \int_a^b g_0(\alpha)(t,\tau)\,ds_0(\varphi)(\tau) + \sum_{a<\tau\le b} g_1(\alpha)(t,\tau)\,d_1\varphi(\tau) + \sum_{a\le\tau<b} g_2(\alpha)(t,\tau)\,d_2\varphi(\tau) \ \text{ for } \ t\in[a,b], \tag{1.1.34}$$

where u_0 is a solution of problem $(1.1.26_0), (1.1.27)$*, and* $g_j(\alpha)$ $(j = 0,1,2)$ *are defined by* (1.1.31)–(1.1.33)*, respectively.*

The algebraic properties of considered problems are investigated in [72,73].

1.1.2 The spectral type necessary and sufficient conditions for the unique solvability of problem (1.1.1), (1.1.4)

In general, it is quite difficult to verify condition (1.1.18) directly even in the case if one is able to write out the fundamental matrix of system $(1.1.1_0)$ explicitly. Therefore, it is important to seek for effective conditions which would guarantee the absence of nontrivial solutions of the homogeneous problem $(1.1.1_0), (1.1.2_0)$. In this subsection, we give the results connected to this topic. Analogous results for ordinary differential equations have been obtained in [47].

To formulate the results, we use the following designations.

For every matrix-function $X \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ such that $\det(I_n - d_1X(t)) \neq 0$ for $t \in [a,b]$ we introduce the matrix-functions $[X]_i$, $(X)_i$ and $V_i(X)$ $(i = 0,1,\dots)$ by the equalities

$$[X]_0(t) = (I_n - d_1X(t))^{-1},$$

$$[X]_i(t) = (I_n - d_1X(t))^{-1} \int_a^t dX_-(\tau)\cdot[X]_{i-1}(\tau) \ \text{ for } \ t \in [a,b] \ \ (i = 1,2,\dots), \tag{$1.1.35_1$}$$

$$(X)_0(t) = O_{n\times n}, \ \ (X)_1(t) = X(t),$$

$$(X)_{i+1}(t) = \int_a^t dX_-(\tau)\cdot(X)_i(\tau) \ \text{ for } \ t \in [a,b] \ \ (i = 1,2,\dots), \tag{$1.1.36_1$}$$

and

$$V_0(X)(t) = X(t), \quad V_1(X)(t) = \big|(I_n - d_1X(t))^{-1}\big| V(X_-)(t),$$

$$V_{i+1}(X)(t) = \big|(I_n - d_1X(t))^{-1}\big| \int_a^t dV(X_-)(\tau)\cdot V_i(X)(\tau) \ \text{ for } \ t \in [a,b] \ \ (i = 1,2,\dots), \tag{$1.1.37_1$}$$

where $X_-(t) \equiv X(t-)$; and for every $X \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ such that $\det(I_n + d_2X(t)) \neq 0$ for $t \in [a,b]$, we put

$$[X]_0(t) = (I_n + d_2X(t))^{-1},$$

$$[X]_i(t) = (I_n + d_2X(t))^{-1} \int_b^t dX_+(\tau)\cdot[X(\tau)]_{i-1} \ \text{ for } \ t \in [a,b] \ \ (i = 1,2,\dots), \tag{$1.1.35_2$}$$

$$(X)_0(t) = O_{n\times n}, \ \ (X(t))_1 = X(b) - X(t),$$

$$(X)_{i+1}(t) = \int_b^t dX_+(\tau)\cdot(X)_i(\tau) \ \text{ for } \ t \in [a,b] \ \ (i = 1,2,\dots) \tag{$1.1.36_2$}$$

and

$$V_0(X)(t) = X(t), \quad V_1(X)(t) = \big|(I_n + d_2X(t))^{-1}\big| (V(X_+)(b) - V(X_+)(t)|,$$

$$V_{i+1}(X)(t) = \big|(I_n + d_2X(t))^{-1}\big| \left|\int_b^t dV(X_+)(\tau)\cdot V_i(X)(\tau)\right| \ \text{ for } \ t \in [a,b] \ \ (i = 1,2,\dots), \tag{$1.1.37_2$}$$

where $X_+(t) \equiv X(t+)$.

In this subsection, along with system (1.1.1), we consider the differential system

$$dx(t) = \varepsilon dA(t)\cdot x(t) + df(t) \tag{1.1.38}$$

which depends on a small positive parameter ε.

Theorem 1.1.2. *The boundary value problem* (1.1.1), (1.1.4) *is uniquely solvable if and only if there exist natural numbers k and m such that the matrix*

$$M_k = -\sum_{i=0}^{k-1} \int_a^b d\mathcal{L}(t) \cdot [A]_i(t) \tag{1.1.39}$$

is nonsingular and

$$r(M_{k,m}) < 1, \tag{1.1.40}$$

where

$$M_{k,m} = V_m(A)(c) + \Big(\sum_{i=0}^{m-1} \big| [A]_i \big|_\infty \Big) \int_a^b dV(M_k^{-1}\mathcal{L})(t) \cdot V_k(A)(t), \tag{1.1.41}$$

the matrix-functions $[A]_i$ $(i = 0, 1, \dots)$ and $V_i(A)$ $(i = 0, 1, \dots)$ are defined, respectively, by $(1.1.35_l)$ *and* $(1.1.37_l)$ *for some $l \in \{1, 2\}$, and $c = b + (a - b)(l - 1)$.*

Theorem 1.1.2$_1$. *Let there exist natural numbers k and m such that the matrix*

$$M_k = \mathcal{L}(a) - \sum_{i=0}^{k-1} \int_a^b d\mathcal{L}(t) \cdot (A)_i(t) \tag{1.1.42}$$

is nonsingular and inequality (1.1.40) *holds, where*

$$M_{k,m} = (V(A))_m(c) + \Big(I_n + \sum_{i=0}^{m-1} |(A)_i|_\infty \Big) \int_a^b dV(M_k^{-1}\mathcal{L})(t) \cdot (V(A))_k(t), \tag{1.1.43}$$

the matrix-functions $(A)_i$ $(i = 0, 1, \dots)$ and $(V(A))_i$ $(i = 0, 1, \dots)$ are defined by $(1.1.36_l)$ *for some $l \in \{1, 2\}$, and $c = b + (a - b)(l - 1)$. Then problem* (1.1.1), (1.1.4) *is uniquely solvable.*

Corollary 1.1.3. *Let either*

$$\det(\mathcal{L}(a)) \neq 0, \tag{1.1.44}$$

or

$$\mathcal{L}(a) = O_{n\times n}, \tag{1.1.45}$$

and the conditions

$$\int_a^b d\mathcal{L}(t) \cdot (A)_i(t) = O_{n\times n} \ \ (i = 0, \dots, j - 1) \tag{1.1.46}$$

and

$$\det\Big(\int_a^b d\mathcal{L}(t) \cdot (A)_j(t) \Big) \neq 0 \tag{1.1.47}$$

hold for some natural j, where the matrix-functions $(A)_i$ $(i = 0, \dots, l)$ are defined by $(1.1.36_1)$ *or* $(1.1.36_2)$*. Then there exists $\varepsilon_0 > 0$ such that problem* (1.1.38), (1.1.4) *is uniquely solvable for every $\varepsilon \in]0, \varepsilon_0[$.*

Theorem 1.1.3. *Let a matrix-function $A_0 \in \mathrm{BV}([a, b]; \mathbb{R}^{n\times n})$ be such that the homogeneous system*

$$dx(t) = dA_0(t) \cdot x(t) \tag{1.1.48}$$

has only the trivial solution satisfying the boundary condition $(1.1.4_0)$*, and let the matrix-function $A \in \mathrm{BV}([a, b]; \mathbb{R}^{n\times n})$ admit the estimate*

$$\int_a^b |\mathcal{G}_0(t, \tau)| \, dV(A - A_0)(\tau) \leq M \ \ \textit{for} \ \ t \in [a, b], \tag{1.1.49}$$

where $\mathcal{G}_0(t,\tau)$ *is the Green matrix of problem* (1.1.48), (1.1.4), *and* $M \in \mathbb{R}_+^{n\times n}$ *is a constant matrix such that*

$$r(M) < 1. \tag{1.1.50}$$

Then problem (1.1.1), (1.1.2) *is uniquely solvable.*

1.1.3 Proof of the main results

Proof of Propositions 1.1.1 and 1.1.2. The first parts of (1.1.12) and (1.1.13) are the well known results (see [44, 73]).

Let us verify the second part of (1.1.12). Using definition of the operator $\mathcal{A}$ (see (0.0.2)) and the integration-by-parts formula (0.0.10), we have

$$f(t) - f(s) + Y(t)\bigg\{Y^{-1}(s)x(s) - \int_s^t dY^{-1}(\tau)\cdot(f(\tau)-f(s))\bigg\}$$
$$= f(t) - f(s) + Y(t)\bigg\{Y^{-1}(s)x(s) - Y^{-1}(t)$$
$$\times\bigg(f(t)-f(s)+\int_s^t dY^{-1}(\tau)\cdot f(\tau) - \sum_{s\le\tau<t} d_1Y^{-1}(\tau)\cdot d_1f(\tau) + \sum_{s<\tau\le t} d_2Y^{-1}(\tau)\cdot d_2f(\tau)\bigg)\bigg\}$$
$$= Y(t)Y^{-1}(s)x(s) + Y(t)\int_s^t Y^{-1}(\tau)\,d\mathcal{A}(A,f)(\tau) \ \text{ for } \ t,s\in[a,b].$$

Similarly, we can show equality (1.1.16). □

Proof of Proposition 1.1.3. Due to (1.1.12), for every solutions of system (1.1.3) we have

$$x(t) = Y(t)c + F(t) \ \text{ for } \ t\in[a,b],$$

where $c \in \mathbb{R}^n$ is a constant vector. So, the vector functions satisfy condition (1.1.2) if and only if c is a solution of the linear algebraic system

$$\ell(Y)c = c_0 - \ell(F).$$

But this system is solvable if and only if the statement of the proposition is valid. □

Proof of Theorem 1.1.1. Let, as above, Y be a fundamental matrix of system ($1.1.1_0$) under the condition $Y(a) = I_n$. According to (1.1.8), such a matrix exists, and by the variation-of-constant formula (1.1.12) we find that

$$x(t) = Y(t)c + \mathcal{B}(q)(t) \tag{1.1.51}$$

for every solution x of system (1.1.3), where $c = x(a)$ and

$$\mathcal{B}(q)(t) = \int_a^t dB(\tau)\cdot q(\tau) - Y(t)\int_a^t dY^{-1}(\tau)\cdot\int_a^\tau dB(s)\cdot q(s). \tag{1.1.52}$$

It is clear that x satisfies condition (1.1.2) if and only if c is a solution of the system of linear algebraic equations

$$\ell(Y)c = c_0 - \ell(\mathcal{B}(q)).$$

But this system and, consequently, problem (1.1.3), (1.1.2) is uniquely solvable if and only if

$$\det(\ell(Y)) \neq 0. \tag{1.1.53}$$

On the other hand, it is clear that (1.1.53) is a necessary and sufficient condition for the absence of the nontrivial solution to problem $(1.1.1_0), (1.1.2_0)$.

If (1.1.53) is fulfilled, then from (1.1.51) and (1.1.52) we get the representation

$$x(t) = x_0(t) + Y(t)h(q) + \mathcal{B}(q)(t) \tag{1.1.54}$$

for the solution x of problem $(1.1.3), (1.1.2)$, where

$$h(q) = -[\ell(Y)]^{-1}\ell(\mathcal{B}(q)) \tag{1.1.55}$$

and

$$x_0(t) = Y(t)[\ell(Y)]^{-1}c_0. \tag{1.1.56}$$

In addition, x_0 is a solution of problem $(1.1.1_0), (1.1.2)$.

Due to (1.1.52) and (1.1.55), $h : L([a,b],\mathbb{R}^n; B) \to \mathbb{R}^n$ is the linear continuous vector-functional. So, in view of Theorem VII.2.1 from [45], we get

$$h(q) = \int\limits_a^b dH(\tau)\cdot q(\tau), \tag{1.1.57}$$

where $H = (h_{ij})_{i,j=1}^n \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$.

By (1.1.52), (1.1.54) and (1.1.57), we have

$$x(t) \equiv x_0(t) + Y(t)\int\limits_a^b dH(\tau)\cdot q(\tau) + \mathcal{B}(q)(t). \tag{1.1.58}$$

Using the integration-by-parts formula (0.0.10) and (0.0.14), we conclude

$$\begin{aligned}\mathcal{B}(q)(t) = Y(t)\bigg(&\int\limits_a^t Y^{-1}(\tau)\,dB(\tau)\cdot q(\tau)\\ &-\sum_{a<\tau\le t} d_1Y^{-1}(\tau)\cdot d_1B(\tau)\cdot q(\tau) + \sum_{a\le\tau<t} d_2Y^{-1}(\tau)\cdot d_2B(\tau)\cdot q(\tau)\bigg)\\ &= Y(t)\bigg(\int\limits_a^t d(Y^{-1}(\tau)B(\tau))\cdot q(\tau) - \int\limits_a^t dY^{-1}(\tau)\cdot B(\tau)q(\tau)\bigg) \ \text{ for } \ a<t\le b\end{aligned}$$

and, consequently,

$$\mathcal{B}(q)(t) \equiv Y(t)\int\limits_a^t dQ_B(\tau)\cdot q(\tau), \tag{1.1.59}$$

where the matrix-function Q_B is defined by (1.1.11). Therefore, due to (1.1.58) and (1.1.59), we find that

$$x(t) = x_0(t) + Y(t)\int\limits_a^b dH(\tau)\cdot q(\tau) + Y(t)\int\limits_a^t dQ_B(\tau)\cdot q(\tau) \ \text{ for } \ t\in[a,b]. \tag{1.1.60}$$

Let the matrix-function $\mathcal{G}_B(t,s)$ be defined by (1.1.10). In view of (0.0.8) and (0.0.9), from (1.1.60) we have

$$\int\limits_a^b d_s\mathcal{G}_B(t,s)\cdot q(s) = Y(t)\int\limits_a^{t-} d(H(s)+Q_B(s))\cdot q(s)$$

$$+\big(\mathcal{G}_B(t,t)-\mathcal{G}_B(t,t-)\big)q(t)+Y(t)\int\limits_{t+}^{b}dH(s)\cdot q(s)+\big(\mathcal{G}_B(t,t+)-\mathcal{G}_B(t,t)\big)q(t)$$

$$=Y(t)\int\limits_{a}^{t}d(H(s)+Q_B(s))\cdot q(s)-Y(t)\,d_1(H(t)+Q_B(t))\cdot q(t)-\mathcal{G}_B(t,t-)q(t)$$

$$+Y(t)\int\limits_{t}^{b}dH(s)\cdot q(s)-Y(t)\,d_2H(t)\cdot q(t)+\mathcal{G}_B(t,t+)q(t)$$

$$=Y(t)\int\limits_{a}^{b}dH(s)\cdot q(s)+Y(t)\int\limits_{a}^{t}dQ_B(s)\cdot q(s)-Y(t)(H(t+)-H(t-)+d_1Q_B(t))q(t)$$

$$+Y(t)\big(H(t+)-H(t-)+(\alpha_1+\alpha_2)Q_B(t)-Q_B(t-)\big)q(t)=x(t)-x_0(t)\ \text{ for }\ t\in[a,b],$$

i.e., (1.1.17) is proved.

Let us show that the matrix-function $\mathcal{G}_B$ satisfies the conditions of Definition 1.1.1. To prove this fact, we use the following

Lemma 1.1.1. *The matrix-function $Z(t)\equiv Y(t)Q_B(t)$, where Y is the fundamental matrix of system* $(1.1.1_0)$ *under the condition $Y(a)=I_n$ and the matrix-function $Q_B(t)$ is defined by* (1.1.11)*, satisfies the generalized differential system*

$$dZ(t)\equiv dA(t)\cdot Z(t)+dB(t)\ \textit{ for }\ t\in[a,b]. \tag{1.1.61}$$

Proof. Let $a\le t_1<t_2<s$. Then, using the equality

$$dY(t)\equiv dA(t)\cdot Y(t) \tag{1.1.62}$$

and the integration-by-parts formula (0.0.10), we conclude

$$Z(t_2)-Z(t_1)-\int\limits_{t_1}^{t_2}dA(t)\cdot Z(t)=Z(t_2)-Z(t_1)-\int\limits_{t_1}^{t_2}dA(t)\cdot Y(t)Q_B(t)$$

$$=Z(t_2)-Z(t_1)-\int\limits_{t_1}^{t_2}dY(t)\cdot Q_B(t)=J_0-J_1+J_2\quad\text{for }\ a\le t_1<t_2\le b,$$

where

$$J_0=\int\limits_{t_1}^{t_2}Y(t)\,dQ_B(t),\quad J_1=\sum_{t_1<t\le t_2}d_1Y(t)\cdot d_1Q_B(t),\quad J_2=\sum_{t_1\le t<t_2}d_2Y(t)\cdot d_2Q_B(t).$$

Moreover, due to (1.1.12), using (0.0.13), (0.0.11) and the general integration-by-parts formula (0.0.11), we find that

$$J_0=\int\limits_{t_1}^{t_2}Y(t)\,d(Y^{-1}(t)B(t))-\int\limits_{t_1}^{t_2}Y(t)\,dY^{-1}(t)\,B(t)$$

$$=B(t_2)-B(t_1)-\sum_{t_1<t\le t_2}Y(t)\,d_1Y^{-1}(t)\cdot d_1B(t)+\sum_{t_1\le t<t_2}Y(t)\,d_2Y(t)\cdot d_2B(t),$$

$$J_1=\sum_{t_1<t\le t_2}d_1Y(t)\cdot\big(d_1(Y^{-1}(t)B(t))-d_1Y(t)\,B(t)\big)=\sum_{t_1<t\le t_2}d_1Y(t)\cdot Y^{-1}(t-)\,d_1B(t),$$

$$J_2=\sum_{t_1\le t<t_2}d_2Y(t)\cdot\big(d_2(Y^{-1}(t)B(t))-d_2Y(t)\,B(t)\big)=\sum_{t_1\le t<t_2}d_2Y(t)\cdot Y^{-1}(t+)\,d_2B(t).$$

Substituting these expressions into (1.1.62), we get

$$Z(t_2)-Z(t_1)-\int\limits_{t_1}^{t_2} dA(t)\cdot Z(t)=B(t_2)-B(t_1) \quad \text{for } \ a\le t_1<t_2\le b,$$

since

$$Y(t)\,d_1Y^{-1}(t)+d_1Y(t)\cdot Y^{-1}(t-)\equiv O_{n\times n}, \quad Y(t)\,d_2Y^{-1}(t)+d_2Y(t)\cdot Y^{-1}(t+)\equiv O_{n\times n}. \qquad \square$$

Now we check that the constructed Green's matrix-function satisfies the conditions of Definition 1.1.1.

Let, as above, $Z(t)\equiv Y(t)Q_B(t)$.

Let $a\le t_1<t_2<s$. Then, due to (1.1.10) and (1.1.61), we have

$$\begin{aligned}&\mathcal{G}_B(t_2,s)-\mathcal{G}_B(t_1,s)=(Y(t_2)-Y(t_1))H(s)+\alpha_1(Z(t_2)-Z(t_1))\\&=\int\limits_{t_1}^{t_2} dA(t)\cdot(Y(t)H(s)+\alpha_1Z(t))+\alpha_1(B(t_2)-B(t_1))=\int\limits_{t_1}^{t_2} dA(t)\cdot\mathcal{G}_B(t,s)+\alpha_1(B(t_2)-B(t_1)).\end{aligned}$$

Analogously, we show that the matrix-function $\mathcal{G}_B$ satisfies the corresponding equation on the interval $]s,b]$ of condition (a) of Definition 1.1.1.

Let us show the equality given in condition (b). By definition of $\mathcal{G}_B$ and the equalities $d_jY(t)=d_jA(t)\cdot Y(t)$ $(j=1,2)$ and $d_jZ(t)=d_jA(t)\cdot Z(t)+d_jB(t)$ $(j=1,2)$, we have

$$\begin{aligned}&\mathcal{G}_B(t-,t)=(I_n-d_1A(t))Y(t)(H(t)+\alpha_1Q_B(t))-\alpha_1d_1B(t),\\&\mathcal{G}_B(t+,t)=(I_n+d_2A(t))Y(t)(H(t)+Q_B(t)-\alpha_2Q_B(t))-\alpha_2d_2B(t) \ \text{ for } \ t\in\,]a,b[\,.\end{aligned}$$

Due to the condition $\alpha_1+\alpha_2=1$, from the last two equalities follows the equality given in condition (b) of Definition 1.1.1. Conditions (c) and (d) are obvious.

Let now $\mathcal{G}_B:[a,b]\times[a,b]\to\mathbb{R}^{n\times n}$ and $\mathcal{G}_{1B}:[a,b]\times[a,b]\to\mathbb{R}^{n\times n}$ be arbitrary matrix-functions corresponding to the common constants α_1 and α_2 satisfying conditions (a)–(c) of Definition 1.1.1. Then, by (a), the columns of the matrix-function $X(t)\equiv\mathcal{G}_B(t,s)-\mathcal{G}_{1B}(t,s)$ satisfy system $(1.1.1_0)$ for every $s\in[a,b]$. So, there exists a matrix-function $H_*\in\mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ such that

$$\mathcal{G}_B(t,s)-\mathcal{G}_{1B}(t,s)\equiv Y(t)H_*(s).$$

From this, due to condition (d), the vector-function $x(t)=Y(t)\int\limits_a^b dH_*(s)\cdot q(s)$ satisfies condition $(1.1.2_0)$ and, therefore,

$$\ell(Y)\int\limits_a^b dH_*(s)\cdot q(s)=0.$$

In addition, owing to (1.1.53), we have

$$\int\limits_a^{t_2} dH_*(s)\cdot q(s)=0 \ \text{ for every } \ q\in\mathrm{BV}([a,b];\mathbb{R}^n),$$

where $H_*\in\mathrm{BV}([a,b];\mathbb{R}^{n\times n})$. According to Proposition I.5.5 from [73],

$$H_*(s+)=H_*(s-)=C=const \ \text{ for } \ s\in[a,b],$$

where $C\in\mathbb{R}^{n\times n}$. Consequently, the Green matrix of problem $(1.1.1_0),(1.1.2_0)$ is unique in the above sense. $\square$

Proof of Corollary 1.1.1. This corollary immediately follows from Theorem 1.1.1, since the vector-function $x \in \mathrm{BV}([a,b];\mathbb{R}^n)$ is a solution of problem (1.1.1), (1.1.2) if and only if the vector-function $y(t) = x(t) - f(t)$ is a solution of system (1.1.2) under the boundary condition $\ell(y) = c_0 - \ell(f)$, where $B(t) \equiv A(t)$ and $q(t) \equiv f(t)$. □

Let us check Remark 1.1.1. Let $\Phi(t) \equiv \int\limits_a^t d\mathcal{L}(\tau)\cdot Y(\tau)$ and let the matrix-functions $\mathcal{B}(q)(t)$ and $Q_B(t)$ be defined by(1.1.59) and (1.1.11), respectively. Then, by virtue of (0.0.10), (0.0.11) and (0.0.13), we have

$$\begin{aligned}
\ell(\mathcal{B}(q)) = &\int\limits_a^b d\mathcal{L}(s)\cdot Y(s)\int\limits_a^s dQ_B(\tau)\cdot q(\tau)\\
&= \int\limits_a^b d\Phi(s)\int\limits_a^s dQ_B(\tau)\cdot q(\tau) = \Phi(b)\int\limits_a^b dQ_B(s)\cdot q(s) - \int\limits_a^b \Phi(t)\,dQ_B(s)\cdot q(s)\\
&\quad + \sum_{a<s\le b} d_1\Phi(s)\,d_1Q_B(s)\cdot q(s) - \sum_{a\le s<b} d_2\Phi(s)\,d_2Q_B(s)\cdot q(s)\\
&= \Phi(s)\int\limits_a^b dQ_B(s)\cdot q(s) - \int\limits_a^b d(\Phi(s)Q_B(s))\cdot q(s) + \int\limits_a^b d\Phi(s)\cdot Q_B(s)q(s)\\
&\quad = \int\limits_a^b d\bigg((\Phi(b)-\Phi(s))Q_B(s) + \int\limits_a^s d\Phi(\tau)\cdot Q_B(\tau)\bigg)\cdot q(s) = -D\int\limits_a^b dH(s)\cdot q(s),
\end{aligned}$$

where the matrix-function H and the constant matrix D are defined by (1.1.22) and (1.1.20), respectively. From this, according to (1.1.55), we get (1.1.57). The remark is proved.

Proof of Corollary 1.1.2. According to Corollary 1.1.1, we find that

$$x(t) \equiv x_0(t) - Y(t)[\ell(Y)]^{-1}\ell(f) + f(t) + \int\limits_a^b d_s\mathcal{G}_A(t,s)\cdot f(s),$$

where $\mathcal{G}_A(t,s)$ and x_0 are defined, respectively, by (1.1.24) and (1.1.56) for $B(t) \equiv A(t)$. From this, due to the equality

$$f(t) \equiv \int\limits_a^b d_s\mathcal{G}_0(t,s)\cdot f(s),$$

where $\mathcal{G}_0(t,s) = O_{n\times n}$ for $s \le t$ and $\mathcal{G}_0(t,s) = I_n$ for $s > t$, it follows that

$$x(t) \equiv x_0(t) + \int\limits_a^b d_s\mathcal{G}_*(t,s)\cdot f(s), \tag{1.1.63}$$

where

$$\mathcal{G}_*(t,s) \equiv \mathcal{G}_0(t,s) + \mathcal{G}_A(t,s) - Y(t)D^{-1}(\mathcal{L}(s) - \mathcal{L}(a)).$$

Now, using equalities (1.1.23) and (1.1.24), it is not difficult to verify that

$$\mathcal{G}_*(t,s) \equiv I_n - Y(t) + \mathcal{G}(t,s).$$

Moreover,

$$\int\limits_a^b d_s\mathcal{G}_*(t,s)\cdot f(s) = \int\limits_a^b d_s\mathcal{G}(t,s)\cdot f(s).$$

Therefore, due to (1.1.63), equality (1.1.25) holds. □

Remark 1.1.2 is evident, since problem $(1.1.1_0),(1.1.2_0)$ has a nontrivial solution if and only if

$$\det(\ell(Y))=0. \tag{1.1.64}$$

Hence, for every $f\in \mathrm{BV}([a,b];\mathbb{R}^n)$, there exists $c_0\in\mathbb{R}^n$ such that the system

$$\ell(Y)c=c_0-\ell(\mathcal{B}(f)),$$

where the operator $\mathcal{B}(f)$ is defined by (1.1.52) for $B(t)\equiv A(t)$, is not solvable and, therefore, problem (1.1.1), (1.1.2) is not solvable, too.

Let us check Remark 1.1.3. In view of (1.1.64), there exists $c_1\in\mathbb{R}^n$ such that the system

$$\ell(Y)c=c_1 \tag{1.1.65}$$

has no solution. Let

$$f(t)\equiv\varphi(t)-\varphi(a)-\int\limits_a^t dA(\tau)\cdot\varphi(\tau),$$

where $\varphi\in\mathrm{BV}([a,b];\mathbb{R}^n)$ is such that

$$\ell(\varphi)=c_0-c_1.$$

If we assume that problem (1.1.1), (1.1.2) has a solution x_* for the above-defined vector-function f, then the vector-function $x(t)=x_*(t)-\varphi(t)$ will be a solution of the homogeneous system $(1.1.1_0)$ under the condition $\ell(x)=c_1$. Consequently, we have $x(t)=Y(t)c$, where $c\in\mathbb{R}^n$ is a constant vector satisfying system (1.1.65). But the system is unsolvable. The obtained contradiction proofs the remark.

By condition (1.1.53), Remark 1.1.4 is evident, since, in the case, the fundamental matrix $Y(t)$, $Y(a)=I_n$, of system $(1.1.1_0)$ has the form

$$Y(t)\equiv\exp(S_0(A)(t))\prod_{a\le\tau<t}(I_n+d_2A(\tau))\prod_{a<\tau\le t}(I_n-d_1A(\tau))^{-1}.$$

Proof of Theorem 1.1.1$_1$. As above, according to the variation-of-constants formula (1.1.12) and (1.1.28), we have

$$\begin{aligned}
u(t)&=\varphi(t)+\lambda(\alpha)(t)\Big(c-\int\limits_a^t\varphi(\tau)\,d\lambda^{-1}(\alpha)(\tau)\Big)\\
&=\lambda(\alpha)(t)c+\varphi(t)-\lambda(\alpha)(t)\Big(\lambda^{-1}(\alpha)(t)\varphi(t)-\int\limits_a^t\lambda^{-1}(\alpha)(\tau)\,d\varphi(\tau)\\
&\qquad+\sum_{a<\tau\le t}d_1\lambda^{-1}(\alpha)(\tau)\cdot d_1\varphi(\tau)-\sum_{a\le\tau<t}d_2\lambda^{-1}(\alpha)(\tau)\cdot d_2\varphi(\tau)\Big)\\
&=\lambda(\alpha)(t)c+\lambda(\alpha)(t)\Big(\int\limits_a^t\lambda^{-1}(\alpha)(\tau)\,ds_0(\varphi)(\tau)\\
&\qquad+\sum_{a<\tau\le t}\lambda^{-1}(\alpha)(\tau)\cdot d_1\varphi(\tau)+\sum_{a\le\tau<t}\lambda^{-1}(\alpha)(\tau)\cdot d_2\varphi(\tau)\\
&\qquad\quad-\sum_{a<\tau\le t}d_1\lambda^{-1}(\alpha)(\tau)\cdot d_1\varphi(\tau)+\sum_{a\le\tau<t}d_2\lambda^{-1}(\alpha)(\tau)\cdot d_2\varphi(\tau)\Big)\ \text{ for }\ t\in[a,b]
\end{aligned}$$

and, therefore,

$$u(t)\equiv\lambda(t)c+\sum_{j=0}^{2}\beta_j(t) \tag{1.1.66}$$

for every solution u of equation $(1.1.26_0)$, where $c = u(0)$,

$$\beta_0(t) \equiv \lambda(\alpha)(t) \int_a^t \lambda^{-1}(\alpha)(\tau)\, ds_0(\alpha)(\tau),$$
$$\beta_1(t) \equiv \lambda(t) \sum_{a<\tau\le t} \lambda^{-1}(\alpha)(\tau-)\, d_1\varphi(\tau),$$
$$\beta_2(t) \equiv \lambda(t) \sum_{a\le\tau<t} \lambda^{-1}(\alpha)(\tau+)\, d_2\varphi(\tau),$$

and $\lambda(\alpha)(t)$ is defined by (1.1.30).

The function u satisfies condition (1.1.27) if and only if c is a solution of the equation

$$(\lambda(\alpha)(b) - 1)c = c_0 - \sum_{j=0}^{2} \beta_j(b). \tag{1.1.67}$$

But this equation and, consequently, problem (1.1.26), (1.1.27) is uniquely solvable if and only if condition (1.1.29) holds. On the other hand, this condition is necessary and sufficient for the absence of the nontrivial solution to problem $(1.1.26_0), (1.1.27_0)$.

If (1.1.29) has been fulfilled, then from (1.1.66) and (1.1.67) we obtain

$$u(t) = \frac{c_0\lambda(\alpha)(t)}{\lambda(\alpha)(b)-1} - \frac{\lambda(\alpha)(t)}{\lambda(\alpha)(b)-1} \sum_{j=0}^{2} \beta_j(b) + \sum_{j=0}^{2} \beta_j(t) \ \text{ for } \ t \in [a,b].$$

Moreover,

$$\begin{aligned}
\beta_0(t) - \frac{\lambda(\alpha)(t)}{\lambda(\alpha)(b)-1} &= \frac{\lambda(\alpha)(t)}{1-\lambda(\alpha)(b)} \Big(\lambda(\alpha)(b) \int_t^b \lambda^{-1}(\alpha)(\tau)\, ds_0(\varphi)(\tau) + \int_a^t \lambda^{-1}(\alpha)(\tau)\, ds_0(\varphi)(\tau) \Big) \\
&= \int_a^b g_0(\alpha)(t,\tau)\, ds_0(\varphi)(\tau),
\end{aligned}$$

$$\begin{aligned}
\beta_1(t) - \frac{\lambda(\alpha)(t)}{\lambda(\alpha)(b)-1}\, \beta_1(b) &= \frac{\lambda(\alpha)(t)}{1-\lambda(\alpha)(b)} \\
&\times \Big(\lambda(\alpha)(b) \sum_{t<\tau\le b} \lambda^{-1}(\alpha)(\tau-)\, d_1\varphi(\tau) + \sum_{a<\tau\le t} \lambda^{-1}(\alpha)(\tau-)\, d_1\varphi(\tau) \Big) \\
&= \sum_{a<\tau\le b} g_1(\alpha)(t,\tau)\, d_1\varphi(\tau)
\end{aligned}$$

and

$$\begin{aligned}
\beta_2(t) - \frac{\lambda(\alpha)(t)}{\lambda(\alpha)(b)-1}\, \beta_2(b) &= \frac{\lambda(\alpha)(t)}{1-\lambda(\alpha)(b)} \\
\times \Big(\lambda(\alpha)(b) \sum_{t\le\tau<b} \lambda^{-1}(\alpha)(\tau+)\, d_2\varphi(\tau) &+ \sum_{a\le\tau<t} \lambda^{-1}(\alpha)(\tau+)\, d_2\varphi(\tau) \Big) = \sum_{a\le\tau<b} g_2(\alpha)(t,\tau)\, d_2\varphi(\tau)
\end{aligned}$$

for $t \in [a,b]$, since

$$\lambda^{-1}(\alpha)(\tau-) = \lambda^{-1}(\alpha)(\tau)(1 - d_1\alpha(\tau))^{-1}$$
$$\lambda^{-1}(\alpha)(\tau+) = \lambda^{-1}(\alpha)(\tau)(1 + d_2\alpha(\tau))^{-1}$$

and, consequently, (1.1.34) holds, where

$$u_0(t) = \frac{c_0\lambda(\alpha)(t)}{\lambda(\alpha)(b)-1}\,. \qquad \square$$

Proof of Theorem 1.1.2. Let $l=1$.

We introduce the following sequence of operators: $p_i:\mathrm{BV}([a,b];\mathbb{R}^{n\times n})\to\mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ ($i=0,1,\dots$):

$$\begin{aligned}&p_0(X)(t)\equiv X(t),\\&p_i(X)(t)\equiv (I_n-d_1A(t))^{-1}\int\limits_a^t dA(\tau-)\cdot p_{i-1}(X)(\tau)\ \ (i=1,2,\dots).\end{aligned}\tag{1.1.68}$$

To prove the theorem, we have to show that the conditions of the theorem are necessary and sufficient for the absence of nontrivial solutions to the homogeneous problem $(1.1.1_0)$, $(1.1.2_0)$.

Let us show the sufficiency. Let $x=(x_i)_{i=1}^n$ be an arbitrary solution of the homogeneous problem $(1.1.1_0)$, $(1.1.2_0)$. Then

$$x(t)=c+\int\limits_a^t dA(\tau)\cdot x(\tau)\ \text{ for }\ t\in[a,b],\tag{1.1.69}$$

where $c=x(a)$. This, by (0.0.6), (1.1.8) and (1.1.68), yields

$$x(t)=c+\int\limits_a^t dA(\tau-)\cdot x(\tau)+d_1A(t)\cdot x(t)$$

and

$$\begin{aligned}x(t)&=(I_n-d_1A(t))^{-1}c+(I_n-d_1A(t))^{-1}\int\limits_a^t dA(\tau-)\cdot x(\tau)=[A]_0(t)\cdot c+p_1(x)(t)\\&=[A]_0(t)\cdot c+p_1([A]_0\cdot c+p_1(x))(t)=[A]_0(t)\cdot c+p_1([A]_0\cdot c)(t)+p_1(p_1(x))(t)=\\&=\big([A]_0(t)+[A]_1\big)(t)\cdot c+p_2(x)(t)=\big([A]_0(t)+[A]_1(t)\big)\cdot c+p_2([A]_0\cdot c+p_1(x))(t)=\\&=\big([A]_0(t)+[A]_1\big)(t)\cdot c+p_2([A]_0\cdot c)(t)+p_2(p_1(x))(t)=\\&=\big([A]_0(t)+[A]_1(t)+[A]_2(t)\big)\cdot c+p_3(x)(t)\ \text{ for }\ t\in[a,b],\end{aligned}$$

etc. Continuing this process infinitely, we obtain

$$x(t)=\Big(\sum_{i=0}^{j-1}[A]_i(t)\Big)c+p_k(x)(t)\ \text{ for }\ t\in[a,b]\tag{$1.1.67_k$}$$

for every natural number k.

According to $(1.1.35_1)$, $(1.1.37_1)$ and (1.1.68), from $(1.1.2_0)$ and $(1.1.67_k)$ we find that

$$M_kc-\int\limits_a^b dL(t)\cdot p_k(x)(t)=0.$$

Thus, in view of the fact that M_k is a nonsingular matrix, we have

$$c=M_k^{-1}\int\limits_a^b dL(t)\cdot p_k(x)(t).$$

Substituting this value of c into $(1.1.67_m)$, we get

$$x(t)=p_m(x)(t)+\Big(\sum_{i=0}^{m-1}[A]_i(t)\Big)\int\limits_a^b d(M_k^{-1}L(t))\cdot p_k(x)(t).\tag{1.1.70}$$

On the other hand, by $(1.1.37_1)$ and (1.1.68), we have

$$|p_j(x)(t)| \le V_j(A)(t)\cdot |x|_\infty \ \text{ for } \ t\in[a,b] \ \ (j=1,2,\dots).$$

From the latter inequality and from (1.1.41), due to (1.1.70), it follows that

$$|x|_\infty \le M_{k,m}|x|_\infty$$

and

$$(I_n - M_{k,m})|x|_\infty \le 0.$$

Hence, according to (1.1.40), we obtain

$$|x|_\infty \le 0.$$

Consequently, $x(t)\equiv 0$. Thus the sufficiency of conditions of the theorem is proved for the absence of nontrivial solutions to the problem $(1.1.1_0), (1.1.2_0)$.

Let us now prove the necessity. Let problem $(1.1.1_0), (1.1.2_0)$ have no nontrivial solutions. Then inequality (1.1.53) holds, where Y is an arbitrary fundamental matrix of system $(1.1.1_0)$. For definiteness, we mean that $Y(a)=I_n$.

Assume

$$Y_k(t)=\sum_{i=0}^{k-1}[A]_i(t) \ \text{ for } \ t\in[a,b] \ \ (k=1,2,\dots). \tag{1.1.71}$$

Analogously to $(1.1.67_k)$, we show that

$$Y(t)=\sum_{i=0}^{k-1}[A]_i(t)+p_k(Y)(t) \ \text{ for } \ t\in[a,b] \ \ (k=1,2,\dots). \tag{1.1.72}$$

We now estimate $\|p_k(Y)\|_\infty$. Let $r_0=\|Y\|_\infty$. It is clear that $(I_n-d_1A(t))^{-1}$ is a bounded matrix-function on $[a,b]$. Therefore,

$$r=\sup\big\{\|(I_n-d_1A(t))^{-1}\|:\ t\in[a,b]\big\}<\infty.$$

Taking into account the fact that A_- is a continuous from the left matrix-function and $V(A_-)$ is nondecreasing, by (0.0.15), we have the estimate

$$\|p_1(Y)(t)\| \le \big\|(I_n-d_1A(t))^{-1}\big\| \int_a^t \|Y(\tau)\|\,d\|V(A_-)(\tau)\| \le rr_0\|V(A_-)(t)\|,$$

$$\begin{aligned}\|p_2(Y)(t)\| &\le \big\|(I_n-d_1A(t))^{-1}\big\| \int_a^t \|p_1(Y)(\tau)\|\,d\|V(A_-)(\tau)\| \\ &\le r^2r_0^2\int_a^t \|V(A_-)(\tau)\|\,d\|V(A_-)(\tau)\| \le \frac{r_0r^2}{2!}\,\|V(A_-)(t)\|^2.\end{aligned}$$

Using the induction method, we obtain

$$\|p_k(Y)(t)\| \le \frac{r_0(r\|V(A_-)(t)\|)^k}{k!} \le \frac{r_0(r\|V(A_-)(b)\|)^k}{k!} \quad \text{for} \quad t\in[a,b] \quad (k=1,2,\dots). \tag{1.1.73}$$

According to (1.1.73), from (1.1.71) and (1.1.72) it follows that

$$\lim_{k\to\infty}\|Y_k-Y\|_\infty=0. \tag{1.1.74}$$

Moreover,

$$\|\ell(Y_k)-\ell(Y)\| \le \int_a^b \|Y_k(t)-Y(t)\|\,d\|V(L)(t)\| \le \|V(L)(b)\|\cdot\|Y_k-Y\|_\infty.$$

Therefore, by (1.1.74)), we have

$$\lim_{k\to\infty} \ell(Y_k) = \ell(Y).$$

But in view of (1.1.39) and (1.1.71),

$$\ell(Y_k) = -M_k,$$

and hence

$$\lim_{k\to\infty} M_k = -\ell(Y).$$

From the above arguments and (1.1.53), there exist a natural number k_0 and a positive number α such that

$$\det(M_k) \neq 0, \quad \|M_k^{-1}\| < \alpha \ \ (k = k_0, k_0+1, \dots). \tag{1.1.75}$$

Moreover, as above, it is easy to verify that

$$\|V_1(A)(t)\| \leq r\|V(A_-)(t)\| \ \text{ for } \ t \in [a,b],$$

$$\|V_2(A)(t)\| \leq \int_a^t \|V_1(A)(\tau)\| \, d\|V(A_-)(\tau) \leq r^2 \int_a^t \|V(A_-)(\tau)\| \, d\|V(A_-)(\tau)\|$$

$$\leq \frac{r^2}{2!} \, \|V(A_-)(t)\|^2 \ \text{ for } \ t \in [a,b],$$

and so on. Thus

$$\|V_k(A)(t)\| \leq \frac{1}{k!} \big(r\|V(A_-)(t)\|\big)^k \leq \frac{1}{k!} \big(r\|V(A_-)(b)\|\big)^k \ \text{ for } \ t \in [a,b] \ \ (k = 1,2,\dots).$$

Taking into account these estimates and (1.1.75), from (1.1.41) we get

$$\lim_{k,m\to\infty} M_{k,m} = O_{n\times n}.$$

Hence inequality (1.1.40) holds for some sufficiently large k and m. The theorem has been proved for $l = 1$.

Let now $l = 2$. For this case we define the operators p_i $(i = 0,1,\dots)$ by

$$p_0(X)(t) = X(t),$$

$$p_i(X)(t) \equiv (I_n + d_2A(t))^{-1} \int_b^t dA(\tau+) \cdot p_{i-1}(X)(\tau) \ \ (i = 1,2,\dots)$$

instead of (1.1.68).

We use the equality

$$x(t) = c + \int_b^t dA(\tau) \cdot x(\tau) \ \text{ for } \ t \in [a,b]$$

instead of (1.1.69).

Acting analogously as in proving the case $l = 1$, we can easily show that the theorem is likewise true in this case. □

Proof of Theorem 1.1.2$_1$. The proof is analogous to that of Theorem 1.1.2.

Let $l = 1$, and let $p_i : \mathrm{BV}([a,b];\mathbb{R}^{n\times n}) \to \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ $(i = 0,1,\dots)$ be the operators defined by

$$p_0(X)(t) \equiv X(t), \quad p_i(X)(t) \equiv \int_a^t dA(\tau) \cdot p_{i-1}(X)(\tau) \ \ (i = 1,2,\dots).$$

Let $x = (x_i)_{i=1}^n$ be an arbitrary solution of problem $(1.1.1_0), (1.1.2_0)$. Then, by virtue of (1.1.69),

$$x(t) = c + p_1(x)(t) = c + \int\limits_a^t dA(\tau)\cdot(c + p_1(x)(\tau)$$

$$= (I_n + (A)_1(t))c + \int\limits_a^t dA(\tau)\cdot p_1(x)(\tau) = (I_n + (A)_1(t))c + p_2(x)(t)$$

$$= (I_n+(A)_1(t))c+\int\limits_a^t dA(\tau)\cdot\int\limits_a^\tau dA(s)(c+p_1(x)(s) = (I_n+(A)_1(t)+(A)_2(t))c+p_3(x)(t) \ \text{ for } \ t\in[a,b],$$

and so on. Continuing this process infinitely, we obtain

$$x(t) = \Big(I_n + \sum_{i=0}^{j-1}(A)_i(t)\Big)c + p_j(x)(t) \ \text{ for } \ t\in[a,b] \ \ (j=1,2,\dots). \tag{1.1.76}$$

According to (1.1.42) and (1.1.43), from $(1.1.2_0)$ and (1.1.76) we can find c as above. Substituting the value of c in (1.1.76) and acting as above, we find that $x(t)\equiv 0$. The theorem has been proved for $l = 1$.

The proof of the theorem is similar for the case $l = 2$. We only note that the operators p_i $(i = 0, 1, \dots)$ are defined by

$$p_0(X)(t)\equiv X(t), \quad p_i(X)(t)\equiv \int\limits_b^t dA(\tau)\cdot p_{i-1}(X)(\tau) \ \ (i=1,2,\dots). \qquad \square$$

Proof of Corollary 1.1.3. Let $A_\varepsilon(t)\equiv \varepsilon A(t)$. It is evident that

$$\lim_{\varepsilon\to 0}\big(I_n + (-1)^j\varepsilon\, d_jA(t)\big) = I_n \ \text{ uniformly on } \ [a,b] \ \ (j=1,2).$$

Therefore, there exists $\varepsilon_1 > 0$ such that

$$\det\big(I_n + (-1)^j\, d_jA_\varepsilon(t)\big)\neq 0 \ \ (t\in[a,b], \ \ j=1,2)$$

for every $\varepsilon\in\,]0,\varepsilon_1]$.

If condition (1.1.44) holds, then we assume $k = 1$, while if conditions (1.1.45)–(1.1.47) hold, we assume $k = l+1$. Moreover, we put

$$M_k(\varepsilon) = L(a) - \sum_{i=0}^{k-1}\int\limits_a^b dL(t)\cdot(\varepsilon A)_i(t)$$

and

$$M_{k,1}(\varepsilon) = (V(\varepsilon A))_1(b) + |M_k^{-1}(\varepsilon)|\int\limits_a^b dV(L)(t)\cdot(V(\varepsilon A))_k(t).$$

In view of condition (1.1.44) (of conditions (1.1.45)–(1.1.47)), we can easily verify that

$$M_k(\varepsilon) = \varepsilon^{k-1}M_k, \quad \det(M_k)\neq 0, \quad M_{k,1}(\varepsilon) = \varepsilon M_{k,1},$$

where M_k and M_{k1} are the matrices defined by (1.1.42) and (1.1.43), respectively. Let

$$\varepsilon_0 = \min\Big\{\frac{1}{r(M_{k,1})}\,,\varepsilon_1\Big\}.$$

Then we have

$$r(M_{k,1}(\varepsilon)) < 1$$

for every $\varepsilon\in\,]0,\varepsilon_0[$. Therefore, according to Theorem $1.1.2_1$, problem (1.1.38), (1.1.2) has one and only one solution for every $\varepsilon\in\,]0,\varepsilon_0[$. $\square$

Proof of Theorem 1.1.3. It suffices for the homogeneous problem $(1.1.1_0), (1.1.2_0)$ to have only the trivial solution. Let $x=(x_i)_{i=1}^n$ be an arbitrary solution of the problem. Since problem $(1.1.48), (1.1.2_0)$ has only the trivial solution, by (1.1.17) and the equality

$$dx(t) = dA_0(t)\cdot x(t) + d\bigg(\int_a^t d(A(\tau)-A_0(\tau))\cdot x(\tau)\bigg) \text{ for } t\in[a,b],$$

we have the representation

$$x(t) = \int_a^t d_\tau \mathcal{G}_0(t,\tau)\int_a^\tau d(A(s)-A_0(s))\cdot x(s) = \int_a^t d\bigg(\int_a^s \mathcal{G}_0(t,\tau)d(A(\tau)-A_0(\tau))\bigg)x(s) \text{ for } t\in[a,b],$$

where $\mathcal{G}_0(t,\tau)$ is the Green matrix of problem $(1.1.48), (1.1.2_0)$.

Therefore, by (1.1.49),

$$|x(t)| \le \int_a^t\int_a^s |\mathcal{G}_0(t,\tau)|\, dV(A-A_0)(\tau)\cdot|x(s)| \le M|x|_\infty \text{ for } t\in[a,b].$$

Hence

$$(I_n - M)|x|_\infty \le 0.$$

From the above, owing to (1.1.50), it follows that $x(t)\equiv 0$. Consequently, problem (1.1.1), (1.1.2) has one and only one solution. □

1.2 The well-posedness of the general linear boundary value problems

1.2.1 Statement of the problem and formulation of the results

Let $A_0\in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$, $f_0\in \mathrm{BV}([a,b];\mathbb{R}^n)$. Consider the system

$$dx = dA_0(t)\cdot x + df_0(t) \text{ for } t\in[a,b] \tag{1.2.1}$$

under the boundary value condition

$$\ell_0(x) = c_0, \tag{1.2.2}$$

where $\ell_0 : \mathrm{BV}_\infty([a,b];\mathbb{R}^n)\to\mathbb{R}^n$ is a linear bounded (with respect to the norm $\|\cdot\|_\infty$) vector-functional, and $c_0\in\mathbb{R}^n$ is an arbitrary constant vector.

Let x_0 be a unique solution of problem $(1.2.1), (1.2.2)$.

Along with problem $(1.2.1), (1.2.2)$, consider the sequence of the problems

$$dx = dA_m(t)\cdot x + df_m(t), \tag{1.2.1$_m$}$$

$$\ell_m(x) = c_m \tag{1.2.2$_m$}$$

$(m=1,2,\dots)$, where $A_m\in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ $(m=1,2,\dots)$, $f_m\in\mathrm{BV}([a,b];\mathbb{R}^n)$ $(m=1,2,\dots)$, $\ell_m : \mathrm{BV}_\infty([a,b];\mathbb{R}^n)\to\mathbb{R}^n$ $(k=1,2,\dots)$ are linear bounded vector-functionals and $c_m\in\mathbb{R}^n$ $(m=1,2,\dots)$.

Let $A_m=(a_{mil})_{i,l=1}^n$ and $f_m=(f_{ml})_{l=1}^n$ $(m=0,1,\dots)$.

Moreover, as above in Section 1.1, without loss of generality we assume $A_m(a)=O_{n\times n}$ and $f_m(a)=0_n$ $(m=0,1,\dots)$.

In this section, we establish the necessary and sufficient and the effective sufficient conditions for the boundary value problem $(1.2.1_m), (1.2.2_m)$ to have a unique solution x_m for any sufficiently large m and prove that

$$\lim_{m\to+\infty} x_m(t) = x_0(t) \tag{1.2.3}$$

uniformly on $[a,b]$.

Along with problems (1.2.1), (1.2.2) and $(1.2.1_m)$, $(1.2.2_m)$, we consider the corresponding homogeneous problems

$$dx = dA_0(t)\cdot x, \tag{$1.2.1_0$}$$
$$\ell_0(x) = 0 \tag{$1.2.2_0$}$$

and

$$dx = dA_m(t)\cdot x, \tag{$1.2.1_{m0}$}$$
$$\ell_m(x) = 0 \tag{$1.2.2_{m0}$}$$

for any natural m.

Definition 1.2.1. We say that the sequence $(A_m, f_m; \ell_m)$ $(m=1,2,\dots)$ belongs to the set $\mathcal{S}(A_0, f_0; \ell_0)$ if for every $c_0\in\mathbb{R}^n$ and for a sequence $c_m\in\mathbb{R}^n$ $(m=1,2,\dots)$ satisfying the condition

$$\lim_{m\to+\infty} c_m = c_0, \tag{1.2.4}$$

problem $(1.2.1_m)$, $(1.2.2_m)$ has the unique solution x_m for any sufficiently large m, and condition (1.2.3) holds uniformly on $[a,b]$.

We assume that

$$\det\big(I_n + (-1)^j\, d_jA_0(t)\big)\neq 0 \ \text{ for } \ t\in[a,b] \ \ (j=1,2). \tag{1.2.5}$$

Theorem 1.2.1. *Let the conditions*

$$\lim_{m\to+\infty}\ell_m(x)=\ell(x) \ \text{ for } \ x\in \mathrm{BV}([a,b];\mathbb{R}^n), \tag{1.2.6}$$
$$\limsup_{m\to+\infty} |||\ell_m||| < +\infty \tag{1.2.7}$$

hold. Then the inclusion

$$\big((A_m, f_m; \ell_m)\big)_{m=1}^{+\infty}\in\mathcal{S}(A_0, f_0; t_0) \tag{1.2.8}$$

holds if and only if there exists a sequence of matrix-functions $H_0, H_m\in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ $(m=1,2,\dots)$ *such that the conditions*

$$\limsup_{m\to+\infty}\bigvee_a^b (H_m + \mathcal{B}(H_m, A_m)) < +\infty \tag{1.2.9}$$

and

$$\inf\big\{|\det(H_0(t))|:\ t\in[a,b]\big\} > 0 \tag{1.2.10}$$

hold, and the conditions

$$\lim_{m\to+\infty} H_m(t) = H_0(t), \tag{1.2.11}$$
$$\lim_{m\to+\infty}\mathcal{B}(H_m, A_m)(t) = \mathcal{B}(H_0, A_0)(t), \tag{1.2.12}$$
$$\lim_{m\to+\infty}\mathcal{B}(H_m, f_m)(t) = \mathcal{B}(H_0, f_0)(t) \tag{1.2.13}$$

hold uniformly on $[a,b]$.

Theorem 1.2.2. *Let conditions* (1.2.6), (1.2.7) *and*

$$\det\big(I_n + (-1)^j\, d_jA_m(t)\big)\neq 0 \ \text{ for } \ t\in[a,b] \ \ (m=0,1,\dots) \tag{1.2.14}$$

hold. Then inclusion (1.2.8) *holds if and only if the conditions*

$$\lim_{m\to+\infty} X_m^{-1}(t) = X_0^{-1}(t) \tag{1.2.15}$$

and

$$\lim_{m\to+\infty} \mathcal{B}(X_m^{-1}, f_m)(t) = \mathcal{B}(X_0^{-1}, f)(t)$$

hold uniformly on $[a,b]$*, where* X_m *is the fundamental matrix of the homogeneous system* $(1.2.1_{m0})$ *for every* $m \in \widetilde{\mathbb{N}}$*.*

Theorem 1.2.3. *Let* $A_0^* \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$*,* $f_0^* \in \mathrm{BV}([a,b];\mathbb{R}^n)$*,* $c_0^* \in \mathbb{R}^n$*, and a* $\ell_0^* : \mathrm{BV}_\infty([a,b];\mathbb{R}^{n\times n}) \to \mathbb{R}^n$ *be a linear bounded vector-functional such that*

$$\det\big(I_n + (-1)^j\, d_j A_0^*(t)\big) \neq 0 \ \ \textit{for} \ \ t \in [a,b] \tag{1.2.16}$$

and the boundary value problem

$$dx = dA_0^*(t)\cdot x + df_0^*(t), \tag{1.2.1*}$$

$$\ell_0^*(x) = c_0^* \tag{1.2.2*}$$

has a unique solution x_0^**. Let, moreover, there exist the sequences of matrix- and vector-functions* $H_m \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ $(m=1,2,\dots)$ *and* $h_m \in \mathrm{BV}([a,b];\mathbb{R}^n)$ $(m=1,2,\dots)$ *such that*

$$\inf\big\{|\det(H_m(t))| : \ t \in [a,b]\big\} > 0 \ \ \textit{for every sufficiently large } m, \tag{1.2.17}$$

and for the sequences

$$\ell_m^*(y) = \ell_m(H_m^{-1}y) \ \ (m=1,2,\dots), \quad A_m^*(t) \equiv \mathcal{I}(H_m, A_m)(t) \ \ (m=1,2,\dots),$$

$$f_m^*(t) \equiv h_m(t) - h_m(a) + \mathcal{B}(H_m, f_m)(t) - \int_a^t dA_m^*(s)\cdot h_m(s) \ \ (m=1,2,\dots)$$

the conditions

$$\lim_{m\to+\infty} \ell_m^*(y) = \ell_0^*(y) \ \ \textit{for} \ \ y \in \mathrm{BV}([a,b];\mathbb{R}^n), \tag{1.2.18}$$

$$\limsup_{m\to+\infty} |||\ell_m^*||| < +\infty, \tag{1.2.19}$$

$$\lim_{m\to+\infty} (c_m + \ell_m^*(h_m)) = c_0^*, \tag{1.2.20}$$

$$\limsup_{m\to+\infty} \bigvee_a^b (A_m^*) < +\infty \tag{1.2.21}$$

hold and the conditions

$$\lim_{m\to+\infty} A_m^*(t) = A_0^*(t), \tag{1.2.22}$$

$$\lim_{m\to+\infty} f_m^*(t) = f_0^*(t) \tag{1.2.23}$$

hold uniformly on $[a,b]$*. Then problem* $(1.2.1_m), (1.2.2_m)$ *has the unique solution* x_m *for any sufficiently large* m *and*

$$\lim_{m\to+\infty} \big(H_m(t)x_m(t) + h_m(t)\big) = x_0^*(t) \tag{1.2.24}$$

uniformly on $[a,b]$*.*

Remark 1.2.1. In Theorem 1.2.3, the vector-function $y_m(t) \equiv H_m(t)\,x_m(t) + h_m(t)$ is a solution of the problem

$$dy = dA_m^*(t)\cdot y + df_m^*(t), \tag{1.2.1$_m^*$}$$

$$\ell_m^*(y) = c_m^* \tag{1.2.2$_m^*$}$$

for every sufficiently large m, where $c_m^* = c_m + \ell_m^*(h_m)$.

Corollary 1.2.1. *Let conditions* (1.2.6), (1.2.7), (1.2.9), (1.2.10) *and*

$$\lim_{m\to+\infty}(c_m - \varphi_k(a)) = c_0 \tag{1.2.25}$$

hold, and conditions (1.2.11), (1.2.12) *and*

$$\lim_{m\to+\infty}\bigg(\mathcal{B}(H_m, f_m-\varphi_m)(t) + \int\limits_a^t d\mathcal{B}(H_m, A_m)(\tau)\cdot\varphi_m(\tau)\bigg) = \mathcal{B}(H_0, f_0)(t) \tag{1.2.26}$$

hold uniformly on $[a,b]$, *where* $H_m \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ $(m=0,1,\dots)$, $\varphi_m \in \mathrm{BV}([a,b];\mathbb{R}^n)$ $(m=1,2,\dots)$. *Then for any sufficiently large* m, *problem* $(1.2.1_m), (1.2.2_m)$ *has the unique solution* x_m *and*

$$\lim_{m\to+\infty}(x_m(t) - \varphi_m(t)) = x_0(t) \tag{1.2.27}$$

uniformly on $[a,b]$.

Theorem 1.2.4. *Let conditions* (1.2.4)–(1.2.7) *and*

$$\limsup_{m\to+\infty}\bigvee_a^b(A_m) < +\infty$$

hold, and the conditions

$$\lim_{m\to+\infty} A_m(t) = A_0(t), \tag{1.2.28}$$

$$\lim_{m\to+\infty} f_m(t) = f_0(t) \tag{1.2.29}$$

hold uniformly on $[a,b]$. *Then the boundary value problem* $(1.2.1_m), (1.2.2_m)$ *has the unique solution* x_m *for any sufficiently large* m *and condition* (1.2.3) *holds uniformly on* $[a,b]$.

Corollary 1.2.2. *Let conditions* (1.2.6), (1.2.7), (1.2.9) *and* (1.2.10) *hold, and conditions* (1.2.11),

$$\lim_{m\to+\infty}\int\limits_a^t H_m(s)\,dA_m(s) = \int\limits_a^t H_0(s)\,dA_0(s), \tag{1.2.30}$$

$$\lim_{m\to+\infty}\int\limits_a^t H_m(s)\,df_m(s) = \int\limits_a^t H_0(s)\,df_0(s), \tag{1.2.31}$$

$$\lim_{m\to+\infty} d_jA_m(t) = d_jA_0(t) \ \ (j=1,2), \tag{1.2.32}$$

$$\lim_{m\to+\infty} d_jf_m(t) = d_jf_0(t) \ \ (j=1,2) \tag{1.2.33}$$

hold uniformly on $[a,b]$, *where* $H_m \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ $(m=0,1,\dots)$. *Let, moreover, either*

$$\limsup_{m\to+\infty}\sum_{t\in[a,b]}\big(\|d_jA_m(t)\| + \|d_jf_m(t)\|\big) < +\infty \ \ (j=1,2), \tag{1.2.34}$$

or

$$\limsup_{m\to+\infty}\sum_{t\in[a,b]}\|d_jH_m(t)\| < +\infty \ \ (j=1,2). \tag{1.2.35}$$

Then inclusion (1.2.8) *holds.*

Corollary 1.2.3. *Let conditions* (1.2.6), (1.2.7) *and* (1.2.9) *hold, and conditions* (1.2.11), (1.2.28), (1.2.29),

$$\lim_{m\to+\infty}\int_a^t dH_m(s)\cdot A_m(s)=A_*(t)-A_*(a) \tag{1.2.36}$$

and

$$\lim_{m\to+\infty}\int_a^t dH_m(s)\cdot f_m(s)=f_*(t)-f_*(a) \tag{1.2.37}$$

hold uniformly on $[a,b]$, *where* $H_0(t)\equiv I_n$, $H_m\in\mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ $(m=1,2,\dots)$, $A_*\in\mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ *and* $f_*\in\mathrm{BV}([a,b];\mathbb{R}^n)$. *Let, moreover, the system*

$$dx=d(A_0(t)-A_*(t))\cdot x+d(f_0(t)-f_*(t))$$

has a unique solution satisfying condition (1.2.2). *Then*

$$\big((A_m,f_m;\ell_m)\big)_{m=1}^{+\infty}\in\mathcal{S}(A_0-A_*,f_0-f_*;\ell_0).$$

Corollary 1.2.4. *Let conditions* (1.2.6) *and* (1.2.7) *hold and let there exist a natural number* μ *and matrix-functions* $B_j\in\mathrm{BV}_{loc}([a,b];\mathbb{R}^{n\times n})$ $(j=0,\dots,\mu-1)$ *such that*

$$\limsup_{m\to+\infty}\bigvee_a^b(A_{m\mu})<+\infty \tag{1.2.38}$$

and the conditions

$$\lim_{m\to+\infty}(A_{mj}(t)-A_{mj}(a))=B_j(t)-B_j(a)\ \ (j=0,\dots,\mu-1), \tag{1.2.39}$$

$$\lim_{m\to+\infty}(A_{m\mu}(t)-A_{m\mu}(a))=A_0(t), \tag{1.2.40}$$

$$\lim_{m\to+\infty}f_{m\mu}(t)=f_0(t) \tag{1.2.41}$$

hold uniformly on $[a,b]$, *where*

$$A_{m0}\equiv A_m(t),\ \ f_{m0}(t)\equiv f_m(t)\ \ (m=1,2,\dots),$$
$$A_{mj}(t)\equiv H_{m\,j-1}(t)+\mathcal{B}(H_{m\,j-1},A_m)(t),\ \ f_{mj}(t)\equiv\mathcal{B}(H_{m\,j-1},f_m)(t)\ \ (j=1,\dots,\mu;\ m=1,2,\dots);$$
$$H_{m0}(t)\equiv I_n,\ \ H_{mj}(t)\equiv\big(I_n-A_{mj}(t)+A_{mj}(a)+B_j(t)-B_j(a)\big)H_{m\,j-1}(t)$$
$$(j=1,\dots,\mu-1;\ m=1,2,\dots).$$

Then inclusion (1.2.8) *holds.*

If $\mu=1$, then Corollary 1.2.4 coincides with Theorem 1.2.4.
If $\mu=2$, then Corollary 1.2.4 has the following form.

Corollary 1.2.4$_1$. *Let conditions* (1.2.6), (1.2.7) *and* (1.2.9) *hold, and the conditions*

$$\lim_{m\to+\infty}A_m(t)=B(t)-B(a),$$
$$\lim_{m\to+\infty}\mathcal{B}(H_m,A_m)(t)=A_0(t),$$
$$\lim_{m\to+\infty}\mathcal{B}(H_m,f_m)(t)=f_0(t)$$

hold uniformly on $[a,b]$, *where* $B\in\mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ *and*

$$H_m(t)\equiv I_n-A_m(t)+B(t)-B(a)\ \ (m=1,2,\dots).$$

Then inclusion (1.2.8) *holds.*

If in Corollary $1.2.4_1$ we choose $B(t) \equiv A_0(t)$, then the corollary has the following simple form.

Corollary $1.2.4_2$. *Let conditions* (1.2.6), (1.2.7) *and*

$$\limsup_{m\to+\infty} \bigvee_a^b \big(A_m - \mathcal{B}(A_m - A_0, A_m)\big) < +\infty$$

hold, and the conditions

$$\lim_{m\to+\infty} A_m(t) = A_0(t),$$

$$\lim_{m\to+\infty} \int_a^t d(A_m(\tau) - A_0(\tau)) \cdot A_m(\tau) = O_{n\times n},$$

$$\lim_{m\to+\infty} \big(f_m(t) - \mathcal{B}(A_m - A_0, f_m)(t)\big) = f_0(t)$$

hold uniformly on $[a, b]$. *Then inclusion* (1.2.8) *holds.*

Remark 1.2.2. in particular, in the above corollary, the last limit condition holds if

$$\lim_{m\to+\infty} f_m(t) = f_0(t) \ \text{ and } \ \lim_{m\to+\infty} \int_a^t d(A_m(\tau) - A_0(\tau)) \cdot f_m(\tau) = 0_n$$

uniformly on $[a, b]$.

Corollary 1.2.5. *Let conditions* (1.2.6) *and* (1.2.7) *hold. Then inclusion* (1.2.8) *holds if and only if there exist a sequence of matrix-functions* $B_m \in \mathrm{BV}([a, b]; \mathbb{R}^{n\times n})$ $(m = 0, 1, \dots)$ *such that*

$$\limsup_{m\to+\infty} \bigvee_a^b (A_m - B_m) < +\infty \tag{1.2.42}$$

and

$$\det\big(I_n + (-1)^j\, d_j B_m(t)\big) \neq 0 \ \textit{ for } \ t \in [a, b] \ \ (j = 1, 2;\ m = 0, 1, \dots), \tag{1.2.43}$$

and the conditions

$$\lim_{m\to+\infty} Z_m^{-1}(t) = Z_0^{-1}(t), \tag{1.2.44}$$

$$\lim_{m\to+\infty} \mathcal{B}(Z_m^{-1}, A_m)(t) = \mathcal{B}(Z_0^{-1}, A_0)(t) \tag{1.2.45}$$

and

$$\lim_{m\to+\infty} \mathcal{B}(Z_m^{-1}, f_m)(t) = \mathcal{B}(Z_0^{-1}, f_0)(t) \tag{1.2.46}$$

hold uniformly on $[a, b]$, *where* Z_m $(Z_m(a) = I_n)$ *is a fundamental matrix of the homogeneous system*

$$dx = dB_m(t) \cdot x \tag{1.2.47}$$

for every $m \in \widetilde{\mathbb{N}}$.

Corollary 1.2.6. *Let conditions* (1.2.6) *and* (1.2.7) *hold and let there exist a sequence of matrix-functions* $B_m \in \mathrm{BV}([a, b]; \mathbb{R}^{n\times n})$ $(m = 0, 1, \dots)$ *satisfying the Lappo–Danilevskiĭ condition at the point* a *such that the conditions*

$$\limsup_{m\to+\infty} \bigg(\bigvee_a^b (A_m - S_c(B_m)) + \bigvee_a^b (S_1(B_m)) + \bigvee_a^b (S_2(B_m))\bigg) < +\infty \tag{1.2.48}$$

and

$$\det\big(I_n + (-1)^j\, d_j B_0(t)\big) \neq 0 \ \ for \ \ t \in [a,b] \ \ (j=1,2) \tag{1.2.49}$$

hold, and the conditions

$$\lim_{m\to+\infty}\big(S_c(B_m(t)) - S_c(B_m(a))\big) = S_c(B_0)(t) - S_c(B_0)(a), \tag{1.2.50}$$

$$\lim_{m\to+\infty} S_j(B_m)(t) = S_j(B_0)(t) \ \ (j=1,2), \tag{1.2.51}$$

$$\lim_{m\to+\infty}\int_a^t Z_m^{-1}(\tau)\,d\mathcal{A}(B_m,A_m)(\tau) = \int_{t_0}^t Z_0^{-1}(\tau)\,d\mathcal{A}(B_0,A_0)(\tau) \tag{1.2.52}$$

and

$$\lim_{m\to+\infty}\int_a^t Z_m^{-1}(\tau)\,d\mathcal{A}(B_m,f_m)(\tau) = \int_{t_0}^t Z_0^{-1}(\tau)\,d\mathcal{A}(B_0,f_0)(\tau) \tag{1.2.53}$$

hold uniformly on $[a,b]$*, where* $\mathcal{A}$ *is the operator defined by* (0.0.2)*, and* Z_m $(Z_m(a) = I_n)$ *is a fundamental matrix of the homogeneous system* (1.2.47) *for any sufficiently large* m*. Then inclusion* (1.2.8) *holds.*

Remark 1.2.3. In Corollary 1.2.6, due to (1.2.51) and (1.2.49), without loss of generality, we can assume that condition (1.2.43) holds for every natural m and, therefore, the fundamental matrices Z_m $(m = 0,1,\dots)$ exist. Hence conditions (1.2.52) and (1.2.53) of the corollary are correct.

Remark 1.2.4. In Corollaries 1.2.5 and 1.2.6, if we assume that the matrix functions B_m $(m = 0,1,\dots)$ are continuous, then conditions (1.2.43) and (1.2.49) are, obviously, valid. Moreover, due to the integration-by-parts formula and definitions of operators $\mathcal{A}$ and $\mathcal{B}$, each of conditions (1.2.45) and (1.2.52) has the form

$$\lim_{m\to+\infty}\int_a^t Z_m^{-1}(\tau)\,dA_m(\tau) = \int_a^t Z_0^{-1}(\tau)\,dA_0(\tau),$$

and each of conditions (1.2.46) and (1.2.53) has the form

$$\lim_{m\to+\infty}\int_a^t Z_m^{-1}(\tau)\,df_m(\tau) = \int_{t_0}^t Z_0^{-1}(\tau)\,df_0(\tau).$$

Remark 1.2.5. If the matrix-function $B \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$ satisfies the Lappo–Danilevskiĭ condition at the point $s \in I$ and $\det(I_n + (-1)^j\, d_j B(t)) \neq 0$ for $t \in I$, $(-1)^j(t-s) < 0$ $(j=1,2)$, then the fundamental matrix Z $(Z(s) = I_n)$ of the homogeneous system

$$dx = dB(t)\cdot x$$

has the form (see [39, 44, 71])

$$Z(t) = \begin{cases} \exp\big(S_c(B)(t) - S_c(B)(s)\big)\displaystyle\prod_{s<\tau\le t}(I_n - d_1B(\tau))^{-1}\prod_{s\le\tau<t}(I_n + d_2B(\tau)) & \text{for } \ t > s,\\ \exp\big(S_c(B)(s) - S_c(B)(t)\big)\displaystyle\prod_{t<\tau\le s}(I_n - d_1B(\tau))\prod_{t\le\tau<s}(I_n + d_2B(\tau))^{-1} & \text{for } \ t < s,\\ I_n & \text{for } \ t = s. \end{cases} \tag{1.2.54}$$

In that paper there is given the form of the fundamental matrix of considered system in the general case, as well, i.e., when the matrix-function B does not satisfy the Lappo–Danilevskiĭ condition. We could get similar results for the general case, but they would be very laborious.

In particular, from Corollary 1.2.6 follows the following

Corollary 1.2.7. *Let conditions* (1.2.6) *and* (1.2.7) *hold and let there exist a sequence of continuous matrix-functions* $B_m \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ $(m=0,1,\dots)$ *satisfying the Lappo–Danilevskiĭ condition at the point a such that condition* (1.2.42) *hold, and the conditions*

$$\lim_{m\to+\infty}(B_m(t)) - B_m(a)) = B_0(t) - B_0(a),$$

$$\lim_{m\to+\infty}\int\limits_a^t \exp(-B_m(\tau)+B_m(a))\,dA_m(\tau) = \int\limits_a^t \exp(-B_0(\tau)+B_0(a))\,dA_0(\tau)$$

and

$$\lim_{m\to+\infty}\int\limits_a^t \exp(-B_m(\tau)+B_m(a))\,df_m(\tau) = \int\limits_a^t \exp(-B_0(\tau)+B_0(a))\,df_0(\tau)$$

hold uniformly on $[a,b]$*. Then inclusion* (1.2.8) *holds.*

Corollary 1.2.8. *Let conditions* (1.2.6), (1.2.7) *and*

$$\limsup_{m\to+\infty}\sum_{t\in[a,b]}\|d_jA_m(t)\| < +\infty \quad (j=1,2) \tag{1.2.55}$$

hold. Let, moreover, the matrix-functions $S_c(A_m)$ $(m=0,1,\dots)$ *satisfy the Lappo–Danilevskiĭ condition at the point a and the conditions*

$$\lim_{m\to+\infty} S_c(A_m)(t) = S_c(A_0)(t), \tag{1.2.56}$$

$$\lim_{m\to+\infty} S_j(A_m)(t) = S_j(A_0)(t) \quad (j=1,2), \tag{1.2.57}$$

$$\lim_{m\to+\infty}\int\limits_a^t \exp(-S_c(A_m)(\tau))\,dA_m(\tau) = \int\limits_a^t \exp(-S_c(A_0)(\tau))\,dA_0(\tau) \tag{1.2.58}$$

and

$$\lim_{m\to+\infty}\int\limits_a^t \exp(-S_c(A_m)(\tau))\,df_m(\tau) = \int\limits_a^t \exp(-S_c(A_0)(\tau))\,df_0(\tau) \tag{1.2.59}$$

hold uniformly on $[a,b]$*. Then inclusion* (1.2.8) *holds.*

Corollary 1.2.9. *Let conditions* (1.2.6), (1.2.7),

$$\limsup_{m\to+\infty}\sum_{i,l=1;\,i\neq l}^{n}\bigvee_a^b(a_{mil}) < +\infty$$

and

$$1+(-1)^j\,d_ja_{0ii}(t)\neq 0 \ \ \textit{for} \ \ t\in[a,b] \quad (j=1,2;\ i=1,\dots,n)$$

hold, and the conditions

$$\lim_{m\to+\infty} a_{mii}(t) = a_{0ii}(t) \quad (i=1,\dots,n),$$

$$\lim_{m\to+\infty}\int\limits_a^t z_{mii}^{-1}(\tau)\,d\mathcal{A}(a_{mii},a_{mil})(\tau) = \int\limits_a^t z_{0ii}^{-1}(\tau)\,d\mathcal{A}(a_{0ii},a_{0il})(\tau) \quad (i\neq l;\ i,l=1,\dots,n)$$

and

$$\lim_{m\to+\infty}\int\limits_a^t z_{mii}^{-1}(\tau)\,d\mathcal{A}(a_{mii},f_{mi})(\tau)=\int\limits_a^t z_{0ii}^{-1}(\tau)\,d\mathcal{A}(a_{0ii},f_{0i})(\tau)\ \ (i=1,\dots,n)$$

hold uniformly on $[a,b]$*, where* $\mathcal{A}$ *is the operator defined by* (0.0.2)*, and* z_{zii} *defined according to* (1.2.54) *is a solution of the initial problem*

$$dz(t)=z(t)\,da_{mii}(t),\ \ z(a)=1\ \ (i=1,\dots,n)$$

for any sufficiently large m*. Then inclusion* (1.2.8) *holds.*

Remark 1.2.6. For Corollary 1.2.8, the remark analogous to Remark 1.2.3 is true, i.e.,

$$1+(-1)^j\,d_ja_{mii}(t)\neq 0\ \text{ for }\ t\in[a,b]\ \ (j=1,2;\ i=1,\dots,n)$$

for every sufficiently large m and, therefore, all conditions of the corollary are correct.

Remark 1.2.7. In theorems and corollaries given above, as well as in the statements below, we may, without loss of generality, assume that $H_0(t)\equiv I_n$. In this case, it is evident that

$$\mathcal{B}(H_0,Y)(t)=\mathcal{I}(H_0,Y)(t)\equiv Y(t)-Y(a)\ \text{ for }\ Y\in \mathrm{BV}([a,b];\mathbb{R}^{n\times n}).$$

Remark 1.2.8. If for some m the matrix-function A_m is such that $A_m(t)=\mathit{const}$ for $t\in I_0$, where $I_0\subset[a,b]$ is an interval, then, due to the proof of the necessity in Theorem 1.2.1, we conclude that $H_m(t)=\mathit{const}$ for $t\in I_0$, as well, since $H_m(t)=X_m^{-1}(t)$, where X_m is the fundamental matrix of the homogeneous system ($1.2.1_{m0}$). Therefore, $X_m(t)=\mathit{const}$ for $t\in I_0$. So, everywhere in the results given above we can assume that the matrix-function H_m has the described property.

Remark 1.2.9. The following example shows that if condition (1.2.34) is violated, then the statement of Corollary 1.2.2 is not true, in general.

Example 1.2.1. Let $a=0$, $b=1$, $n=1$, $A_0(t)\equiv 0$, $f_0(t)\equiv 0$, $\ell_m(x)=x(0)$ $(m=0,1,\dots)$,

$$A_m(t)=\begin{cases} m^{-1} & \text{for }\ t\in\bigcup\limits_{i=1}^{2m^2}\,]t_{2i-1\,m},t_{2i\,m}],\\[2mm] 0 & \text{for }\ t\notin\bigcup\limits_{i=1}^{2m^2}\,]t_{2i-1\,m},t_{2i\,m}],\end{cases}$$

where $t_{im}=(2m^2+1)^{-1}\,i$ $(i=0,\dots,2m^2)$ for every natural m. Then all conditions of Corollary 1.2.2 are fulfilled, except (1.2.34). It is evident that $x_0(t)\equiv 1$. On the other hand, the initial problem ($1.2.1_m$), ($1.2.2_m$) has the unique solution x_m and, in addition, $x_m(1)=(1-\frac{1}{m^2})^{m^2}$. Therefore, condition (1.2.3) is not valid, since

$$\lim_{m\to+\infty}x_m(1)=\exp(-1)\neq x_0(1).$$

The examples concerning the importance of some conditions given in the above statements can be found in [23] (Examples 3.1.1, 3.1.2, 3.1.3).

1.2.2 Auxiliary propositions

Lemma 1.2.1. *The following statements are true:*

(a) *if* $X \in \mathrm{BV}([a,b];\mathbb{R}^{n\times m})$, $Y \in \mathrm{BV}([a,b];\mathbb{R}^{m\times l})$ *and* $Z \in \mathrm{BV}([a,b];\mathbb{R}^{l\times k})$, *then*

$$\mathcal{B}(X,\mathcal{B}(Y,Z))(t) = \mathcal{B}(X\,Y,Z)(t) \ \ \text{for } \ t \in [a,b] \tag{1.2.60}$$

and

$$\mathcal{B}\Big(X, \int_a^{\cdot} dY(s)\cdot Z(s)\Big)(t) = \int_a^t d\mathcal{B}(X,Y)(s)\cdot Z(s) \ \ \text{for } \ t \in [a,b]; \tag{1.2.61}$$

(b) *if* $X \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$, $Y \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ *and* $Z \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$, *then*

$$\mathcal{I}(X,\mathcal{I}(Y,Z))(t) = \mathcal{I}(X\,Y,Z)(t) \ \ \text{for } \ t \in [a,b], \tag{1.2.62}$$

where the operators $\mathcal{B}$ *and* $\mathcal{I}$ *are defined by* (0.0.3) *and* (0.0.4), *respectively.*

Proof. Let us show that (1.2.60) is valid. According to equalities (0.0.10)–(0.0.14), we have

$$\begin{aligned}
\mathcal{B}(X,\mathcal{B}(Y,Z))(t) &= X(t)\mathcal{B}(Y,Z)(t) - \int_a^t dX(s)\cdot \mathcal{B}(Y,Z)(s)\\
&= X(t)\cdot\Big(Y(t)Z(t) - Y(a)Z(a) - \int_a^t dY(s)\cdot Z(s)\Big)\\
&\qquad - \int_a^t dX(s)\cdot\Big(Y(s)Z(s) - Y(a)Z(a) - \int_a^s dY(\tau)\cdot Z(\tau)\Big)\\
&= X(t)Y(t)Z(t) - X(a)Y(a)Z(a) - X(t)\int_a^t dY(s)\cdot Z(s)\\
&\qquad - \int_a^t dX(s)\cdot Y(s)Z(s) + \int_a^t dX(s)\cdot\int_a^s dY(\tau)\cdot Z(\tau)\\
&= X(t)Y(t)Z(t) - X(a)Y(a)Z(a) - \int_a^t dX(s)\cdot Y(s)Z(s)\\
&\quad - \int_a^t X(s)\,dY(s)\cdot Z(s) + \sum_{a<s\le t} d_1X(s)\cdot d_1Y(s)\cdot Z(s) - \sum_{a\le t<s} d_2X(s)\cdot d_2Y(s)\cdot Z(s)\\
&= X(t)Y(t)Z(t) - X(a)Y(a)Z(a) - \int_a^t d\Big(\int_a^s dX(\tau)\cdot Y(\tau)\\
&\quad + \int_a^s X(\tau)\,dY(\tau) - \sum_{a<\tau\le s} d_1X(\tau)\cdot d_1Y(\tau) + \sum_{a\le\tau<s} d_2X(\tau)\cdot d_2Y(\tau)\Big)\cdot Z(s)\\
&= X(t)Y(t)Z(t) - X(a)Y(a)Z(a) - \int_a^t d(X(s)Y(s))\cdot Z(s) = \mathcal{B}(XY,Z)(t).
\end{aligned}$$

Let us verify (1.2.61). By (0.0.13) and (1.2.60), it can be easily shown that

$$\mathcal{B}\Big(X,\int_a^{\cdot} dY(s)\cdot Z(s)\Big)(t)=\mathcal{B}\big(X,YZ-\mathcal{B}(Y,Z)\big)(t)=\mathcal{B}(X,YZ)(t)-\mathcal{B}(XY,Z)(t)$$
$$=\int_a^t d(X(s)Y(s))\cdot Z(s)-\int_a^t dX(s)\cdot Y(s)Z(s)=\int_a^t d\mathcal{B}(X,Y)(s)\cdot Z(s).$$

Finally, using (0.0.13), (1.2.60) and (1.2.61), we have

$$\mathcal{I}(X,\mathcal{I}(Y,Z))(t)=\int_a^t d\big[X(\tau)+\mathcal{B}(X,\mathcal{I}(Y,Z))(\tau)\big]\cdot X^{-1}(\tau)$$
$$=\int_a^t d\bigg(X(\tau)+\mathcal{B}\Big(X,\int_a^{\cdot} d\big[Y(s)+\mathcal{B}(Y,Z)(s)\big]\cdot Y^{-1}(s)\Big)(\tau)\bigg)\cdot X^{-1}(\tau)$$
$$=\int_a^t d\Big(X(\tau)+\int_a^{\tau} d\mathcal{B}(X,Y+\mathcal{B}(Y,Z))(s)\cdot Y^{-1}(s)\Big)\cdot X^{-1}(\tau)$$
$$=\int_a^t d\Big(X(\tau)+\int_a^{\tau} d\mathcal{B}(X,Y)(s)\cdot Y^{-1}(s)+\int_a^{\tau} d\mathcal{B}(X,\mathcal{B}(Y,Z))(s)\cdot Y^{-1}(s)\Big)\cdot X^{-1}(\tau)$$
$$=\int_a^t d\Big(X(\tau)+\int_a^{\tau} d\Big(X(s)Y(s)-\int_a^{s} dX(\sigma)\cdot Y(\sigma)\Big)\cdot Y^{-1}(s)+\int_a^{\tau} d\mathcal{B}(XY,Z)(s)\cdot Y^{-1}(s)\Big)\cdot X^{-1}(\tau)$$
$$=\int_a^t d\Big(\int_a^{\tau} d(X(s)Y(s))\cdot Y^{-1}(s)+\int_a^{\tau} d\mathcal{B}(XY,Z)(s)\cdot Y^{-1}(s)\Big)\cdot X^{-1}(\tau)$$
$$=\int_a^t d\big(X(\tau)Y(\tau)+\mathcal{B}(XY,Z)(\tau)\big)\cdot Y^{-1}(\tau)X^{-1}(\tau)=\mathcal{I}(XY,Z)(t).\qquad\square$$

Lemma 1.2.2. *Let $h\in\mathrm{BV}([a,b];\mathbb{R}^n)$, and $H\in\mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ be a nonsingular matrix-function. Then the mapping*

$$x\to y=Hx+h$$

establishes a one-to-one correspondence between the solutions x and y of systems (1.1.1) *and*

$$dy=dA_*(t)\cdot y+df_*(t),\tag{1.2.63}$$

where the matrix- and vector-functions A_* and f_* are defined, respectively, by

$$A_*(t)\equiv\mathcal{I}(H,A)(t)\ \text{ and }\ f_*(t)\equiv h(t)-h(a)+\mathcal{B}(H,f)(t)-\int_a^t dA_*(s)\cdot h_k(s).$$

Besides,

$$I_n+(-1)^j\,d_jA_*(t)\equiv(H(t)+(-1)^j\,d_jH(t))(I_n+(-1)^j\,d_jA(t))H^{-1}(t)\ \ (j=1,2).\tag{1.2.64}$$

Proof. Let x be a solution of system (1.1.1) and let $y(t)\equiv H(t)x(t)+h(t)$. Due to (1.2.61) and the definition of a solution, we have

$$\int_a^t d\mathcal{B}(H,A)(s)\cdot x(s)=\mathcal{B}(H,x-f)(t)\ \text{ for }\ t\in[a,b].$$

In view of the above said and (0.0.13), we obtain

$$\begin{aligned}
\int_a^t dA_*(s)\cdot y(s)+f_*(t)-f_*(a) &= \int_a^t dA_*(s)\cdot (y(s)-h(s))+\mathcal{B}(H,f)(t)+h(t)-h(a)\\
&= \int_a^t d\bigg(\int_a^t d\big[H(\tau)+\mathcal{B}(H,A)(\tau)\big]\cdot H^{-1}(\tau)\bigg)\cdot H(s)x(s)+\mathcal{B}(H,f)(t)+h(t)-h(a)\\
&= \int_a^t d\big[H(s)+\mathcal{B}(H,A)(s)\big]\cdot x(s)+\mathcal{B}(H,f)(t)+h(t)-h(a)\\
&= \int_a^t dH(s)\cdot x(s)+\mathcal{B}(H,x-f)(t)+\mathcal{B}(H,f)(t)+h(t)-h(a)\\
&= \int_a^t dH(s)\cdot x(s)+\mathcal{B}(H,x)(t)+h(t)-h(a)\\
&= H(t)x(t)-H(a)x(a)+h(t)-h(a)=y(t)-y(a) \ \text{ for } \ t\in[a,b],
\end{aligned}$$

i.e., y is a solution of system (1.2.63).

Let us prove the converse assertion. It suffices to show that

$$\mathcal{I}(H^{-1},A_*)(t)=A(t)-A(a) \ \text{ for } \ t\in[a,b] \tag{1.2.65}$$

and

$$\begin{aligned}
&-H^{-1}(t)h(t)+H^{-1}(a)h(a)+\mathcal{I}(H^{-1},f^*)(t)\\
&\qquad+\int_a^t d\mathcal{I}(H^{-1},A^*)(\tau)\cdot H^{-1}(\tau)h(\tau)=f(t)-f(a) \ \text{ for } \ t\in[a,b].
\end{aligned} \tag{1.2.66}$$

Indeed, by (1.2.62), we have

$$\begin{aligned}
\mathcal{I}(H^{-1},A_*)(t)&=\mathcal{I}(H^{-1},\mathcal{I}(H,A))(t)=\mathcal{I}(I,A)(t)\\
&=\int_a^t d\big[I_n+\mathcal{B}(I_n,A)(s)\big]=\mathcal{B}(I_n,A)(t)=A(t)-f(a) \ \text{ for } \ t\in[a,b].
\end{aligned}$$

Therefore, equality (1.2.65) is proved.

Let us show that (1.2.66) is valid. Let $R(t)$ be the left-hand side of the equality. In view of (1.2.60) and (1.2.61), it is easy to verify that

$$\mathcal{B}\bigg(H^{-1},\int_a^{\cdot} d\mathcal{B}(H,A)(s)\cdot H^{-1}(s)h(s)\bigg)(t)=\int_a^t dA(s)\cdot H^{-1}(s)h(s) \ \text{ for } \ t\in[a,b]$$

and

$$\mathcal{B}\bigg(H^{-1},\int_a^{\cdot} dH(s)\cdot H^{-1}(s)h(s)\bigg)(t)=-\int_a^t dH(s)\cdot h(s) \ \text{ for } \ t\in[a,b].$$

Taking the latter equalities, (0.0.13), (1.2.60), (1.2.61) and (1.2.65) into account, we find that

$$
\begin{aligned}
R(t) = -H^{-1}(t)h(t) + H^{-1}(a)h(a) + \mathcal{B}(H^{-1},h)(t) + \mathcal{B}(H^{-1},\mathcal{B}(H,f))(t) \\
- \mathcal{B}\Big(H^{-1}, \int\limits_a^{\cdot} dA_*(s)\cdot h(s)\Big)(t) + \int\limits_a^t dA(s)\cdot H^{-1}(s)h(s) \\
= \mathcal{B}(I_n,f)(t - \int\limits_a^t dH^{-1}(s)\cdot h(s) - \mathcal{B}\Big(H^{-1}, \int\limits_a^{\cdot} d\mathcal{I}(H,A)\cdot h(s)\Big)(t) \\
+ \int\limits_a^t dA(s)\cdot H^{-1}(s)h(s) = f(t) - f(a) - \int\limits_a^t dH^{-1}(s)\cdot h(s) \\
- \mathcal{B}\Big(H^{-1}, \int\limits_a^{\cdot} dH(s)\cdot H^{-1}h(s)\Big)(t) - \mathcal{B}\Big(H^{-1}, \int\limits_a^{\cdot} d\mathcal{B}(H,A)(s)\cdot H^{-1}(s)h(s)\Big)(t) \\
+ \int\limits_a^t dA(s)\cdot H^{-1}(s)h(s) = f(t) - f(a) \ \text{ for } \ t\in[a,b].
\end{aligned}
$$

Hence (1.2.66) is valid.

Equalities (1.2.64) follow from the equalities

$$d_jA^*(t) = d_j(H(t) + \mathcal{B}(H,A)(t))\cdot H^{-1}(t) \ \text{ for } \ t\in I \ \ (j=1,2)$$

and

$$d_j\mathcal{B}(H,A)(t) = d_j(H(t)A(t))\cdot d_jH(t)\cdot A(t) \ \text{ for } \ t\in I \ \ (j=1,2). \qquad \square$$

Let ε be an arbitrary positive number and $g:[a,b]\to\mathbb{R}$ be a non-decreasing function. We denote

$$\mathcal{D}_j(a,b,\varepsilon;g) = \big\{t\in[a,b]: \ d_jg(t)\geq\varepsilon\big\} \ \ (j=1,2).$$

Let $\mathcal{R}(a,b,\varepsilon;g)$ be a set of all subdivisions $\{\alpha_0,\tau_1,\alpha_1,\dots,\tau_m,\alpha_m\}$ of $[a,b]$ such that

(a) $a=\alpha_0<\alpha_1<\cdots<\alpha_m=b$, $\alpha_0\leq\tau_1\leq\alpha_1\leq\cdots\leq\tau_m\leq\alpha_m$;

(b) if $\tau_i\notin\mathcal{D}_1(a,b,\varepsilon;g)$, then $g(\tau_i)-g(\alpha_{i-1})<\varepsilon$; if $\tau_i\in\mathcal{D}_1(a,b,\varepsilon;g)$, then $\alpha_{i-1}<\tau_i$ and $g(\tau_i-)-g(\alpha_{i-1})<\varepsilon$;

(c) if $\tau_i\notin\mathcal{D}_2(a,b,\varepsilon;g)$, then $g(\alpha_i)-g(\tau_i)<\varepsilon$; if $\tau_i\in\mathcal{D}_2(a,b,\varepsilon;g)$, then $\tau_i<\alpha_i$ and $g(\alpha_i)-g(\tau_i+)<\varepsilon$.

Lemma 1.2.3. *The set $\mathcal{R}(a,b,\varepsilon;g)$ is not empty for an arbitrary positive number ε and a non-decreasing function $g:[a,b]\to\mathbb{R}$.*

We omit the proof of the lemma because it is analogous to that of Lemma 1.1.1 from [52].

Lemma 1.2.4. *Let $\alpha_m,\beta_m\in\mathrm{BV}([a,b];\mathbb{R})$ $(m=0,1,\dots)$ be such that*

$$\lim_{m\to+\infty}\|\beta_m-\beta_0\|_\infty = 0, \tag{1.2.67}$$

$$\lim_{m\to+\infty}\sup\bigvee_a^b(\alpha_m) < +\infty, \tag{1.2.68}$$

and let the condition

$$\lim_{m\to+\infty}(\alpha_m(t)-\alpha_m(a)) = \alpha_0(t)-\alpha_0(a) \tag{1.2.69}$$

be fulfilled uniformly on $[a,b]$. Then

$$\lim_{m\to+\infty}\int\limits_a^t \beta_m(\tau)\,d\alpha_m(\tau) = \int\limits_a^t \beta_0(\tau)\,d\alpha_0(\tau)$$

is fulfilled uniformly on $[a,b]$, as well.

Proof. Let ε be an arbitrary positive number. By Lemma 1.2.3, the set $\mathcal{R}(a,b,\frac{\varepsilon}{5})$, where $g(t)\equiv V(\beta_0)(t)$, is not empty.

Let
$$\big\{\alpha_0,\tau_1,\alpha_1,\dots,\tau_m,\alpha_m\big\}\in\mathcal{R}\Big(a,b,\frac{\varepsilon}{5}\Big)$$
be an arbitrary fixed subdivision. We set
$$\eta(t)=\begin{cases}\beta_0(t) & \text{for } t\in\{\alpha_0,\tau_1,\alpha_1,\dots,\tau_m,\alpha_m\},\\ \beta_0(\tau_i-) & \text{for } t\in\,]\alpha_{i-1},\tau_i[\,,\ \tau_i\in\mathcal{D}_1(a,b,\varepsilon;g),\\ \beta_0(\tau_i) & \text{for } t\in\,]\alpha_{i-1},\tau_i[\,,\ \tau_i\notin\mathcal{D}_1(a,b,\varepsilon;g) \text{ or for } t\in\,]\tau_i,\alpha_i[\,,\ \tau_i\notin\mathcal{D}_2(a,b,\varepsilon;g),\\ \beta_0(\tau_i+) & \text{for } t\in\,]\tau_i,\alpha_i[\,,\ \tau_i\in\mathcal{D}_1(a,b,\varepsilon;g)\\ & \qquad (i=1,\dots,m).\end{cases}$$

It can be easily shown that $\eta\in\mathrm{BV}([a,b];\mathbb{R})$ and
$$|\beta_0(t)-\eta(t)|<2\varepsilon \ \text{ for } \ t\in[a,b]. \tag{1.2.70}$$

For every natural m and $t\in[a,b]$, we assume
$$\gamma_m(t)=\int_a^t\beta_m(\tau)\,d\alpha_k(t)-\int_a^t\beta_0(\tau)\,d\alpha_0(\tau)$$
and
$$\delta_m(t)=\int_a^t\eta(t)\,d(\alpha_m(\tau)-\alpha_0(\tau)).$$

It follows from (1.2.69) that
$$\lim_{m\to+\infty}\|\delta_m\|_\infty=0. \tag{1.2.71}$$
On the other hand, by (1.2.69) and (1.2.70), we have
$$\|\gamma_m\|_\infty\le 4r\varepsilon+r\|\beta_m-\beta_0\|_\infty+\|\delta\|_\infty \ \ (m=1,2,\dots).$$
Hence, in view of (1.2.68) and (1.2.71), we obtain
$$\lim_{m\to+\infty}\|\gamma_m\|_\infty=0,$$
since ε is arbitrary. □

Lemma 1.2.5. *Let condition* (1.2.14) *hold and let*
$$\lim_{m\to+\infty}X_m(t)=X_0(t) \tag{1.2.72}$$
uniformly on $[a,b]$*, where* X_0 *and* X_m $(k=1,2,\dots)$ *are the fundamental matrices of the homogeneous systems* $(1.2.1_0)$ *and* $(1.2.1_{m0})$ $(m=1,2,\dots)$*, respectively. Then*
$$\inf\big\{|\det(X_0(t))|:\ t\in[a,b]\big\}>0, \tag{1.2.73}$$
$$\inf\big\{|\det(X_0^{-1}(t))|:\ t\in[a,b]\big\}>0 \tag{1.2.74}$$
and condition (1.2.15) *holds uniformly on* $[a,b]$*, as well.*

Proof. According to equalities (0.0.14) and the definition of a solution of system $(1.2.1_0)$, we have
$$d_jX_0(t)=d_jA_0(t)\cdot X_0(t) \ \text{ for } \ t\in[a,b] \ \ (j=1,2).$$
From this, by (1.2.14) $(m=0)$, we find that

$$\det\big(X_0(t-)\cdot X_0(t+)\big)$$
$$=\big[\det(X_0(t))\big]^2\cdot\prod_{j=1}^{2}\det\big(I_n+(-1)^j\,d_jA_0(t)\big)\neq 0\ \text{ for }\ t\in[a,b]\ \ (j=1,2).\quad (1.2.75)$$

Let us show that (1.2.73) is valid. Assume the contrary. Then it can be easily shown that there exists a point $t_0\in[a,b]$ such that

$$\det\big(X_0(t_0-)\cdot X_0(t_0+)\big)=0.$$

But this equality contradicts (1.2.75). Thus inequality (1.2.73) is proved.

The proof of inequality (1.2.74) is analogous.

In view of (1.2.72) and (1.2.73), there exists a positive number r such that

$$\inf\big\{|\det(X_m(t))|:\ t\in[a,b]\big\}>r>0$$

for any sufficiently large m. From this and (1.2.72), we obtain (1.2.15). □

Lemma 1.2.6. *Let the sequences of the matrix-functions $B_m\in\mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ $(m=0,1,\dots)$ be such that conditions*

$$\det\big(I_n+(-1)^j\,d_jB_0(t)\big)\neq 0\ \text{ for }\ t\in[a,b]\ \ (j=1,2)\quad (1.2.76)$$

and

$$\lim_{m\to+\infty}\sup\Big\{\|d_jB_m(t)-d_jB_0(t)\|:\ t\in[a,b]\Big\}=0\ \ (j=1,2)\quad (1.2.77)$$

hold. Then there exists a positive number r_0 such that

$$\det\big(I_n+(-1)^j\,d_jB_m(t)\big)\neq 0\ \text{ for }\ t\in[a,b]\ \ (j=1,2)\quad (1.2.78)$$

and

$$\Big\|\big(I_n+(-1)^j\,d_jB_0(t)\big)^{-1}\Big\|+\Big\|\big(I_n+(-1)^j\,d_jB_m(t)\big)^{-1}\Big\|\leq r_0\ \text{ for }\ t\in[a,b]\ \ (j=1,2)\quad (1.2.79)$$

for every sufficiently large m.

Proof. Since $\bigvee\limits_a^b B_0<+\infty$, the series $\sum\limits_{t\in[a,b]}\|d_jB_0(t)\|$ $(j=1,2)$ converge. Thus for any $j\in\{1,2\}$ the inequality

$$\|d_jB_0(t)\|\geq\frac{1}{2}$$

may hold only for some finite number of points $t_{j1},\dots,t_{j\,k_j}$ in I. Therefore,

$$\|d_jB_0(t)\|<\frac{1}{2}\ \text{ for }\ t\in[a,b],\ \ t\neq t_{ji}\ \ (i=1,\dots,k_j).\quad (1.2.80)$$

First, let us consider the case $j=2$.

It follows from (1.2.76), (1.2.77) and (1.2.80) that

$$\det\big(I_n+d_2B_m(t_{2i})\big)\neq 0\ \ (i=1,\dots,k_2)$$

and

$$\|d_2B_m(t)\|<\frac{1}{2}\ \text{ for }\ t\in[a,b],\ \ t\neq t_{2i}\ \ (i=1,\dots,k_2)$$

for every sufficiently large m. The latter inequalities imply that the matrices $I_n+d_2B_m(t)$ are invertible for $t\in[a,b]$, $t\neq t_{2i}$ $(i=1,\dots,k_2)$, too. From this, it is evident that condition (1.2.78) is fulfilled and there exists a positive number r_0 for which estimates (1.2.79) hold. Analogously we prove this estimate for $j=1$. □

1.2.3 Proofs of the results

Proof of Theorem 1.2.3. In view of (1.2.17), $\ell_m^* : \mathrm{BV}_\infty([a,b];\mathbb{R}^n) \to \mathbb{R}^n$ is a linear bounded vector-functional for every sufficiently large m. Moreover, it is not difficult to see that by the mapping

$$x \to y = H_m\, x + h_m$$

is a one-to-one correspondence between solutions of problem $(1.2.1_m), (1.2.2_m)$ and solutions y of problem $(1.2.1_m^*), (1.2.2_m^*)$, where $c_m^* = c_m + \ell_m^*(h_m)$. In fact, according to Lemma 1.2.2, it suffices to show that equality $(1.2.2_m)$ implies equality $(1.2.2_m^*)$. This is obvious by the definition of the functional ℓ_m^*.

Let us show that

$$\det\big(I_n + (-1)^j\, d_j A_m^*(t)\big) \neq 0 \;\text{ for }\; t \in [a,b] \tag{$1.2.5_m^*$}$$

for any sufficiently large m.

By (1.2.22),

$$\lim_{m\to+\infty} d_j A_m^*(t) = d_j A_0^*(t) \;\; (j=1,2)$$

uniformly on $[a,b]$. Therefore, by virtue of Lemma 1.2.6, there exists a positive number r_0 such that condition $(1.2.5_m^*)$ holds and

$$\big\|\big[I_n + (-1)^j\, d_j A_m^*(t)\big]^{-1}\big\| \leq r_0 \;\text{ for }\; t \in [a,b] \;\; (j=1,2) \tag{1.2.81}$$

for any sufficiently large m (i.e., without loss generality, we can assume that for every natural m).

In view of (1.2.16) and $(1.2.5_m^*)$, there exist the fundamental matrices Y_0 and Y_m ($Y_0(a) = Y_m(a) = I_n$) of systems

$$dy = dA_0^*(t)\cdot y$$

and

$$dy = dA_m^*(t)\cdot y \;\; (m=1,2,\dots).$$

Moreover, Y_0^{-1}, $Y_m^{-1} \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ $(m=1,2,\dots)$.

Let us prove

$$\lim_{m\to+\infty} Y_m(t) = Y_0(t) \;\text{ uniformly on }\; [a,b]. \tag{1.2.82}$$

We set

$$Z_m(t) = Y_m(t) - Y_0(t) \;\text{ for }\; t\in[a,b] \;\; (m=1,2,\dots)$$

and

$$B_m(t) = A_m^*(t-) \;\text{ for }\; t\in[a,b] \;\; (m=1,2,\dots).$$

Due (0.0.6), we have

$$\int_a^t d(B_m(\tau) - A_m^*(\tau))\cdot Z_m(\tau) = -d_1 A_m^*(t)\cdot Z_m(t) \;\text{ for }\; t\in[a,b] \;\; (m=1,2,\dots).$$

Consequently,

$$Z_m(t) = (I_n - d_1 A_m^*(t))^{-1}\bigg\{\int_a^t d(A_m^*(\tau) - A^*(\tau))\cdot Y(\tau) + \int_a^t dB_m(\tau)\cdot Z_m(\tau)\bigg\} \;\text{ for }\; t\in[a,b] \;\; (m=1,2,\dots).$$

From this and (1.2.81), we get

$$\|Z_m(t)\| \leq r_0\bigg(\varepsilon_m + \int_a^t d\|V(B_m)(\tau)\|\cdot\|Z_m(\tau)\|\bigg) \;\text{ for }\; t\in[a,b] \;\; (m=1,2,\dots),$$

where

$$\varepsilon_m = \sup\left\{\left\|\int\limits_a^t d(A_m^*(\tau) - A^*(\tau))\cdot Y(\tau)\right\| :\ t\in[a,b]\right\}\ \ (m=1,2,\dots).$$

Hence, according to the Gronwall inequality ([73, Theorem I.4.30]),

$$\|Z_m(t)\| \le r_0\varepsilon_m \exp\Big(r_0\bigvee_a^b B_m\Big) \le r_0\varepsilon_m \exp\Big(r_0\bigvee_a^b A_m^*\Big)\ \text{ for }\ t\in[a,b]\ \ (m=1,2,\dots).$$

By (1.2.21), (1.2.22) and Lemma 1.2.4, this inequality implies (1.2.82).

As we have shown in Subsection 1.1.3, problem $(1.2.1_m^*), (1.2.2_m^*)$ has the unique solution if and only if

$$\det(\ell_m^*(Y_m)) \neq 0 \tag{1.2.83}$$

for every natural m.

Since problem $(1.2.1^*), (1.2.2^*)$ has the unique solution x_0^*, we have

$$\det(\ell^*(Y_0)) \neq 0. \tag{1.2.84}$$

Besides, by (1.2.18), (1.2.19) and (1.2.82),

$$\lim_{m\to+\infty} \ell_m^*(Y_m) = \ell_0^*(Y_0). \tag{1.2.85}$$

Therefore, in view of (1.2.84), there exists a natural number m_0 such that condition (1.2.83) is fulfilled for every $m \ge m_0$. Thus problem $(1.2.1_m^*), (1.2.2_m^*)$ has the unique solution y_m for $m\ge m_0$ and

$$y_m(t) = Y_m(t)\big[\ell_m(Y_m)\big]^{-1}\big(c_m - \ell_m(F_m(f_m^*))\big) + F_m(f_m^*)(t)\ \text{ for }\ t\in[a,b], \tag{1.2.86}$$

where

$$F_m(f_m^*)(t) \equiv f_m^*(t) - f_m^*(a) - Y_m(t)\int\limits_a^t dY_m^{-1}(\tau)\cdot(f_m^*(\tau) - f_m^*(a)).$$

According to Lemma 1.2.5, we have

$$\lim_{m\to+\infty} Y_m^{-1}(t) = Y_0^{-1}(t)\ \text{ uniformly on }\ [a,b] \tag{1.2.87}$$

and

$$\rho = \sup\Big\{\|Y_m^{-1}(t)\| + \|Y_m(t)\| :\ t\in[a,b],\ m\ge m_0\Big\} < +\infty. \tag{1.2.88}$$

The equality

$$Y_m^{-1}(t) - Y_m^{-1}(s) = Y_m^{-1}(s)\int\limits_t^s dA_m^*(\tau)\cdot Y_m(\tau)\,Y_m^{-1}(t)\ \ (t,s\in[a,b])$$

implies

$$\|Y_m^{-1}(t) - Y_m^{-1}(s)\| \le \rho^3 \bigvee_s^t A_m^*\ \text{ for }\ a\le s\le t\le b\ \ (m\ge m_0).$$

This inequality, together with (1.2.21) and (1.2.88), yields

$$\limsup_{m\to+\infty} \bigvee_a^b Y_m^{-1} < +\infty.$$

By this, (1.2.23) and (1.2.87), it follows from Lemma 1.2.4 that

$$\lim_{m\to+\infty}\int\limits_a^t dY_m^{-1}(\tau)\cdot(f_m^*(\tau) - f_m^*(a)) = \int\limits_a^t dY^{-1}(\tau)\cdot(f_0^*(\tau) - f_0^*(a))\ \text{ uniformly on }\ [a,b]. \tag{1.2.89}$$

Using (1.2.18)–(1.2.20), (1.2.23), (1.2.82)–(1.2.85) and (1.2.89), from (1.2.86) we get

$$\lim_{m\to+\infty} y_m(t) = z(t) \ \text{ uniformly on } \ [a,b], \tag{1.2.90}$$

where

$$z(t) \equiv Y(t)\big[\ell(Y)\big]^{-1}\big(c_0 - \ell(F(f_0^*))\big) + F(f_0^*)(t),$$

$$F(f_0^*)(t) \equiv f_0^*(t) - f_0^*(a) - Y(t)\int_a^t dY^{-1}(\tau)\cdot(f_0^*(\tau) - f_0^*(a)).$$

It is easy to verify that the vector-function z is a solution of problem $(1.2.1^*), (1.2.2^*)$. Therefore,

$$x_0^*(t) = z(t) \ \text{ for } \ t\in[a,b].$$

This and (1.2.90) allow us to conclude that condition (1.2.24) holds uniformly on $[a,b]$. □

Proof of Corollary 1.2.1. Verify the conditions of Theorem 1.2.3. Due to (1.2.10) and (1.2.11), conditions (1.2.17) and

$$\lim_{m\to+\infty} \|H_m^{-1} - H_0^{-1}\|_\infty = 0 \tag{1.2.91}$$

hold.

Put

$$h_m(t) = -H_m(t)\varphi_m(t) \ \text{ for } \ t\in[a,b] \ \ (m=1,2,\dots).$$

Then by (1.2.6), (1.2.7), (1.2.25) and (1.2.91), conditions (1.2.18)–(1.2.20), where $c_0^* = c_0$ and $\ell_0^*(y) \equiv \ell_0(H_0^{-1}y)$, are satisfied.

Applying Lemma 1.2.3, from (1.2.9), (1.2.11), (1.2.12) and (1.2.91) we find that (1.2.21) holds and (1.2.22) is fulfilled uniformly on $[a,b]$, where

$$A_0^*(t) \equiv \mathcal{I}(H_0, A_0)(t).$$

On the other hand,

$$f_m^*(t) = \mathcal{B}(H_m, f_m - \varphi_m)(t) + \int_a^t d\mathcal{B}(H_m, A_m)(\tau)\cdot\varphi_m(\tau) \ \text{ for } \ t\in[a,b] \ \ (m=1,2,\dots).$$

Consequently, (1.2.26) implies that the condition (1.2.23), where

$$f_0^*(t) \equiv \mathcal{B}(H_0, f_0)(t),$$

is fulfilled uniformly on $[a,b]$.

Taking into account Lemma 1.2.2 and the equalities

$$\ell_0^*(H_0\,x_0) = \ell_0(x_0) = c_0,$$

it is not difficult to see that problem $(1.2.1^*), (1.2.2^*)$ has a unique solution

$$x_0^*(t) = H_0(t)\,x_0(t) \ \text{ for } \ t\in[a,b].$$

Moreover, it can be easily shown that inequality (1.2.10) is equivalent to the condition

$$\det\big(H_0(t+)\cdot H_0(t-)\big) \neq 0 \ \text{ for } \ t\in[a,b].$$

Thus, by virtue of (1.2.5) and (1.2.64), condition (1.2.16) is fulfilled.

According to Theorem 1.2.3, condition (1.2.24) holds uniformly on $[a,b]$. Hence it follows from (1.2.24) and (1.2.91) that condition (1.2.27) is fulfilled uniformly on $[a,b]$. □

Proof of Theorem 1.2.1. The sufficiency follows from Corollary 1.2.1 if we assume $\varphi_m(t) \equiv 0$ ($m = 1, 2, \dots$) in it.

Let us show the necessity. Let $c_m \in \mathbb{R}^n$ ($m = 0, 1, \dots$) be an arbitrary sequence of constant vectors satisfying (1.2.4) and let $e_j = (\delta_{ij})_{i=1}^n$, where $\delta_{ii} = 1$ and $\delta_{ij} = 0$ if $i \neq j$ ($i, j = 1, \dots, n$) (the Kroneker symbol).

In view of (1.2.8), without loss of generality, we may assume that problem $(1.2.1_m), (1.2.2_m)$ has a unique solution x_m for every natural m.

For any $m \in \{0, 1, \dots\}$ and $j \in \{1, \dots, n\}$, let us denote

$$z_{mj}(t) \equiv x_m(t) - x_{mj}(t),$$

where x_{mj} is a unique solution of system $(1.2.1_m)$ satisfying the condition

$$\ell_m(x) = c_m - e_j.$$

Moreover, let $X_m(t)$ be a matrix-function with columns $z_{m1}(t), \dots, z_{mn}(t)$ ($m = 0, 1, \dots$).

It can be easily shown that X_0 and X_m ($m = 1, 2, \dots$) satisfy, respectively, the homogeneous systems $(1.2.1_0)$ and $(1.2.1_{m0})$ ($m = 1, 2, \dots$) and

$$\ell(z_{mj}) = e_j \quad (m = 0, 1, \dots) \tag{1.2.92}$$

for every $j \in \{1, \dots, n\}$.

If we assume

$$\sum_{j=1}^n \alpha_j z_{mj}(t) \equiv 0$$

for some $m \in \widetilde{\mathbb{N}}$ and $\alpha_j \in \mathbb{R}$ ($j = 1, \dots, n$), then, using (1.2.92), we get

$$\sum_{j=1}^n \alpha_j e_j = 0$$

and, therefore, $\alpha_1 = \dots = \alpha_n = 0$, i.e., X_0 and X_m ($m = 1, 2, \dots$) are the fundamental matrices, respectively, of the homogeneous systems $(1.2.1_0)$ and $(1.2.1_{m0})$ ($m = 1, 2, \dots$).

Owing to (1.2.8), we conclude that

$$\lim_{m \to +\infty} x_m(t) = x_0(t) \ \text{ and } \ \lim_{m \to +\infty} x_{mj}(t) = x_{0j}(t) \quad (j = 1, \dots, n)$$

uniformly on $[a, b]$, where x_{0j} is a unique solution of system (1.2.1) satisfying the condition $\ell_0(x) = c_0 - e_j$. Therefore, condition (1.2.72) holds uniformly on $[a, b]$. Without loss of generality, we assume that

$$X_m(a) = I_n \quad (m = 0, 1, \dots).$$

Now, according to Lemma 1.2.5, we find that condition (1.2.11) holds uniformly on $[a, b]$, where

$$H_m(t) \equiv X_m^{-1}(t) \quad (m = 0, 1, \dots).$$

Let us verify that conditions (1.2.9) and (1.2.10) hold, and conditions (1.2.12), (1.2.13) are fulfilled uniformly on $[a, b]$ for the above-defined matrix-functions H_m ($m = 0, 1, \dots$).

Conditions (1.2.9) and (1.2.10) coincide with conditions (1.2.72) and (1.2.73), respectively.

According to Proposition 1.1.2 (see equality (1.1.13)), we have

$$X_m^{-1}(t) = I_n - \mathcal{B}(X_m^{-1}, A_m)(t) \ \text{ for } \ t \in I \quad (m = 0, 1, \dots). \tag{1.2.93}$$

Therefore,

$$H_m(t) + \mathcal{B}(H_m, A_m)(t) = I_n \ \text{ for } \ t \in [a, b] \quad (m = 0, 1, \dots). \tag{1.2.94}$$

Thus condition (1.2.9) is evident.

On the other hand, by (1.2.94) and the equalities $H_m(a) = I_n$ $(m = 0, 1, \dots)$, according to Lemma 1.2.1 and the definition of the solutions of system $(1.2.1_m)$, we have

$$\mathcal{B}(H_m, f_m)(t) = \mathcal{B}\Big(H_m, x_m - \int_a^{\cdot} dA_k(s)\cdot x_m(s)\Big)(t)$$
$$= \mathcal{B}(H_m, x_m)(t) - \mathcal{B}\Big(H_m, \int_a^{\cdot} dA_m(s)\cdot x_m(s)\Big)(t) = \mathcal{B}(H_m, x_m)(t) - \int_a^t d\mathcal{B}(H_m, A_m)(s)\cdot x_m(s)$$
$$= H_m(t)x_m(t) - x_m(a) - \int_a^t dH_m(s)\cdot x_m(s) - \int_a^t d(I_n - H_m(s))\cdot x_m(s)$$
$$= H_m(t)x_m(t) - x_m(a) \ \text{ for } \ t \in [a,b] \ \ (m = 0, 1, \dots).$$

Hence

$$\mathcal{B}(H_m, f_m)(t) - \mathcal{B}(H_0, f_0)(t)$$
$$= H_m(t)x_m(t) - H_0(t)x_0(t) - (x_m(a) - x_0(a)) \ \text{ for } \ t \in [a,b] \ \ (m = 0, 1, \dots). \quad (1.2.95)$$

By this, (1.2.11) and (1.2.94), conditions (1.2.12) and (1.2.13) hold uniformly on $[a,b]$. □

Proof of Theorem 1.2.2. As it follows from the proof of Theorem 1.2.1, we may assume that $H_m(t) \equiv X_m^{-1}(t)$ $(m = 0, 1, \dots)$. In this case, Theorem 1.2.1 has the form of Theorem 1.2.2. We only note that in view of (1.2.15) and (1.2.93), condition (1.2.12) holds uniformly on $[a,b]$. □

Proof of Theorem 1.2.4. The theorem is a particular case of the sufficiency part of Theorem 1.2.1, where $H_m(t) \equiv I_n$ $(m = 0, 1, \dots)$. □

Proof of Corollary 1.2.2. By (1.2.32), (1.2.33) and (1.2.34) (or (1.2.35)), we have

$$\lim_{m\to+\infty} \sum_{a\le s<t\le b} \big(d_1H_m(s)\cdot d_1A_m(s) - d_1H_0(s)\cdot d_1A_0(s)\big) = O_{n\times n},$$
$$\lim_{m\to+\infty} \sum_{a\le s<t\le b} \big(d_1H_m(s)\cdot d_1f_m(s) - d_1H_0(s)\cdot d_1f_0(s)\big) = 0_n,$$
$$\lim_{m\to+\infty} \sum_{a\le s<t\le b} \big(d_2H_m(s)\cdot d_2A_m(s) - d_2H_0(s)\cdot d_2A_0(s)\big) = O_{n\times n},$$
$$\lim_{m\to+\infty} \sum_{a\le s<t\le b} \big(d_2H_m(s)\cdot d_2f_m(s) - d_2H_0(s)\cdot d_2f_0(s)\big) = 0_n$$

uniformly on $[a,b]$. From this, the integration-by-parts formula, (1.2.30) and (1.2.31), we find that conditions (1.2.12) and (1.2.13) are fulfilled uniformly on $[a,b]$. Condition (1.2.13) coincides with (1.2.26) for $\varphi_m(t) \equiv 0$ $(m = 1, 2, \dots)$.

Therefore, the corollary follows from Corollary 1.2.1. □

Proof of Corollary 1.2.3. Using (1.2.11), (1.2.28) and (1.2.36), we conclude that

$$d_jA^*(t) \equiv O_{n\times n} \ \ (j = 1, 2).$$

Hence, in view of (1.2.5), we have

$$\det\big(I_n + (-1)^j\, d_jA_0^*(t)\big) \neq 0 \ \text{ for } \ t \in [a,b].$$

On the other hand, from (1.2.11), (1.2.28), (1.2.29), (1.2.36) and (1.2.37) we find that the conditions

$$\lim_{m\to+\infty} \mathcal{B}(H_m, A_m)(t) = \mathcal{B}(I_n, A_0^*)(t) \quad \text{and} \quad \lim_{m\to+\infty} \mathcal{B}(H_m, f_m)(t) = \mathcal{B}(I_n, f_0^*)(t)$$

hold uniformly on $[a,b]$. Thus, Corollary 1.2.3 is a direct consequence of the sufficiency part of Theorem 1.2.1. □

Proof of Corollary 1.2.4. By virtue of (1.2.39), we have

$$\lim_{m\to+\infty} C_{mj}(t)=I_n, \quad \lim_{m\to+\infty} H_{mj}(t)=I_n \ \ (j=1,\dots,\mu-1)$$

uniformly on $[a,b]$, where

$$C_{mj}(t)\equiv I_n-(A_{mj}(t)-A_{mj}(a))+(B_j(t)-B_j(a)) \ \ (j=1,\dots,\mu-1;\ m=1,2,\dots).$$

Thus, without loss of generality, we can assume that the matrix-functions H_{mj} $(j=1,\dots,\mu-1)$ and C_{mj} $(j=1,\dots,\mu-1)$ are nonsingular for every natural m. Using now Lemma 1.2.1, we find that

$$\mathcal{B}\big(C_{mj},\mathcal{B}(H_{m\,j-1},A_m)\big)(t)\equiv\mathcal{B}(H_{mj},A_m)(t),$$
$$\mathcal{B}\big(C_{mj},\mathcal{B}(H_{m\,j-1},f_m)\big)(t)\equiv\mathcal{B}(H_{mj},f_m)(t)$$

and

$$\mathcal{I}\big(C_{mj},\mathcal{I}(H_{m\,j-1},A_m)\big)(t)\equiv\mathcal{I}(H_{mj},A_m)(t) \ \ (j=1,\dots,\mu-1;\ m=1,2,\dots).$$

In addition, by conditions (1.2.38)–(1.2.41), according to Lemma 1.2.4 and the definition of the operator $\mathcal{I}$, we find that conditions (1.2.11)–(1.2.13) hold uniformly on $[a,b]$, where $H_0(t)\equiv I_n$ and $H_m(t)\equiv H_{m\,\mu-1}(t)$ $(m=1,2,\dots)$. The corollary follows from Theorem 1.2.1. □

Proof of Corollary 1.2.5. Let us show the sufficiency. Let $H_m(t)=Z_m^{-1}(t)$ $(m=0,1,\dots)$ in Theorem 1.2.1. In view of (1.2.44), there exists a positive number r such that

$$\|Z_m^{-1}(t)\|\le r \ \text{ for } \ t\in[a,b] \ \ (m=0,1,\dots).$$

Using this estimate, by (1.1.13), the definition of the operator $\mathcal{B}$ and the integration-by-parts formula, we have

$$\begin{aligned}
&\big\|Z_m^{-1}(t)+\mathcal{B}(Z_m^{-1},A_m)(t)-Z_m^{-1}(s)-\mathcal{B}(Z_m^{-1},A_m)(s)\big\| \\
&\qquad=\big\|\mathcal{B}(Z_m^{-1},A_m-B_m)(t)-\mathcal{B}(Z_m^{-1},A_m-B_m)(s)\big\| \\
&\qquad=\bigg\|\int_s^t Z_m^{-1}(\tau)\,d(A_m(\tau)-B_m(\tau))-\sum_{s<\tau\le t} d_1Z_m^{-1}(\tau)\cdot d_1(A_m(\tau)-B_m(\tau)) \\
&\qquad\qquad+\sum_{s\le\tau<t} d_2Z_m^{-1}(\tau)\cdot d_2(A_m(\tau)-B_m(\tau))\bigg\| \\
&\qquad\le r\bigvee_s^t(A_m-B_m)+2r\sum_{s<\tau\le t}\|d_1(A_m(\tau)-B_m(\tau))\|+2r\sum_{s\le\tau<t}\|d_2(A_m(\tau)-B_m(\tau))\| \\
&\qquad\qquad\le 5r\bigvee_s^t(A_m-B_m) \ \text{ for } \ s<t \ \ (k=m,1,\dots).
\end{aligned}$$

Consequently,

$$\bigvee_a^b(H_m+\mathcal{B}(H_m,A_m))\le 5r\bigvee_a^b(A_m-B_m) \ \ (m=0,1,\dots)$$

and, due to (1.2.42), estimate (1.2.9) holds. Conditions (1.2.12) and (1.2.13) coincide with (1.2.45) and (1.2.46), respectively. Hence the sufficiency follows from Theorem 1.2.1.

Let us show the necessity. Let $B_m(t)=A_m(t)$ $(m=0,1,\dots)$. Then $Z_m(t)\equiv X_m(t)$ $(m=0,1,\dots)$, where X_0 and X_m $(m=1,2,\dots)$ are the fundamental matrices of systems $(1.1.1_0)$ and $(1.1.1_{m0})$, respectively. Analogously, just as in the proof of Theorem 1.2.1, conditions (1.2.44) and (1.2.95) are valid, where $H_m(t)\equiv Z_m^{-1}(t)$ $(m=0,1,\dots)$. In addition, condition (1.2.45) coincides with (1.2.12), and condition (1.2.46) follows from (1.2.95). □

Proof of Corollary 1.2.6. Let us prove that condition (1.2.44) holds uniformly on $[a,b]$, where Z_m ($Z_m(a)=I_n$) is the fundamental matrix of system (1.2.47) for every $m\in\widetilde{\mathbb{N}}$. In view of (1.2.54), we have

$$Z_m(t)=Z_{mc}(t)Z_{m1}(t)^{-1}Z_{m2}(t)\ \text{ for }\ t\in[a,b]\ \ (m=0,1,\dots), \tag{1.2.96}$$

where

$$Z_{mc}(t)\equiv\exp\big(S_c(B)(t)-S_c(B)(a)\big),\ \ Z_{m1}(t)\equiv\prod_{a<\tau\le t}(I_n-d_1B(\tau)),$$
$$Z_{m2}(t)\equiv\prod_{a\le\tau<t}(I_n+d_2B(\tau))\ \ (m=0,1,\dots).$$

It is evident that Z_{m0}, Z_{m1} and Z_{m2} are the fundamental matrices of systems

$$dx=dS_c(B_m)(t)\cdot x,\ \ dx=dS_1(B_m)(t)\cdot x\ \text{ and }\ dx=dS_2(B_m)(t)\cdot x,$$

respectively $(m=0,1,\dots)$.

Applying Theorem 1.2.4 to these system for the case $\ell_m(x)=x(a)$ (the Cauchy problem) ($m=0,1\dots$), we conclude that conditions

$$\lim_{m\to+\infty}Z_{mc}(t)=Z_{0c}(t),\ \ \lim_{m\to+\infty}Z_{m1}(t)=Z_{01},\ \ \lim_{m\to+\infty}Z_{m2}(t)=Z_{02}(t)$$

hold uniformly on $[a,b]$. From this and Lemma 1.2.5, we get that condition (1.2.44) holds uniformly on $[a,b]$.

Let us show that other conditions of Corollary 1.2.4 hold.

We verify condition (1.2.45). Using the integration-by-parts formula, we find that

$$\mathcal{B}(Z_m^{-1},A_m)(t)-\mathcal{B}(Z_m^{-1},A_m)(s)=\int_s^t Z_m^{-1}(\tau)\,dA_m(\tau)$$
$$-\sum_{s<\tau\le t}d_1Z_m^{-1}(\tau)\cdot d_1A_m(\tau)+\sum_{s\le\tau<t}d_2Z_m^{-1}(\tau)\cdot d_2A_m(\tau)\ \text{ for }\ a\le s<t\le b\ \ (m=0,1,\dots).$$

In addition, by equalities (1.1.14), we have

$$d_jZ_m^{-1}(t)\equiv-Z_m^{-1}(t)\,d_jB_m(t)\cdot\big(I_n+(-1)^j\,d_jB_m(t)\big)^{-1}\ \ (j=1,2;\ m=0,1,\dots).$$

Consequently, due to (1.1.16), we get

$$\mathcal{B}(Z_m^{-1},A_m)(t)-\mathcal{B}(Z_m^{-1},A_m)(s)=\int_s^t Z_m^{-1}(\tau)\,d\mathcal{A}(B_m,A_m)(\tau)\ \ (m=0,1,\dots)$$

for $a\le s<t\le b$. In the same way, we establish the last equalities for the case $a\le t<s\le b$.

Analogously, we check the equalities

$$\mathcal{B}(Z_m^{-1},f_m)(t)-\mathcal{B}(Z_m^{-1},f_m)(s)=\int_s^t Z_m^{-1}(\tau)\,d\mathcal{A}(B_m,f_m)(\tau)\ \text{ for }\ s,t\in[a,b]\ \ (m=0,1,\dots).$$

Therefore, equalities (1.2.45) and (1.2.46) coincide with equalities (1.2.52) and (1.2.53), respectively. The corollary follows from Corollary 1.2.5. □

Proof of Corollary 1.2.8. The corollary follows from Corollary 1.2.6 if we assume that $B_m(t)\equiv S_c(A_m)(t)$ $(m=0,1,\dots)$ in it. In addition, we note that condition (1.2.48) is of the form (1.2.55), condition (1.2.50) is equivalent to conditions (1.2.56) and (1.2.57), and by (1.2.54), conditions (1.2.52) and (1.2.53) coincide with (1.2.58) and (1.2.59), respectively. □

Proof of Corollary 1.2.9. The corollary follows from Corollary 1.2.6 if we assume that $B_m(t)\equiv\operatorname{diag}(A_m(t))$ $(m=0,1,\dots)$ in it. □

Chapter 2

Multi-point boundary value problems for systems of generalized ordinary differential equations

2.1 General multi-point boundary value problem

In this chapter, we consider a linear system of generalized ordinary differential equations

$$dx = dA(t) \cdot x + df(t) \text{ for } t \in [a,b]. \tag{2.1.1}$$

Below, unless otherwise stated, we assume that

$$A = (a_{ik})_{i,k=1}^{n} \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n}), \quad f = (f_k)_{k=1}^{n} \in \mathrm{BV}([a,b];\mathbb{R}^{n}).$$

We investigate the question on the existence of solutions of system (2.1.1) under the following general multi-point boundary value condition

$$\sum_{j=1}^{\nu} L_j x(t_j) = c_0, \tag{2.1.2}$$

where $t_j \in [a,b]$ $(j=1,\dots,\nu)$, $L_j \in \mathbb{R}^{n\times n}$ $(j=1,\dots,\nu)$ are constant matrixes, and ν is a fixed natural number.

In the section, we realize the results given in Subsections 1.1.1 and 1.1.2 to problem (2.1.1), (2.1.2).

Along with problem (2.1.1), (2.1.2), we consider the corresponding homogeneous problem

$$dx = dA(t) \cdot x, \tag{2.1.1$_0$}$$

$$\sum_{j=1}^{\nu} L_j x(t_j) = 0. \tag{2.1.2$_0$}$$

Below, we use the definition of the operators given in Subsection 1.1.2.

Theorem 2.1.1. *The boundary value problem* (2.1.1), (2.1.2) *is uniquely solvable if and only if the corresponding homogeneous problem* $(2.1.1_0), (2.1.2_0)$ *has only the trivial solution, i.e., if and only if*

$$\det\Big(\sum_{j=1}^{\nu} L_j Y(t_j)\Big) \neq 0, \tag{2.1.3}$$

where Y is a fundamental matrix of system $(2.1.1_0)$*. If the latter condition holds, then the solution x of problem* (2.1.1), (2.1.2) *admits the representation*

$$x(t) = x_0(t) + \int_a^b d_s\mathcal{G}(t,s) \cdot f(s) \ \textit{ for } \ t \in [a,b],$$

where x_0 is a solution of problem $(2.1.1_0), (2.1.2)$*, and $\mathcal{G}$ is the Green matrix of problem* $(2.1.1_0)$, $(2.1.2_0)$.

In condition (2.1.2), without loss generality, we can assume that $a \le t_1 < t_2 < \dots < t_\nu \le b$. This condition is a particular case of condition (1.1.4), where the matrix-function $\mathcal{L}$ is defined as

$$\mathcal{L}(t) = -\sum_{j=1}^{\nu} \chi_j(t) L_j \ \text{ for } \ t \in [a,b],$$

where χ_j is the characteristic function of the interval $[a, t_j[$ $(j = 1, \dots, \nu)$.

It is evident that $\mathcal{L}(b) = O_{n\times n}$. Moreover, it is not difficult to verify that

$$\int_a^t d\mathcal{L}(\tau)\cdot X(\tau) = \sum_{j=1}^{\nu} (1-\chi_j(t)) L_j\, X(t_j) \ \text{ for } \ X \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n}) \ \ (t \in [a,b]).$$

Hence, in view of (1.1.21), the Green matrix of problem $(2.1.1_0), (2.1.2_0)$ has the form

$$\mathcal{G}(t,s) = \begin{cases} -Y(t)\displaystyle\sum_{j=1}^{\nu} (1-\chi_j(s)) Z_j Y^{-1}(s) & \text{for } a \le s < t \le b, \\ Y(t)\displaystyle\sum_{j=1}^{\nu} \chi_j(s) Z_j Y^{-1}(s) & \text{for } a \le t < s \le b, \\ O_{n\times n} & \text{for } a \le t = s \le b, \end{cases} \tag{2.1.4}$$

where

$$Z_j = \Big(\sum_{i=1}^{\nu} L_i Y(t_i)\Big)^{-1} L_j Y(t_j) \ \ (j = 1, \dots, \nu).$$

Proposition 1.1.3 has the following form for the case under consideration.

Proposition 2.1.1. *Let the matrix-function $A \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ be such that condition* (1.1.8) *hold. Then the boundary value problem* (2.1.1), (2.1.2) *is solvable if and only if the condition*

$$\Big(c_0 - \sum_{j=1}^{\nu} L_j F(t_j)\Big)^{\top} \gamma = 0 \tag{2.1.5}$$

holds for every $\gamma \in \mathbb{R}^n$ such that

$$\Big(\sum_{j=1}^{\nu} L_j Y(t_j)\Big)^{\top} \gamma = 0_n,$$

where

$$F(t) \equiv Y(t) \int_a^t Y^{-1}(\tau)\, d\mathcal{A}(A,f)(\tau).$$

So, if condition (2.1.3) holds, then only the vector $\gamma = 0_n$ satisfies the homogeneous system appearing in Proposition 2.1.1 and, therefore, condition (2.1.5) holds evidently. If condition (2.1.3) is violated, then problem (2.1.1), (2.1.2) is solvable only for c_0, that satisfies the conditions of the proposition.

Remark 2.1.1. Let the matrix-function A satisfy the Lappo–Danilevskiĭ condition at the point a. Then problem (2.1.1), (2.1.2) is uniquely solvable if and only if

$$\det\Big(\sum_{j=1}^{\nu} L_j \exp(S_c(A)(t_j)) \prod_{a\le\tau<t_j} (I_n + d_2A(\tau)) \prod_{a<\tau\le t_j} (I_n - d_1A(\tau))^{-1}\Big) \ne 0.$$

Theorem 2.1.2. *The boundary value problem* (2.1.1), (2.1.2) *is uniquely solvable if and only if there exist natural numbers k and m such that the matrix*

$$M_k = \sum_{i=0}^{k-1} \sum_{j=1}^{\nu} L_j \, [A]_i(t_j)$$

is nonsingular and the inequality

$$r(M_{k,m}) < 1 \tag{2.1.6}$$

holds, where

$$M_{k,m} = V_m(A)(c) + \Big(\sum_{i=0}^{m-1} \big| \, [A]_i \big|_\infty \Big) \sum_{j=1}^{\nu} |M_k^{-1} L_j| V_k(A)(t_j),$$

and the operators $[A]_i$ $(i = 0, 1, \dots)$ and $V_i(A)$ $(i = 0, 1, \dots)$ are defined, respectively, by $(1.1.35_l)$ *and* $(1.1.37_l)$ *for some $l \in \{1, 2\}$, and $c = b + (a - b)(l - 1)$.*

Theorem 2.1.2$_1$. *Let there exist natural numbers k and m such that the matrix*

$$M_k = \sum_{j=1}^{\nu} L_j \Big(\sum_{i=0}^{k-1} (A)_i(t_j) - 1 \Big)$$

is nonsingular and inequality (2.1.6) *holds, where*

$$M_{k,m} = (V(A))_m(c) + \Big(I_n + \sum_{i=0}^{m-1} |(A)_i|_\infty \Big) \sum_{j=1}^{\nu} |M_k^{-1} L_j| (V(A))_k(t_j),$$

the operators $(A)_i$ $(i = 0, 1 \dots)$ and $(V(A))_i$ $(i = 0, 1, \dots)$ are defined by $(1.1.36_l)$ *and* $(1.1.37_l)$*, respectively for some $l \in \{1, 2\}$, and $c = b + (a - b)(l - 1)$. Then problem* (2.1.1), (2.1.2) *is uniquely solvable.*

The following corollary is a special case of Theorem 2.1.2$_1$, where $k = 1$ and $m = 1$.

Corollary 2.1.1. *Let*

$$\det \Big(\sum_{j=1}^{\nu} L_j \Big) \neq 0 \tag{2.1.7}$$

and

$$r\Big(L_0 \bigvee_a^b (A) \Big) < 1,$$

where

$$L_0 = I_n + \bigg| \Big(\sum_{j=1}^{\nu} L_j \Big)^{-1} \bigg| \sum_{j=1}^{\nu} |L_j|.$$

Then problem (2.1.1), (2.1.2) *is uniquely solvable.*

For the system

$$dx(t) = \varepsilon \, dA(t) \cdot x(t) + df(t) \tag{2.1.8}$$

with small parameter ε, from Theorem 2.1.2 it follows

Corollary 2.1.2. *Let either condition* (2.1.7) *hold, or there exist a natural number k such that the conditions*

$$\sum_{j=1}^{\nu} L_j = O_{n\times n}, \quad \det \Big(\sum_{j=1}^{\nu} L_j (A)_i(t_j) \Big) = 0 \ \ (i = 0, \dots, k-1)$$

and

$$\det \Big(\sum_{j=1}^{\nu} L_j (A)_k(t_j) \Big) \neq 0$$

hold. Then there exists $\varepsilon_0 > 0$ such that problem (2.1.8), (2.1.2) *is uniquely solvable for every $\varepsilon \in \,]0, \varepsilon_0[$.*

The results of this subsection are the particular cases of those given above in the previous section.

2.2 The Cauchy–Nicoletti type multi-point boundary value problems

In this section, we consider a linear system of generalized ordinary differential equations

$$dx = dA(t) \cdot x + df(t) \ \text{ for } \ t \in [a, b]. \tag{2.2.1}$$

Below we assume that

$$A = (a_{ik})_{i,k=1}^{n} \in \mathrm{BV}([a, b]; \mathbb{R}^{n\times n}), \ \ f = (f_k)_{k=1}^{n} \in \mathrm{BV}([a, b]; \mathbb{R}^n).$$

We investigate the question on the existence of solutions of system (2.2.1) under the following boundary value conditions:

(i) the Cauchy–Nicoletti type problem

$$x_i(t_i) = \ell_i(x_1, \dots, x_n) + c_{0i} \ \ (i = 1, \dots, n), \tag{2.2.2}$$

where $\ell_i : \mathrm{BV}_\infty([a, b]; \mathbb{R}^n) \to \mathbb{R}$ $(i = 1, \dots, n)$ are linear bounded functional.

(ii) the Cauchy–Nicoletti problem

$$x_i(t_i) = c_{0i} \ \ (i = 1, \dots, n), \tag{2.2.3}$$

where $c_{0i} \in \mathbb{R}$, and x_i is the i-th component of the vector-function x for every $i \in \{1, \dots, n\}$.

Along with problems (2.2.1), (2.2.2) and (2.2.1), (2.2.3), we consider the corresponding homogeneous system

$$dx = dA(t) \cdot x \tag{2.2.1$_0$}$$

under the homogeneous boundary value conditions

$$x_i(t_i) = \ell_i(x_1, \dots, x_n) \ \ (i = 1, \dots, n), \tag{2.2.2$_0$}$$

and

$$x_i(t_i) = 0 \ \ (i = 1, \dots, n). \tag{2.2.3$_0$}$$

Before we proceed to formulate the results, we introduce the following

Definition 2.2.1. Let $t_1, \dots, t_n \in [a, b]$. We say that a pair (C, ℓ_0) consisting of a matrix-function $C = (c_{il})_{i,l=1}^{n} \in \mathrm{BV}([a, b]; \mathbb{R}^{n\times n})$ and a bounded vector-functional $\ell_0 = (\ell_{0i})_{i=1}^{n} : \mathrm{BV}_\infty([a, b]; \mathbb{R}_+^{n\times n}) \to \mathbb{R}_+^n$ belongs to the set $\mathbb{U}(t_1, \dots, t_n)$ if:

(i) the matrix-function C is quasi-nondecreasing, i.e., the functions c_{il} $(i \neq l;\ i, l = 1, \dots, n)$ are nondecreasing on $[a, b]$;

(ii) ℓ_0 is a positive homogeneous, bounded and nondecreasing vector-functional;

(iii) the system of generalized differential inequalities

$$\begin{gathered} \operatorname{sgn}(t - t_i) dx_i(t) \le \sum_{l=1}^{n} x_l(t)\, dc_{il}(t) \ \text{ for } \ t \in [a, b], \ \ t \neq t_i \ \ (i = 1, \dots, n), \\ (-1)^j\, d_j x_i(t_i) \le \sum_{l=1}^{n} x_l(t_i)\, d_j c_{il}(t_i) \ \ (j = 1, 2;\ i = 1, \dots, n) \end{gathered} \tag{2.2.4}$$

has no nontrivial, nonnegative solution satisfying the condition

$$x_i(t_i) \le \ell_{0i}(x_1, \dots, x_n) \ \ (i = 1, \dots, n). \tag{2.2.5}$$

The set $\mathbb{U}(t_1, \dots, t_n)$ was introduced by I. Kiguradze for ordinary differential equations (see [46,47]).

2.2.1 Formulation of the results

Theorem 2.2.1. *Let the conditions*

$$\big(s_c(a_{ii})(t)-s_c(a_{ii})(s)\big)\operatorname{sgn}(t-s)\leq s_c(c_{ii})(t)-s_c(c_{ii})(s) \ \text{ for } \ (t-s)(s-t_i)>0 \ \ (i=1,\dots,n), \tag{2.2.6}$$

$$\big|s_c(a_{il})(t)-s_c(a_{il})(s)\big|\leq s_c(c_{il})(t)-s_c(c_{il})(s) \ \text{ for } \ s<t \ \ (i\neq l;\ i,l=1,\dots,n), \tag{2.2.7}$$

$$|d_j a_{ii}(t)|\leq |d_j c_{ii}(t)|, \ \ |d_j a_{il}(t)|\leq d_j c_{il}(t) \ \ (j=1,2;\ i\neq l;\ i,l=1,\dots,n) \tag{2.2.8}$$

hold on $[a,b]$, *and*

$$|\ell_i(x_1,\dots,x_n)|\leq \ell_{0i}\big(|x_1|,\dots,|x_n|\big) \ \text{ for } \ x_l\in \mathrm{BV}([a,b];\mathbb{R}) \ \ (i,l=1,\dots,n), \tag{2.2.9}$$

where a matrix-function $C=(c_{il})_{i,l=1}^n\in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ *and a vector-functional* $\ell_0=(\ell_{0i})_{i=1}^n$ *are such that*

$$(C,\ell_0)\in \mathbb{U}(t_1,\dots,t_n). \tag{2.2.10}$$

Then problem (2.2.1), (2.2.2) *has one and only one solution.*

Theorem 2.2.2. *Let the conditions*

$$\big(s_c(a_{ii})(t)-s_c(a_{ii})(s)\big)\operatorname{sgn}(t-s)\leq \int_s^t h_{ii}(\tau)\,ds_c(\alpha_i)(\tau) \ \text{ for } \ (t-s)(s-t_i)>0 \ \ (i=1,\dots,n), \tag{2.2.11}$$

$$\big|s_c(a_{il})(t)-s_c(a_{il})(s)\big|\leq \int_s^t h_{il}(\tau)\,ds_c(\alpha_l)(\tau) \ \text{ for } \ s<t \ \ (i\neq l;\ i,l=1,\dots,n) \tag{2.2.12}$$

and

$$|d_j a_{ii}(t)|\leq |h_{ii}(t)|\,d_j\alpha_l(t), \ \ |d_j a_{il}(t)|\leq h_{il}(t)\,d_j\alpha_l(t) \ \ (j=1,2;\ i\neq l;\ i,l=1,\dots,n) \tag{2.2.13}$$

hold on $[a,b]$, *where* α_l $(l=1,\dots,n)$ *are the functions nondecreasing on* $[a,b]$ *and having not more than a finite number of discontinuity points,* $h_{ii}\in L^\mu([a,b],\mathbb{R};\alpha_i)$, $h_{il}\in L^\mu([a,b],\mathbb{R}_+;\alpha_l)$ $(i\neq l;$ $l=1,\dots,n)$, $1\leq\mu\leq+\infty$. *Let, moreover,*

$$|\ell_i(x_1,\dots,x_n)|\leq \sum_{m=0}^{2}\sum_{k=1}^{n} l_{mik}\|x_k\|_{\nu,s_m(\alpha_k)} \ \text{ for } \ x_k\in \mathrm{BV}([a,b];\mathbb{R}) \ \ (i,k=1,\dots,n) \tag{2.2.14}$$

and

$$r(\mathcal{H})<1, \tag{2.2.15}$$

where $l_{mik}\in\mathbb{R}_+$ $(m=0,1,2;\, i,k=1,\dots,n)$, $\frac{1}{\mu}+\frac{2}{\nu}=1$, *and the* $3n\times 3n$*-matrix* $\mathcal{H}=(\mathcal{H}_{j+1\,m+1})_{j,m=0}^2$ *is defined by*

$$\mathcal{H}_{j+1\,m+1}=\big(\xi_{ij}l_{mik}+\lambda_{kmij}\|h_{ik}\|_{\mu,S_m(\alpha_i)}\big)_{i,k=1}^n \ \ (j,m=0,1,2),$$

$$\xi_{ij}=\big(s_j(\alpha_i)(b)-s_j(\alpha_i)(a)\big)^{\frac{1}{\nu}} \ \ (j=0,1,2,;\ i=1,\dots,n);$$

$$\lambda_{k0i0}=\begin{cases}\Big(\dfrac{4}{\pi^2}\Big)^{\frac{1}{\nu}}\xi_{k0}^2 & \text{if } \ s_c(\alpha_i)(t)\equiv s_c(\alpha_k)(t),\\ \xi_{k0}\xi_{i0} & \text{if } \ s_c(\alpha_i)(t)\not\equiv s_c(\alpha_k)(t) \ \ (i,k=1,\dots,n);\end{cases}$$

$$\lambda_{kmij}=\xi_{km}\xi_{ij} \ \text{ if } \ m^2+j^2>0, \ \ mj=0 \ \ (j,m=0,1,2;\ i,k=1,\dots,n),$$

$$\lambda_{kmij}=\Big(\frac{1}{4}\,\mu_{\alpha_k m}\nu_{\alpha_k m\alpha_i j}\sin^{-2}\frac{\pi}{4n_{\alpha_k m}+2}\Big)^{\frac{1}{\nu}} \ \ (j,m=1,2;\ i,k=1,\dots,n).$$

Then problem (2.2.1), (2.2.2) *has one and only one solution.*

Remark 2.2.1. The $3n\times 3n$-matrix $\mathcal{H}$ appearing in Theorem 2.2.2 can be replaced by the $n\times n$-matrix

$$\bigg(\max\Big\{\sum_{j=0}^{2}\big(\xi_{ij}l_{mik}+\lambda_{kmij}\|h_{ik}\|_{\mu,S_m(\alpha_k)}\big):\ m=0,1,2\Big\}\bigg)_{i,k=1}^{n}.$$

Corollary 2.2.1. *Let conditions* (2.2.11)–(2.2.13) *hold on* $[a,b]$*, where* α_l $(l=1,\dots,n)$ *are the functions nondecreasing on* $[a,b]$ *and having not more than a finite number of discontinuity points,* $h_{ii}\in L^{\mu}([a,b],\mathbb{R};\alpha_i)$*,* $h_{il}\in L^{\mu}([a,b],\mathbb{R}_+;\alpha_l)$ $(i\neq l;\ i,l=1,\dots,n)$*,* $1\leq\mu\leq+\infty$*. Let, moreover,*

$$r(\mathcal{H}_0)<1, \tag{2.2.16}$$

where $\mathcal{H}_0=((\lambda_{kmij}\|h_{ik}\|_{\mu,s_m(\alpha_i)})_{i,k=1}^{n})_{m,j=0}^{2}$ *is a* $3n\times 3n$*-matrix, and* λ_{kmij}*,* ξ_{ij} $(j,m=0,1,2;$ $i,k=1,\dots,n)$ *and* ν *are defined as in Theorem 2.2.2. Then problem* (2.2.1), (2.2.3) *has one and only one solution.*

Remark 2.2.2. The $3n\times 3n$-matrix $\mathcal{H}_0$ appearing in Corollary 2.2.1 can be replaced by the $n\times n$-matrix

$$\bigg(\max\Big\{\sum_{j=0}^{2}\lambda_{kmij}\|h_{ik}\|_{\mu,S_m(\alpha_k)}:\ m=0,1,2\Big\}\bigg)_{i,k=1}^{n}.$$

By Remark 2.2.2, Corollary 2.2.1 has the following form for $h_{il}(t)\equiv h_{il}=const$ $(i,l=1,\dots,n)$ and $\mu=+\infty$.

Corollary 2.2.2. *Let the conditions*

$$\begin{gathered}\big(s_c(a_{ii})(t)-s_c(a_{ii})(s)\big)\operatorname{sgn}(t-s)\leq h_{ii}\big|s_c(\alpha)(t)-s_c(\alpha)(s)\big|\ \text{ for }\ (t-s)(s-t_i)>0\\ \big|s_c(a_{il})(t)-s_c(a_{il})(s)\big|\leq h_{il}\big(s_c(\alpha)(t)-s_c(\alpha)(s)\big)\ \text{ for }\ s<t\ \ (i\neq l;\ i,l=1,\dots,n)\end{gathered}$$

and

$$|d_ja_{ii}(t)|\leq h_{ii}\,d_j\alpha(t),\ \ |d_ja_{il}(t)|\leq h_{il}\,d_j\alpha(t)\ \ (j=1,2;\ i\neq l;\ i,l=1,\dots,n)$$

hold on $[a,b]$*, where* α *is a function nondecreasing on* $[a,b]$ *and having not more than a finite number of discontinuity points,* $h_{ii}\in\mathbb{R}$*,* $h_{il}\in\mathbb{R}_+$ $(i\neq l;\ i,l=1,\dots,n)$*. Let, moreover,*

$$\rho_0\,r(\mathcal{H})<1, \tag{2.2.17}$$

where

$$\begin{gathered}\mathcal{H}=(h_{ik})_{i,k=1}^{n},\ \ \rho_0=\max\Big\{\sum_{j=0}^{2}\lambda_{mj}:\ m=0,1,2\Big\},\\ \lambda_{00}=\frac{2}{\pi}\big(s_c(\alpha)(b)-s_c(\alpha)(a)\big),\\ \lambda_{0j}=\lambda_{j0}=\big(s_c(\alpha)(b)-s_c(\alpha)(a)\big)^{\frac12}\big(s_j(\alpha)(b)-s_j(\alpha)(a)\big)^{\frac12}\ \ (j=1,2),\\ \lambda_{mj}=\frac12\big(\mu_{\alpha m}\nu_{\alpha m\alpha j}\big)^{\frac12}\sin^{-1}\frac{\pi}{4n_{\alpha_m+2}+2}\ \ (m,j=1,2).\end{gathered}$$

Then problem (2.2.1), (2.2.3) *has one and only one solution.*

Remark 2.2.3. Condition (2.2.17) is optimal in the sense that it cannot be replaced by the nonstrict inequality

$$\rho_0 r(\mathcal{H})\leq 1.$$

The corresponding example is constructed for ordinary differential equations in [47]. For the sake of completeness, we present here this example.

Consider the problem

$$\frac{dx_1}{dt}=x_2,\quad \frac{dx_2}{dt}=-\frac{\pi^2}{4(b-a)^2}\,x_1, \tag{2.2.18}$$

$$x_1(a) = 0, \quad x_2(b) = 0. \tag{2.2.19}$$

In this case,

$$n = 2, \;\; t_1 = a, \;\; t_2 = b, \;\; a_{11}(t) = a_{22}(t) \equiv 0, \;\; a_{12}(t) \equiv t, \;\; a_{21}(t) \equiv -\frac{\pi}{4(b-a)^2}\, t,$$

and conditions (1.2.31)–(1.2.34) are fulfilled for

$$h_{11} = h_{22} = 0, \;\; h_{12} = 1, \;\; h_{21} = \frac{\pi^2}{4(b-a)^2}\,, \;\; \alpha(t) \equiv t.$$

Moreover,

$$\rho_0 = \frac{2(b-a)}{\pi}\,,$$

and

$$\lambda_1 = \frac{\pi}{2(b-a)} \;\text{ and }\; \lambda_2 = -\frac{\pi}{2(b-a)}$$

are the eigenvalues of the matrix

$$H = \begin{pmatrix} 0 & 1 \\ \dfrac{\pi^2}{4(b-a)^2} & 0 \end{pmatrix}.$$

Therefore,

$$\rho_0 r(H) = 1. \tag{2.2.20}$$

Thus for problem (2.2.18), (2.2.19) all conditions of Corollary 2.2.2 are fulfilled except for condition (2.2.17), instead of which equality (2.2.20) is fulfilled. On the other hand, the problem is not uniquely solvable, since it has a nontrivial solution

$$x_1(t) = \sin\frac{\pi(t-a)}{2(b-a)}\,, \;\; x_2(t) = \frac{\pi}{2(b-a)}\,\cos\frac{\pi(t-a)}{2(b-a)}$$

along with the trivial one.

Theorem 2.2.3. *Let conditions* (2.2.6)–(2.2.8)*,*

$$1 + (-1)^j\, d_j c_{ii}(t) > 0 \;\; (j = 1,2; \; i = 1,\dots,n), \tag{2.2.21}$$

$$|\ell_i(x_1,\dots,x_n)| \le |\mu_i|\, |x_i(\tau_i)| \;\; \textit{for} \;\; x_l \in \mathrm{BV}([a,b];\mathbb{R}) \;\; (i,l = 1,\dots,n) \tag{2.2.22}$$

and

$$|\mu_i| \gamma_i(\tau_i) < 1 \;\; (i = 1,\dots,n) \tag{2.2.23}$$

hold on $[a,b]$, where the functions c_{ii} $(i = 1,\dots,n)$ are non-increasing on $[a,b]$, $\mu_i \in \mathbb{R}$, $\tau_i \in [a,b]$, $\tau_i \ne t_i$ $(i = 1,\dots,n)$; $\lambda_i(t) \equiv \gamma_{a_i}(t,t_i)$, the function $\gamma_{a_i}(t,t_i)$ is defined according to (1.1.9)*, and $a_i(t) \equiv (c_{ii}(t) - c_{ii}(t_i))\operatorname{sgn}(t - t_i)$ $(i = 1,\dots,n)$. Let, moreover, the functions $c_{il} \in \mathrm{BV}([a,b];\mathbb{R})$ $(i \ne l;\, i,l = 1,\dots,n)$ be such that*

$$r(\mathcal{M}) < 1, \tag{2.2.24}$$

where $\mathcal{M} = (\mu_{il})_{i,l=1}^n$,

$$\mu_{ii} = 0, \quad \mu_{il} = \mu_i (1 - \mu_i \lambda_i(\tau_i))^{-1} |f_{il}(\tau_i) - f_{il}(t_i)| + (f_{il}(b) - f_{il}(t_i)),$$

$$f_{il}(t) \equiv \bigvee_a^t (s_c(c_{il})) + \sum_{a<\tau\le t} |d_1 c_{il}(\tau)| + \sum_{a\le\tau<t} |d_2 c_{il}(\tau)| \;\; (i \ne l; \; i,l = 1,\dots,n).$$

Then problem (2.2.1)*,* (2.2.2) *has one and only one solution.*

Remark 2.2.4. In particular, the statement of Theorem 2.2.3 is true for the boundary value condition

$$x(t_i) = \mu_i x_i(\tau_i) + c_{0i} \;\; (i = 1,\dots,n). \tag{2.2.25}$$

Theorem 2.2.4. *Let conditions* (2.2.6)–(2.2.8), (2.2.22) *and* (2.2.23) *hold on* $[a,b]$, *where* $\mu_i \geq 0$, $\tau_i \in [a,b]$, $\tau_i \neq t_i$ $(i=1,\dots,n)$, *and let the functions* c_{ii} $(i=1,\dots,n)$ *be such that the functions* $\lambda_i(t) \equiv \gamma_{c_{ii}}(t,t_i)$ $(i=1,\dots,n)$, *defined according to* (1.1.9), *are monotone on the intervals* $[a,t_i[$ *and* $]t_i,b]$. *Let, moreover, the functions* $c_{il} \in \mathrm{BV}([a,b];\mathbb{R})$ $(i \neq l;\ i,l=1,\dots,n)$ *be nondecreasing on* $[a,b]$ *and condition* (2.2.24) *hold, where* $\mathcal{M} = (\mu_{il})_{i,l=1}^n$,

$$\mu_{ii}=0, \quad \mu_{il} = \zeta_{il}(1+\xi_i) + \nu_i\big(1+|\lambda_i(\tau_i)-1|\big)\Big|\bigvee_{t_i}^{\tau_i}(c_{il})\Big|, \ \ \zeta_{il} = \max\Big\{\bigvee_a^{t_i}(c_{il}), \bigvee_{t_i}^{b}(c_{il})\Big\},$$

$$\nu_i = \mu_i(1-\mu_i\lambda_i(\tau_i))^{-1}\|\lambda_i\|_\infty, \ \ \eta_i = \sup\big\{|\lambda_i(t)-1| :\ t\in[a,b]\big\},$$

$$f_{il}(t) \equiv \bigvee_a^t(s_c(c_{il})) + \sum_{a<\tau\leq t}|d_1c_{il}(\tau)| + \sum_{a\leq\tau<t}|d_2c_{il}(\tau)| \ \ (i\neq l;\ i,l=1,\dots,n).$$

Then problem (2.2.1), (2.2.25) *has one and only one solution.*

Below, we give a general theorem on the unsolvability of problem (2.2.1), (2.2.2) in the case where condition (2.2.10) is violated.

Theorem 2.2.5. *Let* $\ell_{0i} : \mathrm{BV}_\infty([a,b];\mathbb{R}_+^n) \to \mathbb{R}_+$ $(i=1,\dots,n)$ *be linear bounded functionals, the matrix-function* $C = (c_{il})_{i,l=1}^n \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ *be such that the functions* c_{il} $(i\neq l;\ i,l=1,\dots,n)$ *are nondecreasing on* $[a,b]$ *and problem* (2.2.4), (2.2.5) *has a nontrivial nonnegative solution* $x = (x_i)_{i=1}^n$, *i.e., condition* (2.2.10) *is violated. Let, moreover,*

$$d_jc_{ii}(t) \geq 0 \ \ \text{for} \ \ t\in[a,b] \ \ (j=1,2;\ i=1,\dots,n). \tag{2.2.26}$$

Then there exist a matrix-function $A = (a_{il})_{i,l=1}^n \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$, *linear bounded functionals* $\ell_i : \mathrm{BV}_\infty([a,b];\mathbb{R}^n) \to \mathbb{R}$ $(i=1,\dots,n)$ *and numbers* $c_{0i}\in\mathbb{R}$ $(i=1,\dots,n)$ *such that conditions* (2.2.6)–(2.2.9) *hold, but problem* $(2.2.1_0)$, (2.2.2) *is unsolvable. In addition, if the matrix-function* $C = (c_{il})_{i,l=1}^n$ *is such that*

$$\det\Big(I_n + (-1)^j\,\mathrm{diag}(\mathrm{sgn}(t-t_1),\dots,\mathrm{sgn}(t-t_n))\,d_jC(t) \times \mathrm{diag}(\varepsilon_1,\dots,\varepsilon_n)\Big) \neq 0 \ \ \text{for} \ \ t\in[a,b] \ \ (j=1,2), \tag{2.2.27}$$

where $\varepsilon_i \in [0,1]$ $(i=1,\dots,n)$, *then the matrix-function* $A = (a_{il})_{i,l=1}^n$ *satisfies condition* (1.1.8).

Remark 2.2.5. Condition (2.2.27) holds, for example, if either

$$\sum_{l=1}^n |d_jc_{il}(t)| < 1 \ \ \text{for} \ \ t\in[a,b] \ \ (j=1,2;\ i=1,\dots,n), \tag{2.2.28}$$

or

$$d_jc_{ii}(t) \leq 1 \ \ \text{for} \ \ (-1)^j(t-t_i)<0 \ \ (j=1,2;\ i=1,\dots,n) \tag{2.2.29}$$

and

$$\sum_{l=1,\,l\neq i}^n |d_jc_{il}(t)| < \big|1+(-1)^j\,\mathrm{sgn}(t-t_i)\,d_jc_{ii}(t)\big| \ \ \text{for} \ \ t\in[a,b] \ \ (j=1,2;\ i=1,\dots,n)$$
$$\bigg(\sum_{l=1,\,l\neq i}^n |d_jc_{li}(t)| < \big|1+(-1)^j\,\mathrm{sgn}(t-t_i)\,d_jc_{ii}(t)\big| \ \ \text{for} \ \ t\in[a,b] \ \ (j=1,2;\ i=1,\dots,n)\bigg). \tag{2.2.30}$$

2.2.2 Auxiliary propositions

We give here the following lemma dealing with the differential inequalities.

Lemma 2.2.1. *Let $t_1,\dots,t_n$, $B=(b_{il})_{i,l=1}^n\in\mathrm{BV}([a,b];\mathbb{R}^{n\times n})$, $q=(q_i)_{i=1}^n\in\mathrm{BV}([a,b];\mathbb{R}^n)$, $C=(c_{il})_{i,l=1}^n\in\mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ be such that the functions $c_{il}\operatorname{sgn}(t-t_i)$ $(i\neq l;\ i,l=1,\dots,n)$ are nondecreasing on $[a,b]$ and the conditions*

$$\big(s_c(b_{ii})(t)-s_c(b_{ii})(s)\big)\operatorname{sgn}(t-t_i)\le s_c(c_{ii})(t)-s_c(c_{ii})(s)\ \text{ for }\ (t-s)(s-t_i)>0\quad (i=1,\dots,n),\tag{2.2.31}$$

$$\big|s_c(b_{il})(t)-s_c(b_{il})(s)\big|\le s_c(c_{il})(t)-s_c(c_{il})(s)\ \text{ for }\ s<t\ \ (i\neq l;\ i,l=1,\dots,n)\tag{2.2.32}$$

and

$$|d_jb_{ii}(t)|\le|d_jc_{ii}(t)|,\ \ |d_jb_{il}(t)|\le d_jc_{il}(t)\ \ (j=1,2;\ i\neq l;\ i,l=1,\dots,n)\tag{2.2.33}$$

hold on $[a,b]$. Then every solution $x=(x_i)_{i=1}^n$ of the system

$$dx=dB(t)\cdot x+dq(t)\ \text{ for }\ t\in[a,b]\tag{2.2.34}$$

will be a solution of the system of generalized differential inequalities

$$\begin{aligned}\operatorname{sgn}(t-t_i)\,d|x_i(t)|\le\sum_{l=1}^n|x_l(t)|\,dc_{il}(t)&\\ +\operatorname{sgn}x_i(t)\,ds_c(q_i)(t)+d_jg_i(t)\ \text{ for }\ t\in[a,b]\ \ (i=1,\dots,n),&\qquad(2.2.35)\\ (-1)^j\,d_j|x_i(t_i)|\le\sum_{l=1}^n|x_l(t_i)|\,d_jc_{il}(t_i)+d_jg_i(t_i)\ \ (j=1,2;\ i=1,\dots,n),&\end{aligned}$$

where

$$g_i(t)\equiv\sum_{j=1}^2\bigvee_a^t s_j(q_i)\ \ (i=1,\dots,n).$$

Proof. Let $i\in\{1,\dots,n\}$ be fixed. Using (0.0.16) for $\int_s^t\operatorname{sgn}x_i(\tau)\,dx_i(\tau)$ $(s<t)$ and the definition of the solution of system (2.2.34), it can be easily shown that

$$\begin{aligned}|x_i(t)|-|x_i(s)|=\sum_{l=1}^n\int_s^t x_l(\tau)\operatorname{sgn}x_i(\tau)\,ds_c(b_{il})(\tau)+\sum_{s<\tau\le t}\big(|x_i(\tau)|-|x_i(\tau-)|\big)&\\ +\sum_{s\le\tau<t}\big(|x_i(\tau+)|-|x_i(\tau)|\big)+\int_s^t\operatorname{sgn}x_i(\tau)\,ds_c(q_i)(\tau)\ \text{ for }\ t_i<s<t\le b.&\qquad(2.2.36)\end{aligned}$$

By (2.2.31) and (2.2.32), from the above equality we have

$$\begin{aligned}|x_i(t)|-|x_i(s)|\le\int_s^t|x_i(\tau)|\,ds_c(c_{ii})(\tau)+\sum_{l\neq i;\ l=1}^n\int_s^t|x_l(\tau)|\,ds_c(c_{il})(\tau)+\sum_{s<\tau\le t}\big(|x_i(\tau)|-|x_i(\tau-)|\big)&\\ +\sum_{s\le\tau<t}\big(|x_i(\tau+)|-|x_i(\tau)|\big)+\int_s^t\operatorname{sgn}x_i(\tau)\,ds_c(q_i)(\tau)\ \text{ for }\ t_i<s<t\le b.&\qquad(2.2.37)\end{aligned}$$

Moreover, it is evident that

$$\begin{aligned}\int_s^t|x_l(\tau)|\,ds_c(c_{il})(\tau)=\int_s^t|x_l(\tau)|\,dc_{il}(\tau)&\\ -\sum_{s<\tau\le t}|x_l(\tau)|\,d_1c_{il}(\tau)-\sum_{s\le\tau<t}|x_l(\tau)|\,d_2c_{il}(\tau)\ \text{ for }\ t_i<s<t\le b\ \ (l=1,\dots,n).&\qquad(2.2.38)\end{aligned}$$

In addition, due to (2.2.31)–(2.2.33), it is not difficult to verify the inequalities

$$|x_i(\tau)| - |x_i(\tau-)| \leq \Big| \sum_{l=1}^{n} x_l(\tau)\, d_1 b_{il}(\tau) + d_1 q_i(\tau) \Big|$$
$$\leq \sum_{l=1}^{n} |x_l(\tau)|\, |d_1 c_{il}(\tau)| + |d_1 q_i(\tau)| \ \text{ for } \ t_i < t \leq b \tag{2.2.39}$$

and

$$|x_i(\tau+)| - |x_i(\tau)| \leq \Big| \sum_{l=1}^{n} x_l(\tau)\, d_2 b_{il}(\tau) + d_2 q_i(\tau) \Big|$$
$$\leq \sum_{l=1}^{n} |x_l(\tau)|\, |d_2 c_{il}(\tau)| + |d_2 q_i(\tau)| \ \text{ for } \ t_i \leq t < b. \tag{2.2.40}$$

Inserting (2.2.38)–(2.2.40) into (2.2.37) and using (2.2.33), we get

$$|x_i(t)| - |x_i(s)| \leq \sum_{l=1}^{n} \int_s^t |x_l(t)|\, dc_{il}(t) + \int_s^t \operatorname{sgn} x_i(\tau)\, ds_c(q_i)(\tau) + g_i(t) - g_i(s)$$
$$\text{for } \ t_i < s < t \leq b.$$

Similarly, we show

$$-\big(|x_i(t)| - |x_i(s)|\big) \leq \sum_{l=1}^{n} \int_s^t |x_l(t)|\, dc_{il}(t) + \int_s^t \operatorname{sgn} x_i(\tau)\, ds_c(q_i)(\tau) + g_i(t) - g_i(s)$$
$$\text{for } \ a \leq s < t < t_i.$$

Therefore, the first estimate of (2.2.35) holds. As to the second estimate, it follows from the first one. □

Lemma 2.2.2. *Let $t_1, \dots, t_n;\ b_{il}, q_i \in \mathrm{BV}([a,b];\mathbb{R})$ $(i,l = 1,\dots,n)$ be such that the functions $b_{il} \operatorname{sgn}(t - t_i)$ $(i \neq l;\ i,l = 1,\dots,n)$ are nondecreasing on $[a,b]$. Then every solution $x = (x_i)_{i=1}^n$ of the system*

$$dx_i(t) = \sum_{l=1}^{n} x_l(t)\, db_{il}(t) + dq_i(t) \ \ \textit{for} \ \ t \in [a,b] \ \ (i = 1,\dots,n)$$

will be a solution of the system of generalized differential inequalities

$$\operatorname{sgn}(t - t_i)\, d|x_i(t)| \leq \sum_{l=1}^{n} |x_l(t)|\, db_{il}(t) + \operatorname{sgn} x_i(t)\, ds_c(q_i)(t) + dg_i(t) \ \ \textit{for} \ \ t \in [a,b] \ \ (i = 1,\dots,n),$$
$$(-1)^j d|x_i(t_i)| \leq \sum_{l=1}^{n} |x_l(t_i)|\, db_{il}(t_i) + \operatorname{sgn} x_i(t_i)\, ds_c(q_i)(t_i) + dg_i(t_i) \ \ (j = 1,2;\ i = 1,\dots,n),$$

where the functions g_i $(i = 1,\dots,n)$ are defined as in Lemma 2.2.1.

Proof. The lemma follows from Lemma 2.2.1 if we assume therein that $c_{il}(t) \equiv b_{il}(t)$ $(i,l = 1,\dots,n)$. In this case, conditions (2.2.31)–(2.2.33) are fulfilled automatically. □

Lemma 2.2.3. *Let $g \in \mathrm{BV}([a,b];\mathbb{R}^{\ltimes})$ and let $C \in \mathrm{BV}([a,b];\mathbb{R}^{\ltimes\times\ltimes})$ be such that*

$$1 + (-1)^j\, d_j C(t) \neq 0 \ \ \textit{for} \ \ t \in [a,b]. \tag{2.2.41}$$

Let, moreover, $\xi \in \mathrm{BV}([a,b];\mathbb{R}^n)$ be a solution of the system

$$dx = dC(t) \cdot x + dg(t). \tag{2.2.42}$$

Then

$$Y^{-1}(t)\xi(t)-Y^{-1}(s)\xi(s)=\mathcal{B}(Y^{-1},g)(t)-\mathcal{B}(Y^{-1},g)(s)\ \ \textit{for}\ \ a\le s<t\le b, \tag{2.2.43}$$

where $Y\in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ *is a fundamental matrix of the system*

$$dx=dC(t)\cdot x. \tag{2.2.42$_0$}$$

Proof. By (2.2.41), the fundamental matrix of system (2.2.42$_0$) exists.

Let $a\le s<t\le b$. Due to (1.1.13), (2.2.42) and the integration-by-parts formula, we have

$$\begin{aligned}
&Y^{-1}(t)\xi(t)-Y^{-1}(s)\xi(s)\\
&\quad=\int_s^t Y^{-1}(\tau)\,d\xi(\tau)+\int_s^t \xi(\tau)\,dY^{-1}(\tau)-\sum_{s<\tau\le t}d_1Y^{-1}(\tau)\cdot d_1g(\tau)+\sum_{s\le\tau<t}d_2Y^{-1}(\tau)\cdot d_2g(\tau)\\
&\qquad=\int_s^t Y^{-1}(\tau)\xi(\tau)\,dC(\tau)+\int_s^t Y^{-1}(\tau)\,dg(\tau)+\int_s^t \xi(\tau)\,dY^{-1}(\tau)\\
&\qquad\quad-\sum_{s<\tau\le t}d_1Y^{-1}(\tau)\cdot\big(\xi(\tau)\,d_1C(\tau)+d_1g(\tau)\big)+\sum_{s\le\tau<t}d_2Y^{-1}(\tau)\cdot\big(\xi(\tau)\,d_2C(\tau)+d_2g(\tau)\big)
\end{aligned}$$

and

$$\begin{aligned}
Y^{-1}(\tau)=Y^{-1}(s)-\int_s^\tau Y^{-1}(\sigma)\,dC(\sigma)&\\
+\sum_{s<\sigma\le\tau}d_1Y^{-1}(\sigma)\cdot d_1C(\sigma)-\sum_{s\le\sigma<\tau}d_2Y^{-1}(\sigma)\cdot d_2C(\sigma)&\ \ \text{for}\ \ s<\tau\le t.
\end{aligned}$$

By the latter equality,

$$\begin{aligned}
\int_s^t dY^{-1}(\tau)\cdot\xi(\tau)=-\int_s^t Y^{-1}(\tau)\,dC(\tau)\cdot\xi(\tau)&\\
+\sum_{s<\tau\le t}d_1Y^{-1}(\tau)\cdot d_1C(\tau)\cdot\xi(\tau)-\sum_{s\le\tau<t}d_2Y^{-1}(\tau)\cdot d_2C(\tau)\cdot\xi(\tau)&\ \ \text{for}\ \ s<t.
\end{aligned}$$

From this, using the integration-by-partes formulae, we conclude that

$$\begin{aligned}
Y^{-1}(t)\xi(t)-Y^{-1}(s)\xi(s)=\int_s^t Y^{-1}(\tau)\,dg(\tau)&\\
-\sum_{s<\tau\le t}d_1Y^{-1}(\tau)\cdot d_1g(\tau)+\sum_{s\le\tau<t}d_2Y^{-1}(\tau)\cdot d_2g(\tau)&\ \ \text{for}\ \ a\le s<t\le b.
\end{aligned}$$

So, by definition of the operator $\mathcal{B}$, equality (2.2.43) holds. □

We use the following lemma which is a particular case of Theorem 1.1.9 from [23]. For the completeness, we give some modification of the proof of the lemma.

Lemma 2.2.4. *Let* $t_0\in[a,b]$, $C=(c_{ik})_{i,k=1}^n\in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ *be such that the conditions*

$$\det\big(I_n+(-1)^j\,d_jC(t)\big)\ne 0\ \ \textit{for}\ \ (-1)^j(t-t_0)<0\ \ (j=1,2), \tag{2.2.44}$$

$$1+(-1)^j\,d_jc_{ii}(t)>0\ \ \textit{for}\ \ (-1)^j(t-t_0)\ge 0\ \ (j=1,2;\ i=1,\dots,n) \tag{2.2.45}$$

and

$$\big(I_n + (-1)^j\, d_j C(t)\big)^{-1} > O_{n\times n} \ \ for \ \ (-1)^j (t-t_0) < 0 \ \ (j=1,2). \tag{2.2.46}$$

hold on $[a,b]$. *Let, moreover, for every* $j \in \{1,2\}$, *the functions* $(-1)^{j+1} c_{il}$ $(i \neq l;\ i,l = 1,\dots,n)$ *be non-decreasing on the set* $J_j = \{t \in [a,b] : (-1)^j (t-t_0) < 0\}$. *Then*

$$U(t,s) \geq 0 \ \ for \ \ t \leq s \leq t_0 \ \ or \ \ t_0 \leq s \leq t, \tag{2.2.47}$$

where U $(U(s,s) \equiv I_n)$ *is the Cauchy matrix of system* $(2.2.42_0)$.

Proof. Let $s \in [a,b]$ $(s \neq t_0)$ be fixed and $j \in \{1,2\}$ be such that $s \in J_j$. Let $k \in \{1,\dots,n\}$ be fixed and let $u_k(t,s) = (u_{ik}(t,s))_{i=1}^n$ be the k-th column of the matrix $U(t,s)$.

Assume

$$y(t) = (y_i(t))_{i=1}^n \ \text{ for } \ t \in [a,b],$$

where $y_i(t) \equiv \gamma_i^{-1}(t,s)\, u_{ik}(t,s)$ $(i=1,\dots,n)$, $\gamma_i(t,s) = \gamma_i(t)\,\gamma_i^{-1}(s)$, and γ_i is a solution of the Cauchy problem $d\gamma(t) = \gamma(t)\, dc_{ii}(t)$, $\gamma(s) = 1$; here, in view of (1.1.9) and (2.2.45), $\gamma_i(t) > 0$ for $t \in [a,b]$ $(i=1,\dots,n)$.

Since $U(t,s) = (u_{il})_{i,l=1}^n$ is the Cauchy matrix of system $(2.2.42_0)$, we conclude that for every $i \in \{1,\dots,n\}$, the function u_{ik} is a solution of the equation

$$dx = dc_{ii}(t) \cdot x + dg_i(t),$$

where

$$g_i(t) \equiv \sum_{l\neq i;\, l=1}^{n} \int_a^t u_{lk}(\tau)\, dc_{il}(\tau).$$

So, according to Lemma 2.2.3 and the integration-by-parts formula, we find that

$$\begin{aligned}
y_i(t) - y_i(r) &= \mathcal{B}(\gamma_i^{-1}, g_i)(t) - \mathcal{B}(\gamma_i^{-1}, g_i)(r) \\
&= \int_r^t \gamma_i^{-1}(\tau,s)\, dg_i(\tau) - \sum_{r<\tau\leq t} d_1\gamma_i^{-1}(\tau,s)\, d_1 g_i(\tau) + \sum_{r\leq\tau<t} d_2\gamma_i^{-1}(\tau,s)\, d_2 g_i(\tau) \\
&= \sum_{l\neq i,\, l=1}^{n} \bigg(\int_r^t \gamma_i^{-1}(\tau,s) u_{lk}(\tau,s)\, ds_c(c_{il})(\tau) \\
&\quad + \sum_{r<\tau\leq t} \gamma_i^{-1}(\tau-,s) u_{lk}(\tau,s)\, d_1 c_{il}(\tau) + \sum_{r\leq\tau<t} \gamma_i^{-1}(\tau+,s) u_{lk}(\tau,s)\, d_2 c_{il}(\tau) \bigg) \\
&= \sum_{l\neq i,\, l=1}^{n} \bigg(\int_r^t \gamma_i^{-1}(\tau,s)\gamma_l(\tau,s)(\tau) y_l(\tau)\, ds_c(c_{il})(\tau) + \sum_{r<\tau\leq t} \gamma_i^{-1}(\tau-,s) y_l(\tau)\gamma_l(\tau,s)\, d_1 c_{il}(\tau) \\
&\quad + \sum_{r\leq\tau<t} \gamma_i^{-1}(\tau+,s) y_l(\tau)\gamma_l(\tau,s)\, d_2 c_{il}(\tau) \bigg) \ \text{ for } \ a \leq r \leq t \leq b \ \ (i=1,\dots,n).
\end{aligned}$$

Hence $y = (y_i)_{i=1}^n$ is a solution of the Cauchy problem

$$dy = dC^*(t) \cdot y, \ \ y(s) = e_k,$$

where $e_k = (\delta_{ik})_{i=1}^n$, $C^*(t) = (c^*_{il}(t))_{i,l=1}^n$, $c^*_{ii}(t) \equiv 0$ and

$$c^*_{il}(t) \equiv \int_a^t \gamma_i^{-1}(\tau,s)\gamma_l(\tau,s)(\tau)\, ds_c(c_{il})(\tau)$$

$$+\sum_{a<\tau\le t}\gamma_i^{-1}(\tau-,s)\gamma_l(\tau,s)\,d_1c_{il}(\tau)+\sum_{a\le\tau<t}\gamma_i^{-1}(\tau+,s)\gamma_l(\tau,s)\,d_2c_{il}(\tau)\ \ (i\neq l;\ i,l=1,\dots,n).$$

In view of the conditions of the lemma, the functions $(-1)^{j+1}c^*_{il}$ $(i\neq l;\ i,l=1,\dots,n)$ are nondecreasing on J_j $(j=1,2)$.

Let

$$\Lambda_s(t)=\operatorname{diag}(\gamma_1(t,s),\dots,\gamma_n(t,s))$$

and

$$Q(t)=\operatorname{diag}(c_{11}(t),\dots,c_{nn}(t))\ \text{ for }\ t\in[a,b].$$

Using (1.1.14), we have

$$\begin{aligned}I_n+(-1)^j\,d_jC^*(t)&=I_n+(-1)^j\big(\Lambda_s^{-1}(t)+(-1)^j\,d_j\Lambda_s^{-1}(t)\big)(d_jC(t)-d_jQ(t))\Lambda_s(t)\\&=\big(\Lambda_s^{-1}(t)+(-1)^j\,d_j\Lambda_s^{-1}(t)\big)\Big(\big(I_n+(-1)^j\,d_jQ(t)\big)\Lambda_s(t)+(-1)^j\big(d_jC(t)-d_jQ(t)\big)\Lambda_s(t)\Big)\\&\qquad\text{for }\ t\in[a,b]\ \ (j=1,2)\end{aligned}$$

and

$$\begin{aligned}I_n&+(-1)^j\,d_jC^*(t)\\&=\big(\Lambda_s^{-1}(t)+(-1)^j\,d_j\Lambda_s^{-1}(t)\big)(I_n+(-1)^j\,d_jC(t))\Lambda_s(t)\ \text{ for }\ t\in[a,b]\ \ (j=1,2).\end{aligned}$$

Hence, due to (2.2.44) and (2.2.45), we obtain

$$\det\big(I_n+(-1)^j\,d_jC^*(t)\big)\neq 0\ \text{ for }\ t\in[a,b]\setminus\{t_0\}\ \ (j=1,2)$$

and

$$\big(I_n+(-1)^j\,d_jC^*(t)\big)^{-1}\ge O_{n\times n}\ \text{ for }\ (-1)^j(t-t_0)<0\ \ (j=1,2),\tag{2.2.48}$$

since

$$\Lambda_s(t)\ge O_{n\times n}\ \text{ for }\ t\in[a,b].\tag{2.2.49}$$

According to Theorem 1.1.8 from [23],

$$\lim_{m\to+\infty}z_m(t)=y(t)\ \text{ uniformly on }\ [a,b],\tag{2.2.50}$$

where

$$\begin{gathered}z_m(s)=e_k\ \ (m=0,1,\dots),\\ z_0(t)=\big(I_n+(-1)^j\,d_jC^*(t)\big)^{-1}e_k\ \text{ for }\ (-1)^j(t-s)<0\ \ (j=1,2),\end{gathered}$$

$$\begin{aligned}z_m(t)=\big(I_n+(-1)^j\,d_jC^*(t)\big)^{-1}&\bigg(e_k+\int_s^t dC^*(\tau)\cdot z_{m-1}(\tau)\\&+(-1)^j\,d_jC^*(t)\cdot z_{m-1}(t)\bigg)\ \text{ for }\ (-1)^j(t-s)<0\ \ (j=1,2;\ m=1,2,\dots).\end{aligned}\tag{2.2.51}$$

From (2.2.48), (2.2.50) and (2.2.51) we get

$$z_m(t)\ge\big(I_n+(-1)^j\,d_jC^*(t)\big)^{-1}e_k\ \text{ for }\ (-1)^j(t-s)<0\ \ (j=1,2;\ m=0,1,\dots)$$

and

$$y(s)\ge e_k,\ \ y(t)\ge\big(I_n+(-1)^j\,d_jC^*(t)\big)^{-1}e_k\ \text{ for }\ (-1)^j(t-s)<0\ \ (j=1,2).\tag{2.2.52}$$

On the other hand, equalities

$$y(t)=\Lambda_s^{-1}(t)u_k(t,s)\ \text{ for }\ t\in J_j\ \ (j=1,2)$$

and inequalities (2.2.52) imply that

$$u_k(t,s) \geq \Lambda_s(t)\big(I_n + (-1)^j\, d_j C^*(t)\big)^{-1} e_k \ \text{ for } \ (-1)^j(t-s) < 0, \ \ (-1)^j(t-t_0) < 0 \ \ (j=1,2).$$

Since the latter inequalities are fulfilled for every $k \in \{1,\dots,n\}$, we have

$$U(t,s) \geq \Lambda_s(t)\big(I_n + (-1)^j\, d_j C^*(t)\big)^{-1} \ \text{ for } \ (-1)^j(t-s) < 0 \ \ (j=1,2). \tag{2.2.53}$$

By (2.2.48) and (2.2.49), condition (2.2.53) implies (2.2.47). □

Remark 2.2.6. In fact, we have proved estimate (2.2.53) which is stronger than (2.2.47). Note also that the condition

$$\|d_j C(t)\| < 1 \ \text{ for } \ t \in [a,b] \ \ (j=1,2)$$

guarantees conditions (2.2.44), (2.2.45).

Lemma 2.2.5. *Let $t_0 \in [a,b]$, $c_0 \in \mathbb{R}^n$, $q \in \mathrm{BV}([a,b];\mathbb{R}^n)$, and a matrix-function $C = (c_{ik})_{i,k=1}^n \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$, where c_{ik} $(i \neq k;\ i,k=1,\dots,n)$ are nondecreasing functions on $[a,b]$, be such that the conditions*

$$\det(I_n - d_j C(t)) \neq 0 \ \ \textit{for} \ \ (-1)^j(t-t_0) < 0 \ \ (j=1,2), \tag{2.2.54}$$

$$1 + d_j c_{ii}(t) > 0 \ \ \textit{for} \ \ (-1)^j(t-t_0) > 0 \ \ (j=1,2;\ i=1,\dots,n) \tag{2.2.55}$$

and

$$(I_n - d_j C(t))^{-1} > O_{n\times n} \ \ \textit{for} \ \ (-1)^j(t-t_0) < 0 \ \ (j=1,2). \tag{2.2.56}$$

hold on $[a,b]$. Let, moreover, a vector-function $x \in \mathrm{BV}_{loc}([a,t_0[\,;\mathbb{R}^n) \cap \mathrm{BV}_{loc}(]t_0,b];\mathbb{R}^n)$ be a solution of the system of linear differential inequalities

$$\operatorname{sgn}(t-t_0)\, dx \leq dC(t)\cdot x + dq(t) \tag{2.2.57}$$

on the intervals $[a,t_0[$ and $]t_0,b]$ satisfying the condition

$$x(t_0) + (-1)^j\, d_j x(t_0) \leq c_0 + d_j C(t_0)\cdot c_0 + d_j q(t_0) \ \ (j=1,2). \tag{2.2.58}$$

Then the estimate

$$x(t) \leq y(t) \ \ \textit{for} \ \ t \in [a,b] \setminus \{t_0\} \tag{2.2.59}$$

holds, where $y \in \mathrm{BV}([a,b];\mathbb{R}^n)$ is a solution of the system

$$\operatorname{sgn}(t-t_0)\, dy = dC(t)\cdot y + dq(t) \tag{2.2.60}$$

on the intervals $[a,t_0[$ and $]t_0,b]$ satisfying the conditions

$$(-1)^j\, d_j y(t_0) = d_j C(t_0)\cdot y(t_0) + d_j q(t_0) \ \ (j=1,2) \tag{2.2.61}$$

and

$$y(t_0) = c_0. \tag{2.2.62}$$

Proof. Assume $t_0 < b$ and consider the closed interval $[t_0,b]$. Then problem (2.2.60)–(2.2.62) has the form

$$dy(t) = dC(t)\cdot y(t) + dq(t), \ \ y(t_0) = c_0.$$

Let Z $(Z(t_0) = I_n)$ be a fundamental matrix of system $(2.2.42_0)$. Then, by the variation-of-constants formula, we have

$$y(t) = q(t) - q(s) + Z(t)\bigg\{Z^{-1}(s)y(s) - \int\limits_s^t dZ^{-1}(\tau)\cdot\big(q(\tau)-q(s)\big)\bigg\} \ \text{ for } \ s,\, t \in [t_0,b]. \tag{2.2.63}$$

Put

$$g(t)=-x(t)+x(t_0)+\int\limits_{t_0}^{t} dC(\tau)\cdot x(\tau)+q(t)-q(t_0) \ \text{ for } \ t\in[t_0,b]. \tag{2.2.64}$$

Evidently,

$$dx(t)=dC(t)\cdot x(t)+d(q(t)-g(t)) \ \text{ for } \ t\in[t_0,b].$$

Let ε be an arbitrary positive number. Then

$$x(t)=q(t)-q(t_0+\varepsilon)-g(t)+g(t_0+\varepsilon)+Z(t)\bigg\{Z^{-1}(t_0+\varepsilon)x(t_0+\varepsilon) \\ -\int\limits_{t_0+\varepsilon}^{t} dZ^{-1}(\tau)\cdot\big(q(\tau)-q(t_0+\varepsilon)-g(\tau)+g(t_0+\varepsilon)\big)\bigg\} \ \text{ for } \ t\in[t_0+\varepsilon,b].$$

Hence, by (2.2.63), we get

$$x(t)=y(t)+Z(t)Z^{-1}(t_0+\varepsilon)\big(x(t_0+\varepsilon)-y(t_0+\varepsilon)\big)+g_\varepsilon(t) \ \text{ for } \ t\in[t_0+\varepsilon,b], \tag{2.2.65}$$

where

$$g_\varepsilon(t)\equiv -g(t)+g(t_0+\varepsilon)+Z(t)\int\limits_{t_0+\varepsilon}^{t} dZ^{-1}(\tau)\cdot(g(\tau)-g(t_0+\varepsilon)).$$

Using the integration-by-parts formula, we have

$$g_\varepsilon(t)=-\int\limits_{t_0+\varepsilon}^{t} U(t,\tau)\,ds_c(g)(\tau) \\ -\sum_{t_0+\varepsilon<\tau\le t} U(t,\tau-)\,d_1g(\tau)-\sum_{t_0+\varepsilon\le\tau<t} U(t,\tau+)\,d_2g(\tau) \ \text{ for } \ t\in[t_0+\varepsilon,b], \tag{2.2.66}$$

where $U(t,\tau)=Z(t)Z^{-1}(\tau)$ is the Cauchy matrix of system $(2.2.42_0)$.

On the other hand, conditions (2.2.54)–(2.2.56) guarantee conditions (2.2.44)–(2.2.46). Hence, according to Lemma 2.2.4, estimate (2.2.47) holds, and by (2.2.66),

$$g_\varepsilon(t)\le 0 \ \text{ for } \ t\in[t_0+\varepsilon,b],$$

since by (2.2.57) and (2.2.64) the function g is nondecreasing on $]t_0,b]$. From this and (2.2.65), we conclude

$$x(t)\le y(t)+U(t,t_0+\varepsilon)(x(t_0+\varepsilon)-y(t_0+\varepsilon)) \ \text{ for } \ t\in[t_0+\varepsilon,b].$$

Passing to the limit as $\varepsilon\to 0$ in the latter inequality and taking into account (2.2.47) and (2.2.58), we get

$$x(t)\le y(t) \ \text{ for } \ t\in\,]t_0,b],$$

since, by (2.2.61) and (2.2.62),

$$y(t_0+)=c_0+d_2C(t_0)\cdot c_0+d_2q(t_0).$$

Analogously, we can show the validity of inequality (2.2.59) for $t\in[a,t_0[$. □

Lemma 2.2.5 has the following form for $n=1$.

Lemma 2.2.6. *Let $t_0 \in [a,b]$, α and $q \in \mathrm{BV}_{loc}([a,t_0[\,;\mathbb{R}) \cap \mathrm{BV}_{loc}(]t_0,b];\mathbb{R})$ be such that*

$$1 - d_j\alpha(t) \neq 0 \ \text{ for } \ (-1)^j(t-t_0) < 0 \ \ (j=1,2),$$
$$1 + d_j\alpha(t) > 0 \ \text{ for } \ t \in [a,b] \setminus \{t_0\} \ \ (j=1,2).$$

Let, moreover, $x \in \mathrm{BV}_{loc}([a,t_0[\,;\mathbb{R}) \cap \mathrm{BV}_{loc}(]t_0,b];\mathbb{R})$ satisfy the linear generalized differential inequality

$$\operatorname{sgn}(t-t_0)\, dx \leq x d\alpha(t) + dq(t)$$

on the intervals $[a,t_0[$ and $]t_0,b]$, and

$$x(t_0+) \leq y(t_0+) \ \text{ and } \ x(t_0-) \leq y(t_0-),$$

where $y \in \mathrm{BV}_{loc}([a,t_0[\,;\mathbb{R}) \cap \mathrm{BV}_{loc}(]t_0,b];\mathbb{R})$ is a solution of the general differential equality

$$\operatorname{sgn}(t-t_0)\, dy = y\, d\alpha(t) + dq(t).$$

Then

$$x(t) \leq y(t) \ \text{ for } \ t \in [a,t_0[\cup]t_0,b].$$

The following lemma is analogous to the Wirtinger inequality (see [43, 58]) for the discontinuity case.

Lemma 2.2.7. *Let α and β be nondecreasing functions on $[a,b]$, and let α have not more than a finite number of discontinuity points. Then the estimates*

$$\int_a^b \bigg(\int_{t_0}^t v(\tau)\, ds_c(\alpha)(\tau)\bigg)^2 ds_c(\alpha)(t) \leq \gamma_0 \int_a^b v^2(t)\, ds_c(\alpha)(t) \tag{2.2.67}$$

and

$$\int_a^b \bigg(\int_{t_0}^t v(\tau)\, ds_m(\alpha)(\tau)\bigg)^2 ds_j(\beta)(t) \leq \gamma_{mj} \int_a^b v^2(t)\, ds_m(\alpha)(t) \ \ (j,m=1,2) \tag{2.2.68}$$

hold for every $v \in \mathrm{BV}([a,b];\mathbb{R})$ and $t_0 \in [a,b]$, where

$$\gamma_0 = \bigg(\frac{2}{\pi}\,(s_c(\alpha)(b) - s_c(a)(a))\bigg)^2, \quad \gamma_{mj} = \frac{1}{4}\,\mu_{\alpha m}\nu_{\alpha m\beta j} \sin^{-2}\frac{\pi}{4n_{\alpha m}+2} \ \ (j,m=1,2).$$

In addition, these estimates are unimprovable.

Proof. Obviously, it suffices to verify inequalities (2.2.67) and (2.2.68) for $t_0 = a$ and $t_0 = b$. Assume $t_0 = b$. Let us show (2.2.67). Without loss of generality, we may assume that

$$s_c(\alpha)(t) < s_c(\alpha)(b) \ \text{ for } \ a \leq t < b.$$

Put

$$u(t) = \int_b^t v(\tau)\, ds_c(\alpha)(\tau), \quad \widetilde{\alpha}(t) = \frac{1}{\sqrt{\gamma_0}}\,\big(s_c(\alpha)(t) - s_c(\alpha)(b)\big) \ \text{ for } \ t \in [a,b].$$

Let ε be a small positive number. It is easily seen that

$$\int_a^{b-\varepsilon} \big(u(t)\operatorname{ctg}\widetilde{\alpha}(t) - \sqrt{\gamma_0}\, v(t)\big)\, ds_c(\alpha)(t)$$
$$= -\int_a^{b-\varepsilon} u^2(t)\, ds_c(\alpha)(t) + \sqrt{\gamma_0}\int_a^{b-\varepsilon} v^2(t)\, ds_c(\alpha)(t) - \sqrt{\gamma_0}\, u^2(b-\varepsilon)\operatorname{ctg}\widetilde{\alpha}(b-\varepsilon).$$

Consequently,

$$\int\limits_a^{b-\varepsilon} u^2(t)\,ds_c(\alpha)(t) \le \sqrt{\gamma_0}\int\limits_a^{b-\varepsilon} v^2(t)\,ds_c(\alpha)(t) - \sqrt{\gamma_0}\,u^2(b-\varepsilon)\operatorname{ctg}\widetilde{\alpha}(b-\varepsilon).$$

Passing in the latter inequality to the limit as $\varepsilon \to 0$, we obtain (2.2.67).

Let us show (2.2.68). We have

$$\begin{aligned}\int\limits_a^b\bigg(\int\limits_b^t v(\tau)\,ds_m(\alpha)(\tau)\bigg)^2 ds_j(\beta)(t) &= \sum_{a\le t\le b}\bigg(\int\limits_t^b v(\tau)\,ds_m(\alpha)(\tau)\bigg)^2 ds_j(\beta)(t)\\ &= \sum_{l=1}^{n_{\alpha m}}\bigg(d_j\beta(t_{\alpha mk\,l+m-2}) + \sum_{t_{\alpha mk\,l-1}<t<t_{\alpha mkl}} d_j\beta(t)\bigg)\omega_m^2(l)\\ &\le \gamma_{mj}\sum_{l=1}^{n_{\alpha m}+1}\omega_m^2(l)\ \ (j,m=1,2),\end{aligned} \tag{2.2.69}$$

where

$$\omega_m(l) = \sum_{k=l}^{n_{\alpha m}} v(t_{\alpha mk})\,ds_m(t_{\alpha mk})\ \ (l=1,\dots,n_{\alpha m}),\ \ \omega_m(n_{\alpha m}+1)=0\ \ (m=1,2).$$

According to the discrete analogue of the Wirtinger inequality [58], we obtain

$$\begin{aligned}\sum_{l=1}^{n_{\alpha m}+1}\omega_m^2(l) &\le \frac14\,\sin^{-2}\frac{\pi}{4n_{\alpha m}+2}\sum_{l=1}^{n_{\alpha m}+1}\big(\omega_m(l)-\omega_m(l-1)\big)^2\\ &\le \frac14\,\sin^{-2}\frac{\pi}{4n_{\alpha m}+2}\sum_{l=1}^{n_{\alpha m}+1}\big(v(t_{\alpha mk\,l-1})\,ds_m(t_{\alpha mk\,l-1})\big)^2\\ &\le \frac14\,\sin^{-2}\frac{\pi}{4n_{\alpha m}+2}\sum_{l=1}^{n_{\alpha m}+1}\int\limits_a^b v^2(t)\,ds_m(\alpha)(t)\ \ (m=1,2),\end{aligned}$$

Using this, from (2.2.69) we deduce (2.2.68). The proof of (2.2.67) and (2.2.68) is similar for $t_0 = a$.

Finally, it should be noted that equality (2.2.67) transforms into the equality for $t_0 = a$ and $v(t) = \gamma^{-1}\cos(\gamma_0(s_c(\alpha)(t) - s_c(\alpha)(a)))$. As for inequality (2.2.68), it transforms into the equality if $a = t_0 = 0$, $b = l$, $\alpha(t) = \beta(t) = k-1$ for $k-1 \le t < k$ $(k = 1,\dots,l)$, $\alpha(l) = \beta(l) = l$, $v(k) = \sin\frac{\pi k}{2l+1} - \sin\frac{\pi(k-1)}{2l+1}$ and $v(t) = 0$ for $t \in [0,l]\setminus\{1,\dots,m\}$. □

2.2.3 On the set $\mathbb{U}(t_1,\dots,t_n)$. The lemmas on the a priory estimates

The following lemmas make more precise the ones given in [18] (see Lemma 6.3).

Lemma 2.2.8. *Let conditions* (2.2.14), (2.2.15) *and*

$$\big|c_{il}(t) - c_{il}(s)\big| \le \int\limits_s^t h_{il}(\tau)\,d\alpha_l(\tau)\ \ \text{for}\ \ s<t\ \ (i,l=1,\dots,n) \tag{2.2.70}$$

hold on $[a,b]$, *where* $c_{il} \in \mathrm{BV}([a,b];\mathbb{R})$ $(i,l=1,\dots,n)$, α_l $(l=1,\dots,n)$ *are the functions nondecreasing on* $[a,b]$ *and having not more than a finite number of discontinuity points;* $h_{il} \in L^\mu([a,b],\mathbb{R}_+;\alpha_l)$ $(i\ne l)$, $h_{ii} \in L^\mu([a,b],\mathbb{R};\alpha_l)$ $(i,l=1,\dots,n)$, $1\le\mu\le+\infty$; $l_{mik}\in\mathbb{R}_+$ $(m=0,1,2;\ i,k=1,\dots,n)$, $\frac1\mu+\frac2\nu=1$, *and* $\mathcal{H} = (\mathcal{H}_{j+1,m+1})_{j,m=0}^2$ *is the* $3n\times3n$*-matrix defined as in Theorem 2.2.2. Then problem*

(2.2.4), (2.2.5) *has no nontrivial non-negative solution. In addition, if* c_{il} $(i \neq l;\ i,l = 1,\dots,n)$ *are the functions nondecreasing on* $[a,b]$, *then condition* (2.2.10) *holds for* $C = (c_{il})_{i,l=1}^n$ *and* $\ell_0 = (\ell_{0i})_{i=1}^n$,

$$\ell_{0i}(x_1,\dots,x_n) = \sum_{m=0}^{2}\sum_{k=1}^{n} l_{mik}\|x\|_{\nu,s_m(\alpha_k)} \ \textit{for}\ x = (x_l)_{l=1}^n \in \mathrm{BV}([a,b];\mathbb{R}^n) \ \ (i = 1,\dots,n).$$

Proof. Let $x = (x_i)_{i=1}^n$ be an arbitrary solution of problem (2.2.4), (2.2.5). By (2.2.70) and Hölder's inequality, we have

$$x_i(t) \le \sum_{m=0}^{2}\sum_{k=1}^{n}\bigg(l_{mik}\|x\|_{\nu,s_m(\alpha_k)} + \|h_{ik}\|_{\mu,s_m(\alpha_k)}\bigg|\int_{t_i}^{t} |x_k(\tau)|^{\frac{\nu}{2}}\,ds_m(\alpha_k)(\tau)\bigg|^{\frac{2}{\nu}}\bigg)$$
$$\text{for}\ \ t \in [a,b]\ \ (i = 1,\dots,n). \tag{2.2.71}$$

This, in view of Minkovski's inequality, implies

$$\|x\|_{\nu,s_j(\alpha_k)} \le \sum_{m=0}^{2}\sum_{k=1}^{n}\bigg(l_{mik}\big(s_j(\alpha_i)(b) - s_j(\alpha_i)(a)\big)^{\frac{1}{\nu}}\|x\|_{\nu,s_m(\alpha_k)}$$
$$+ \|h_{ik}\|_{\mu,s_m(\alpha_k)}\bigg[\int_a^b\bigg|\int_{t_i}^{t} |x_k(\tau)|^{\frac{\nu}{2}}\,ds_m(\alpha_k)(\tau)\bigg|^2 ds_j(\alpha_i)(\tau)\bigg]^{\frac{1}{\nu}}\bigg) \ \text{for}\ \ t \in [a,b]\ \ (i = 1,\dots,n). \tag{2.2.72}$$

On the other hand, by virtue of Hölder's inequality in the case $m^2 + j^2 + (i-k)^2 > 0$, $j = 0$, and by (2.2.67) and (2.2.68) in the other cases, we have

$$\bigg[\int_a^b\bigg|\int_{t_i}^{t} |x_k(\tau)|^{\frac{\nu}{2}}\,ds_m(\alpha_k)(\tau)\bigg|^2 ds_j(\alpha_i)(\tau)\bigg]^{\frac{1}{\nu}}$$
$$\le \lambda_{kmij}\bigg[\int_a^b |x_k(\tau)|^2\,ds_m(\alpha_k)(\tau)\bigg]^{\frac{1}{\nu}} \ \ (j,m = 1,2;\ i,k = 1,\dots,n).$$

The latter inequality and (2.2.72) yield

$$\|x_i\|_{\nu,s_j(\alpha_i)} \le \sum_{m=0}^{2}\sum_{k=1}^{n}\big(\xi_{ij}l_{mik} + \lambda_{kmij}\|h_{ik}\|_{\nu,s_m(\alpha_k)}\big)\|x_k\|_{\nu,s_m(\alpha_k)} \ \ (j = 0,1,2;\ i = 1,\dots,n).$$

Therefore

$$(I_{3n} - \mathcal{H})r \le 0_{3n}, \tag{2.2.73}$$

where $r \in \mathbb{R}^{3n}$ is the vector with components

$$r_{i+nj} = \|x_k\|_{\nu,s_j(\alpha_i)} \ \ (j = 0,1,2;\ i = 1,\dots,n).$$

From (2.2.73), we find that $r = 0_{3n}$, since the module of the characteristic value of the matrix $\mathcal{H}$ is less than 1. Using (2.2.71), we can see that $x_i(t) \equiv 0$ $(i = 1,\dots,n)$. Consequently, problem (2.2.4), (2.2.5) has no nontrivial nonnegative solution. □

Lemma 2.2.9. *Let conditions* (2.2.6)–(2.2.8), (2.2.21) *and* (2.2.23) *hold on* $[a,b]$, *where the functions* c_{ii} $(i = 1,\dots,n)$ *are nonincreasing on* $[a,b]$, $\mu_i \ge 0$, $\tau_i \in [a,b]$, $\tau_i \neq t_i$ $(i = 1,\dots,n)$; $\lambda_i(t) \equiv \gamma_{a_i}(t,t_i)$, *the function* $\gamma_{a_i}(t,t_i)$ *is defined according to* (1.1.9), *and* $a_i(t) \equiv (c_{ii}(t) - c_{ii}(t_i))\,\mathrm{sgn}(t - t_i)$ $(i = 1,\dots,n)$. *Let, moreover, the functions* $c_{il}\,\mathrm{BV}([a,b];\mathbb{R})$ $(i \neq l;\ i,l = 1,\dots,n)$ *be such that condition* (2.2.24) *hold, where the constant matrix* $\mathcal{M}$ *is defined as in Theorem 2.2.3. Then problem* (2.2.4), (2.2.5), *where*

$$\ell_{0i}(x_1,\dots,x_n) = \mu_i x_i(\tau_i) \ \ \textit{for}\ \ x_l \in \mathrm{BV}([a,b];\mathbb{R}_+) \ \ (i,l = 1,\dots,n), \tag{2.2.74}$$

has no nontrivial nonnegative solution.

Proof. Let $(x_i)_{i=1}^n$ be an arbitrary nonnegative solution of problem (2.2.4), (2.2.5). Let $i \in \{1,\dots,n\}$ be fixed. Due to (2.2.4), it is evident that

$$\begin{gathered}\operatorname{sgn}(t-t_i)\,dx_i \le x_i(t)\,dc_{ii}(t) + df_i(t) \ \text{ for } \ t\in[a,b], \ \ t\neq t_i,\\ (-1)^j\,d_jx_i(t_i) \le x_i(t_i)\,d_jc_{ii}(t_i) + d_jf_i(t_i) \ \ (j=1,2),\end{gathered}$$

where

$$f_i(t) \equiv \sum_{l\neq i;\,l=1}^n \int\limits_{t_i}^t x_l(\tau)\,dc_{il}(\tau).$$

So, by (2.2.21), the function x_i satisfies the conditions of Lemma 2.2.6, where $\alpha(t) \equiv c_{ii}(t)$, $q(t)\equiv f_i(t)$, $t_0 = t_i$.

According to the lemma and (2.2.74), the estimate

$$x_i(t) \le y_i(t) \ \text{ for } \ t\in[a,b] \tag{2.2.75}$$

holds, where $y_i \in \mathrm{BV}([a,b];\mathbb{R})$ is a solution of the system

$$\operatorname{sgn}(t-t_i)\,dy = y\,d\alpha(t) + dq(t) \tag{2.2.76}$$

on the intervals $[a,t_i[$ and $]t_i,b]$ satisfying the conditions

$$(-1)^j\,d_jy(t_i) = y(t_i)\,d_j\alpha(t_i) + d_jq(t_i) \ \ (j=1,2) \tag{2.2.77}$$

and

$$y(t_i) = \mu_i x(\tau_i). \tag{2.2.78}$$

Problem (2.2.76)–(2.2.78) is equivalent to the initial problem

$$dy = y\,da_i(t) + df_i(t), \ \ y(t_i) = \mu_i x(\tau_i) \ \text{ for } \ t\in[a,b].$$

Let us show that

$$x_i(t) \le \mu_i\lambda_i(t)x_i(\tau_i) + \sum_{l\neq i;\,l=1}^n |f_{il}(t) - f_{il}(t_i)|\,\rho_l \ \text{ for } \ t\in[a,b], \tag{2.2.79}$$

where

$$\rho_l = \sup\big\{|x_l(t)| : \ t\in[a,b]\big\} \ \ (l=1,\dots,n).$$

First, consider the case $t\in\,]t_i,b]$. According to the variation-of-constant and integration-by-parts formulaes, we have

$$\begin{aligned} y_i(t) &= f_i(t) + \lambda_i(t)\bigg\{y_i(t_i) - \int\limits_{t_i}^t f_i(\tau)\,d\lambda_i^{-1}(\tau)\bigg\} = \lambda_i(t)y_i(t_i)\\ &+ \lambda_i(t)\sum_{l\neq i;\,l=1}^n \bigg\{\int\limits_{t_i}^t \lambda_i^{-1}(\tau)x_l(\tau)\,dc_{il}(\tau) - \sum_{t_i<\tau\le t} d_1\lambda_i^{-1}(\tau)x_l(\tau)\,d_1c_{il}(\tau) + \sum_{t_i\le\tau<t} d_2\lambda_i^{-1}(\tau)x_l(\tau)\,d_2c_{il}(\tau)\bigg\}\\ &\qquad \text{for } \ t\in[t_i,b]\end{aligned}$$

and, later, by (0.0.12), we can conclude that

$$\begin{aligned} x_i(t) &\le \lambda_i(t)y_i(t_i) + \lambda_i(t)\sum_{l\neq i;\,l=1}^n \bigg\{\int\limits_{t_i}^t \lambda_i^{-1}(\tau)x_l(\tau)\,ds_c(c_{il})(\tau)\\ &+ \sum_{t_i<\tau\le t} \lambda_i^{-1}(\tau-)\,x_l(\tau)\,d_1c_{il}(\tau) + \sum_{t_i\le\tau<t} \lambda_i^{-1}(\tau+)\,x_l(\tau)\,d_2c_{il}(\tau)\bigg\} \ \text{ for } \ t\in[t_i,b]. \end{aligned} \tag{2.2.80}$$

The function $\lambda_i^{-1}(t)$ is nondecreasing on $]t_i,b]$, since the function c_{ii} is nonincreasing on the same interval. So, by (2.2.78), from (2.2.80) follows estimate (2.2.79).

Analogously, we verify inequality (2.2.79) for $t\in[a,t_i]$, as well.

In view of (2.2.79) due to (2.2.23), we find that

$$x_i(\tau_i)\leq(1-\mu_i\lambda_i(\tau_i))^{-1}\sum_{l\neq i;\,l=1}^{n}|f_{il}(\tau_i)-f_{il}(t_i)|\,\rho_l$$

and, consequently,

$$\rho_i\leq\sum_{l\neq i;\,1}^{n}\mu_{il}\rho_l\ \ (i=1,\dots,n).$$

Thus the constant vector $\rho=(\rho_i)_{i=1}^n$ satisfies the system

$$(I_n-M)\rho\leq 0_n.$$

From this, using (2.2.24), we obtain $\rho=0_n$. Consequently, $x_i(t)\equiv 0$ $(i=1,\dots,n)$. □

Lemma 2.2.10. *Let conditions* (2.2.6)–(2.2.8) *and* (2.2.23) *hold on* $[a,b]$*, where* $\mu_i\geq 0$*,* $\tau_i\in[a,b]$*,* $\tau_i\neq t_i$ $(i=1,\dots,n)$*; and let the functions* c_{ii} $(i=1,\dots,n)$ *be such that the functions* $\lambda_i(t)\equiv\gamma_{c_{ii}}(t,t_i)$ $(i=1,\dots,n)$ *defined according to* (1.1.9) *be monotone on the intervals* $[a,t_i]$ *and* $]t_i,b]$*. Let, moreover, the functions* $c_{il}\in\mathrm{BV}([a,b];\mathbb{R})$ $(i\neq l;\ i,l=1,\dots,n)$ *be such that condition* (2.2.24) *hold, where the constant matrix* $\mathcal{M}$ *is defined as in Theorem 2.2.4. Then problem* (2.2.4)*,* (2.2.5)*, where functionals* ℓ_{0i} $(i=1,\dots,n)$ *are defined by* (2.2.74)*, has no nontrivial nonnegative solution.*

Proof. Here we use the designations given in the proof of Lemma 2.2.9. As in that proof we show the estimate

$$y_i(t)=f_i(t)+\lambda_i(t)\bigg\{y_i(t_i)-\int_{t_i}^{t}f_i(\tau)\,d\lambda_i^{-1}(\tau)\bigg\}\ \text{ for }\ t\in[a,b].$$

From this, by (2.2.75) and (2.2.78), we obtain

$$x_i(t)\leq f_i(t)+\mu_i\lambda_i(t)x_i(\tau_i)-\lambda_i(t)\int_{t_i}^{t}f_i(\tau)\,d\lambda_i^{-1}(\tau)\ \text{ for }\ t\in[a,b]. \tag{2.2.81}$$

Therefore, using (2.2.23), we conclude that

$$x_i(\tau_i)\leq(1-\mu_i\lambda_i(\tau_i))^{-1}\bigg(f_i(\tau_i)-\lambda_i(\tau_i)\int_{t_i}^{\tau_i}f_i(\tau)\,d\lambda_i^{-1}(\tau)\bigg)\ \text{ for }\ t\in[a,b].$$

Substituting the obtained estimate into (2.2.81), we get

$$\begin{aligned}x_i(t)&\leq f_i(t)+\mu_i(1-\mu_i\lambda_i(\tau_i))^{-1}\lambda_i(t)\bigg(f_i(\tau_i)-\lambda_i(\tau_i)\int_{t_i}^{\tau_i}f_i(\tau)\,d\lambda_i^{-1}(\tau)\bigg)-\lambda_i(t)\int_{t_i}^{t}f_i(\tau)\,d\lambda_i^{-1}(\tau)\\&\leq\sum_{l\neq i;\,l=1}^{n}\bigg(\big(1+|\lambda_i(t)-1|\big)\Big|\bigvee_{t_i}^{t}(c_{il})\Big|+\nu_i\big(1+|\lambda_i(\tau_i)-1|\big)\Big|\bigvee_{t_i}^{\tau_i}(c_{il})\Big|\bigg)\rho_l\\&\leq\sum_{l\neq i;\,l=1}^{n}\bigg(\zeta_{il}(1+\xi_i)+\nu_i\big(1+|\lambda_i(\tau_i)-1|\big)\Big|\bigvee_{t_i}^{\tau_i}(c_{il})\Big|\bigg)\rho_l\ \text{ for }\ t\in[a,b].\end{aligned}$$

So,

$$\rho_i\leq\sum_{l\neq i;\,1}^{n}\mu_{il}\rho_l\ \ (i=1,\dots,n).$$

Later, as in Lemma 2.2.9, we conclude that $x_i(t)\equiv 0$ $(i=1,\dots,n)$. □

Lemma 2.2.11. *Let* $t_{k1},\dots,t_{kn}\in[a,b]$ $(k=1,2)$*,* $\ell_{0ki}:\mathrm{BV}([a,b];\mathbb{R}^n_+)\to\mathbb{R}_+$ $(k=1,2;\ i=1,\dots,n)$ *be linear bounded functionals, and* $C_{kj}=(c_{kjil})_{i,l=1}^n\in\mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ $(k,j=1,2)$ *be such that the system*

$$\begin{gathered}\operatorname{sgn}(t-t_{1i})\,dx_{1i}(t)\le\sum_{l=1}^n x_{1l}(t)\,dc_{11il}(t)+\sum_{l=1}^n x_{2l}(t)\,dc_{12il}(t)\ \text{ for } t\in[a,b],\ \ t\neq t_{1i}\ \ (i=1,\dots,n),\\ (-1)^j\,d_jx_{1i}(t_{1i})\le\sum_{l=1}^n x_{1l}(t_{1i})\,d_jc_{11il}(t_{1i})+\sum_{l=1}^n x_{2l}(t_{1i})\,d_jc_{12il}(t_{1i})\ \ (j=1,2;\ i=1,\dots,n),\\ dx_{2i}(t)=\sum_{l=1}^n x_{1l}(t)\,dc_{21il}(t)+\sum_{l=1}^n x_{2l}(t)\,dc_{22il}(t)\ \text{ for } t\in[a,b]\ \ (i=1,\dots,n)\end{gathered}\tag{2.2.82}$$

has a nontrivial nonnegative solution satisfying the condition

$$x_{ki}(t_{1i})\le l_{0ki}(x_{11},\dots,x_{1n},x_{21},\dots,x_{2n})\ \ (k=1,2;\ i=1,\dots,n).\tag{2.2.83}$$

Then there exist a matrix-function $A\in\mathrm{BV}([a,b];\mathbb{R}^{n\times n})$*, linear bounded functionals* $\ell_i:\mathrm{BV}_\infty([a,b];\mathbb{R}^{2n})\to\mathbb{R}$ $(i=1,\dots,2n)$ *and numbers* $c_{0i}\in\mathbb{R}$ $(i=1,\dots,2n)$ *such that the* $2n$*-system*

$$dx(t)=d\widetilde{A}(t)\cdot x(t)\tag{2.2.84}$$

under the $2n$*-condition* (1.1.4) *is unsolvable, here* $t_i=t_{1i}$ $(i=1,\dots,n)$*,* $t_{n+i}=t_{2i}$ $(i=1,\dots,n)$*, and*

$$\widetilde{A}(t)\equiv\begin{pmatrix}A(t) & C_{12}(t)\\ C_{21}(t) & C_{22}(t)\end{pmatrix}.\tag{2.2.85}$$

Proof. Let $x=(x_k)_{k=1}^2$, $x_k=(x_{ki})_{i=1}^n$ $(k=1,2)$ be the nonnegative solution of problem (2.2.82), (2.2.83).

Let $\alpha_i,q_i\in\mathrm{BV}([a,b];\mathbb{R})$ $(i=1,\dots,n)$ be the functions defined by

$$\alpha_i(t)\equiv\big(s_c(c_{11\,ii})(t)-s_c(c_{11\,ii})(s)\big)\operatorname{sgn}(t-t_i)\ \ (i=1,\dots,n)$$

and

$$q_i(t)\equiv\bigg(\sum_{l=1}^n\bigg(\int_{t_{1i}}^t x_{1l}(\tau)\,dc_{11\,il}(\tau)+\int_{t_{1i}}^t x_{2l}(\tau)\,dc_{12\,il}(\tau)\bigg)-\int_{t_{1i}}^t x_{1i}(\tau)\,ds_c(c_{11\,ii})(\tau)\bigg)\ \ (i=1,\dots,n).$$

It is evident that the Cauchy problem

$$dy(t)=y(t)\,d\alpha_i(t)+dq_i(t),\tag{2.2.86}$$
$$y(t_{1i})=x_{1i}(t_{1i})\tag{2.2.87}$$

has a unique solution y_{1i} for every $i\in\{1,\dots,n\}$.

Moreover, it is easy to verify that the function $z(t)=z_i(t)$,

$$z_i(t)\equiv x_{1i}(t)-y_{1i}(t),$$

satisfies the conditions of Lemma 2.2.6 and the problem

$$du(t)=u(t)\,d\alpha_i(t),\quad u(t_{1i})=0$$

has only the trivial solution for every $i\in\{1,\dots,n\}$.

According to this lemma, we have

$$x_{1i}(t)\le y_{1i}(t)\ \text{ for } t\in[a,b]\ \ (i=1,\dots,n)$$

and, therefore,

$$x_{1i}(t) = \eta_i(t) y_{1i}(t) \ \text{ for } \ t \in [a,b] \ \ (i = 1, \dots, n),$$

where, by Theorem I.4.25 from [73], $\eta_i : [a,b] \to [0,1]$ $(i = 1, \dots, n)$ are the functions such that the integrals $\int\limits_{t_i}^{t} \eta_i(\tau)\, dc_{11\,il}(\tau)$ $(i, l = 1, \dots, n)$ exist for every $t \in [a,b]$.

Let us introduce the notation

$$\begin{aligned} a_{ii}(t) &\equiv \alpha_i(t) + \operatorname{sgn}(t - t_{1i}) \int\limits_{t_{1i}}^{t} \eta_i(\tau)\, d\big(c_{11\,ii}(\tau) - s_c(c_{11\,ii})(\tau)\big) \ \ (i = 1, \dots, n), \\ a_{il}(t) &\equiv \operatorname{sgn}(t - t_{1i}) \int\limits_{t_{1i}}^{t} \eta_l(\tau)\, dc_{11\,il}(\tau) \ \ (i \neq l, \ \ i, l = 1, \dots, n). \end{aligned} \tag{2.2.88}$$

Due to (2.2.83), (2.2.86) and (2.2.87), the vector-function $y = (y_i)_{i=1}^{2n}$, $y_i(t) = x_{1i}(t)$ $(i = 1, \dots, n)$, $y_{n+i}(t) = x_{2i}(t)$ $(i = 1, \dots, n)$, is a nontrivial nonnegative solution of the $2n$-problem

$$dy(t) = d\widetilde{A}(t) \cdot y(t), \tag{2.2.89}$$

$$y_i(t_i) = \delta_i \ell_{0i}(y_1, \dots, y_{2n}) \ \ (i = 1, \dots, 2n), \tag{2.2.90}$$

where $\delta_i \in [0,1]$ $(i = 1, \dots, n)$, $\delta_{n+i} = 1$ $(i = 1, \dots, n)$, $A(t) = (a_{il}(t))_{i,l=1}^{n}$,

$$\begin{aligned} &\ell_{0i}(y_1, \dots, y_{2n}) = l_{0ki}(y_1, \dots, y_{2n}) \\ &\qquad \text{for } \ (y_l)_{l=1}^{2n} \in \mathrm{BV}([a,b]; \mathbb{R}^{2n}) \ \ (k = 1, 2; \ i = (k-1)n + 1, \dots, kn). \end{aligned}$$

Let $\ell_i : \mathrm{BV}_\infty([a,b]; \mathbb{R}^{2n}) \to \mathbb{R}$ $(i = 1, \dots, 2n)$ be linear functionals defined by

$$\begin{aligned} &\ell_i(x_1, \dots, x_{2n}) = \delta_i \big(\ell_{0i}\big([x_1]_+, \dots, [x_{2n}]_+\big) - \ell_{0i}\big([x_1]_-, \dots, [x_{2n}]_-\big)\big) \\ &\qquad\qquad \text{for } \ (x_l)_{l=1}^{2n} \in \mathrm{BV}([a,b]; \mathbb{R}^{2n}) \ \ (i = 1, \dots, 2n), \end{aligned} \tag{2.2.91}$$

where

$$[x_i]_+(t) = \frac{1}{2}\big(|x_i(t)| + x_i(t)\big) \ \text{ and } \ [x_i]_-(t) = \frac{1}{2}\big(|x_i(t)| - x_i(t)\big) \ \ (i = 1, \dots, 2n)$$

are the positive and negative parts of the function x_i, respectively.

By (2.2.88)–(2.2.90), $y = (y_i)_{i=1}^{2n}$ is a nontrivial, nonnegative solution of system (2.2.84) under the boundary condition $(2.2.2_0)$.

On the other hand, by Remark 1.1.2, there exist numbers $c_{0i} \in \mathbb{R}$ $(i = 1, \dots, 2n)$ such that problem (2.2.84), (2.2.2) is unsolvable, where the matrix-function $\widetilde{A}(t)$ is defined by (2.2.85), (2.2.88), and the linear functionals ℓ_i $(i = 1, \dots, 2n)$ are defined by (2.2.91). □

2.2.4 Proof of the main results

Proof of Theorem 2.2.1. According to Corollary 1.1.1, to prove the theorem it suffices to verify that the homogeneous problem $(2.2.1_0), (2.2.2_0)$ has only the trivial solution.

Let $(x_i)_{i=1}^{n}$ be an arbitrary solution of the problem. We assume

$$\overline{x}_i(t) = |x_i(t)| \ \ (i = 1, \dots, n).$$

Let $i \in \{i = 1, \dots, n\}$ be fixed.

Due to (2.2.36),

$$\begin{aligned} |x_i(t)| - |x_i(s)| &= \sum_{l=1}^{n} \int\limits_{s}^{t} x_l(\tau) \operatorname{sgn} x_i(\tau)\, ds_c(a_{il})(\tau) \\ &\quad + \sum_{s<\tau\le t} \big(|x_i(\tau)| - |x_i(\tau-)|\big) + \sum_{s\le\tau<t} \big(|x_i(\tau+)| - |x_i(\tau)|\big) \ \text{ for } \ a \le s < t \le b. \end{aligned}$$

From this it follows that

$$\operatorname{sgn}(s-t_i)\big(|x_i(t)|-|x_i(s)|\big) = \sum_{l=1}^{n}\int_s^t x_l(\tau)\operatorname{sgn} x_i(\tau)\,d\big(\operatorname{sgn}(\tau-t_i)\,s_c(a_{il})(\tau)\big) + \sum_{s<\tau\le t}\operatorname{sgn}(\tau-t_i)\big(|x_i(\tau)|-|x_i(\tau-)|\big) + \sum_{s\le\tau<t}\operatorname{sgn}(\tau-t_i)\big(|x_i(\tau+)|-|x_i(\tau)|\big) \text{ for } (t-s)(s-t_i)>0.$$

Then, by (2.2.6)–(2.2.8) and Lemma 2.2.1, we have

$$\operatorname{sgn}(t-t_i)\,d\overline{x}_i(t)\le\sum_{l=1}^{n}\overline{x}_l(t)\,dc_{il}(t) \text{ for } t\in[a,b],\ \ t\ne i_i \ \ (i=1,\dots,n)$$

and

$$(-1)^j\,d_j\overline{x}_i(t_i)\le\sum_{l=1}^{n}\overline{x}_l(t_i)\,d_jc_{il}(t_i) \ \ (j=1,2;\ i=1,\dots,n).$$

In addition, (2.2.9) yields

$$\overline{x}_i(t_i)\le\ell_{0i}(\overline{x}_1,\dots,\overline{x}_n) \ \ (i=1,\dots,n).$$

Hence $(\overline{x}_i)_{i=1}^n$ is a nonnegative solution of problem (2.2.4), (2.2.5). Therefore, by (2.2.10), $\overline{x}_i(t)\equiv 0$ $(i=1,\dots,n)$ and

$$x_i(t)\equiv 0 \ \ (i=1,\dots,n).$$

□

Proof of Theorem 2.2.2. By Lemma 2.2.8, condition (2.2.10) holds for $C=(c_{il})_{i,l=1}^n$ and $\ell_0=(\ell_{0i})_{i=1}^n$, where

$$c_{il}(t)=\int_a^t h_{il}(\tau)d\alpha_l(\tau) \text{ for } t\in[a,b] \ \ (i,l=1,\dots,n)$$

and

$$\ell_{0i}(x_1,\dots,x_n)=\sum_{m=0}^{2}\sum_{k=1}^{n} l_{mik}\|x_k\|_{\nu,s_m(\alpha_k)} \text{ for } (x_l)_{l=1}^n\in \mathrm{BV}([a,b];\mathbb{R}^n) \ \ (i=1,\dots,n).$$

Therefore, the theorem follows from Theorem 2.2.1. □

Remark 2.2.1 follows from the fact that Lemma 2.2.8 is likewise true for the $n\times n$-matrix described in this remark.

Corollary 2.2.1 is a particular case of Theorem 2.2.2, when $l_{mki}=0$ $(m=0,1,2;\ i,k=1,\dots,n)$.

Proof of Theorem 2.2.3. By Lemma 2.2.9, condition (2.2.10) holds for $C=(c_{il})_{i,l=1}^n$ and $\ell=(\ell_{0i})_{i=1}^n$, where the functionals ℓ_{0i} $(i=1,\dots,n)$ are defined by (2.2.74). Therefore, the theorem follows from Theorem 2.2.1. □

Proof of Theorem 2.2.4. By Lemma 2.2.10, condition (2.2.10) holds for $C=(c_{il})_{i,l=1}^n$ and $\ell=(\ell_{0i})_{i=1}^n$, where the functionals ℓ_{0i} $(i=1,\dots,n)$ are defined by (2.2.74). Therefore, the theorem follows from Theorem 2.2.1. □

Proof of Theorem 2.2.5. Note that problem (2.2.4), (2.2.5) is a particular case of problem (2.2.82), (2.2.83) if we assume in it $C_{11}(t)\equiv C(t)$, $C_{12}(t)=C_{21}(t)=C_{22}(t)\equiv O_{n\times n}$ and $\ell_{01i}(x_1,\dots x_{2n})\equiv \ell_{0i}(x_1,\dots,x_n)$ $(i=1,\dots,n)$, $\ell_{02i}(x_1,\dots,x_{2n})\equiv 0$ $(i=1,\dots,n)$.

By Lemma 2.2.11, there exist a matrix-function $A=(a_{il})_{i,l=1}^n\in\mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ and linear bounded functionals ℓ_i $(i=1,\dots,2n)$ defined by (2.2.88) and (2.2.91), respectively, and numbers c_{0i} $(i=1,\dots,2n)$ such that the $2n$-system (2.2.84) is unsolvable under the $2n$-condition (2.2.2) (defined in Lemma 2.2.11), and $\widetilde{A}(t)$ is defined by (2.2.85) and (2.2.88). Moreover, it is evident that system

(2.2.84) is equivalent to system $(2.2.1_0)$. Therefore, the problem $(2.2.1_0), (2.2.2)$ is unsolvable for the matrix-function A and linear functionals ℓ_i $(i = 1, \dots, n)$.

Due to (2.2.26), (2.2.88) and (2.2.91), it is not difficult to verify that conditions (2.2.6)–(2.2.9) are fulfilled.

Let now condition (2.2.27) hold. By (2.2.88), we get

$$d_j A(t) = \operatorname{diag}\big(\operatorname{sgn}(t - t_1), \dots, \operatorname{sgn}(t - t_n)\big)\, d_j C(t) \operatorname{diag}(\eta_1(t), \dots, \eta_n(t)) \ \text{ for } \ t \in [a, b] \ \ (j = 1, 2).$$

Therefore, in view of (2.2.27), condition (1.1.8) holds. □

Consider Remark 2.2.5. The first case is evident. Indeed, by (2.2.88),

$$d_j a_{il}(t) = \operatorname{sgn}(t - t_i)\eta_l(t)\, d_j c_{il}(t) \ \text{ for } \ t \in [a, b] \ \ (j = 1, 2;\ i, l = 1, \dots, n)$$

and

$$|d_j a_{il}(t)| \le |d_j c_{il}(t)| \ \text{ for } \ t \in [a, b] \ \ (j = 1, 2;\ i, l = 1, \dots, n).$$

Taking this into account, by (2.2.28), we have

$$\sum_{l=1}^{n} |d_j a_{il}(t)| < 1 \ \text{ for } \ t \in [a, b] \ \ (j = 1, 2;\ i = 1, \dots, n).$$

Hence condition (1.1.8) holds.

Let now conditions (2.2.29) and (2.2.30) be valid. Then from (2.2.30) we have

$$\sum_{l=1,\, l \ne i}^{n} \big|\operatorname{sgn}(t - t_l) \cdot \varepsilon_i\, d_j c_{il}(t)\big| \\ \le \big|\varepsilon_i + (-1)^j \operatorname{sgn}(t - t_i) \cdot \varepsilon_i\, d_j c_{ii}(t)\big| \ \text{ for } \ t \in [a, b] \ \ (j = 1, 2;\ i = 1, \dots, n). \quad (2.2.92)$$

Using (2.2.29), we obtain

$$\big|\varepsilon_i + (-1)^j \operatorname{sgn}(t - t_i) \cdot \varepsilon_i\, d_j c_{ii}(t)\big| \\ \le 1 + (-1)^j \operatorname{sgn}(t - t_i) \cdot \varepsilon_i\, d_j c_{ii}(t) \ \text{ for } \ t \in [a, b] \ \ (j = 1, 2;\ i = 1, \dots, n).$$

This and (2.2.92) yield

$$\sum_{l=1,\, l \ne i}^{n} \big|\operatorname{sgn}(t - t_l) \cdot \varepsilon_i\, d_j c_{il}(t)\big| \\ < 1 + (-1)^j \operatorname{sgn}(t - t_i) \cdot \varepsilon_i\, d_j c_{ii}(t) \ \text{ for } \ t \in [a, b] \ \ (j = 1, 2;\ i = 1, \dots, n).$$

Therefore, by Hadamard's theorem (see [36, p. 382]), condition (1.1.8) holds. Remark 2.2.5 is proved analogously for the second case of (2.2.30).

2.3 Nonnegativity of solutions of the Cauchy–Nicoletti type multi-point boundary value problems

In this section, we give some propositions on the existence of nonnegative solutions of problems (2.2.1), (2.2.2) and (2.2.1), (2.2.3).

2.3.1 Formulation of the results

Theorem 2.3.1. *Let the matrix-function* $A=(a_{il})_{i,l=1}^n\in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ *be such that the functions* $a_{il}(t)\operatorname{sgn}(t-t_i)$ $(i\neq l;\ i,l=1,\dots,n)$ *are nondecreasing, the conditions*

$$(a_{il}(t)-a_{il}(s))\operatorname{sgn}(t-s)\leq (c_{il}(t)-c_{il}(s)) \ \text{ for } \ (t-s)(s-t_i)>0 \ \ (i,l=1,\dots,n), \tag{2.3.1}$$

and

$$f_i(t)\operatorname{sgn}(t-t_i) \ \text{ are nondecreasing}, \ \ c_{0i}\geq 0 \ \ (i=1,\dots,n) \tag{2.3.2}$$

hold on $[a,b]$, *and*

$$0\leq \ell_i(x_1,\dots,x_n)\leq \ell_{0i}(x_1,\dots,x_n) \ \text{ for } \ x_l\in \mathrm{BV}([a,b];\mathbb{R}_+) \ \ (i,l=1,\dots,n), \tag{2.3.3}$$

where a matrix-function $C=(c_{il})_{i,l=1}^n\in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ *and a vector-functional* $\ell_0=(\ell_{0i})_{i=1}^n$ *are such that*

$$(C,\ell_0)\in \mathbb{U}(t_1,\dots,t_n). \tag{2.3.4}$$

Then problem (2.2.1), (2.2.2) *has one and only one solution and it is nonnegative.*

From the results of Subsection 2.2.1 we have the following results.

Corollary 2.3.1. *Let the matrix-function* $A=(a_{il})_{i,l=1}^n\in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ *be such that the functions* $a_{il}(t)\operatorname{sgn}(t-t_i)$ $(i\neq l;\ i,l=1,\dots,n)$ *are nondecreasing and conditions* (2.3.2) *and*

$$(a_{il}(t)-a_{il}(s))\operatorname{sgn}(t-s)\leq \int\limits_s^t h_{il}(\tau)\,d\alpha_i(\tau) \ \text{ for } \ (t-s)(s-t_i)>0 \ \ (i,l=1,\dots,n) \tag{2.3.5}$$

hold on $[a,b]$, *where* α_i $(i=1,\dots,n)$ *are the functions nondecreasing on* $[a,b]$ *and having not more than a finite number of discontinuity points,* $h_{ii}\in L^\mu([a,b],\mathbb{R};\alpha_i)$, $h_{il}\in L^\mu([a,b],\mathbb{R}_+;\alpha_l)$ $(i\neq l;$ $l=1,\dots,n)$, $1\leq\mu\leq+\infty$. *Let, moreover, condition* (2.2.15) *hold and*

$$0\leq \ell_i(x_1,\dots,x_n)\leq \sum_{m=0}^{2}\sum_{k=1}^{n} l_{mik}\|x_k\|_{\nu,s_m(\alpha_k)} \ \text{ for } \ x_k\in \mathrm{BV}([a,b];\mathbb{R}_+) \ \ (i,k=1,\dots,n),$$

where the constant matrix $\mathcal{H}$ *is defined as in Theorem 2.2.2. Then problem* (2.2.1), (2.2.2) *has one and only one solution and it is nonnegative.*

Corollary 2.3.2. *Let the matrix-function* $A=(a_{il})_{i,l=1}^n\in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ *be such that the functions* $a_{il}(t)\operatorname{sgn}(t-t_i)$ $(i\neq l;\ i,l=1,\dots,n)$ *are nondecreasing and conditions* (2.3.2) *and* (2.3.5) *hold on* $[a,b]$, *where* α_l $(l=1,\dots,n)$ *are the functions nondecreasing on* $[a,b]$ *and having not more than a finite number of discontinuity points,* $h_{ii}\in L^\mu([a,b],\mathbb{R};\alpha_i)$, $h_{il}\in L^\mu([a,b],\mathbb{R}_+;\alpha_l)$ $(i\neq l;\ i,l=1,\dots,n)$, $1\leq\mu\leq+\infty$. *Let, moreover, condition* (2.2.16) *hold, where* $\mathcal{H}_0=((\lambda_{kmij}\|h_{ik}\|_{\mu,s_m(\alpha_i)})_{i,k=1}^n)_{m,j=0}^2$ *is a* $3n\times 3n$*-matrix, and* λ_{kmij}, ξ_{ij} $(j,m=0,1,2;\ i,k=1,\dots,n)$ *and* ν *are defined as in Corollary 2.2.1. Then problem* (2.2.1), (2.2.3) *has one and only one solution and it is nonnegative.*

By Remark 2.2.2, Corollary 2.3.2 has the following form for $h_{il}(t)\equiv h_{il}=const$ $(i,l=1,\dots,n)$ and $\mu=+\infty$.

Corollary 2.3.3. *Let the matrix-function* $A=(a_{il})_{i,l=1}^n\in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ *be such that the functions* $a_{il}(t)\operatorname{sgn}(t-t_i)$ $(i\neq l;\ i,l=1,\dots,n)$ *are nondecreasing and the conditions* (2.3.2) *and*

$$(a_{ii})(t)-a_{ii}(s))\operatorname{sgn}(t-s)\leq h_{ii}(\alpha(t)-\alpha(s)) \ \text{ for } \ (t-s)(s-t_i)>0 \ \ (i=1,\dots,n)$$

hold on $[a,b]$, *where* α *is a function nondecreasing on* $[a,b]$ *and having not more than a finite number of discontinuity points,* $h_{ii}\in\mathbb{R}$, $h_{il}\in\mathbb{R}_+$ $(i\neq l;\ i,l=1,\dots,n)$. *Let, moreover, condition* (2.2.17) *hold, where* ρ_0 *and a constant matrix* $\mathcal{H}=(h_{ik})_{i,k=1}^n$ *are defined as in Corollary 2.2.2. Then problem* (2.2.1), (2.2.3) *has one and only one solution and it is nonnegative.*

Corollary 2.3.4. *Let the matrix-function* $A = (a_{il})_{i,l=1}^n \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ *be such that the functions* $a_{il}(t)\operatorname{sgn}(t-t_i)$ $(i \neq l;\ i,l = 1,\dots,n)$ *are nondecreasing, the conditions* (2.2.21), (2.2.23), (2.3.1), (2.3.2) *and*

$$0 \le \ell_i(x_1,\dots,x_n) \le \mu_i x_i(\tau_i) \ \text{ for } \ x_l \in \mathrm{BV}([a,b];\mathbb{R}_+) \ \ (i,l = 1,\dots,n) \tag{2.3.6}$$

hold on $[a,b]$*, where the functions* c_{ii} $(i = 1,\dots,n)$ *are nonincreasing on* $[a,b]$*,* $\mu_i \in \mathbb{R}$*,* $\tau_i \in [a,b]$*,* $\tau_i \neq t_i$ $(i = 1,\dots,n)$*;* $\lambda_i(t) \equiv \gamma_{a_i}(t,t_i)$*, the function* $\gamma_{a_i}(t,t_i)$ *is defined according to* (1.1.9)*, and* $a_i(t) \equiv (c_{ii}(t) - c_{ii}(t_i))\operatorname{sgn}(t-t_i)$ $(i = 1,\dots,n)$*. Let, moreover, the functions* $c_{il} \in \mathrm{BV}([a,b];\mathbb{R})$ $(i \neq l;\ i,l = 1,\dots,n)$ *be nondecreasing on* $[a,b]$ *and condition* (2.2.24) *hold, where* $\mathcal{M} = (\mu_{il})_{i,l=1}^n$ *is the constant matrix defined as in Theorem 2.2.3. Then problem* (2.2.1), (2.2.2) *has one and only one solution and it is nonnegative.*

Remark 2.3.1. In particular, the statement of Corollary 2.3.4 is true for the boundary value condition (2.2.25).

Corollary 2.3.5. *Let the matrix-function* $A = (a_{il})_{i,l=1}^n \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ *be such that the functions* $a_{il}(t)\operatorname{sgn}(t-t_i)$ $(i \neq l;\ i,l = 1,\dots,n)$ *are nondecreasing, conditions* (2.2.21), (2.2.23), (2.3.1), (2.3.2) *and* (2.3.6) *hold on* $[a,b]$*, where* $\mu_i \in \mathbb{R}$*,* $\tau_i \in [a,b]$*,* $\tau_i \neq t_i$ $(i = 1,\dots,n)$*; and the functions* c_{ii} $(i = 1,\dots,n)$ *are such that the functions* $\lambda_i(t) \equiv \gamma_{c_{ii}}(t,t_i)$ $(i = 1,\dots,n)$*, defined according to* (1.1.9)*, are monotone on the intervals* $[a,t_i[$ *and* $]t_i,b]$*. Let, moreover, the functions* $c_{il} \in \mathrm{BV}([a,b];\mathbb{R})$ $(i \neq l;\ i,l = 1,\dots,n)$ *be nondecreasing on* $[a,b]$ *and condition* (2.2.24) *hold, where* $\mathcal{M} = (\mu_{il})_{i,l=1}^n$ *is the constant matrix defined as in Theorem 2.2.3. Then problem* (2.2.1), (2.2.25) *has one and only one solution and it is nonnegative.*

Remark 2.3.2. Let problem (2.2.1), (2.2.2) have a unique solution, the functions $a_{il}(t)\operatorname{sgn}(t-t_i)$ $(i \neq l;\ i,l = 1,\dots,n)$ be nondecreasing, condition (2.3.2) and the inequalities

$$0 \le \ell_i(x_1,\dots,x_n) \ \text{ for } \ x_l \in \mathrm{BV}([a,b];\mathbb{R}_+) \ \ (i,l = 1,\dots,n)$$

hold. These conditions solely fail to imply that the solutions of the problem, in general, are nonnegative (a corresponding example is given below).

2.3.2 Proof of the results

Proof of Theorem 2.3.1. Due to (2.3.1), conditions (2.2.6)–(2.2.8) hold. Moreover, from (2.3.3) follows estimate (2.2.9), since ℓ_{0i} $(i = 1,\dots,n)$ are nondecreasing functionals. So, in view of Theorem 2.2.1, problem (2.2.1), (2.2.2) is uniquely solvable.

Let $x = (x_i)_{i=1}^n$ be the solution of problem (2.2.1), (2.2.2) and let $i \in \{1,\dots,n\}$ be fixed. Due to (2.2.36), we have

$$\begin{aligned} |x_i(t)| - |x_i(s)| &= \sum_{l=1}^n \int_s^t x_l(\tau)\operatorname{sgn} x_i(\tau)\,ds_c(a_{il})(\tau) + \sum_{s<\tau\le t}\big(|x_i(\tau)| - |x_i(\tau-)|\big) \\ &\quad + \sum_{s\le\tau<t}\big(|x_i(\tau+)| - |x_i(\tau)|\big) + \int_s^t \operatorname{sgn} x_i(\tau)\,ds_c(f_i)(\tau) \ \text{ for } \ a \le s < t \le b. \end{aligned} \tag{2.3.7}$$

From the definition of solutions of system (2.2.1), properties of the integral and the equalities

$$d_j x(t) \equiv d_j A(t)\cdot x(t) + d_j f(t) \ \ (j = 1,2)$$

result in

$$\begin{aligned} x_i(t) - x_i(s) &= \sum_{l=1}^n \int_s^t x_l(\tau)\,ds_c(a_{il})(\tau) + \sum_{s<\tau\le t}\big(x_i(\tau) - x_i(\tau-)\big) \\ &\quad + \sum_{s\le\tau<t}\big(x_i(\tau+) - x_i(\tau)\big) + (f_i(t) - f_i(s)) \ \text{ for } \ a \le s < t \le b \ \ (j = 1,2). \end{aligned}$$

Subtracting the last equality from (2.3.7), we can conclude that

$$y_(t)-y_i(s)=\sum_{l=1}^{n}\int_{s}^{t}\big(x_l(\tau)\operatorname{sgn}x_i(\tau)-x_l(\tau)\big)\,ds_c(a_{il})(\tau)+\sum_{s<\tau\le t}(y_i(\tau)-y_i(\tau-))$$
$$+\sum_{s\le\tau<t}(y_i(\tau+)-y_i(\tau))+\int_{s}^{t}\operatorname{sgn}x_i(\tau)\,ds_c(f_i)(\tau)-(f_i(t)-f_i(s))\ \text{ for }\ a\le s<t\le b,$$

where

$$y_i(t)\equiv|x_i(t)|-x_i(t).$$

From this, using conditions of Lemma 2.2.6, just as in the proof of Lemma 2.2.1, we obtain

$$y_i(t)-y_i(s)\le\sum_{l=1}^{n}\int_{s}^{t}y_l(t)\,dc_{il}(t)\ \text{ for }\ (t-s)(s-t_i)>0. \tag{2.3.8}$$

In addition, by (2.3.2) and (2.3.3), we have

$$|x_i(t_i)|\le\ell_i(|x_1|,\dots,|x_n|)+c_{0i}\ \ (i=1,\dots,n),$$

From this, because ℓ_i $(i=1,\dots,n)$ are linear functionals, and (2.3.8), it is clear that

$$\operatorname{sgn}(t-t_i)dy_i(t)\le\sum_{l=1}^{n}y_l(t)\,dc_{il}(t)\ \text{ for }\ t\in[a,b],\ \ t\ne t_i\ \ (i=1,\dots,n),$$
$$(-1)^j\,d_jy_i(t_i)\le\sum_{l=1}^{n}y_l(t_i)\,d_jc_{il}(t_i)\ \ (j=1,2;\ i=1,\dots,n);$$
$$y_i(t_i)\le\ell_{0i}(y_1,\dots,y_n)\ \ (i=1,\dots,n).$$

So, since y_i $(i=1,\dots,n$ are the nonnegative functions, according to condition (2.3.4), we conclude that

$$y_i(t)\equiv 0\ \ (i=1,\dots,n).$$

Consequently,

$$|x_i(t)|=x_i(t)\ge 0\ \text{ for }\ t\in[a,b]\ \ (i=1,\dots,n). \qquad\square$$

Corollaries 2.3.1–2.3.5 follow from Theorem 2.3.1 by virtue of Lemmas 2.2.8–2.2.10.

In connection to Remark 2.3.2, for the completeness, we give the corresponding example from [47] for ordinary differential case.

Consider the problem

$$\frac{dx_1}{dt}=x_2,\ \ \frac{dx_2}{dt}=-\mu^2x_1-1;\ \ x_1(a)=0,\ x_2(b)=0, \tag{2.3.9}$$

where

$$\mu=\frac{7\pi}{3}\,(b-a)^{-1}.$$

For this example, the conditions of Remark 2.3.2 are valid for

$$n=2,\ \ t_1=a,\ \ t_2=b;\ \ a_{11}(t)\equiv a_{22}(t)\equiv 0,\ \ a_{12}(t)\equiv t,\ \ a_{21}(t)\equiv-\mu^2t,$$
$$f_1(t)\equiv 0,\ \ f_2(t)\equiv -t;\ \ \ell_i(x_1,x_2)\equiv 0\ \ (i=1,2)$$

and problem (2.3.9) has the unique solution

$$x_1(t)=\mu^{-2}\big(2\cos\mu(b-a)-1\big),\ \ x_1(t)=2\mu^{-1}\sin\mu(b-a).$$

In addition, the values of functions x_1 and x_2 at the point $t_0=b-3\pi(2mu)^{-1}$ are negative.

2.4 On a method for constructing solutions of the Cauchy–Nicoletti type multi-point boundary value problems

In this section, we give a method for constructing solutions of the system

$$dx(t) = dA(t) \cdot x(t) + df(t) \ \text{ for } \ t \in [a,b] \tag{2.4.1}$$

satisfying one of the conditions

$$x_i(t_i) = \ell_i(x_1,\dots,x_n) + c_{0i} \ \ (i=1,\dots,n), \tag{2.4.2}$$

$$x_i(t_i) = \mu_i x_i(\tau_i) + c_{0i} \ \ (i=1,\dots,n) \tag{2.4.3}$$

and

$$x_i(t_i) = c_{0i} \ \ (i=1,\dots,n), \tag{2.4.4}$$

where $c_{oi} \in \mathbb{R}$, $\mu_i \in \mathbb{R}$ and $\tau_i \in [a,b]$ $(i=1,\dots,n)$.

As the zero approximation to the solution of problem (2.4.1), (2.4.2), we choose an arbitrary function $(x_{0i})_{i=1}^n \in \mathrm{BV}(I;\mathbb{R}^n)$. If the $(m-1)$-th approximation $(x_{m-1\,i})_{i=1}^n$ is constructed, then by the m-th approximation we take $(x_{mi})_{i=1}^n$, i-th component of which is defined by

$$x_{mi}(t_i) = \ell_i(x_{m-1\,1},\dots,x_{m-1\,n}) + c_{0i} \ \ (i=1,\dots,n), \tag{2.4.5}$$

$$x_{mi}(t) = \gamma_i(t,t_i)x_{mi}(t_i) + \omega_i(x_{m-1\,1},\dots,x_{m-1\,n},f_i)(t) \ \text{ for } \ t \in [a,b] \ \ (i=1,\dots,n), \tag{2.4.6}$$

where the operators $\omega_i : \mathrm{BV}(I;\mathbb{R}^{n+1}) \to \mathrm{BV}(I;\mathbb{R})$ $(i=1,\dots,n)$ are defined as

$$\begin{aligned}
&\omega_i(y_1,\dots,y_{n+1})(t) = g_i(y_1,\dots,y_{n+1})(t) - \gamma_i(t,t_i)\int_{t_i}^{t} g_i(y_1,\dots,y_{n+1})(s)\, d\gamma_i^{-1}(s,t_i),\\
&g_i(y_1,\dots,y_{n+1})(t) = \sum_{l=1}^{n}\int_{t_i}^{t} y_l(s)\, d(a_{il}(s) - \delta_{il}\widetilde{a}_i(s)) + y_{n+1}(t) - y_{n+1}(t_i)\\
&\qquad\qquad \text{for } \ t \in [a,b] \ \ (i=1,\dots,n);\\
&\gamma_i(t,t_i) = \gamma_{\widetilde{a}_i}(t,t_i), \ \ \widetilde{a}_i(t) = s_c(a_{ii})(t) \ \text{ for } \ t \in [a,b] \ \ (i=1,\dots,n),
\end{aligned} \tag{2.4.7}$$

and the function $\gamma_{\widetilde{a}_i}(t,t_i)$ is defined by (1.1.9).

2.4.1 Formulation of the results

Theorem 2.4.1. *Let conditions* (2.2.6)–(2.2.9) *hold on* $[a,b]$, *where a matrix-function* $C = (c_{il})_{i,l=1}^n \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ *and a vector-functional* $\ell_0 = (\ell_{0i})_{i=1}^n$ *are such that condition* (2.3.4) *hold. Then problem* (2.4.1), (2.4.2) *has one and only one solution and there exist* $\rho_0 > 0$ *and* $\delta \in]0,1[$ *such that*

$$\sum_{i=1}^{n} \|x_i - x_{mi}\|_\infty \le \rho_0\delta^m \ \ (m=1,2\dots), \tag{2.4.8}$$

where the vector-functions $(x_{mi})_{i=1}^n$ $(m=1,2,\dots)$ *are defined by* (2.4.5)–(2.4.7).

Remark 2.4.1. The described above process of constructing a solution of problem (2.4.1), (2.4.2) is stable in the sense given below.

Corollary 2.4.1. *Let the conditions of Theorem 2.2.2 hold. Then the statement of Theorem 2.4.1 is true.*

Corollary 2.4.2. *Let the conditions of Theorem 2.2.4 hold. Then problem* (2.4.1), (2.4.3) *has the unique solution* $(x_i)_{i=1}^n$ *and for an arbitrary function* $(x_{0i})_{i=1}^n \in \mathrm{BV}([a,b];\mathbb{R}^n)$ *estimate* (2.4.8) *holds, where*

$$x_{mi}(t) = \mu_i\gamma_i(t,t_i)x_{m-1\,i}(\tau_i) + c_{0i} + \omega_i(x_{m-1\,1},\dots,x_{m-1\,n},f_i)(t) \ \text{ for } \ t \in [a,b] \ \ (i=1,\dots,n),$$

the operators $\omega_i : \mathrm{BV}([a,b];\mathbb{R}^{n+1}) \to \mathrm{BV}([a,b];\mathbb{R})$ $(i=1,\dots,n)$ *are defined by* (2.4.7), *and* $\rho_0 > 0$ *and* $\delta \in]0,1[$ *are the constants that do not depend on* m.

Corollary 2.4.3. *Let the conditions of Corollaries 2.2.1 or 2.2.2 hold. Then problem* (2.4.1), (2.4.4) *has the unique solution* $(x_i)_{i=1}^n$ *and for an arbitrary function* $(x_{0i})_{i=1}^n \in \mathrm{BV}([a,b];\mathbb{R}^n)$ *estimate* (2.4.8) *holds, where*

$$x_{mi}(t) = \gamma_i(t,t_i)c_{0i} + \omega_i(x_{m-1\,1},\dots,x_{m-1\,n},f_i)(t) \ \text{ for } \ t \in [a,b] \ \ (i=1,\dots,n),$$

the operators $\omega_i : \mathrm{BV}([a,b];\mathbb{R}^{n+1}) \to \mathrm{BV}([a,b];\mathbb{R})$ $(i=1,\dots,n)$ *are defined by* (2.4.7), *and* $\rho_0 > 0$ *and* $\delta \in]0,1[$ *are the constants that do not depend on* m.

2.4.2 Auxiliary propositions

Lemma 2.4.1. *Let* $y, y_k \in \mathrm{BV}([a,b];\mathbb{R}^n)$ $(k=1,2,\dots)$ *be vector-functions such that the conditions*

$$\lim_{k\to+\infty} y_k(t) = y(t) \ \text{ for } \ t \in [a,b]$$

and

$$\|y_k(t) - y_k(s)\| \le l_k + \|g(t) - g(s)\| \ \text{ for } \ a \le s < t \le b \ \ (k=1,2,\dots) \tag{2.4.9}$$

hold, where $l_k \ge 0$, $l_k \to 0$ *as* $k \to +\infty$, *and* $g : [a,b] \to R^n$ *is a nondecreasing vector-function. Then*

$$\lim_{k\to+\infty} \|y_k - y\|_\infty = 0.$$

Proof. Let ε be an arbitrary positive number, and let $R(a,b,\varepsilon;g)$ and $D_j(a,b,\varepsilon;g)$ $(j=1,2)$ be the sets defined in Subsection 1.2.2. Due to Lemma 1.2.3, the set $R(a,b,\varepsilon;g)$ is not empty. Let $\{\alpha_0,\tau_1,\alpha_1,\dots,\tau_m,\alpha_m\} \in R(a,b,\varepsilon/5;g)$, and let n_ε be a natural number such that

$$l_i < \frac{\varepsilon}{5} \ \text{ and } \ \|y_i(\tau) - y_k(\tau)\| < \frac{\varepsilon}{5} \ \text{ for } \ \tau \in \{\alpha_0,\tau_1,\alpha_1,\dots,\tau_m,\alpha_m\} \ \ (i,k \ge n_\varepsilon).$$

Assume that $\alpha_{j-1} < t < \tau_j$ $(j=1,\dots,m)$. Then, in view of (2.4.9), we have

$$\begin{aligned}\|y_i(t) - y_k(t)\| &\le \|y_i(t) - y_i(\tau_j)\| + \|y_i(\tau_j) - y_k(\tau_j)\| + \|y_k(\tau_j) - y_k(t)\| \\ &\le l_i + l_k + 2\|g(t) - g(\tau_j)\| + \|y_i(\tau_j) - y_k(\tau_j)\| \\ &< \frac{3\varepsilon}{5} + 2\|g(\tau_j) - g(\alpha_{j-1})\| < \varepsilon \ \text{ for } \ \tau_j \notin D_1\Big(a,b,\frac{\varepsilon}{5};g\Big) \ \ (i,k \ge n_\varepsilon)\end{aligned}$$

and

$$\begin{aligned}\|y_i(t) - y_k(t)\| &\le \|y_i(t) - y_i(\alpha_{j-1})\| + \|y_i(\alpha_{j-1}) - y_k(\alpha_{j-1})\| + \|y_k(\alpha_{j-1}) - y_k(t)\| \\ &\le l_i + l_k + 2\|g(t) - g(\alpha_{j-1})\| + \|y_i(\alpha_{j-1}) - y_k(\alpha_{j-1})\| \\ &< \frac{3\varepsilon}{5} + 2\|g(\tau_j-) - g(\alpha_{j-1})\| < \varepsilon \ \text{ for } \ \tau_j \in D_1\Big(a,b,\frac{\varepsilon}{5};g\Big) \ \ (i,k \ge n_\varepsilon).\end{aligned}$$

The case $\tau_j < t < \alpha_j$ $(j=1,\dots,m)$ is considered analogously, where we use the set $D_2(a,b,\varepsilon;g)$. $\square$

Lemma 2.4.2. *Let the matrix-function* $C = (c_{il})_{i,l=1}^n \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ *and a vector-functional* $\ell_0 = (\ell_{0i})_{i=1}^n$ *be such that conditions* (2.3.4) *and*

$$1 + d_j c_{ii}(t) > 0 \ \ for \ \ t \in [a,b] \ \ (j = 1,2; \ i = 1,\dots,n) \tag{2.4.10}$$

hold. Then there exists a positive number ρ_* *such that every solution of the problem*

$$\begin{gathered}\operatorname{sgn}(t-t_i)\, d|x_i(t)| \le \sum_{l=1}^n |x_l(t)|\, dc_{il}(t) + du_i(t) \ \ for \ \ t\in[a,b] \ \ (i=1,\dots,n),\\ (-1)^j\, d_j|x_i(t_i)| \le \sum_{l=1}^n |x_l(t_i)|\, dc_{il}(t_i) + d_j u_i(t_i) \ \ (j=1,2; \ i=1,\dots,n);\end{gathered} \tag{2.4.11}$$

$$|x_i(t_i)| \le \ell_{0i}\big(|x_1|,\dots,|x_n|\big) + \gamma \ \ (i=1,\dots,n) \tag{2.4.12}$$

admits an estimate

$$\sum_{i=1}^n \|x_i\|_\infty \le \rho_*\big(\gamma + \|u(\,\cdot\,) - u(a)\|_\infty\big), \tag{2.4.13}$$

where $\gamma \in \mathbb{R}_+$ *and* $u = (u_i)_{i=1}^n$ *are an arbitrary number and a nondecreasing vector-function, respectively.*

Proof. Let $x = (x_i)_{i=1}^n$ be an arbitrary solution of problem (2.4.11), (2.4.12). In view of (2.4.11), we conclude that

$$\begin{gathered}\operatorname{sgn}(t-t_i)\, d|x_i(t)| \le |x_l(t)|\, dc_{ii}(t) + \sum_{l\neq i;\, l=1}^n |x_l(t)|\, dc_{il}(t) + du_i(t) \text{ for } t\in[a,b] \ (i=1,\dots,n),\\ (-1)^j\, d_j|x_i(t_i)| \le |x_l(t_i)|\, dc_{ii}(t_i) + \sum_{l\neq i;\, l=1}^n |x_l(t_i)|\, dc_{il}(t_i) + d_j u_i(t_i) \ (j=1,2; \ i=1,\dots,n).\end{gathered} \tag{2.4.14}$$

According to Lemma 2.2.6, from (2.4.14) it follows that

$$|x_i(t)| \le y_i(t) \ \text{ for } \ t\in[a,b] \ \ (i=1,\dots,n),$$

where y_i, for every $i \in \{1,\dots,n\}$, is a solution of the Cauchy problem

$$\begin{gathered}\operatorname{sgn}(t-t_i)\, dy_i(t) = y_l(t)\, dc_{ii}(t) + \sum_{l\neq i;\, l=1}^n |x_l(t)|\, dc_{il}(t) + du_i(t) \ \text{ for } \ t\in[a,b],\\ y(t_i) = |x_i(t_i)|.\end{gathered} \tag{2.4.15}$$

If along with this we take into account that ℓ_{0i} is a nondecreasing functional and c_{il} and u_i ($i \neq l$; $i,l = 1,\dots,n$) are nondecreasing functions, from (2.4.12) and (2.4.15) we find that

$$\begin{gathered}\big|dy_i(t) - y_i(t)\, d\alpha_i(t)\big| \le \sum_{l\neq i;\, l=1}^n y_l(t)\, dc_{il}(t) + du_i(t) \ \text{ for } \ t\in[a,b] \ \ (i=1,\dots,n),\\ y_i(t_i) \le \ell_{0i}(y_1,\dots,y_n) + \gamma \ \ (i=1,\dots,n),\end{gathered} \tag{2.4.16}$$

where

$$\alpha_i(t) \equiv c_{ii}(t)\operatorname{sgn}(t-t_i) \ \ (i=1,\dots,n). \tag{2.4.17}$$

Thus, for the proof of the lemma, it suffices to prove the existence of a positive number ρ such that every nonnegative solution of problem (2.4.16) would admit an estimate

$$\sum_{i=1}^n \|y_i\|_\infty \le \rho_*\big(\gamma + \|u(\,\cdot\,) - u(a)\|_\infty\big).$$

Let us say the opposite, i.e., there does not exist such a number ρ. Then for every natural m there exists a vector-function $u_0=(u_{0im})_{i=1}^n$ (nondecreasing on $[a,b]$), $\gamma_m\in\mathbb{R}_+$ and $(y_{im})_{i=1}^n\in \mathrm{BV}([a,b];\mathbb{R}^n)$ such that

$$\big|dy_{im}(t)-y_{im}(t)\,d\alpha_i(t)\big|\le\sum_{l\neq i;\,l=1}^n y_{lm}(t)\,dc_{il}(t)+du_{0im}(t)\ \text{ for }\ t\in[a,b]\ \ (i=1,\dots,n),$$
$$y_{im}(t_i)\le \ell_{0i}(y_{1m},\dots,y_{nm})+\gamma_m\ \ (i=1,\dots,n)$$

and

$$\sum_{i=1}^n\|y_{im}\|_\infty>m\big(\gamma_m+\|u_{0m}(\cdot)-u_{0m}(a)\|_\infty\big).$$

If we put

$$\overline{y}_{im}(t)\equiv\Big(\sum_{k=1}^n\|y_{km}\|_\infty\Big)^{-1}y_{im}(t)\ \ (i=1,\dots,n)$$

and

$$\overline{u}_{0im}(t)\equiv\Big(\sum_{k=1}^n\|y_{km}\|_\infty\Big)^{-1}u_{0im}(t)\ \ (i=1,\dots,n),$$

then, taking into account that ℓ_{0i} is positive homogeneous, we get

$$\big|d\overline{y}_{im}(t)-\overline{y}_{im}(t)\,d\alpha_i(t)\big|\le\sum_{l\neq i;\,l=1}^n \overline{y}_{lm}(t)\,dc_{il}(t)+d\overline{u}_{0im}(t)\ \text{ for }\ t\in[a,b]\ \ (i=1,\dots,n),\tag{2.4.18}$$
$$\overline{y}_{im}(t_i)\le \ell_{0i}(\overline{y}_{1m},\dots,\overline{y}_{nm})+\frac{1}{m}\ \ (i=1,\dots,n),\tag{2.4.19}$$
$$\overline{u}_{0im}(b)-\overline{u}_{0im}(a)\le\frac{1}{m}\ \ (i=1,\dots,n),\sum_{i=1}^n\|\overline{y}_{im}\|_\infty=1\tag{2.4.20}$$

and

$$\sum_{i=1}^n\|\overline{y}_{im}\|_\infty=1.\tag{2.4.21}$$

According to (2.4.18), (2.4.20) and (2.4.21), there exists a positive number r such that

$$\bigvee_a^b\overline{y}_{im}<r\ \ (i=1,\dots,n;\ m=1,2,\dots).\tag{2.4.22}$$

Therefore, due to the Hally choice theorem, without loss of generality, we may assume that

$$\lim_{m\to+\infty}\overline{y}_{im}(t)=y_i(t)<r\ \text{ for }\ t\in[a,b]\ \ (i=1,\dots,n),$$

where $y_i\in\mathrm{BV}([a,b];\mathbb{R})$ $(i=1,\dots,n)$.

By (2.4.18) and (2.4.22), there exists nondecreasing vector-function g such that condition (2.4.9) holds for $l_m=0$ $(m=1,2,\dots)$. Using Lemma 2.4.1, we get

$$\lim_{m\to+\infty}\|\overline{y}_{im}-y_i\|_\infty=0\ \ (i=1,\dots,n).\tag{2.4.23}$$

Consequently, by this, (2.4.19) and (2.4.21), we have

$$y_i(t_i)\le\ell_{0i}(y_1,\dots,y_n)\ \ (i=1,\dots,n),\tag{2.4.24}$$
$$\sum_{i=1}^n\|y_i\|_\infty=1.\tag{2.4.25}$$

Let us put

$$q_{im}(t) \equiv f_{im}(t) + \overline{u}_{0im}(t) - \overline{u}_{0im}(a) \ \ (i = 1, \dots, n; \ m = 1, 2, \dots),$$

where

$$f_{im}(t) \equiv \sum_{l \neq i;\, l=1}^{n} \int_a^t \overline{y}_{lm}(t)\, dc_{il}(t) \ \ (i = 1, \dots, n; \ m = 1, 2, \dots).$$

By virtue of (2.4.18), Lemma 2.2.6 and the variation-of-constants formula (1.1.12), we have

$$\overline{y}_{im}(t) \le \gamma_i(t, t_i)\overline{y}_{im}(t_i) + \left| \int_{t_i}^t \gamma_i(t, \tau)\, d\mathcal{A}(\alpha_i, q_{im})(\tau) \right| \ \text{ for } \ t \in [a, b] \ \ (i = 1, \dots, n), \tag{2.4.26}$$

where $\gamma_i(t, \tau) \equiv \gamma_{\alpha_i}(t, \tau)$ is defined by (1.1.9).

Due to (2.4.20) and (2.4.26), we find that

$$\begin{aligned} \overline{y}_{im}(t) &\le \gamma_i(t, t_i)\overline{y}_{im}(t_i) + \left| \int_{t_i}^t \gamma_i(t, \tau)\, d\mathcal{A}(\alpha_i, f_{im})(\tau) \right| + \left| \int_{t_i}^t \gamma_i(t, \tau)\, d\mathcal{A}(\alpha_i, u_{im})(\tau) \right| \\ &\le \gamma_i(t, t_i)\overline{y}_{im}(t_i) + \left| \sum_{l \neq i;\, l=1}^{n} \int_{t_i}^t \gamma_i(t, \tau)\overline{y}_{lm}(t)\, d\mathcal{A}(\alpha_i, c_{il})(\tau) \right| + \frac{r}{m} \ \text{ for } \ t \in [a, b] \ \ (i = 1, \dots, n), \end{aligned}$$

where r is some positive number.

If in the last inequality we pass to the limit as $m \to +\infty$, then by (2.4.23) we conclude that

$$y_i(t) \le x_i(t) \ \text{ for } \ t \in [a, b] \ \ (i = 1, \dots, n), \tag{2.4.27}$$

where

$$x_i(t) \equiv \gamma_i(t, t_i)y_i(t_i) + \left| \sum_{l \neq i;\, l=1}^{n} \int_{t_i}^t \gamma_i(t, \tau)y_l(\tau)\, d\mathcal{A}(\alpha_i, c_{il})(\tau) \right| \ \ (i = 1, \dots, n). \tag{2.4.28}$$

Put

$$g_i(t) = \left| \sum_{l \neq i;\, l=1}^{n} \int_{t_i}^t \gamma_i(t_i, \tau)y_l(\tau)\, d\mathcal{A}(\alpha_i, c_{il})(\tau) \right| \ \text{ for } \ t \in [a, b] \ \ (i = 1, \dots, n).$$

Then by (2.4.28) we have

$$x_i(t) \equiv \gamma_i(t, t_i)(y_i(t_i) + g_i(t)) \ \text{ for } \ t \in [a, b] \ \ (i = 1, \dots, n). \tag{2.4.29}$$

Using the equalities

$$\gamma_i(t, s) \equiv \gamma_i(t, t_i)\gamma_i(t_i, s), \ \ d\gamma_i(t, t_i) \equiv \gamma_i(t, t_i)\, d\alpha_i(t)$$

and

$$d_j\gamma_i(t, t_i) \equiv \gamma_i(t, t_i)\, d_j\alpha_i(t) \ \ (j = 1, 2; \ i = 1, \dots, n),$$

the general integration-by-parts formulas (0.0.11) and (2.4.17), from (2.4.29) we conclude that

$$x_i(t)-x_i(s)=\int_s^t dx_i(\tau)$$
$$=\int_s^t (y_i(t_i)+g_i(\tau))\,d\gamma_i(\tau,t_i)+\int_s^t \gamma_i(\tau,t_i)\,dg_i(\tau)-\sum_{s<\tau\le t} d_1\gamma_i(\tau,t_i)\,d_1g_i(\tau)+\sum_{s\le\tau t} d_2\gamma_i(\tau,t_i)\,d_2g_i(\tau)$$
$$=\int_s^t x_i(\tau)\,d\alpha_i(\tau)+\sum_{l\ne i;\,l=1}^n\int_s^t y_l(\tau)\,d\mathcal{A}(\alpha_i,c_{il})(\tau)-\sum_{l\ne i;\,l=1}^n\sum_{s<\tau\le t} d_1\gamma_i(\tau,t_i)\cdot\gamma_i(t_i,\tau)y_l(\tau)\,d_1\mathcal{A}(\alpha_i,c_{il})(\tau)$$
$$+\sum_{l\ne i;\,l=1}^n\sum_{s\le\tau<t} d_2\gamma_i(\tau,t_i)\cdot\gamma_i(t_i,\tau)y_l(\tau)\,d_2\mathcal{A}(\alpha_i,c_{il})(\tau)\ \text{ for }\ a\le s<t\le b\ \ (i=1,\dots,n).$$

Hence

$$x_i(t)-x_i(s)=\int_s^t x_i(\tau)\,d\alpha_i(\tau)+\sum_{l\ne i;\,l=1}^n\int_s^t y_l(\tau)\,dc_{il}(\tau)\ \text{ for }\ s,t\in[a,b]\ \ (i=1,\dots,n).$$

Consequently, owing to (2.4.27), we have

$$x_i(t)-x_i(s)\le\sum_{l=1}^n\int_s^t x_l(\tau)\,dc_{il}(\tau)\ \text{ for }\ s,t\in[a,b]\ \ (i=1,\dots,n).$$

By virtue of this, (2.4.24), (2.4.25) and (2.4.27), we obtain

$$\sum_{i=1}^n\|x_i\|_\infty\ge 1$$

and thus the function $(x_i)_{=1}^n$ is a nonnegative nontrivial solution of problem (2.2.4), (2.2.5), which contradicts condition (2.2.10). □

Lemma 2.4.3. *Let the matrix-function* $C=(c_{il})_{i,l=1}^n\in\mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ *and a vector-functional* $\ell_0=(\ell_{0i})_{i=1}^n$ *be such that conditions* (2.2.10) *and* (2.4.10) *hold. Then there exists a number* $\delta\in]0,1[$ *such that*

$$(\widetilde{C},\widetilde{\ell}_0)\in\mathbb{U}(t_1,\dots,t_n), \tag{2.4.30}$$

where $\widetilde{C}=(\widetilde{c}_{il})_{i,l=1}^n$, $\widetilde{\ell}_0=(\widetilde{\ell}_{0i})_{i=1}^n$,

$$\widetilde{c}_{ii}(t)\equiv c_{ii}(t),\ \ \widetilde{c}_{il}(t)\equiv\frac{1}{\delta}\,c_{il}(t)\ \ (i\ne l;\ i,l=1,\dots,n) \tag{2.4.31}$$

and

$$\widetilde{\ell}_{0i}(y_1,\dots,y_n)\equiv\frac{1}{\delta}\,\ell_{0i}(y_1,\dots,y_n)\ \ (i=1,\dots,n). \tag{2.4.32}$$

Proof. According to Lemma 2.4.2, there exists a positive number ρ_* such that every solution of problem (2.4.11), (2.4.12) admits estimate (2.4.13), where $\gamma\in\mathbb{R}_+$ and $u=(u_i)_{i=1}^n$ are an arbitrary number and a nondecreasing vector-function, respectively.

Let

$$\gamma_0=\sum_{l=1}^n\ell_{0l}(1,\dots,1),\ \ u_{0i}(t)\equiv\sum_{l\ne i;\,l=1}^n\big(c_{il}(t)-c_{il}(a)\big)\ \ (i=1,\dots,n),$$

$\delta \in]0,1[$ be a number such that

$$\frac{1-\delta}{\delta}\,\rho_*\Big(\gamma_0+\sum_{l=1}^{n}u_{il}(b)\Big)<\frac{1}{2}\ \ (i=1,\dots,n) \tag{2.4.33}$$

and let $\widetilde{c}_{il}$ and $\widetilde{\ell}_{0i}$ $(i,l=1,\dots,n)$ be, respectively, the functions and the functionals given by (2.4.31) and (2.4.32).

Consider an arbitrary nonnegative solution $x=(x_i)_{i=1}^n$ of the problem

$$\begin{gathered}\operatorname{sgn}(t-t_i)\,dx_i(t)\le\sum_{l=1}^{n}x_l(t)\,d\widetilde{c}_{il}(t)\ \text{ for }\ t\in[a,b]\ \ (i=1,\dots,n),\\ (-1)^j\,d_jx_i(t_i)\le\sum_{l=1}^{n}x_l(t_i)\,d\widetilde{c}_{il}(t_i)\ \ (j=1,2;\ i=1,\dots,n);\end{gathered} \tag{2.4.34}$$

$$x_i(t_i)\le\ell_{0i}(x_1,\dots,x_n)\ \ (i=1,\dots,n). \tag{2.4.35}$$

It is not difficult to verify that $(x_i)_{i=1}^n$ will be the solution of problem (2.4.11), (2.4.12), where

$$\gamma=\frac{1-\delta}{\delta}\,\gamma_0\sum_{l=1}^{n}\|x_l\|_\infty,\ \ u_i(t)\equiv\frac{1-\delta}{\delta}\,u_{0i}(t)\sum_{l=1}^{n}\|x_l\|_\infty\ \ (i=1,\dots,n).$$

By the choose of ρ_*, estimate (2.4.13) holds, which, in view of (2.4.33), implies

$$\sum_{l=1}^{n}\|x_l\|_\infty\le\frac{1}{2}\sum_{l=1}^{n}\|x_l\|_\infty.$$

Consequently, $x_i(t)\equiv0$ $(i=1,\dots,n)$. So, condition (2.4.30) holds. □

Lemma 2.4.4. *Let the matrix-function $C=(c_{il})_{i,l=1}^n\in\mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ and a vector-functional $\ell_0=(\ell_{0i})_{i=1}^n$ be such that conditions* (2.2.10) *and*

$$1+(-1)^j\operatorname{sgn}(t-t_i)\,d_jc_{ii}(t)>0\ \ \textit{for}\ \ (-1)^j(t-t_i)<0\ \ (j=1,2;\ i=1,\dots,n) \tag{2.4.36}$$

hold. Then there exist a positive number ρ and $\delta\in]0,1[$ such that for an arbitrary $(y_{0i})_{i=1}^m\in\mathrm{BV}([a,b];\mathbb{R}_+^n)$ and any sequences of numbers $\gamma_m\in\mathbb{R}_+$ $(m=1,2,\dots)$, the vector-functions $(y_{mi})_{i=1}^m\in\mathrm{BV}([a,b];\mathbb{R}_+^n)$ $(m=1,2,\dots)$ and nondecreasing vector-functions $u_m=(u_{mi})_{i=1}^m\in\mathrm{BV}([a,b];\mathbb{R}^n)$ $(m=1,2,\dots)$ satisfying the inequalities

$$\begin{gathered}\operatorname{sgn}(t-t_i)\,dy_{mi}(t)\le y_{mi}(t)\,d\widetilde{c}_i(t)\\ +\sum_{l=1}^{n}y_{m-1\,l}(t)\,d\big(c_{il}(t)-\delta_{il}\widetilde{c}_i(t)\big)+du_{mi}(t)\ \textit{for}\ t\in[a,b]\ \ (i=1,\dots,n),\\ (-1)^j\,d_jy_{mi}(t_i)\le\sum_{l=1}^{n}y_{m-1\,l}(t_i)\,d_jc_{il}(t_i)+d_ju_{mi}(t_i)\ \ (j=1,2;\ i=1,\dots,n),\end{gathered} \tag{2.4.37}$$

$$y_{mi}(t_i)\le\ell_{0i}(y_{m-1\,l},\dots,y_{m-1\,n})+\gamma_m\ \ (i=1,\dots,n) \tag{2.4.38}$$

for every natural m, the estimates

$$\sum_{i=1}^{n}\|y_{mi}\|_\infty\le\rho\bigg\{\sum_{l=1}^{m}\delta^{m-l}\big(\gamma_l+\|u_l(b)-u_l(a)\|\big)+\delta^m\sum_{i=1}^{n}\|y_{0i}\|_\infty\bigg\}\ \ (m=1,2,\dots) \tag{2.4.39}$$

hold.

Proof. By Lemma 2.4.3, there exists a number $\delta\in\,]0,1[$ such that the functions $\widetilde{c}_{il}$ and the functionals $\widetilde{\ell}_{0i}$ $(i,l=1,\dots,n)$ given, respectively, by (2.4.31) and (2.4.32), satisfy condition (2.4.30).

On $\mathrm{BV}([a,b];\mathbb{R}^n_+)$, we introduce the operators

$$\omega_i(z_1,\dots,z_n)(t)=\gamma_i(t)\bigg\{\widetilde{\ell}_{0i}(z_1,\dots,z_n)+\sum_{l\neq i;\,l=1}^{n}\bigg(\bigg|\int_{t_i}^{t}\gamma_i^{-1}(\tau)z_l(\tau)\,d\widetilde{c}_{il}(\tau)$$
$$+\sum_{t_{*i}<\tau\le t_i^*}\gamma_i^{-1}(\tau-)z_l(\tau)\,d_1\widetilde{c}_{il}(\tau)+\sum_{t_{*i}\le\tau<t_i^*}\gamma_i^{-1}(\tau+)z_l(\tau)\,d_2\widetilde{c}_{il}(\tau)\bigg)\bigg\}\ \ (i=1,\dots,n),$$

where $t_{*i}=\min\{t_i,t\}$, $t_i^*=\max\{t_i,t\}$ $(i=1,\dots,n)$, and $\gamma_i(t)$ is a solution of the problem

$$d\gamma_i(t)=\gamma_i(t)\,d\alpha_i(t),\ \ \gamma_i(t_i)=1.$$

By (2.4.36), the latter problem has the unique solution which is strongly positive on the whole $[a,b]$.

Let

$$z_{0i}(t)\equiv 1\ \ (i=1,\dots,n),\ \ \eta=\sum_{i=1}^{n}\|\gamma_i\|_\infty\big(1+n\|\gamma_i^{-1}\|_\infty\big)$$

and $(z_{mi})_{i=1}^n$ $(m=1,2,\dots)$ be a sequence of vector-functions defined by

$$z_{mi}(t)=\omega_i(z_{m-11},\dots,z_{m-1n})(t)+\eta\ \ (i=1,\dots,n;\ m=1,2,\dots). \tag{2.4.40}$$

Clearly, $\omega_i(\,\cdot\,)$ $(i=1,\dots,n)$ are nondecreasing operators and, therefore,

$$1\le z_{m-1i}(t)\le z_{mi}(t)\ \text{ for }\ t\in[a,b]\ \ (i=1,\dots,n;\ m=1,2,\dots).$$

Consequently,

$$\rho_m=\sum_{i=1}^{n}\|z_{mi}\|_\infty\ \ (m=1,2,\dots)$$

is a nondecreasing sequence of positive numbers.

Let us show that

$$\rho=\lim_{m\to\infty}\rho_m<\infty. \tag{2.4.41}$$

Assuming to the contrary that $\rho_m\to\infty$ as $m\to\infty$, we put

$$x_{mi}(t)=\frac{z_{mi}(t)}{\rho_m}\,,\ \ \overline{x}_{mi}(t)=\omega_i(x_{m-1\,1},\dots,x_{m-1\,n})(t),\ \ \eta_m=\frac{\eta}{\rho_m}\ \ (m=1,2,\dots).$$

Then

$$\lim_{m\to\infty}\eta_m=0, \tag{2.4.42}$$
$$\sum_{i=1}^{n}\|x_{mi}\|_\infty=1\ \ (m=1,2,\dots). \tag{2.4.43}$$

Taking into account the Hally choice theorem and Lemma 2.4.1, it is not difficult to verify that

$$\lim_{m\to\infty}\sup\overline{x}_{mi}(t)=\overline{x}_i(t)\ \text{ uniformly on }\ [a,b]\ \ (i=1,\dots,n), \tag{2.4.44}$$

where $(\overline{x}_i)_{i=1}^n$ is a vector-function from $\mathrm{BV}([a,b];\mathbb{R}^n_+)$. On the other hand, from (2.4.40) we have

$$x_{mi}(t)\le\overline{x}_{mi}(t)+\eta_m\ \text{ for }\ t\in[a,b]\ \ (i=1,\dots,n;\ m=1,2,\dots)$$

and

$$\overline{x}_{mi}(t) \le \omega_i(\overline{x}_{m-1\,1} + \eta_{m-1}, \dots, \overline{x}_{m-1\,n} + \eta_{m-1})(t) \ \text{ for } \ t \in [a,b] \ \ (i = 1, \dots, n; \ m = 1, 2, \dots).$$

By (2.4.42)–(2.4.44) and the latter inequalities, it follows that

$$\sum_{i=1}^{n} \|\overline{x}_i\|_\infty > 1$$

and

$$\overline{x}_i(t) \le x_i(t) \ \text{ for } \ t \in [a,b] \ \ (i = 1, \dots, n),$$

where

$$x_i(t) \equiv \omega_i(\overline{x}_1, \dots, \overline{x}_n)(t) \ \ (i = 1, \dots, n).$$

Now, for every $i \in \{1, \dots, n\}$, using the integration-by-parts formula (1.1.12), from definition of the function ω_i we get

$$\operatorname{sgn}(t - t_i)(x_i(t) - x_i(s)) - \sum_{l=1}^{n} \int_s^t x_l(\tau)\, d\widetilde{c}_{il}(\tau) = \frac{1}{\delta} \sum_{l \ne i;\, l=1}^{n} \int_s^t (\overline{x}_l(\tau) - x_l(\tau))\, dc_{il}(\tau) \le 0$$
$$\text{for } \ a \le s \le t < t_i \ \text{ and } \ t_i < s \le t \le b \ \ (i = 1, \dots, n),$$
$$(-1)^j\, d_j x_i(t_i) \le \sum_{l=1}^{n} x_l(t_i)\, d_j \widetilde{c}_{il}(t_i) \ \ (j = 1, 2; \ i = 1, \dots, n);$$
$$x_i(t_i) = \widetilde{\ell}_{0i}(\overline{x}_1, \dots, \overline{x}_n) \le \widetilde{\ell}_{0i}(x_1, \dots, x_n) \ \ (i = 1, \dots, n).$$

So, $(x_i)_{i=1}^n$ is nonnegative nonzero (due to (2.4.43)) solution of problem (2.4.34), (2.4.35). But this contradicts condition (2.4.30). The obtained contradiction proves inequality (2.4.41).

Let

$$(y_{0i})_{i=1}^n \in \mathrm{BV}([a,b]; \mathbb{R}_+^n), \ \ \zeta_0 = \sum_{i=1}^{n} \|y_{0i}\|_\infty > 0,$$

and $\gamma_m \in \mathbb{R}_+$, $(y_{mi})_{i=1}^n \in \mathrm{BV}([a,b]; \mathbb{R}_+^n)$ and $u_m = (u_{mi})_{i=1}^n \in \mathrm{BV}([a,b]; \mathbb{R}^n)$ $(m = 1, 2, \dots)$ be arbitrary sequences satisfying conditions (2.4.37) and (2.4.38) for every natural m.

Put

$$\zeta_m = \sum_{k=1}^{m} \delta^{m-k} (\gamma_k + \|u_k(b) - u_k(a)\|) + \delta^m \sum_{i=1}^{n} \|y_{0i}\|_\infty,$$
$$\overline{y}_{0i}(t) \equiv 1, \ \ \overline{y}_{mi}(t) \equiv \frac{y_{mi}(t)}{\zeta_m} \ \ (i = 1, \dots, n).$$

Regarding the inequalities

$$\zeta_m \ge \delta \zeta_{m-1}, \ \ \zeta_m > \gamma_m, \ \ \gamma_m > \|u_m(b) - u_m(a)\| \ \ (m = 1, 2, \dots),$$

from (2.4.37) and (2.4.38) we discover that

$$\begin{gathered} \operatorname{sgn}(t - t_i)\, d\overline{y}_{mi}(t) \le \overline{y}_{mi}(t)\, d\widetilde{c}_i(t) \\ + \sum_{l \ne i;\, 1=1}^{n} \overline{y}_{m-1l}(t)\, d\widetilde{c}_{il}(t) + d\overline{u}_{mi}(t) \ \text{ for } \ t \in [a,b] \ \ (i = 1, \dots, n), \end{gathered} \tag{2.4.45}$$
$$(-1)^j\, d_j \overline{y}_{mi}(t_i) \le \sum_{l=1}^{n} \overline{y}_{m-11}(t_i)\, d_j c_{il}(t_i) + d_j \overline{u}_{mi}(t_i) \ \ (j = 1, 2; \ i = 1, \dots, n),$$
$$\overline{y}_{mi}(t_i) \le \widetilde{\ell}_{0i}(\overline{y}_{m-1l}, \dots, \overline{y}_{m-1n}) + 1 \ \ (i = 1, \dots, n) \tag{2.4.46}$$

for every natural m, where $\overline{u}_{mi}(t)=u_{mi}(t)/\zeta_m$.

Let now
$$q^*_{mi}(t)\equiv q_i(\overline{y}_{m-1\,1},\dots,\overline{y}_{m-1\,n})(t)+|\overline{u}_{mi}(t)-\overline{u}_{mi}(t_i)|\ \ (i=1,\dots,n)$$
and
$$y^*_{mi}(t)\equiv \overline{y}_{mi}(t)-q^*_{mi}(t)\ \ (i=1,\dots,n)$$
for every natural m.

Then, by (2.4.45), (2.4.46) and the equalities
$$d_j\alpha_i(t_i)=(-1)^j\,d_jc_{ii}(t_i),\ \ d_jq^*_{im}(t_i)=(-1)^j\,d_j\overline{u}_{mi}(t_i)\ \ (j=1,2;\ i=1,\dots,n),$$
we find that
$$\begin{gathered}\operatorname{sgn}(t-t_i)\,dy^*_{mi}(t)\le (y^*_{mi}(t)+q^*_{mi}(t))\,dc_{ii}(t)\ \text{ for }\ t\in[a,b]\ \ (i=1,\dots,n),\\ (-1)^j\,d_jy^*_{mi}(t_i)\le (y^*_{mi}(t_i)+q^*_{mi}(t_i))\,d_jc_{ii}(t_i)\ \ (j=1,2;\ i=1,\dots,n)\end{gathered}$$
and
$$y^*_{mi}(t_i)\le c_{0i}\ \ (j=1,2;\ i=1,\dots,n),$$
where
$$c_{0i}=\widetilde{\ell}_{0i}(\overline{y}_{m-1\,1},\dots,\overline{y}_{m-1\,n})+1\ \ (j=1,2;\ i=1,\dots,n)$$
for every natural m.

By Lemma 2.2.6, we have
$$y^*_{mi}(t)\le x^*_{mi}(t)\ \text{ for }\ t\in[a,b]\ \ (i=1,\dots,n;\ m=1,2,\dots),\tag{2.4.47}$$
where x^*_{mi} is a solution of the problem
$$dx^*_{mi}(t)=(x^*_{mi}(t)+q^*_{mi}(t))\,d\alpha_i(t),\quad x^*_{mi}(t_i)=c_{0i}.$$

Due to condition (2.4.36), the latter problem has the unique solution x^*_{mi} and by the variation-of-constant formula (1.1.12), this solution is given by the equality
$$x^*_{mi}(t)=\int\limits_{t_i}^{t}q^*_{mi}(\tau)\,d\alpha_i(\tau)+\gamma_i(t)\bigg\{c_{0i}-\int\limits_{t_i}^{t}\bigg(\int\limits_{t_i}^{\tau}q^*_{mi}(s)\,d\alpha_i(s)\bigg)\,d\gamma_i^{-1}(\tau)\bigg\}$$
$$\text{for }\ t\in[a,b]\ \ (i=1,\dots,n;\ m=1,2,\dots).$$

Hence, by the integration-by-parts formula and equalities (0.0.13) and (0.0.14), we conclude that
$$\begin{aligned}x^*_{mi}(t)&=\gamma_i(t)\bigg\{c_{0i}+\int\limits_{t_i}^{t}q^*_{mi}(\tau)\gamma_i^{-1}(\tau)\,d\alpha_i(\tau)\\&\qquad-\sum_{t_i<\tau\le t}q^*_{mi}(\tau)\,d_1\alpha_i(\tau)\,d_1\gamma_i^{-1}(\tau)+\sum_{t_i\le\tau<t}q^*_{mi}(\tau)\,d_2\alpha_i(\tau)\,d_2\gamma_i^{-1}(\tau)\bigg\}\\&=\gamma_i(t)\bigg\{c_{0i}+\int\limits_{t_i}^{t}q^*_{mi}(\tau)\,d\bigg(\int\limits_{t_i}^{\tau}\gamma_i^{-1}(s)\,d\alpha_i(s)-\sum_{t_i<s\le\tau}d_1\alpha_i(s)\,d_1\gamma_i^{-1}(s)+\sum_{t_i\le s<\tau}d_2\alpha_i(s)\,d_2\gamma_i^{-1}(s)\bigg)\bigg\}\\&=\gamma_i(t)\bigg\{c_{0i}+\int\limits_{t_i}^{t}q^*_{mi}(\tau)\,d\bigg(\alpha_i(\tau)\gamma_i^{-1}(\tau)-\int\limits_{t_i}^{\tau}\alpha_i(s)\,d\gamma_i^{-1}(s)\bigg)\bigg\}\ \text{ for }\ t>t_i\ \ (i=1,\dots,n;\ m=1,2,\dots).\end{aligned}$$

In addition, by Proposition 1.1.2 (see equality (1.1.13)), we get
$$\gamma_i^{-1}(t)=1-\gamma_i^{-1}(t)\alpha_i(t)+\int\limits_{t_i}^{t}\alpha_i(s)\,d\gamma_i^{-1}(s)\ \text{ for }\ t\in[a,b]\ \ (i=1,\dots,n)$$

and, therefore, we obtain

$$x^*_{mi}(t) = \gamma_i(t)\bigg(c_{0i} - \int\limits_{t_i}^{t} q^*_{mi}(\tau)\, d\gamma_i^{-1}(\tau)\bigg) \ \text{ for } \ t > t_i \ \ (i = 1, \dots, n; \ m = 1, 2, \dots). \tag{2.4.48}$$

The inequality (2.4.48) for $t < t_i$ can be verified analogously.

By definitions of y^*_{mi}, q^*_{mi}, ω_i and η_i, from (2.4.47) and (2.4.48) it follows that

$$\begin{aligned}
&\overline{y}_{mi}(t) \le \omega_i\big(\overline{y}_{m-11}, \dots, \overline{y}_{m-1n}\big)(t) + \gamma_i(t) \\
&\quad + \gamma_i(t)\bigg\{\bigg|\int\limits_{t_i}^{t} \gamma_i^{-1}(\tau)\, ds_c(\overline{u}_{mi})(\tau)\bigg| + \sum_{t_{*i}<\tau\le t_i^*} \gamma_i^{-1}(\tau-)\, d_1\overline{u}_{mi}(\tau) + \sum_{t_{*i}\le\tau<t_i^*} \gamma_i^{-1}(\tau+)\, d_2\overline{u}_{mi}(\tau)\bigg\} \\
&\qquad \le \omega_i(\overline{y}_{m-11}, \dots, \overline{y}_{m-1n})(t) + \|\gamma_i\|_\infty \\
&\quad + \|\gamma_i\|_\infty \|\gamma_i^{-1}\|_\infty \big(s_c(\overline{u}_{mi})(b) - s_c(\overline{u}_{mi})(a)\big) + \sum_{a<\tau\le b} d_1\overline{u}_{mi}(\tau) + \sum_{a\le\tau<b} d_2\overline{u}_{mi}(\tau) \\
&\qquad \le \omega_i(\overline{y}_{m-11}, \dots, \overline{y}_{m-1n})(t) + \|\gamma_i\|_\infty \big(1 + \|\gamma_i^{-1}\|_\infty \big(\overline{u}_{mi}(b) - \overline{u}_{mi}(a)\big)\big).
\end{aligned}$$

So,

$$\overline{y}_{mi}(t) \le \omega_i(\overline{y}_{m-11}, \dots, \overline{y}_{m-1n})(t) + \eta \ \text{ for } \ t \in [a,b] \ \ (i = 1, \dots, n; \ m = 1, 2, \dots).$$

This, according to (2.4.41), implies

$$\overline{y}_{mi}(t) \le z_{mi}(t) \ \text{ for } \ t \in [a,b] \ \ (i = 1, \dots, n; \ m = 1, 2, \dots)$$

and

$$\sum_{i=1}^{n} \|\overline{y}_{mi}\|_\infty \le \sum_{i=1}^{n} \|z_{mi}\|_\infty = \rho_m \le \rho \ \ (m = 1, 2, \dots).$$

Hence, estimates (2.4.39) are valid. Since ρ does not depend on $(y_{0i})_{i=1}^n$, these estimates will be also valid if $y_{0i} \equiv 0$ $(i = 1, \dots, n)$. □

2.4.3 Proof of the results

Proof of Theorem 2.4.1. According to Theorem 2.2.1, problem (2.4.1), (2.4.2) has the unique solution $(x_i)_{i=1}^n$. On the other hand, due (2.4.6) and (2.4.7), for every $i \in \{1, \dots, n\}$ and every natural m, by variation-of-constant formula (1.1.12), the function x_{mi} is a solution of the Cauchy problem

$$dx_{mi}(t) = x_{mi}(t)\, d\widetilde{a}_i(t) + \sum_{l=1}^{n} x_{m-1\,l}(t)\, d(a_{il}(t) - \delta_{il}\widetilde{a}_i(t)) + df_i(t),$$
$$x_{mi}(t_i) = \ell_i(x_{m-1\,1}, \dots, x_{m-1\,n}) + c_{0i}$$

and, therefore,

$$d(x_i(t) - x_{mi}(t)) = (x_i(t) - x_{mi}(t))\, d\widetilde{a}_i(t) + \sum_{l=1}^{n} (x_l(t) - x_{m-1,l}(t))\, d(a_{il}(t) - \delta_{il}\widetilde{a}_i(t)), \tag{2.4.49}$$

$$x_i(t_i) - x_{mi}(t_i) = \ell_i(x_1 - x_{m-1\,1}, \dots, x_n - x_{m-1n}). \tag{2.4.50}$$

Put

$$y_{mi}(t) \equiv |x_{mi}(t) - x_i(t)| \ \ (i = 1, \dots, n; \ m = 1, 2, \dots).$$

Then with regard for (2.2.6)–(2.2.8), from (2.4.49) and (2.4.50), by Lemma 2.2.1, it follows that for every natural m the functions y_{mi} $(i = 1, \dots, n)$ satisfy the inequalities

$$\operatorname{sgn}(t - t_i)\, dy_{mi}(t) \le y_{mi}(t)\, d\widetilde{c}_i(t) + \sum_{l=1}^{n} y_{m-1\,l}(t)\, d(c_{il}(t) - \delta_{il}\widetilde{c}_i(t)) \ \text{ for } \ t \in [a,b] \ \ (i = 1, \dots, n),$$

$$(-1)^j\, d_j y_{mi}(t_i) \le \sum_{l=1}^{n} y_{m-1\,l}(t_i)\, d_j c_{il}(t_i)(t_i) \ \ (j=1,2;\ i=1,\dots,n),$$
$$y_{mi}(t_i) \le \ell_{0i}(y_{m-1\,l},\dots,y_{m-1\,n}) \ \ (i=1,\dots,n).$$

Thus, by virtue of Lemma 2.4.4, we find that

$$\sum_{i=1}^{n} \|y_{mi}\|_\infty \le \rho\,\delta^m \sum_{i=1}^{n} \|y_{0i}\|_\infty \ \ (m=1,2,\dots),$$

where $\rho > 0$ and $\delta \in]0,1[$ are the constants independent of m and $(y_{0i})_{i=1}^n$. Hence estimate (2.4.8) holds, where $\rho_0 = \rho \sum\limits_{i=1}^{n} \|y_{0i}\|_\infty$. □

Corollary 2.4.1 immediately follows from Theorems 2.2.2 and 2.4.1.

Corollary 2.4.2 follows from Theorems 2.2.3, 2.4.1 and Remark 2.2.4.

Corollary 2.4.3 follows from Theorem 2.4.1 and Corollaries 2.2.1 or 2.2.2.

Now, consider Remark 2.4.1. Let the conditions of Theorem 2.4.1 be fulfilled. Let $\gamma_{mi} \in \mathbb{R}$ and $\Delta_{mi} \in \mathrm{BV}([a,b];\mathbb{R})$ $(i=1,\dots;\ m=1,2,\dots)$ be arbitrary sequences of numbers and functions satisfying the condition

$$\lim_{m\to+\infty} \varepsilon_m = 0, \tag{2.4.51}$$

where

$$\varepsilon_m = \sum_{i=1}^{n} \Big(|\eta_{mi}| + \bigvee_a^b (\Delta_{mi}) \Big).$$

Let $(\overline{x}_{0i})_{i=1}^n \in \mathrm{BV}([a,b];\mathbb{R}^n)$ be arbitrary. Consider the sequence

$$\overline{x}_{mi}(t_i) = \ell_i(\overline{x}_{m-1\,1},\dots,\overline{x}_{m-1\,n}) + c_{0i} + \gamma_{mi} \ \ (i=1,\dots,n),$$
$$\overline{x}_{mi}(t) = \gamma_i(t,t_i)\overline{x}_{mi}(t_i) + \omega_i(\overline{x}_{m-1\,1},\dots,\overline{x}_{m-1\,n}, f_i+\Delta_{mi})(t) \ \text{ for } \ t\in[a,b] \ \ (i=1,\dots,n),$$

where the operators $\omega_i : \mathrm{BV}([a,b];\mathbb{R}^{n+1}) \to \mathrm{BV}([a,b];\mathbb{R})$ $(i=1,\dots,n)$ are defined by (2.4.7).

As above, in proving of Theorem 2.4.1, we can conclude that the function $\overline{x}_{mi}$ is a solution of the Cauchy problem

$$d\overline{x}_{mi}(t) = \overline{x}_{mi}(t)\, d\widetilde{a}_i(t) + \sum_{l=1}^{n} \overline{x}_{m-1\,l}(t)\, d(a_{il}(t) - \delta_{il}\widetilde{a}_i(t)) + d(f_i(t) + \Delta_{mi}(t)),$$
$$\overline{x}_{mi}(t_i) = \ell_i(\overline{x}_{m-1\,1},\dots,\overline{x}_{m-1\,n}) + c_{0i} + \gamma_{mi}$$

for every $i \in \{1,\dots,n\}$ and natural m. From this we find that

$$\begin{aligned} d(\overline{x}_{mi}(t) - x_{mi}(t)) &= (\overline{x}_{mi}(t) - x_{mi}(t))\, d\widetilde{a}_i(t) \\ &\quad + \sum_{l=1}^{n} (\overline{x}_{m-1l}(t) - x_{m-1l}(t))\, d(a_{il}(t) - \delta_{il}\widetilde{a}_i(t)) + d\Delta_{mi}(t), \\ \overline{x}_{mi}(t_i) - x_{mi}(t_i) &= \ell_i\big(\overline{x}_{m-11} - x_{m-11},\dots,\overline{x}_{m-1n} - x_{m-1n}\big) + \eta_{mi} \end{aligned}$$

for every $i \in \{1,\dots,n\}$ and every natural m.

If we put

$$\overline{y}_{mi}(t) \equiv |\overline{x}_{mi}(t) - x_i(t)| \ \ (i=1,\dots,n;\ m=1,2,\dots),$$

then, as above, for every natural m, we find that

$$\operatorname{sgn}(t-t_i)\,dy_{mi}(t) \le y_{mi}(t)\,d\,\widetilde{c}_i(t) \\ + \sum_{l=1}^{n} y_{m-1\,l}(t)\,d(c_{il}(t)-\delta_{il}\widetilde{c}_i(t)) + du_{mi}(t) \ \text{ for } \ t\in[a,b] \ \ (i=1,\dots,n),$$

$$(-1)^j\,d_j y_{mi}(t_i) \le \sum_{l=1}^{n} y_{m-1\,l}(t_i)\,d_j c_{il}(t_i) + d_j u_{mi}(t_i) \ \ (j=1,2;\ i=1,\dots,n);$$

$$y_{mi}(t_i) \le \ell_{0i}(y_{m-1\,l},\dots,y_{m-1\,n}) + \gamma_m \ \ (i=1,\dots,n),$$

where $u_{mi}(t)\equiv \bigvee\limits_a^t(\Delta_{mi})$ and $\eta_m=\sum\limits_{i=1}^{n}|\eta_{mi}|$ $(i=1,\dots,n)$.

By this and Lemma 2.4.4, there exists a positive number ρ and $\delta\in\,]0,1[$ such that

$$\sum_{i=1}^{n}\|\overline{x}_{mi}-x_{mi}\|_\infty \le \rho\sum_{l=0}^{m}\varepsilon_l\delta^{m-l} \ \ (m=1,2,\dots),$$

where $\varepsilon_0=\sum\limits_{i=1}^{n}\|\overline{x}_{0i}-x_{0i}\|_\infty$. Thus, in view of (2.4.51), we obtain

$$\lim_{m\to\infty}\sum_{i=1}^{n}\|x_i-\overline{x}_{mi}\|_\infty = 0.$$

Consequently, the sequence $(\overline{x}_{mi})_{i=1}^n$ $(m=1,2,\dots)$ approximates the solution $(x_i)_{i=1}^n$, as well.

Chapter 3

Two-point boundary value problems for systems of generalized ordinary differential equations

3.1 Statement of the problem. Unique solvability

This section is devoted to the investigation of the problem of existence of solutions of a linear system of generalized ordinary differential equations

$$dx = dA(t) \cdot x + df(t) \ \text{ for } \ t \in [a, b] \tag{3.1.1}$$

satisfying the two-point boundary value condition

$$L_1 x(a) + L_2 x(b) = c_0 \tag{3.1.2}$$

and, in particular, the condition

$$x_i(a) = \mu_i x_i(b) + c_{0i} \ \ (i = 1, \dots, n). \tag{3.1.3}$$

Below, unless otherwise stated, we assume

$$A = (a_{ik})_{i,k=1}^n \in \mathrm{BV}([a, b]; \mathbb{R}^{n\times n}), \ \ f = (f_k)_{k=1}^n \in \mathrm{BV}([a, b]; \mathbb{R}^n);$$
$$L_1, L_2 \in \mathbb{R}^{n\times n}, \ \ c_0 \in \mathbb{R}^n, \ \ c_{0i} \in \mathbb{R}, \ \ \mu_i \in \mathbb{R} \ \ (i = 1, \dots, n).$$

Along with problem (3.1.1), (3.1.2), we consider the corresponding homogeneous problem

$$dx = dA(t) \cdot x, \tag{$3.1.1_0$}$$
$$L_1 x(a) + L_2 x(b) = 0. \tag{$3.1.2_0$}$$

In this section we realize the results given in Chapter 2 to problems (3.1.1), (3.1.2) and (3.1.1), (3.1.3).

3.1.1 Formulation of the results

Theorem 3.1.1. *The boundary value problem* (3.1.1), (3.1.2) *is uniquely solvable if and only if the corresponding homogeneous problem* ($3.1.1_0$), ($3.1.2_0$) *has only the trivial solution, i.e., if and only if*

$$\det\big(L_1 Y(a) + L_2 Y(b)\big) \neq 0, \tag{3.1.4}$$

where Y is a fundamental matrix of system $(3.1.1_0)$. If the latter condition holds, then the solution x of problem $(3.1.1), (3.1.2)$ admits the representation

$$x(t) = x_0(t) + \int_a^b d_s\mathcal{G}(t,s)\cdot f(s) \ \text{ for } \ t\in[a,b],$$

where x_0 is a solution of problem $(3.1.1_0), (3.1.2)$, and $\mathcal{G}$ is the Green matrix of problem $(3.1.1_0)$, $(3.1.2_0)$.

Hence, in view of (2.1.4), the Green matrix of problem $(3.1.1_0), (3.1.2_0)$ has the form

$$\mathcal{G}(t,s) = \begin{cases} -Y(t)\big(L_1Y(a)+L_2Y(b)\big)^{-1}L_1Y(a)Y^{-1}(s) & \text{for } \ a\le s<t\le b,\\ Y(t)\big(L_1Y(a)+L_2Y(b)\big)^{-1}L_2Y(b)Y^{-1}(s) & \text{for } \ a\le t<s\le b,\\ O_{n\times n} & \text{for } \ a\le t=s\le b. \end{cases}$$

Proposition 2.1.1 for the case under consideration has the following form.

Proposition 3.1.1. *Let the matrix-function $A\in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ be such that condition (1.1.8) hold. Then the boundary value problem $(3.1.1), (3.1.2)$ is solvable if and only if the condition*

$$(c_0 - L_2\,F(b))^\top\gamma = 0$$

holds for every $\gamma\in\mathbb{R}^n$ such that

$$(L_2\,F(b))^\top\gamma = 0_n,$$

where

$$F(t)\equiv Y(t)\int_a^t Y^{-1}(\tau)\,d\mathcal{A}(A,f)(\tau).$$

So, if condition (3.1.4) holds, then only the vector $\gamma = 0_n$ satisfies the homogeneous system appearing in Proposition 3.1.1 and, therefore, condition (1.1.18) holds evidently. If condition (3.1.4) is violated, then problem $(3.1.1), (3.1.2)$ is solvable only for c_0, which satisfies the conditions of the proposition.

Remark 3.1.1. Let the matrix-function A satisfy the Lappo–Danilevskiĭ condition at the point a. Then problem $(3.1.1), (3.1.2)$ is uniquely solvable if and only if

$$\det\bigg(L_1 + L_2\exp(S_c(A)(b))\prod_{a\le\tau<b}(I_n+d_2A(\tau))\prod_{a<\tau\le b}(I_n-d_1A(\tau))^{-1}\bigg)\neq 0.$$

Theorem 3.1.2. *The boundary value problem $(3.1.1), (3.1.2)$ is uniquely solvable if and only if there exist natural numbers k and m such that the matrix*

$$M_k = L_1 + \sum_{i=0}^{k-1} L_2[A]_i(b) \ \ \bigg(M_k = L_2 + \sum_{i=0}^{k-1} L_1[A]_i(a)\bigg)$$

is nonsingular and the inequality

$$r(M_{k,m}) < 1 \tag{3.1.5}$$

holds, where

$$M_{k,m} = V_m(A)(b) + \sum_{i=0}^{m-1}\big|\,[A]_i\big|_\infty |M_k^{-1}L_2|V_k(A)(b)$$

$$\bigg(M_{k,m} = V_m(A)(a) + \sum_{i=0}^{m-1}\big|\,[A]_i\big|_\infty |M_k^{-1}L_1|V_k(A)(a)\bigg),$$

the operators $[A]_i$ $(i=0,1,\dots)$ and $V_i(A)$ $(i=0,1,\dots)$ are defined, respectively, by $(1.1.35_1)$ and $(1.1.37_2)$ (resp. $(1.1.35_2)$ and $(1.1.37_2)$).

Theorem 3.1.2$_1$. *Let there exist natural numbers k and m such that the matrix*

$$M_k=-L_1+L_2\Big(\sum_{i=0}^{k-1}(A)_i(b)-1\Big)\ \ \bigg(M_k=-L_2+L_1\Big(\sum_{i=0}^{k-1}(A)_i(a)-1\Big)\bigg)$$

is nonsingular and inequality (3.1.5) *holds, where*

$$M_{k,m}=(V(A))_m(b)+\Big(I_n+\sum_{i=0}^{m-1}|(A)_i|_\infty\Big)\,|M_k^{-1}L_2|(V(A))_k(b)$$

$$\bigg(M_{k,m}=(V(A))_m(a)+\Big(I_n+\sum_{i=0}^{m-1}|(A)_i|_\infty\Big)\,|M_k^{-1}L_1|(V(A))_k(a)\bigg),$$

the matrix-functions $(A)_i$ $(i=0,1\dots)$ and $(V(A))_i$ $(i=0,1,\dots)$ are defined by $(1.1.36_1)$ (*resp.* $(1.1.36_2)$). *Then problem* (3.1.1), (3.1.2) *is uniquely solvable.*

Corollary 3.1.1. *Let*

$$\det(L_1+L_2)\neq 0 \tag{3.1.6}$$

and

$$r\Big(L_0\bigvee_a^b(A))\Big)<1, \tag{3.1.7}$$

where

$$L_0=I_n+\big|(L_1+L_2)^{-1}\big|\,\big(|L_1|+|L_2|\big).$$

Then problem (3.1.1), (3.1.2) *is uniquely solvable.*

For the system

$$dx(t)=\varepsilon\,dA(t)\cdot x(t)+df(t) \tag{3.1.8}$$

with a small parameter ε from Theorem 3.1.2 follows

Corollary 3.1.2. *Let either condition* (3.1.6) *hold, or there exist a natural number k such that the conditions*

$$L_1+L_2=O_{n\times n},\quad \det\big(L_1(A)_i(a)+L_2(A)_i(b)\big)=0\ \ (i=0,\dots,k-1)$$

and

$$\det\big(L_1(A)_k(a)+L_2(A)_k(b)\big)\neq 0$$

hold. Then there exists $\varepsilon_0>0$ such that problem (3.1.8), (3.1.4) *is uniquely solvable for every $\varepsilon\in\,]0,\varepsilon_0[$.*

The results given above are the particular cases of the results given in the previous section.
Let us consider the specific theorems for the case $\nu=2$ in (2.1.2).

Theorem 3.1.3. *Let conditions* (3.1.6), (3.1.7) *and*

$$\big|(L_1+L_2)^{-1}L_i\big|\le L_0\ \ (i=1,2) \tag{3.1.9}$$

hold, where $L_0\in\mathbb{R}^{n\times n}$. Then problem (3.1.1), (3.1.2) *is uniquely solvable.*

Definition 3.1.1. Let $m,r_1,\dots,r_m$ and $n_1,\dots,n_m$ $(0=n_0<n_1<\dots<n_m=n)$ be natural numbers; $\mathfrak{A}=(\alpha_{lj})_{l,j=1}^{r_j,m}$, where α_{lj} $(l=1,\dots,r_j;\ j=1,\dots,m)$ be nondecreasing on $[a,b]$ functions; and let $\mathcal{P}=(P_{lj})_{l,j=1}^{r_j,m}$, where $P_{lj}=(p_{ljik})_{i,k=1}^n$ $(l=1,\dots,r_j;\ j=1,\dots,m)$ be such that $p_{ljik}\in L([a,b],\mathbb{R};\alpha_{lj})$ $(i,k=n_{j-1}+1,\dots,n_j)$. Then by $Q_m(\mathfrak{A},\mathcal{P})$ we denote the set of all matrix-functions $A\in\mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ such that

$$a_{ik}(t)\equiv 0\ \text{ for }\ t\in[a,b]\ \ (i=n_{j-1}+1,\dots,n_j;\ k=n_j+1,\dots,n;\ j=1,\dots,m) \tag{3.1.10}$$

and

$$b_{jik}(t)=\sum_{l=1}^{r_j}\int_a^t p_{ljik}(\tau)\,d\alpha_{lj}(\tau)\ \text{ for } t\in[a,b]\ \ (i\neq k;\ i,k=n_{j-1}+1,\dots,n_j;\ j=1,\dots,m),\quad (3.1.11)$$

where

$$b_{jik}(t)\equiv a_{ik}(t)-\bigg(\frac{1}{2}\sum_{a<\tau\leq t}\sum_{r=n_{j-1}+1}^{n_j} d_1a_{ri}(\tau)\cdot d_1a_{rk}(\tau)-\sum_{a\leq\tau<t}\sum_{r=n_{j-1}+1}^{n_j} d_2a_{ri}(\tau)\cdot d_2a_{rk}(\tau)\bigg)$$
$$(i,k=n_{j-1}+1,\dots,n_j;\ j=1,\dots,m).$$

Theorem 3.1.4. *Let there exist* $\sigma\in\{1,2\}$, *natural* $m,r_1,\dots,r_m$ *and* $n_1,\dots,n_m$ $(0=n_0<n_1<\dots<n_m=n)$ *such that* $A=(a_{ik})_{i,k=1}^n\in Q_m(\mathcal{P},\mathfrak{A})$,

$$(-1)^\sigma\bigg(b_{jii}(t)-b_{jii}(s)-\sum_{l=1}^{r_j}\int_s^t p_{ljii}(\tau)\,d\alpha_{lj}(\tau)\bigg)\leq 0\ \text{ for } a\leq s<t\leq b\ \ (j=1,\dots,m),\quad (3.1.12)$$

and

$$(-1)^\sigma\bigg(\sum_{i,k=n_{j-1}+1}^{n_j} p_{ljik}(t)x_ix_k-h_{\sigma lj}(t)\sum_{i=n_{j-1}+1}^{n_j} x_i^2\bigg)\leq 0$$
$$\text{for } t\in[a,b],\ (x_i)_{i=1}^n\in\mathbb{R}^n\ \ (l=1,\dots,r_j;\ j=1,\dots,m),\quad (3.1.13)$$

where $h_{\sigma lj}\in L([a,b],\mathbb{R}_+;\alpha_{lj})$. *Let, moreover,* $L_1\in\mathbb{R}^{n\times n}$ *be a nonsingular matrix such that the conditions*

$$(-1)^\sigma\big(L_0x*x-l_\sigma\,(x*x)\big)\leq 0\ \text{ for } x\in\mathbb{R}^n,\quad (3.1.14)$$

$$1+(-1)^l d_l\beta_{\sigma j}(t)>0\ \text{ for } t\in[a,b]\ \ (l=1,2;\ j=1,\dots,m)\quad (3.1.15)$$

and

$$(-1)^\sigma\big(l_\sigma\gamma_{\beta_{\sigma j}}(b,a)-1\big)<0\ \ (j=1,\dots,m)\quad (3.1.16)$$

hold, where $L_0=(L_1^{-1}L_2)^\top L_1^{-1}L_2$, $l_\sigma>0$,

$$\beta_{\sigma j}(t)\equiv 2\sum_{l=1}^{r_j}\int_a^t h_{\sigma lj}(\tau)\,d\alpha_{lj}(\tau)\ \ (j=1,\dots,m),$$

and the functions $\gamma_{\beta_{\sigma j}}(t,a)$ $(j=1,\dots,m)$ *are defined by* (1.1.9). *Then problem* (3.1.1), (3.1.2) *is uniquely solvable.*

If $m=1$ and $r_m=1$, we use designation $Q(P,\alpha)$ instead of $Q_m(\mathcal{P},\mathfrak{A})$. In this case we have the following

Definition 3.1.2. Let α be a function nondecreasing on $[a,b]$, and $P=(p_{ik})_{i,k=1}^n$, where $p_{ik}\in L([a,b];\mathbb{R}_+;\alpha)$ $(i,k=1,\dots,n)$. Then by $Q(P;\alpha)$ we denote the set of all matrix-functions $A=(a_{ik})_{i,k=1}^n\in\mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ such that

$$b_{ik}(t)=\int_a^t p_{ik}(\tau)\,d\alpha(\tau)\ \text{ for } t\in[a,b]\ \ (i\neq k;\ i,k=1,\dots,n),$$

where

$$b_{ik}\equiv a_{ik}(t)-\frac{1}{2}\sum_{l=1}^{n}\bigg(\sum_{a<\tau\leq t} d_1a_{li}(\tau)\cdot d_1a_{lk}(\tau)-\sum_{a\leq\tau<t} d_2a_{li}(\tau)\cdot d_2a_{lk}(\tau)\bigg)\ \ (i,k=1,\dots,n).\quad (3.1.17)$$

In this case Theorem 3.1.4 takes the following form.

Theorem 3.1.4$_1$. *Let there exist $\sigma \in \{1,2\}$ such that $A=(a_{ik})_{i,k=1}^n \in Q(P;\alpha)$,*

$$(-1)^\sigma\bigg(b_{ii}(t)-b_{ii}(s)-\int\limits_s^t p_{ii}(\tau)\,d\alpha(\tau)\bigg)\le 0 \ \ \text{for} \ \ a\le s<t\le b,$$

$$(-1)^\sigma\Big(\sum_{i,k=1}^n p_{ik}(t)x_ix_k-h_\sigma(t)\sum_{i=1}^n x_i^2\Big)\le 0 \ \ \text{for} \ \ t\in[a,b], \ \ (x_i)_{i=1}^n\in\mathbb{R}^n,$$

where α is a function nondecreasing on $[a,b]$, $p_{ik}\in L([a,b];\mathbb{R};\alpha)$ $(i,k=1,\dots,n)$; $h_\sigma\in L([a,b];\mathbb{R}_+;\alpha)$. Let, moreover, $L_1\in\mathbb{R}^{n\times n}$ be a nonsingular matrix such that conditions (3.1.14),

$$1+(-1)^l d_l\beta_\sigma(t)>0 \ \ \text{for} \ \ t\in[a,b] \ \ (l=1,2)$$

and

$$(-1)^\sigma\big(l_\sigma\gamma_{\beta_\sigma}(b,a)-1\big)<0$$

hold, where $L_0=(L_1^{-1}L_2)^\top L_1^{-1}L_2$, $l_\sigma>0$, $\beta_\sigma(t)\equiv 2\int\limits_0^t h_\sigma(\tau)\,d\alpha(\tau)$, and the function $\gamma_{\beta_\sigma}(t,a)$ is defined by (1.1.9). *Then problem* (3.1.1), (3.1.2) *is uniquely solvable.*

Under $\lambda_0(H)$ and $\lambda^0(H)$ we understand, respectively, the minimum and maximum eigenvalues of the symmetric matrix $H\in\mathbb{R}^{n\times m}$.

Corollary 3.1.3. *Let $A=(a_{ik})_{i,k=1}^n\in Q(P;\alpha)$, $P=(p_{ik})_{i,k=1}^n$ and a nonsingular matrix $L_1\in\mathbb{R}^{n\times n}$ be such that the conditions*

$$1+(-1)^j 2\lambda_0(C(t))\,d_j\alpha(t)>0 \ \ \text{for} \ \ t\in[a,b] \ \ (j=1,2) \ \ \text{and} \ \ \lambda_0(L_0)\,\gamma_{\beta_1}(b,a)>1$$

or

$$1+(-1)^j 2\lambda^0(C(t))\,d_j\alpha(t)>0 \ \ \text{for} \ \ t\in[a,b] \ \ (j=1,2) \ \ \text{and} \ \ \lambda^0(L_0)\,\gamma_{\beta_2}(b,a)<1$$

hold, where α is a function nondecreasing on $[a,b]$, $p_{ik}\in L([a,b];\mathbb{R};\alpha)$ $(i,k=1,\dots,n)$, $C(t)\equiv P(t)+P^\top(t)$, $L=(L_1^{-1}L_2)^\top L_1^{-1}L_2$, $\beta_1(t)\equiv 2\int\limits_0^t\lambda_0(C(\tau))\,d\alpha(\tau)$, $\beta_2(t)\equiv 2\int\limits_0^t\lambda^0(C(\tau))\,d\alpha(\tau)$, and the functions $\gamma_{\beta_l}(t,a)$ $(l=1,2)$ are defined by (1.1.9). *Then problem* (3.1.1), (3.1.2) *is uniquely solvable.*

We assume that neither the matrix L_1, nor L_2 is nonsingular.

We consider the case $\det(L_1)\ne 0$. The second case will be considered analogously.

Let $L_1^{-1}=(\beta_{il})_{i,l=1}^n$ and $(-L_1^{-1})\,L_2=(\alpha_{il})_{i,l=1}^n$. In this case, the two-point boundary value problem (3.1.1), (3.1.2) is equivalent to the following Cauchy–Nicoletti type problem:

$$\begin{gathered} dx=dA(t)\cdot x+df(t) \ \ \text{for} \ \ t\in[a,b],\\ x_i(t_i)=\ell_i^*(x_1,\dots,x_n)+c_{0i}^* \ \ (i=1,\dots,n), \end{gathered} \tag{3.1.18}$$

where $x=(x_i)_{i=1}^n$,

$$t_i=a, \ \ \ell_i^*(x_1,\dots,x_n)\equiv\sum_{l=1}^n\alpha_{il}x_l(b), \ \ c_{0i}^*=\sum_{l=1}^n\beta_{il}c_{0l} \ \ (i=1,\dots,n).$$

For the case under consideration Definition 2.2.1 takes the following form.

Definition 3.1.3. We say that a pair (C,ℓ_0) consisting of a matrix-function $C=(c_{il})_{i,l=1}^n\in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ and a positive homogeneous nondecreasing bounded vector-functional $\ell_0=(\ell_{0i})_{i=1}^n$:

$\mathrm{BV}_\infty([a,b];\mathbb{R}_+^{n\times n}) \to \mathbb{R}_+^n$ belongs to the set $\mathbb{U}(a)$ if the functions c_{il} $(i \neq l;\ i,l = 1,\dots,n)$ are nondecreasing, and the system

$$dx_i(t) \leq \sum_{l=1}^n x_l(t)\,dc_{il}(t) \ \text{ for } \ t \in]a,b], \ \ t \neq t_i \ \ (i = 1,\dots,n),$$
$$d_2x_i(a) \leq \sum_{l=1}^n x_l(a)\,d_2c_{il}(a) \ \ (i = 1,\dots,n)$$

has no nontrivial, nonnegative solution satisfying the condition

$$x_i(a) \leq \ell_{0i}(x_1,\dots,x_n) \quad (i = 1,\dots,n).$$

Remark 3.1.2. In the case where the matrix L_1 is nonsingular, we can assume

$$\ell_{0i}(x_1,\dots,x_n) \equiv \sum_{k=1}^n |\alpha_{il}|\,|x_l(b)|.$$

Similarly, we can construct the functionals $\ell_{0i}(x_1,\dots,x_n)$ $(i = 1,\dots,n)$ when the matrix L_2 is nonsingular. In the latter case we define the set $\mathbb{U}(b)$.

So, we can use the results of Section 2.2. They have the following forms.

Theorem 3.1.5. *Let the conditions*

$$s_c(a_{ii})(t) - s_c(a_{ii})(s) \leq s_c(c_{ii})(t) - s_c(c_{ii})(s) \ \textit{ for } \ a < s < t \leq b \ \ (i = 1,\dots,n), \tag{3.1.19}$$
$$\big|s_c(a_{il})(t) - s_c(a_{il})(s)\big| \leq s_c(c_{il})(t) - s_c(c_{il})(s) \ \textit{ for } \ a \leq s < t \leq b \ \ (i \neq l;\ i,l = 1,\dots,n), \tag{3.1.20}$$
$$|d_ja_{ii}(t)| \leq |d_jc_{ii}(t)|, \ \ |d_ja_{il}(t)| \leq d_jc_{il}(t) \ \ t \in [a,b] \ \ (j = 1,2;\ i \neq l;\ i,l = 1,\dots,n) \tag{3.1.21}$$

and

$$\Big|\sum_{l=1}^n \alpha_{il}x_l(b)\Big| \leq \ell_{0i}\big(|x_1|,\dots,|x_n|\big) \ \textit{ for } \ x_l \in \mathrm{BV}([a,b];\mathbb{R}) \ \ (i,l = 1,\dots,n)$$

hold, where a matrix-function $C = (c_{il})_{i,l=1}^n \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ and a vector-functional $\ell_0 = (\ell_{0i})_{i=1}^n$ are such that

$$(C,\ell_0) \in \mathbb{U}(a).$$

Then problem (3.1.1), (3.1.2) *is uniquely solvable.*

Corollary 3.1.4. *Let the conditions*

$$s_c(a_{ii})(t) - s_c(a_{ii})(s) \leq \int_s^t h_{ii}(\tau)\,ds_c(\alpha_i)(\tau) \ \textit{ for } \ a < s < t \leq b \ \ (i = 1,\dots,n),$$
$$\big|s_c(a_{il})(t) - s_c(a_{il})(s)\big| \leq \int_s^t h_{il}(\tau)\,ds_c(\alpha_l)(\tau) \ \textit{ for } \ a \leq s < t \leq b \ \ (i \neq l;\ i,l = 1,\dots,n)$$

and

$$|d_ja_{ii}(t)| \leq |h_{ii}(t)|\,d_j\alpha_l(t), \ \ |d_ja_{il}(t)| \leq h_{il}(t)\,d_j\alpha_l(t) \ \textit{ for } \ t \in [a,b] \ \ (j = 1,2;\ i \neq l;\ i,l = 1,\dots,n)$$

hold, where α_l $(l = 1,\dots,n)$ are functions nondecreasing on $[a,b]$ and having not more than a finite number of discontinuity points, $h_{ii} \in L^\mu([a,b],\mathbb{R};\alpha_i)$, $h_{il} \in L^\mu([a,b],\mathbb{R}_+;\alpha_l)$ $(i \neq l;\ l = 1,\dots,n)$, $1 \leq \mu \leq +\infty$. Let, moreover,

$$\Big|\sum_{l=1}^n \alpha_{il}x_l(b)\Big| \leq \sum_{m=0}^2\sum_{k=1}^n l_{mik}\|x_k\|_{\nu,s_m(\alpha_k)} \ \textit{ for } \ x_k \in \mathrm{BV}([a,b];\mathbb{R}) \ \ (i,k = 1,\dots,n)$$

and

$$r(\mathcal{H})<1, \tag{3.1.22}$$

where $l_{mik}\in\mathbb{R}_+$ $(m=0,1,2;\ i,k=1,\dots,n)$, $\frac{1}{\mu}+\frac{2}{\nu}=1$, *and the* $3n\times 3n$*-matrix* $\mathcal{H}=(\mathcal{H}_{j+1\,m+1})_{j,m=0}^{2}$ *is defined as in Theorem 2.2.2. Then the statement of Theorem 3.1.5 is true.*

Corollary 3.1.5. *Let conditions* (3.1.19)–(3.1.21),

$$1+(-1)^j\,d_jc_{ii}(t)>0 \ \ \text{for}\ \ t\in[a,b]\ \ (j=1,2;\ i=1,\dots,n), \tag{3.1.23}$$

and

$$|\mu_i|\gamma_i(b)<1 \ \ (i=1,\dots,n) \tag{3.1.24}$$

hold, where c_{ii} $(i=1,\dots,n)$ *are nonincreasing functions and* c_{il} $(i\neq l;\ i,l=1,\dots,n)$ *are nondecreasing functions;* $\lambda_i(t)\equiv\gamma_{a_i}(t,a)$*, the function* $\gamma_{a_i}(t,a)$ *is defined according to* (1.1.9)*, and* $a_i(t)\equiv c_{ii}(t)-c_{ii}(a)$ $(i=1,\dots,n)$*. Let, moreover,*

$$r(\mathcal{M})<1, \tag{3.1.25}$$

where $\mathcal{M}=(\mu_{il})_{i,l=1}^{n}$ *is the constant matrix defined as in Theorem 2.2.3. Then problem* (3.1.1), (3.1.3) *is uniquely solvable.*

Remark 3.1.3. The results similar to Theorems 3.1.4 and $3.1.4_1$, Corollary 3.1.3, and so on, are likewise true for the case where the matrix L_2 is nonsingular.

3.1.2 Proof of the results

Proof of Theorem 3.1.3. Sinve the matrix L_1+L_2 is nonsingular, the system

$$dx=dO_{n\times n}\cdot x \tag{3.1.26}$$

has only the trivial solution satisfying the boundary value condition $(3.1.2_0)$. So, the Green matrix of problem (3.1.26), (3.1.2) has the form

$$\mathcal{G}(t,s)=\begin{cases}-(L_1+L_2)^{-1}L_1 & \text{for}\ \ a\le s<t\le b,\\ (L_1+L_2)^{-1}L_2 & \text{for}\ \ a\le t<s\le b,\\ O_{n\times n} & \text{for}\ \ a\le t=s\le b.\end{cases}$$

Now, taking into account (3.1.7) and (3.1.9), we obtain

$$\int_a^b |\mathcal{G}(t,\tau)|\,dV(A)(\tau)\le M \ \ \text{for}\ \ t\in[a,b],$$

where

$$M=L_0\bigvee_a^b(A) \quad\text{and}\quad r(M)<1.$$

These conditions, due to Theorem 1.1.3, guarantee the unique solvability of problem (3.1.1), (3.1.2). ☐

Proof of Theorem 3.1.4. Due to Theorem 3.1.1, it suffices to show that problem $(3.1.1_0)$, $(3.1.2_0)$ has only the trivial solution. Assume the contrary, i.e., the problem has nontrivial solution $x=(x_i)_{i=1}^{n}$.

We put

$$u_j(t)\equiv\sum_{i=n_{j-1}+1}^{n_j}x_i^2(t).$$

Then by the definition of the solution of system $(3.1.1_0)$, taking into account (3.1.10) and using (0.0.12), we conclude

$$u_1(t)-u_1(s)=\sum_{i=1}^{n_1}\bigg(2\int_s^t x_i(\tau)\,dx_i(\tau)-\sum_{s<\tau\le t}(d_1x_i(\tau))^2+\sum_{s\le\tau<t}\big(d_2x_i(\tau)\big)^2\bigg)$$
$$=\sum_{i=1}^{n_1}\bigg(2\sum_{k=1}^{n_1}\int_s^t x_i(\tau)x_k(\tau)\,da_{ik}(\tau)$$
$$+\sum_{s<\tau\le t}\big(x_i^2(\tau)-x_i^2(\tau-)-2x_i(\tau)\,d_1x_i(\tau)\big)+\sum_{s\le\tau<t}\big(x_i^2(\tau+)-x_i^2(\tau)-2x_i(\tau)\,d_2x_i(\tau)\big)\bigg)$$
$$=2\sum_{i,k=1}^{n_1}\bigg(\int_s^t x_i(\tau)x_k(\tau)\,da_{ik}(\tau)-\sum_{s<\tau\le t}x_i(\tau)x_k(\tau)\,d_1a_{ik}(\tau)-\sum_{s\le\tau<t}x_i(\tau)x_k(\tau)\,d_2a_{ik}(\tau)\bigg)$$
$$+\sum_{j=1}^{2}\big(s_j(u)(t)-s_j(u)(\tau)\big)\ \text{ for} s\ \ a\le s\le t\le b.$$

Hence

$$u_1(t)-u_1(s)=2\sum_{i,k=1}^{n_1}\int_s^t x_i(\tau)x_k(\tau)\,ds_c(a_{ik})(\tau)+\sum_{j=1}^{2}\big(s_j(u)(t)-s_j(u)(s)\big)\ \text{ for }\ a\le s\le t\le b.$$

On the other hand,

$$\sum_{j=1}^{2}\big(s_j(u_1)(t)-s_j(u_1)(s)\big)$$
$$=\sum_{i=1}^{n_1}\bigg(\sum_{s<\tau\le t}d_1x_i(\tau)\big(2x_i(\tau)-d_1x_i(\tau)\big)+\sum_{s\le\tau<t}d_2x_i(\tau)\big(2x_i(\tau)+d_2x_i(\tau)\big)\bigg)$$
$$=2\sum_{i,k=1}^{n_1}\bigg(\sum_{s<\tau\le t}x_i(\tau)x_k(\tau)\Big(d_1a_{ik}(\tau)-\frac12\sum_{l=1}^{n_1}d_1a_{li}(\tau)\cdot d_1a_{lk}(\tau)\Big)$$
$$+\sum_{s\le\tau<t}x_i(\tau)x_k(\tau)\Big(d_2a_{ik}(\tau)-\frac12\sum_{l=1}^{n_1}d_2a_{li}(\tau)\cdot d_2a_{lk}(\tau)\Big)\bigg)\ \text{ for }\ a\le s\le t\le b.$$

Thus, due to (3.1.17), we get

$$u_1(t)-u_1(s)=2\sum_{i=1}^{n_1}\int_s^t x_i^2(\tau)\,db_{1ii}(\tau)+2\sum_{i\ne k;\,i,k=1}^{n_1}\int_s^t x_i(\tau)x_k(\tau)\,db_{1ik}(\tau)\ \text{ for }\ a\le s\le t\le b.$$

Let us consider the case $\sigma_1=1$. Then from the last equalities, taking into account (3.1.11) and (3.1.12), we find that

$$u_1(t)-u_1(s)\ge 2\sum_{l=1}^{r_1}\sum_{i=1}^{n_1}\int_s^t p_{l1ii}(\tau)x_i^2(\tau)\,d\alpha_{l1}(\tau)+2\sum_{i\ne k;\,i,k=1}^{n_1}\int_s^t x_i(\tau)x_k(\tau)\,db_{1ik}(\tau)$$
$$=2\sum_{l=1}^{r_1}\sum_{i,k=1}^{n_1}\int_s^t p_{l1ik}(\tau)x_i(\tau)x_k(\tau)\,d\alpha_{l1}(\tau)\ \text{ for }\ a\le s\le t\le b$$

and, consequently, due to (3.1.13),

$$u_1(t) - u_1(s) \geq 2 \sum_{l=1}^{r_1} \int_s^t h_{1l1}(\tau) \sum_{i=1}^{n_1} x_i^2(\tau)\, d\alpha_{l1}(\tau) = \int_s^t u_1(\tau)\, d\beta_{11}(\tau) \ \text{ for } \ a \leq s \leq t \leq b.$$

From this, it is evident that the function $v(t) \equiv u_1(t)$ satisfies the conditions of Lemma 2.2.6, where $t_0 = a$ and $\alpha(t) \equiv \beta_{11}(t)$, which is nondecreasing. Thus, by the lemma, we have

$$u_1(a)\gamma_{\beta_{11}}(t,a) \leq u_1(t) \ \text{ for } \ t \in [a,b].$$

If we observe that by (3.1.15) the function $\gamma_{\beta_{11}}$ is positive, we get

$$u_1(a) \leq \gamma_{\beta_{11}}^{-1}(b,a)u_1(b).$$

We show in the same way that

$$u_j(a) \leq \gamma_{\beta_{1j}}^{-1}(b,a)u_j(b) \ \ (j = 1, \dots, m). \tag{3.1.27}$$

Let

$$u(t) = \sum_{j=1}^{n} u_j(t) \ \text{ for } \ t \in [a,b].$$

It is evident that $u(t) \equiv x(t) * x(t)$. In addition, due $(3.1.2_0)$, we get

$$u(a) = L_0 x(b) * x(b),$$

and, due to (3.1.14), we find that

$$l_1\left(x(b) * x(b)\right) \leq L_0 x(b) * x(b).$$

Therefore, using (3.1.16), we conclude

$$u(b) \leq \sum_{j=1}^{m} (l_1 \gamma_{\beta_{1j}}(b,a))^{-1} u_j(b). \tag{3.1.28}$$

From this, in view of (3.1.16), we obtain $u(b) = 0$. Indeed, in the opposite case, $u_j(b) \neq 0$ for some $j \in \{1, \dots, m\}$. So, due to (3.1.16), from (3.1.28) we get the contradiction $u(b) < u(b)$.

Thus, x will be a solution of system $(3.1.1_0)$ satisfying the Cauchy condition $x(b) = 0_n$. But this Cauchy problem has only the trivial solution. The obtained contradiction proves the theorem. Hence $x(t) \equiv 0_n$.

Similarly, we establish that $x(t) \equiv 0_n$ in the case $\sigma = 2$, as well. We only note that we have the inequalities

$$u_j(a) \geq \gamma_{\beta_{2j}}^{-1}(b,a)u_j(b) \ \ (j = 1, \dots, m)$$

instead of (3.1.27). □

We use the following lemma from [47, Lemma 1.9].

Lemma 3.1.1. *Let $H \in \mathbb{R}^{n \times n}$ be the symmetric matrix. Then*

$$\lambda_0(H)\,(x * x) \leq Hx * x \ \text{ for } \ x \in \mathbb{R}^n \ \ \big(Hx * x \leq \lambda^0(H)\,(x * x) \ \text{ for } \ x \in \mathbb{R}^n \ \big).$$

Proof of Corollary 3.1.3. It is evident that

$$\sum_{i,k=1}^{n} p_{ik}(t) x_i x_k = P(t)x * x = \frac{1}{2}\,(C(t)x * x) \ \text{ for } \ t \in [a,b], \ \ x \in \mathbb{R}^n.$$

Due to Lemma 3.1.1,

$$\lambda_0(C(t))\,(x*x)\le C(t)x*x \ \text{ for } \ t\in[a,b],\ \ x\in\mathbb{R}^n$$
$$\big(\, C(t)x*x\le\lambda^0(C(t))\,(x*x) \ \text{ for } \ t\in[a,b],\ \ x\in\mathbb{R}^n \,\big)$$

and

$$\lambda_0(L_0)\,(x*x)\le L_0x*x \ \text{ for } \ x\in\mathbb{R}^n \ \ \big(\, L_0x*x\le\lambda^0(L_0)\,(x*x) \ \text{ for } \ x\in\mathbb{R}^n \,\big).$$

So, conditions (3.1.13) and (3.1.14) are fulfilled for $h_1(t)\equiv\frac12\,\lambda_0(C(t))$, $h_2(t)\equiv\frac12\,\lambda^0(C(t))$ and $l_1=\lambda_0(L_0)$, $l_2=\lambda^0(L_0)$, respectively. As to the other conditions of Theorem 3.1.4, they are fulfilled evidently. Therefore, the corollary follows from the theorem. □

Finally, we note that the algebraic properties of problem (3.1.1), (3.1.2) are investigated in [73].

3.2 Nonnegativity of solutions of two-point boundary value problems

In this section, we consider the question on the existence of nonnegative solutions of problems (3.1.1), (3.1.2) and (3.1.1), (3.1.3). We assume that the suppositions of the previous section are valid. In particular, we assume that the matrix L_1 is nonsingular, $L_1^{-1}=(\beta_{il})_{i,l=1}^n$ and $(-L_1^{-1})L_2=(\alpha_{il})_{i,l=1}^n$. As above, each of these problems can be rewritten in the form (2.2.1), (2.2.2).

We realize the results of Section 2.3 for the considered two-point boundary value problems.

Theorem 3.2.1. *Let the matrix-function $A=(a_{il})_{i,l=1}^n\in\mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ and the matrices L_1 and L_2 be such that the functions a_{il} $(i\ne l;\ i,l=1,\dots,n)$ are nondecreasing, and the conditions*

$$a_{il}(t)-a_{il}(s)\le c_{il}(t)-c_{il}(s) \ \text{ for } \ a<s<t\le b \ \ (i,l=1,\dots,n), \tag{3.2.1}$$

$$f_i \ \text{ are nondecreasing, } \ \sum_{l=1}^n\alpha_{il}c_{0l}\ge0 \ \ (i=1,\dots,n) \tag{3.2.2}$$

and

$$0\le\sum_{l=1}^n\alpha_{il}x_l(b)\le\ell_{0i}(x_1,\dots,x_n) \ \text{ for } \ x_l\in\mathrm{BV}([a,b];\mathbb{R}_+) \ \ (i,l=1,\dots,n)$$

hold, where a matrix-function $C=(c_{il})_{i,l=1}^n\in\mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ and a vector-functional $\ell_0=(\ell_{0i})_{i=1}^n$ are such that

$$(C,\ell_0)\in\mathbb{U}(a).$$

Then problem (3.1.1), (3.1.2) has one and only one solution and it is nonnegative.

Corollary 3.2.1. *Let the matrix-function $A=(a_{il})_{i,l=1}^n\in\mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ be such that the functions a_{il} $(i\ne l;\ i,l=1,\dots,n)$ are nondecreasing and conditions (3.2.2) and*

$$a_{il}(t)-a_{il}(s)\le\int_s^t h_{il}(\tau)\,d\alpha_i(\tau) \ \text{ for } \ a<s<t\le b \ \ (i,l=1,\dots,n) \tag{3.2.3}$$

hold, where α_i $(i=1,\dots,n)$ are functions nondecreasing on $[a,b]$ and having not more than a finite number of discontinuity points, $h_{ii}\in L^\mu([a,b],\mathbb{R};\alpha_i)$, $h_{il}\in L^\mu([a,b],\mathbb{R}_+;\alpha_l)$ $(i\ne l;\ l=1,\dots,n)$, $1\le\mu\le+\infty$. Let, moreover, condition (3.1.22) hold and the matrices L_1 and L_2 be such that

$$0\le\sum_{l=1}^n\alpha_{il}x_l(b)\le\sum_{m=0}^2\sum_{k=1}^n l_{mik}\|x_k\|_{\nu,s_m(\alpha_k)} \ \text{ for } \ x_k\in\mathrm{BV}([a,b];\mathbb{R}_+) \ \ (i,k=1,\dots,n),$$

where the constant matrix $\mathcal{H}$ is defined as in Theorem 2.2.2. Then problem (3.1.1), (3.1.2) has one and only one solution and it is nonnegative.

Corollary 3.2.2. *Let the matrix-function $A=(a_{il})_{i,l=1}^n\in\mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ be such that the functions a_{il} $(i\neq l;\ i,l=1,\dots,n)$ are nondecreasing and conditions* (3.2.2) *and* (3.2.3) *hold, where α_l $(l=1,\dots,n)$ are the functions nondecreasing on $[a,b]$ and having not more than a finite number of discontinuity points, $h_{ii}\in L^\mu([a,b],\mathbb{R};\alpha_i)$, $h_{il}\in L^\mu([a,b],\mathbb{R}_+;\alpha_l)$ $(i\neq l;\ i,l=1,\dots,n)$, $1\le\mu\le+\infty$. Let, moreover,*

$$r(\mathcal{H}_0)<1,$$

where $\mathcal{H}_0=((\lambda_{kmij}\|h_{ik}\|_{\mu,s_m(\alpha_i)})_{i,k=1}^n)_{m,j=0}^2$ is a $3n\times 3n$-matrix, and λ_{kmij}, ξ_{ij} $(j,m=0,1,2;\ i,k=1,\dots,n)$ and ν are defined as in Corollary 2.2.1. Then problem (3.1.1), (3.1.2), *where $L_2=O_{n\times n}$, has one and only one solution and it is nonnegative.*

By Remark 2.2.2, Corollary 3.2.2 has the following form for $h_{il}(t)\equiv h_{il}=const$ $(i,l=1,\dots,n)$ and $\mu=+\infty$.

Corollary 3.2.3. *Let the matrix-function $A=(a_{il})_{i,l=1}^n\in\mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ and the matrix L_1 be such that the functions a_{il} $(i\neq l;\ i,l=1,\dots,n)$ are nondecreasing and conditions* (3.2.2) *and*

$$a_{ii}(t)-a_{ii}(s)\le h_{ii}(\alpha(t)-\alpha(s))\ \text{ for }\ a<s<t\le b\ \ (i=1,\dots,n)$$

hold, where α is a function nondecreasing on $[a,b]$ and having not more than a finite number of discontinuity points, $h_{ii}\in\mathbb{R}$, $h_{il}\in\mathbb{R}_+$ $(i\neq l;\ i,l=1,\dots,n)$. Let, moreover,

$$\rho_0\,r(\mathcal{H})<1,$$

where ρ_0 and the constant matrix $\mathcal{H}=(h_{ik})_{i,k=1}^n$ are defined as in Corollary 2.2.2. Then problem (3.1.1), (3.1.2), *where $L_2=O_{n\times n}$, has one and only one solution and it is nonnegative.*

Corollary 3.2.4. *Let the matrix-function $A=(a_{il})_{i,l=1}^n\in\mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ and the matrices L_1 and L_2 be such that the functions a_{il} $(i\neq l;\ i,l=1,\dots,n)$ are nondecreasing, the conditions* (3.1.23), (3.1.24), (3.2.1), (3.2.2) *and*

$$0\le\sum_{l=1}^n\alpha_{il}x_l(b)\le|\mu_i|x_i(\tau_i)\ \text{ for }\ x_l\in\mathrm{BV}([a,b];\mathbb{R}_+)\ \ (i,l=1,\dots,n)$$

hold, where the functions c_{ii} $(i=1,\dots,n)$ are nonincreasing, and c_{il} $(i\neq l;\ i,l=1,\dots,n)$ are nondecreasing; $\mu_i\in\mathbb{R}$, $\tau_i\in[a,b]$, $\tau_i\neq a$ $(i=1,\dots,n)$; $\lambda_i(t)\equiv\gamma_{a_i}(t,a)$, the function $\gamma_{a_i}(t,a)$ is defined according to (1.1.9), *and $a_i(t)\equiv c_{ii}(t)-c_{ii}(a)$ $(i=1,\dots,n)$. Let, moreover, condition* (3.1.25) *hold, where $\mathcal{M}=(\mu_{il})_{i,l=1}^n$ is the constant matrix defined as in Theorem 2.2.3. Then problem* (3.1.1), (3.1.2) *has one and only one solution and it is nonnegative.*

Remark 3.2.1. In particular, the statement of Corollary 3.2.4 is true for condition (3.1.3).

Corollary 3.2.5. *Let the matrix-function $A=(a_{il})_{i,l=1}^n\in\mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ be such that the functions a_{il} and f_i $(i\neq l;\ i,l=1,\dots,n)$ are nondecreasing, conditions* (3.1.23), (3.1.24) *and* (3.2.1) *hold, where c_{ii} $(i=1,\dots,n)$ are nonincreasing functions, and c_{il} $(i\neq l;\ i,l=1,\dots,n)$ are nondecreasing ones; $\mu_i\in\mathbb{R}$, the functions $\lambda_i(t)\equiv\gamma_{c_{ii}}(t,a)$ $(i=1,\dots,n)$ defined according to* (1.1.9) *are monotone on the intervals $[a,t_i[$ and $]t_i,b]$ $(i=1,\dots,n)$. Let, moreover,*

$$c_{0i}\ge 0\ \ (i=1,\dots,n)$$

and condition (3.1.25) hold, where $\mathcal{M}=(\mu_{il})_{i,l=1}^n$ is the constant matrix defined as in Theorem 2.2.4. Then problem (3.1.1), (3.1.3) *has one and only one solution and it is nonnegative.*

3.3 On a method for constructing solutions

In this section, we present a method for constructing solutions of problems (3.1.1), (3.1.2) and (3.1.1), (3.1.3).

We assume that neither the matrix L_1, nor L_2 is nonsingular.

We consider the case $\det(L_1) \neq 0$. The second case will be considered analogously.

Let $L_1^{-1} = (\beta_{il})_{i,l=1}^n$ and $(-L_1^{-1})\,L_2 = (\alpha_{il})_{i,l=1}^n$. In this case, as in Section 3.1, the two-point boundary value problem (3.1.1), (3.1.2) is equivalent to the Cauchy–Nicoletti type problem (3.1.18), where

$$t_i = a, \quad \ell_i^*(x_1,\dots,x_n) \equiv \sum_{l=1}^n \alpha_{il} x_l(b), \quad c_{0i}^* = \sum_{l=1}^n \beta_{il} c_{0l} \ \ (i=1,\dots,n).$$

As in Section 3.1, we use the set $\mathbb{U}(a)$ and if the matrix L_1 is nonsingular, we can assume

$$\ell_{0i}(x_1,\dots,x_n) \equiv \sum_{k=1}^n |\alpha_{il}|\,|x_l(b)|.$$

Similarly, we can construct the functionals $\ell_{0i}(x_1,\dots,x_n)$ $(i=1,\dots,n)$ when the matrix L_2 is nonsingular. In the latter case, we define the set $\mathbb{U}(b)$.

So, we can use the results of Section 3.1.

As the zero approximation to the solution of problem (3.1.1), (3.1.2), we choose an arbitrary function $(x_{0i})_{i=1}^n \in \mathrm{BV}([a,b];\mathbb{R}^n)$. If the $(m-1)$-th approximation $(x_{m-1\,i})_{i=1}^n$ is constructed, then by the m-th approximation we take $(x_{mi})_{i=1}^n$, i-th components of which are defined by

$$x_{mi}(a) = \sum_{l=1}^n \alpha_{il} x_{m-1\,l}(b) + \sum_{l=1}^n \beta_{il} c_{0l} \ \ (i=1,\dots,n), \tag{3.3.1}$$

$$x_{mi}(t) = \gamma_i(t,a) x_{mi}(a) + \omega_i(x_{m-1\,1},\dots,x_{m-1\,n},f_i)(t) \ \text{ for } \ t\in[a,b] \ \ (i=1,\dots,n), \tag{3.3.2}$$

where the operators $\omega_i : \mathrm{BV}([a,b];\mathbb{R}^{n+1}) \to \mathrm{BV}([a,b];\mathbb{R})$ $(i=1,\dots,n)$ are defined as

$$\begin{gathered}
\omega_i(y_1,\dots,y_{n+1})(t) = g_i(y_1,\dots,y_{n+1})(t) - \gamma_i(t,a)\int_a^t g_i(y_1,\dots,y_{n+1})(s)\,d\gamma_i^{-1}(s,a) \\
\text{for } \ t\in[a,b] \ \ (i=1,\dots,n); \\
g_i(y_1,\dots,y_{n+1})(t) = \sum_{l=1}^n \int_a^t y_l(s)\,d(a_{il}(s) - \delta_{il}\widetilde{a}_i(s)) + y_{n+1}(t) - y_{n+1}(a) \\
\text{for } \ t\in[a,b] \ \ (i=1,\dots,n); \\
\gamma_i(t,a) \equiv \gamma_{\widetilde{a}_i}(t,a), \ \ \widetilde{a}_i(t) \equiv s_c(a_{ii})(t) \ \ (i=1,\dots,n),
\end{gathered} \tag{3.3.3}$$

and the function $\gamma_{\widetilde{a}_i}(t,a)$ is defined by (1.1.9).

Theorem 3.3.1. *Let the conditions of Theorem 3.1.5 hold. Then problem* (3.1.1), (3.1.2) *has the unique solution $x=(x_i)_{i=1}^n$ and there exist $\rho_0>0$ and $\delta\in\,]0,1[$ such that*

$$\sum_{i=1}^n \|x_i - x_{mi}\|_\infty \le \rho_0\delta^m \ \ (m=1,2\dots), \tag{3.3.4}$$

where the vector-functions $(x_{mi})_{i=1}^n$ $(m=1,2,\dots)$ are defined by (3.3.1), (3.3.2).

Corollary 3.3.1. *Let the conditions of Corollary 3.1.4 hold. Then the statement of Theorem 3.3.1 is true.*

Corollary 3.3.2. *Let the conditions of Corollary 3.1.5 hold. Then problem* (3.1.1), (3.1.3) *has the unique solution $x=(x_i)_{i=1}^n$ and for an arbitrary function $(x_{0i})_{i=1}^n \in \mathrm{BV}([a,b];\mathbb{R}^n)$ estimate* (3.3.4) *holds, where*

$$x_{mi}(t) = \mu_i\gamma_i(t,a)x_{m-1\,i}(b) + c_{0i} + \omega_i(x_{m-1\,1},\dots,x_{m-1\,n},f_i)(t) \ \textit{ for } \ t\in[a,b] \ \ (i=1,\dots,n),$$

the operators $\omega_i : \mathrm{BV}([a,b];\mathbb{R}^{n+1}) \to \mathrm{BV}([a,b];\mathbb{R})$ $(i=1,\dots,n)$ are defined by (3.3.3), *and $\rho_0>0$ and $\delta\in\,]0,1[$ are the constants independent of m.*

Remark 3.3.1. The above process of constructing the solutions of problems (3.1.1), (3.1.2) and (3.1.1), (3.1.3) is stable in the sense given above, in Section 2.4.

Chapter 4

The periodic problem for systems of generalized ordinary differential equations

4.1 Statement of the problem. Formulations of the theorems on the existence and uniqueness of solutions

In this section, we investigate the solvability for the linear generalized system

$$dx(t) = dA(t) \cdot x(t) + df(t) \ \text{ for } \ t \in \mathbb{R} \tag{4.1.1}$$

with the $\omega > 0$-periodic condition

$$x(t+\omega) = x(t) \ \text{ for } \ t \in \mathbb{R}, \tag{4.1.2}$$

where ω is a fixed positive number, $A=(a_{ik})_{i,k=1}^{n} \in \mathrm{BV}_\omega(\mathbb{R};\mathbb{R}^{n\times n})$ and $f=(f_i)_{i=1}^{n} \in \mathrm{BV}_\omega(\mathbb{R};\mathbb{R}^{n})$, i.e.,

$$A(t+\omega) = A(t) + C \ \text{ and } \ f(t+\omega) = f(t) + c \ \text{ for } \ t \in \mathbb{R}, \tag{4.1.3}$$

where $C \in \mathbb{R}^{n\times n}$ and $c \in \mathbb{R}^{n}$ are, respectively, some constant matrix and constant vector.

Moreover, we assume that

$$\det\big(I_n + (-1)^j \, d_j A(t)\big) \neq 0 \ \text{ for } \ t \in \mathbb{R} \ \ (j=1,2). \tag{4.1.4}$$

We establish a Green type theorem on the solvability of problem (4.1.1), (4.1.2) and represent a solution of the problem. In addition, we give the effective necessary and sufficient conditions (of spectral type) for the unique solvability of the problem.

Along with (4.1.1), we consider the corresponding homogeneous system

$$dx(t) = dA(t) \cdot x(t). \tag{4.1.1$_0$}$$

Moreover, along with condition (4.1.2), we consider the condition

$$x(0) = x(\omega). \tag{4.1.5}$$

Definition 4.1.1. Let condition (4.1.4) hold and there exist a fundamental matrix Y of problem $(4.1.1_0), (4.1.5)$ such that

$$\det(D) \neq 0, \tag{4.1.6}$$

where $D = Y(\omega) - Y(0)$. A matrix-function $\mathcal{G} : [0,\omega]\times[0,\omega] \to \mathbb{R}^{n\times n}$ is said to be the Green matrix of problem $(4.1.1_0), (4.1.5)$ if:

(a) the matrix-function $\mathcal{G}(\,\cdot\,,s)$ satisfies the matrix equation

$$dX(t) = dA(t)\cdot X(t)$$

on both $[0,s[$ and $]s,\omega]$ for every $s\in\,]0,\omega[\,$;

(b) for $t\in\,]a,b[$,

$$\mathcal{G}(t,t+)-\mathcal{G}(t,t-) = Y(t)D^{-1}\Big\{Y(\omega)Y^{-1}(t)(I_n+d_2A(t))^{-1} - Y(0)Y^{-1}(t)(I_n-d_1A(t))^{-1}\Big\};$$

(c) $\mathcal{G}(t,\,\cdot\,)\in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ for every $t\in[0,\omega]$;

(d) the equality

$$\int\limits_0^\omega d_s(\mathcal{G}(\omega,s)-\mathcal{G}(0,s))\cdot f(s) = 0_n$$

holds for every $f\in \mathrm{BV}([0,\omega],\mathbb{R}^n)$.

The Green matrix of problem $(4.1.1_0),(4.1.5)$ exists and is unique in the following sense. If $\mathcal{G}(t,s)$ and $\mathcal{G}_1(t,s)$ are two matrix-functions satisfying conditions (a)–(d) of Definition 4.1.1, then

$$\mathcal{G}(t,s)-\mathcal{G}_1(t,s)\equiv Y(t)H_*(s),$$

where $H_*\in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ is a matrix-function such that

$$H_*(s+) = H_*(s-) = C = const \ \text{ for } \ s\in[0,\omega],$$

and $C\in\mathbb{R}^{n\times n}$ is a constant matrix.

In particular,

$$\mathcal{G}(t,s) = \begin{cases} Y(t)D^{-1}Y(0)Y^{-1}(s) & \text{for } \ 0\le s<t\le\omega,\\ Y(t)D^{-1}Y(\omega)Y^{-1}(s) & \text{for } \ 0\le t<s\le\omega,\\ \text{arbitrary} & \text{for } \ t=s. \end{cases}$$

Theorem 4.1.1. *System* (4.1.1) *has a unique ω-periodic solution x if and only if the corresponding homogeneous system* $(4.1.1_0)$ *has only the trivial solution satisfying condition* (4.1.5)*, i.e., when condition* (4.1.6) *holds, where Y is a fundamental matrix of system* $(4.1.1_0)$*. If the last condition holds, then the solution x can be written in the form*

$$x(t) = \int\limits_0^\omega d_s\mathcal{G}(t,s)\cdot f(s) \ \ \textit{for} \ \ t\in[0,\omega], \tag{4.1.7}$$

where $\mathcal{G}:[a,b]\times[a,b]\to\mathbb{R}^{n\times n}$ is the Green matrix of problem $(4.1.1_0),(4.1.5)$.

Corollary 4.1.1. *Let conditions* (4.1.3) *and* (4.1.4) *hold, and the matrix-function A satisfy the Lappo–Danilevskiĭ condition at the point* 0*. Then system* (4.1.1) *has a unique ω-periodic solution if and only if*

$$\det\bigg(\exp(S_c(A)(\omega))\prod_{0\le\tau<\omega}(I_n+d_2A(\tau))\prod_{0<\tau\le\omega}(I_n-d_1A(\tau))^{-1}-I_n\bigg)\ne 0. \tag{4.1.8}$$

Note that if the matrix-function A satisfies the Lappo–Danilevskiĭ condition at the point 0, then the matrix-function Y defined by $Y(0)=I_n$ and

$$Y(t) = \exp(S_c(A)(t))\prod_{0\le\tau<t}(I_n+d_2A(\tau))\prod_{0<\tau\le t}(I_n-d_1A(\tau))^{-1} \ \text{ for } \ t\in\,]0,\omega] \tag{4.1.9}$$

is the fundamental matrix of system $(4.1.1_0)$.

Remark 4.1.1. Let system $(4.1.1_0)$ have a nontrivial ω-periodic solution. Then there exists $f \in \mathrm{BV}_\omega(\mathbb{R},\mathbb{R}^n)$ such that system (4.1.1) has no ω-periodic solution (see Remark 1.1.2).

In general, it is rather difficult to verify condition (4.1.6) directly even in the case if the fundamental matrix of system $(4.1.1_0)$ is written explicitly. Therefore, it is important to look for effective conditions which would guarantee the absence of nontrivial ω-periodic solutions of the homogeneous system $(4.1.1_0)$. Below, we will give the results dealing with this question. Analogous results for ordinary differential equations have been obtained in [47].

Theorem 4.1.2. *System* (4.1.1) *has a unique ω-periodic solution if and only if there exist natural numbers k and m such that the matrix*

$$M_k = -\sum_{i=0}^{k-1}\big([A]_i(\omega)-[A]_i(0)\big)$$

is nonsingular and

$$r(M_{k,m})<1, \tag{4.1.10}$$

where

$$M_{k,m}=V_m(A)(c)+\Big(\sum_{i=0}^{m-1}\big|\,[A]_i\big|_\infty\Big)\,|M_k^{-1}|\,\big(V_k(A)(\omega)-V_k(A)(0)\big),$$

the functions $[A]_i$ $(i=0,\dots,m-1)$ and $V_i(A)$ $(i=0,\dots,m-1)$ are defined, respectively, by $(1.1.35_l)$ *and* $(1.1.37_l)$ *for some $l\in\{1,2\}$, and $c=(2-l)\omega$.*

Corollary 4.1.2. *System* (4.1.1) *has a unique ω-periodic solution if and only if there exist natural numbers k and m such that the matrix*

$$M_k = -\sum_{i=0}^{k-1}\big((A)_i(\omega)-(A)_i(0)\big)$$

is nonsingular and inequality (4.1.10) *holds, where*

$$M_{k,m}=(V(A))_m(c)+\Big(I_n+\sum_{i=0}^{m-1}|(A)_i|_\infty\Big)\,|M_k^{-1}|\,\big((V(A))_k(\omega)-(V(A))_k(0)\big),$$

the functions $(A)_i$ $(i=0,\dots,m-1)$ and $(V(A))_i$ $(i=0,\dots,m-1)$ are defined by $(1.1.36_l)$ *for some $l\in\{1,2\}$, and $c=(2-l)\omega$.*

Corollary 4.1.3. *Let there exist a natural number j such that*

$$(A)_i(0)=(A)_i(\omega)\ \ (i=1,\dots,j-1)$$

and

$$\det\big((A)_j(\omega)-(A)_j(0)\big)\neq 0,$$

where the functions $(A)_i$ $(i=0,\dots,j)$ are defined by $(1.1.36_l)$ *for some $l\in\{1,2\}$. Then there exists $\varepsilon_0>0$ such that the system*

$$dx(t)=\varepsilon dA(t)\cdot x(t)+df(t)$$

has one and only one ω-periodic solution for every $\varepsilon\in\,]0,\varepsilon_0[\,$.

Theorem 4.1.3. *Let a matrix-function $A_0\in\mathrm{BV}_\omega(\mathbb{R};\mathbb{R}^{n\times n})$ be such that*

$$\det\big(I_n+(-1)^j\,d_jA_0(t)\big)\neq 0\ \ \textit{for}\ \ t\in[0,\omega]\ \ (j=1,2)$$

and the homogeneous system

$$dx(t)=dA_0(t)\cdot x(t) \tag{4.1.11}$$

has only the trivial ω-periodic solution. Let, moreover, the matrix-function $A \in \mathrm{BV}_\omega(\mathbb{R};\mathbb{R}^{n\times n})$ admit the estimate

$$\int_0^\omega |\mathcal{G}_0(t,\tau)|\,dV(A-A_0)(\tau) \le M \ \ for\ \ t\in[0,\omega],$$

where $\mathcal{G}_0(t,\tau)$ is the Green matrix of problem (4.1.11), (4.1.5), *and $M \in \mathbb{R}_+^{n\times n}$ is a constant matrix such that*

$$r(M) < 1.$$

Then system (4.1.1) *has one and only one ω-periodic solution.*

Formula (4.1.7) can be written in a simpler form if we introduce the concept of the Green matrix for problem $(4.1.1_0), (4.1.2)$.

Definition 4.1.2. A matrix-function $\mathcal{G}_\omega : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^{n\times n}$ is said to be the Green matrix of problem $(4.1.1_0), (4.1.2)$ if:

(a)
$$\mathcal{G}_\omega(t+\omega,\tau+\omega) = \mathcal{G}_\omega(t,\tau), \ \ \mathcal{G}_\omega(t,t+\omega)-\mathcal{G}_\omega(t,t) = I_n \ \text{ for } \ t,\tau\in\mathbb{R}; \tag{4.1.12}$$

(b) the matrix-function $\mathcal{G}_\omega(\,\cdot\,,\tau) : \mathbb{R}\to\mathbb{R}^{n\times n}$ is a fundamental matrix of system $(4.1.1_0)$ for every $\tau\in\mathbb{R}$.

Theorem 4.1.4. *Let conditions* (4.1.3),

$$\det\big(I_n \pm d_j A(t)\big) \ne 0 \ \ for\ \ t\in\mathbb{R} \ \ (j=1,2) \tag{4.1.13}$$

hold and system $(4.1.1_0)$ have only the trivial ω-periodic solution. Then system (4.1.1) *has a unique ω-periodic solution x and it is written in the form*

$$x(t) = \int_t^{t+\omega} \mathcal{G}_\omega(t,\tau)\,d\mathcal{A}(A,\mathcal{A}(-A,f))(\tau) \ \ for\ \ t\in\mathbb{R}, \tag{4.1.14}$$

where $\mathcal{G}_\omega$ is the Green matrix of problem $(4.1.1_0)$, (4.1.2).

For the periodic problem, we give the definition of the set $Q_{\omega m}(\mathfrak{A},\mathcal{P})$ analogous to the set $Q_m(\mathfrak{A},\mathcal{P})$ (see Definition 3.1.1).

Definition 4.1.3. Let $m, r_1,\dots,r_m$ and $n_1,\dots,n_m$ $(0=n_0<n_1<\cdots<n_m=n)$ be natural numbers; $\mathfrak{A} = (\alpha_{lj})_{l,j=1}^{r_j,m}$, where $\alpha_{lj}\in \mathrm{BV}_\omega(\mathbb{R};\mathbb{R})$ $(l=1,\dots,r_j;\ j=1,\dots,m)$ be the functions nondecreasing on $[0,\omega]$; and let $\mathcal{P} = (P_{lj})_{l,j=1}^{r_j,m}$, where $P_{lj} = (p_{ljik})_{i,k=1}^n$ $(l=1,\dots,r_j;\ j=1,\dots,m)$ be such that $p_{ljik}\in L_\omega(\mathbb{R};\mathbb{R};\alpha_{lj})$ $(i,k=n_{j-1}+1,\dots,n_j)$. Then by $Q_{\omega m}(\mathfrak{A},\mathcal{P})$ we denote the set of all matrix-functions $A\in \mathrm{BV}_\omega(\mathbb{R};\mathbb{R}^{n\times en})$ such that

$$a_{ik}(t) = 0 \ \text{ for } \ t\in\mathbb{R} \ \ (i=n_{j-1}+1,\dots,n_j;\ k=n_j+1,\dots,n;\ j=1,\dots,m),$$

and

$$b_{jik}(t) = \sum_{l=1}^{r_j}\int_0^t p_{ljik}(\tau)\,d\alpha_{lj}(\tau) \ \text{ for } \ t\in\mathbb{R} \ \ (i\ne k;\ i,k=n_{j-1}+1,\dots,n_j;\ j=1,\dots,m),$$

where

$$b_{jik}(t) \equiv a_{ik}(t) - \bigg(\frac12 \sum_{0<\tau\le t}\ \sum_{r=n_{j-1}+1}^{n_j} d_1a_{ri}(\tau)\cdot d_1a_{rk}(\tau) - \sum_{0\le\tau<t}\ \sum_{r=n_{j-1}+1}^{n_j} d_2a_{ri}(\tau)\cdot d_2a_{rk}(\tau)\bigg)$$
$$(i,k=n_{j-1}+1,\dots,n_j;\ j=1,\dots,m).$$

If $m=1$ and $r_m=1$, we use the designation $Q_\omega(P,\alpha)$ instead of $Q_{\omega m}(\mathcal{P},\mathfrak{A})$. In this case, we have the following

Definition 4.1.4. Let $\alpha\in\mathrm{BV}_\omega(\mathbb{R};\mathbb{R})$ be a function nondecreasing on $[0,\omega]$ and $P=(p_{ik})_{i,k=1}^n$, where $p_{ik}\in L_\omega(\mathbb{R};\mathbb{R}_+;\alpha)$ $(i,k=1,\dots,n)$. Then by $Q_\omega(P;\alpha)$ we denote the set of all matrix-functions $A=(a_{ik})_{i,k=1}^n\in\mathrm{BV}_\omega(\mathbb{R};\mathbb{R}^{n\times n})$ such that

$$b_{ik}(t)=\int\limits_0^t p_{ik}(\tau)\,d\alpha(\tau)\ \text{ for }\ t\in\mathbb{R}\ \ (i\neq k;\ i,k=1,\dots,n),$$

where

$$b_{ik}\equiv a_{ik}(t)-\frac12\sum_{l=1}^n\Big(\sum_{a<\tau\le t}d_1a_{li}(\tau)\cdot d_1a_{lk}(\tau)-\sum_{a\le\tau<t}d_2a_{li}(\tau)\cdot d_2a_{lk}(\tau)\Big)\ \ (i,k=1,\dots,n).$$

Theorem 4.1.5. *Let there exist natural numbers m, $r_1,\dots,r_m$ and $n_1,\dots,n_m$ $(0=n_0<n_1<\cdots<n_m=n)$; $\sigma_j\in\{-1,1\}$ $(j=1,\dots,m)$; nondecreasing on $[0,\omega]$ functions $\alpha_{lj}\in\mathrm{BV}_\omega(\mathbb{R};\mathbb{R})$ $(l=1,\dots,r_j;\ j=1,\dots,m)$ and matrix-functions $\mathcal{P}_{lj}=(p_{ljik})_{i,k=1}^n$ $(l=1,\dots,r_j;\ j=1,\dots,m)$, $p_{ljik}\in L_\omega(\mathbb{R};\mathbb{R};\alpha_{lj})$ $(i,k=n_{j-1}+1,\dots,n_j)$, such that $A=(a_{ik})_{i,k=1}^n\in Q_{\omega m}(\mathcal{P},\mathfrak{A})$,*

$$\sigma_j\bigg(b_{jii}(t)-b_{jii}(s)-\sum_{l=1}^{r_j}\int\limits_s^t p_{ljii}(\tau)\,d\alpha_{lj}(\tau)\bigg)\le0\ \text{ for }\ s<t;\ \ s,t\in\mathbb{R}\ \ (j=1,\dots,m)\tag{4.1.15}$$

and

$$\sigma_j\Big(\sum_{i,k=n_{j-1}+1}^{n_j}p_{ljik}(t)x_ix_k-h_{\sigma_jlj}(t)\sum_{i=n_{j-1}+1}^{n_j}x_i^2\Big)\le0$$
$$\text{for }\ t\in\mathbb{R},\ \ (x_i)_{i=1}^n\in\mathbb{R}^n\ \ (l=1,\dots,r_j;\ j=1,\dots,m),\tag{4.1.16}$$

where $h_{\sigma_jlj}\in L_\omega(\mathbb{R};\mathbb{R}_+;\alpha_{lj})$ $(j=1,\dots,m)$. Let, moreover,

$$1+(-1)^ld_l\beta_j>0\ \text{ for }\ t\in[0,\omega]\ \ (l=1,2,;\ j=1,\dots,m)\tag{4.1.17}$$

and

$$\gamma_{\beta_j}(\omega-t_j,t_j)<1\ \ (j=1,\dots,m),\tag{4.1.18}$$

where $t_j=\frac12\,(1+\sigma_j)\omega$, the functions $\gamma_{\beta_j}(t,t_j)$ $(j=1,\dots,m)$ are defined by (1.1.9)*, and*

$$\beta_j(t)=2\sigma_j\sum_{l=1}^{r_j}\int\limits_0^t h_{\sigma_jlj}(\tau)\,d\alpha_{lj}(\tau)\ \text{ for }\ t\in[0,\omega]\ \ (j=1,\dots,m).$$

Then system (4.1.1) *has a unique ω-periodic solution.*

Remark 4.1.2. In the above theorem, due to (1.1.9), inequality (4.1.18) is equivalent to

$$\exp(s_c(\beta_j)(\omega))>-\frac12\bigg((1+\sigma_j)\prod_{0<\tau\le\omega}(1-d_1\beta_j(\tau))\prod_{0\le\tau<\omega}(1+d_1\beta_j(\tau))^{-1}$$
$$+(1-\sigma_j)\prod_{0<\tau\le\omega}(1+d_1\beta_j(\tau))^{-1}\prod_{0\le\tau<\omega}(1-d_2\beta_j(\tau))\bigg)\ \text{ for }\ t\in[0,\omega]\ \ (j=1,\dots,m).$$

If $m = 1$ and $r_m = 1$, then Theorem 4.1.5 has the following form.

Theorem 4.1.5$_1$. *Let there exist $\sigma \in \{-1, 1\}$ such that $A = (a_{ik})_{i,k=1}^n \in Q_\omega(P; \alpha)$,*

$$\sigma\bigg(b_{ii}(t) - b_{ii}(s) - \int_s^t p_{ii}(\tau)\, d\alpha(\tau)\bigg) \le 0 \;\; \text{for} \;\; s < t, \;\; s, t \in \mathbb{R},$$

$$\sigma\Big(\sum_{i,k=1}^n p_{ik}(t) x_i x_k - h_\sigma(t) \sum_{i=1}^n x_i^2\Big) \le 0 \;\; \text{for} \;\; t \in \mathbb{R}, \;\; (x_i)_{i=1}^n \in \mathbb{R}^n,$$

where $\alpha \in \mathrm{BV}_\omega(\mathbb{R}; \mathbb{R})$ is a nondecreasing function on $[0, \omega]$, $p_{ik} \in L_\omega(\mathbb{R}; \mathbb{R}; \alpha)$ $(i, k = 1, \dots, n)$; $h_\sigma \in L_\omega(\mathbb{R}; \mathbb{R}_+; \alpha)$. Let, moreover,

$$1 + (-1)^l d_l \beta_\sigma > 0 \;\; \text{for} \;\; t \in [0, \omega] \;\; (l = 1, 2)$$

and

$$\gamma_{\beta_\sigma}(\omega - t_\sigma, t_\sigma) < 1,$$

where

$$t_\sigma = \frac{1}{2}\,(1 + \sigma)\omega, \quad \beta_\sigma(t) \equiv 2 \int_0^t h_\sigma(\tau)\, d\alpha(\tau),$$

and the function $\gamma_{\beta_\sigma}(t, t_\sigma)$ is defined by (1.1.9)*. Then system* (4.1.1) *has a unique ω-periodic solution.*

Corollary 4.1.4. *Let the conditions of Theorem 4.1.5 hold, where $h_{\sigma_j l j}(t) \equiv \lambda_0(H)(t)$ and $h_{\sigma_j l j}(t) \equiv \lambda^0(H)(t)$ if $j \in \{1, \dots, m\}$ is such that $\sigma_j = -1$ and $\sigma_j = 1$, respectively, and $P^*(t) \equiv P(t) + P^\top(t)$. Then system* (4.1.1) *has a unique ω-periodic solution.*

Definition 4.1.5. Let $\sigma_i \in \{-1, 1\}$ $(i = 1, \dots, n)$. We say that a matrix-function $C = (c_{il})_{i,l=1}^n \in \mathrm{BV}_\omega(\mathbb{R}; \mathbb{R}^{n\times n})$ belongs to the set $\mathbb{U}_\omega^{\sigma_1, \dots, \sigma_n}$ if it is quasi-nondecreasing on $[0, \omega]$,

$$1 + (-1)^j \sigma_i\, d_j c_{ii}(t) > 0 \;\; \text{for} \;\; t \in \mathbb{R} \;\; (j = 1, 2; \;\; i = 1, \dots, n) \tag{4.1.19}$$

and the system

$$\sigma_i\, dx_i(t) \le \sum_{l=1}^n x_l(t)\, dc_{il}(t) \;\; \text{for} \;\; t \in \mathbb{R} \;\; (i = 1, \dots, n) \tag{4.1.20}$$

has no nontrivial, nonnegative ω-periodic solution.

One of the relations between $\mathbb{U}_\omega^{\sigma_1, \dots, \sigma_n}$ and $\mathbb{U}(t_1, \dots, t_n)$ (see Subsection 2.2.1) is given below, in Subsection 4.2.1.

Theorem 4.1.6. *Let the conditions*

$$\sigma_i\big(s_c(a_{ii})(t) - s_c(a_{ii})(s)\big) \le s_c(c_{ii})(t) - s_c(c_{ii})(s) \;\; \text{for} \;\; (t - s)\sigma_i > 0 \;\; (i = 1, \dots, n), \tag{4.1.21}$$

$$\big|s_c(a_{il})(t) - s_c(a_{il})(s)\big| \le s_c(c_{il})(t) - s_c(c_{il})(s) \;\; \text{for} \;\; s < t \;\; (i \ne l; \; i, l = 1, \dots, n), \tag{4.1.22}$$

$$|d_j a_{ii}(t)| \le |d_j c_{ii}(t)|, \;\; |d_j a_{il}(t)| \le d_j c_{il}(t) \;\; (j = 1, 2; \; i \ne l; \; i, l = 1, \dots, n) \tag{4.1.23}$$

hold on $[0, \omega]$, and

$$C = (c_{il})_{i,l=1}^n \in \mathbb{U}_\omega^{\sigma_1, \dots, \sigma_n}. \tag{4.1.24}$$

Then system (4.1.1) *has a unique ω-periodic solution.*

Corollary 4.1.5. *Let conditions* (4.1.19)*,* (4.1.21)*–*(4.1.23) *and*

$$\sigma_i \lambda(\sigma_i c_{ii})(\omega) < 1 \;\; (i = 1, \dots, n)$$

hold on $[0,\omega]$, *where* $\sigma_i \in \{-1,1\}$ $(i=1,\dots,n)$, *a matrix-function* $C=(c_{il})_{i,l=1}^n \in \mathrm{BV}_\omega(\mathbb{R};\mathbb{R}^{n\times n})$ *is quasi-nondecreasing on* $[0,\omega]$, *and the functions* $\lambda(\sigma_i c_{ii})(t)$ $(i=1,\dots,n)$ *are defined by* (1.1.30) *for* $t\in[0,\omega]$. *Let, moreover,*

$$r(S)<1,$$

where the matrix $S=\big(s_{il}\big)_{i,l=1}^n$ *is defined by*

$$s_{ii}=0,\ \ s_{il}=\sup\bigg\{\sum_{j=0}^{2}\int\limits_0^\omega \sigma_i g_j(\sigma_i c_{ii})(t,\tau)\,ds_j(c_{il})(\tau):\ t\in[0,\omega]\bigg\}\ \ (i\neq l;\ i,l=1,\dots,n), \tag{4.1.25}$$

under the operator s_0 *we understand the operator* s_c, *and* g_j $(j=0,1,2)$ *are the operators defined by* (1.1.31)–(1.1.33), *respectively. Then the conclusion of Theorem 4.1.6 is true.*

Corollary 4.1.6. *Let conditions* (4.1.19), (4.1.21)–(4.1.23) *hold, where*

$$c_{il}(t)=\eta_{il}\alpha_i(t)\ \ \text{for}\ \ t\in\mathbb{R}\ \ (i,l=1,\dots,n), \tag{4.1.26}$$

$\sigma_i \in \big\{-1,1\big\}$, $\eta_{il}\in\mathbb{R}_+$ $(i\neq l; i,l=1,\dots,n)$, α_i $(\alpha_i(\omega)\neq 0;\ i=1,\dots,n)$ *are the functions nondecreasing on* $[0,\omega]$. *Let, moreover,*

$$\eta_{ii}<0\ \ (i=1,\dots,n) \tag{4.1.27}$$

and

$$r(\mathcal{H})<1, \tag{4.1.28}$$

where $\mathcal{H}=(h_{il})_{i,l=1}^n$,

$$h_{ii}=0,\ \ h_{il}=-\frac{\eta_{il}}{\eta_{ii}}\ \ (i\neq l;\ i,l=1,\dots,n). \tag{4.1.29}$$

Then the conclusion of Theorem 4.1.6 is true.

Corollary 4.1.7. *Let conditions* (4.1.19), (4.1.21)–(4.1.23) *hold, where* $\sigma_1=\sigma_2=\dots=\sigma_n=\sigma_0$, $\sigma_0\in\{-1;1\}$, *and a matrix-function* $C=(c_{il})_{i,l=1}^n\in\mathrm{BV}_\omega(\mathbb{R},\mathbb{R}^{n\times n})$ *is quasi-nondecreasing on* $[0,\omega]$. *Let, moreover, the module of every multiplicator of the system*

$$dy(t)=dC_{\sigma_0}(t)\cdot y(t), \tag{4.1.30}$$

where $C_{\sigma_0}(t)=\sigma_0 C(\sigma_0 t+\frac{1-\sigma_0}{2}\,\omega)$, *be less than* 1. *Then the conclusion of Theorem 4.1.6 is true.*

4.2 Auxiliary propositions and proof of the results

Lemma 4.2.1. *The following statements are valid:*

(a) *if* x *is a solution of system* (4.1.1), *then the vector-function* $y(t)=x(t+\omega)$ $(t\in\mathbb{R})$ *is a solution of system* (4.1.1), *as well;*

(b) *problem* (4.1.1), (4.1.2) *is solvable if and only if system* (4.1.1) *has, on the closed interval* $[0,\omega]$, *a solution satisfying the boundary condition* (4.1.5). *Moreover, the set of restrictions of the solutions of problem* (4.1.1), (4.1.2) *on* $[0,\omega]$ *coincides with that of solutions of problem* (4.1.1), (4.1.5).

Proof. Let x be an arbitrary solution of system (4.1.1). Assume $y(t)=x(t+\omega)$ for $t\in\mathbb{R}$. Then by

(4.1.3) we have

$$\begin{aligned} y(t) &= x(0) + \int\limits_0^{t+\omega} dA(\tau)\cdot x(\tau) + f(t+\omega) - f(0) \\ &= x(0) + \int\limits_0^{\omega} dA(\tau)\cdot x(\tau) + f(\omega) - f(0) + \int\limits_{\omega}^{t+\omega} dA(\tau)\cdot x(\tau) + f(t+\omega) - f(\omega) \\ &= x(\omega) + \int\limits_0^{t} dA(\tau+\omega)\cdot x(\tau+\omega) + f(t+\omega) - f(\omega) \\ &= y(0) + \int\limits_0^{t} dA(\tau)\cdot y(\tau) + f(t) - f(0) \ \text{ for } \ t\in\mathbb{R}. \end{aligned}$$

Therefore, y is a solution of system (4.1.1), as well. Statement (a) of the lemma is proved.

Let us show statement (b). It is evident that the restrictions of every solution of problem (4.1.1), (4.1.2) on the interval $[0,\omega]$ will be a solution of problem (4.1.1), (4.1.5). Consider now an arbitrary solution x of problem (4.1.1), (4.1.5). Any continuation of this solution we also denote by x. According to statement (a), the vector-function $y(t)=x(t+\omega)$ is a solution of system (4.1.1), too. On the other hand, in view of (4.1.5), we have

$$y(0)=x(\omega)=x(0).$$

This implies that the functions x and y are the solutions of system (4.1.1) under the common initial value condition. Thus $x(t)\equiv y(t)$. Therefore, x is a solution of problem (4.1.1), 4.1.2. □

Lemma 4.2.2. *An arbitrary fundamental matrix Y of system* $(4.1.1_0)$ *satisfies the identity*

$$Y(t+\omega)=Y(t)M \ \text{ for } \ t\in\mathbb{R}, \tag{4.2.1}$$

where $M=Y^{-1}(0)Y(\omega)$ is the monodromy matrix of system $(4.1.1_0)$.

Proof. By Lemma 4.2.1, the columns of the matrix-function $Z(t)=Y(t+\omega)$ are the solutions of system $(4.1.1_0)$. Therefore, there exists a constant matrix $C\in\mathbb{R}$ such that

$$Z(t)=Y(t)C \ \text{ for } \ t\in\mathbb{R}.$$

Obviously,

$$C=Y^{-1}(0)Z(0)=Y^{-1}(0)Y(\omega).$$

Thus equality (4.2.1) holds. □

Lemma 4.2.3. *Let problem* $(4.1.1_0)$, (4.1.2) *have only the trivial solution. Then there exists a unique Green matrix of the problem having the form*

$$\mathcal{G}_\omega(t,\tau)=Y(t)\big(Y^{-1}(\omega)Y(0)-I_n\big)^{-1}Y^{-1}(\tau) \ \text{ for } \ t,\tau\in\mathbb{R}, \tag{4.2.2}$$

where Y is a fundamental matrix of system $(4.1.1_0)$.

Proof. Let Y be an arbitrary fundamental matrix of system $(4.1.1_0)$. Then, by Lemma 4.2.1, condition (4.1.6) holds, since the lemma guarantees the validity of Theorem 4.1.1 (see the proof of Theorem 4.1.1 below). According to Definition 4.1.2, the matrix-function $\mathcal{G}_\omega:\mathbb{R}\times\mathbb{R}\to\mathbb{R}^{n\times n}$ is a Green matrix if and only if

$$\mathcal{G}_\omega(t,\tau)=Y(t)C(\tau) \ \text{ for } \ t,\tau\in\mathbb{R},$$

where the matrix-function $C:\mathbb{R}\to\mathbb{R}^{n\times n}$ is such that equalities (4.1.12) hold, i.e.,

$$Y(t+\omega)C(\tau+\omega)=Y(t)C(\tau), \ \ Y(t)(C(t+\omega)-C(t))=I_n \ \text{ for } \ t,\tau\in\mathbb{R}. \tag{4.2.3}$$

By equality (4.2.1), equalities (4.2.3) hold if and only if

$$Y^{-1}(0)Y(\omega)C(\tau+\omega)=C(\tau) \text{ and } C(\tau+\omega)-C(\tau)=Y^{-1}(\tau) \text{ for } \tau\in\mathbb{R}.$$

Clearly, this implies that

$$\big(I_n-Y^{-1}(0)Y(\omega)\big)C(\tau)=Y^{-1}(0)Y(\omega)Y^{-1}(\tau) \text{ for } \tau\in\mathbb{R}.$$

Therefore, taking into account condition (4.1.6), we conclude that

$$C(\tau)=(Y^{-1}(\omega)Y(0)-I_n)^{-1}Y^{-1}(\tau) \text{ for } \tau\in\mathbb{R}.$$

Putting the obtained value of $C(t)$ in (4.2.3), we obtain equality (4.2.2). □

Lemma 4.2.4. *If $X\in\mathrm{BV}_\omega(\mathbb{R},\mathbb{R}^{n\times n})$ and $Y\in\mathrm{BV}_\omega(\mathbb{R},\mathbb{R}^{n\times m})$, then:*

(a)
$$d_jX(t+\omega)=d_jX(t) \textit{ for } t\in\mathbb{R} \ (j=1,2); \tag{4.2.4}$$

(b)
$$\mathcal{A}(X,Y)\in\mathrm{BV}_\omega(\mathbb{R},\mathbb{R}^{n\times m}) \textit{ i.e., } \mathcal{A}(X,Y)(t+\omega)=\mathcal{A}(X,Y)(t)+C \textit{ for } t\in\mathbb{R}, \tag{4.2.5}$$

where C is some constant $n\times n$-matrix.

Proof. Consider equality (4.2.4). Let $j=1$. Then, by the definition of the set $\mathrm{BV}_\omega(\mathbb{R},\mathbb{R}^{n\times m})$, we have

$$\begin{aligned}d_1X(t+\omega)&=\lim_{\varepsilon\to0,\,\varepsilon>0}\big(X(t+\omega)-X(t+\omega-\varepsilon)\big)\\&=\lim_{\varepsilon\to0,\,\varepsilon>0}(X(t)-X(t-\varepsilon))=d_1X(t) \text{ for } t\in\mathbb{R}.\end{aligned}$$

Analogously, we show equality (4.2.4) for $j=2$.

Let us now show (4.2.5). From the definition of the operator $\mathcal{A}$ and equalities (4.2.4), we conclude that

$$\begin{aligned}\mathcal{A}(X,Y)(t+\omega)&=Y(t+\omega)-Y(0)\\&\quad+\sum_{0<\tau\le t+\omega}d_1X(\tau)\cdot(I_n-d_1X(\tau))^{-1}\,d_1Y(\tau)-\sum_{0\le\tau<t+\omega}d_2X(\tau)\cdot(I_n+d_2X(\tau))^{-1}\,d_2Y(\tau)\\&=Y(t+\omega)-Y(0)+C_0+\sum_{\omega<\tau\le t+\omega}d_1X(\tau)\cdot(I_n-d_1X(\tau))^{-1}\,d_1Y(\tau)\\&\quad-\sum_{0\le\tau<t}d_2X(\tau+\omega)\cdot(I_n+d_2X(\tau+\omega))^{-1}\,d_2Y(\tau+\omega)\\&=Y(t+\omega)-Y(0)+C_0+\sum_{0<\tau\le t}d_1X(\tau+\omega)\cdot(I_n-d_1X(\tau+\omega))^{-1}\,d_1Y(\tau+\omega)\\&\quad-\sum_{0\le\tau<t}d_2X(\tau)\cdot(I_n+d_2X(\tau+\omega))^{-1}\,d_2Y(\tau+\omega)=\mathcal{A}(X,Y)(t)+C \text{ for } t\in\mathbb{R},\end{aligned}$$

where

$$C_0=\sum_{0<\tau\le\omega}d_1X(\tau)\cdot(I_n-d_1X(\tau))^{-1}\,d_1Y(\tau)-\sum_{0\le\tau<\omega}d_2X(\tau)\cdot(I_n+d_2X(\tau))^{-1}\,d_2Y(\tau),$$

and C is some constant matrix. □

4.2.1 On the set $\mathbb{U}_{\omega}^{\sigma_1,\dots,\sigma_n}$

Lemma 4.2.5. *Let condition* (4.1.24) *hold. Then*

$$c_{ii}(\omega)-\sigma_i\sum_{0<\tau\le\omega}\big(\ln(1-\sigma_i\,d_1c_{ii}(\tau))+\sigma_i\,d_1c_{ii}(\tau)\big)\\ +\sigma_i\sum_{0\le\tau<\omega}\big(\ln(1+\sigma_i\,d_2c_{ii}(\tau))-\sigma_i\,d_1c_{ii}(\tau)\big)<0\ \ (i=1,\dots,n). \quad (4.2.6)$$

Proof. Suppose the contrary that condition (4.2.6) is violated. Let $k\in\{1,\dots,n\}$ be such that

$$u(c_{kk})(\omega)\ge 0, \quad (4.2.7)$$

where the operator $u:\mathrm{BV}([0,\omega];\mathbb{R})\to\mathrm{BV}([0,\omega];\mathbb{R})$ is defined by

$$u(c)(t)\equiv c(t)-\sigma_k\sum_{0<\tau\le t}\big(\ln(1-\sigma_k\,d_1c(\tau))+\sigma_k\,d_1c(\tau)\big)\\ +\sigma_i\sum_{0\le\tau<t}\big(\ln(1+\sigma_k\,d_2c(\tau))-\sigma_k\,d_2c(\tau)\big).$$

Let $v_j:[0,\omega]\to\mathbb{R}$ $(j=1,2)$ be the nondecreasing functions such that

$$v_1(0)=0\ \text{ and }\ v_1(t)-v_2(t)=u(c_{kk})(t)\ \text{ for }\ t\in[0,\omega]. \quad (4.2.8)$$

By (4.2.7), we have

$$v_1(\omega)-v_2(\omega)\ge 0. \quad (4.2.9)$$

Consider the functions

$$\xi_j(t)=0\ \text{ if }\ v_j(\omega)=0\ \text{ and }\ \xi_j(t)=\frac{v_1(t)v_2(\omega)-v_1(\omega)v_2(t)}{g_j(\omega)}\ \text{ if }\ g_j(\omega)\ne0\ \text{ for }\ t\in[0,\omega]\ \ (j=1,2)$$

and

$$c_j(t)=\xi_j(t)-\sum_{0<\tau\le t}\alpha_{1j}(\tau)+\sum_{0\le\tau<t}\alpha_{2j}(\tau)\ \text{ for }\ t\in[0,\omega]\ \ (j=1,2),$$

where

$$\alpha_{mj}(t)\equiv(-1)^{m-1}\,d_m\xi_j(t)-\sigma_k\big(1-\exp((-1)^m\,d_m\xi_j(t))\big)\ \ (m,j=1,2).$$

Note that by the inequalities

$$|\alpha_{mj}(t)|\le M|d_m\xi_j(t)|\ \text{ for }\ t\in[0,\omega]\ \ (m,j=1,2),$$

where M is a constant independent of t, we have

$$\Big|\sum_{0\le t\le\omega}\alpha_{mj}(t)\Big|<+\infty\ \ (m,j=1,2).$$

From (4.1.24) and (4.2.8), we conclude $c_j(0)=0$ $(j=1,2)$. Moreover,

$$\ln(1+(-1)^m\sigma_k\,d_mc_j(t))-(-1)^m\sigma_k\,d_mc_j(t)=-\sigma_k\alpha_{mj}(t)\ \text{ for }\ t\in[0,\omega]\ \ (m,j=1,2).$$

Therefore,

$$u(c_j)(\omega)=0\ \ (j=1,2). \quad (4.2.10)$$

On the other hand, by (4.2.8), (4.2.9) and the definition of c_j, we have

$$s_c(c_j)(t)-s_c(c_j)(s)\le s_c(c_{kk})(t)-s_c(c_{kk})(s)=0\ \text{ for }\ 0\le s\le t\le\omega\ \ (j=1,2) \quad (4.2.11)$$

and

$$d_m c_j(t) \le d_m c_{kk}(t) = 0 \ \text{ for } \ t \in [0,\omega] \ \ (m,j=1,2). \tag{4.2.12}$$

Thus

$$1 - \sigma_k \, d_1 c_j(t) > 0 \ \text{ for } \ t \in [0,\omega] \ \ (j=1,2)$$

and, consequently, the Cauchy problem

$$dy = \sigma_k y \, dc_j(t), \ \ y(0) = 1$$

has the unique solution $\gamma_{\sigma_k c_j}(t,0)$ defined according to (1.1.9). In addition, by virtue of (4.2.10), we find that y_j is the positive and ω-periodic function for every $j \in \{1,2\}$. Hence, in view of (4.2.11) and (4.2.12), the vector-function $(y_i)_{i=1}^n$, where

$$y_i(t) \equiv 0, \ \ y_k(t) \equiv \gamma_{\sigma_k c_j}(t,0) \ \ (i \ne k; \ i=1,\dots,n),$$

is a nontrivial, nonnegative ω-periodic solution of system (4.1.20) for every $j \in \{1,2\}$. But this contradicts (4.1.24). □

Lemma 4.2.6. *Condition* (4.1.24) *holds if and only if the matrix-function C is ω-periodic and*

$$(C, \ell_0) \in \mathbb{U}(t_1,\dots,t_n) \tag{4.2.13}$$

on the closed interval $[0,\omega]$, *where*

$$t_i = \frac{1-\sigma_i}{2}\,\omega \ \ (i=1,\dots,n) \tag{4.2.14}$$

$$\ell_0(x_1,\dots,x_n) = \big(\ell_{0i}(x_1,\dots,x_n)\big)_{i=1}^n, \ \ \ell_{0i}(x_1,\dots,x_n) \equiv x_i(\omega - t_i) \ \ (i=1,\dots,n). \tag{4.2.15}$$

Proof. Let (4.1.24) hold. Then, according to Lemma 4.2.5, inequality (4.2.6) is fulfilled. From this and (4.1.19), the ω-periodic problem

$$du = \sigma_i u \, dc_{ii}(t), \ \ u(0) = u(\omega) \tag{4.2.16}$$

has only the trivial solution and

$$\sigma_i g_j(\sigma_i c_{ii})(t,\tau) \ge 0 \ \text{ for } \ t,\tau \in [0,\omega] \ \ (j=0,1,2; \ i=1,\dots,n), \tag{4.2.17}$$

where g_j $(j=0,1,2)$ are the operators defined by (1.1.31)–(1.1.33), and $\lambda(\alpha)(t) \equiv \gamma_\alpha(t,0)$, $\alpha(t) \equiv \sigma_i c_{ii}(t)$, is defined due to (1.1.9) for $i \in \{1,\dots,n\}$.

Assume now the contrary that condition (4.2.13) is violated, i.e., the problem

$$\begin{gathered} \operatorname{sgn}(t-t_i)\, dx_i(t) \le \sum_{l=1}^n x_l(t)\, dc_{il}(t) \ \text{ for } \ t \in [0,\omega], \ \ t \ne t_i \ \ (i=1,\dots,n), \\ (-1)^j \, d_j x_i(t_i) \le \sum_{l=1}^n x_l(t_i)\, d_j c_{il}(t_i) \ \ (j=1,2; \ i=1,\dots,n); \end{gathered} \tag{4.2.18}$$

$$x_i(t_i) \le x_i(\omega - t_i) \ \ (i=1,\dots,n) \tag{4.2.19}$$

has a nontrivial nonnegative solution $(x_i)_{i=1}^n$.

Put

$$q_i(t) \equiv \sigma_i(x_i(t_i) - x_i(t)) + \sum_{l=1}^n \int_{t_i}^t x_l(\tau)\, dc_{il}(\tau) \ \ (i=1,\dots,n) \tag{4.2.20}$$

and

$$\alpha_i = x_i(\omega - t_i) - x_i(t_i) \ \ (i=1,\dots,n). \tag{4.2.21}$$

By (4.2.18), the functions q_i $(i=1,\dots,n)$ are nondecreasing on $[0,\omega]$ and problem (4.2.18), (4.2.19) can be rewritten in the form

$$dx_i(t)=\sigma_i\sum_{l=1}^{n}x_l(t)\,dc_{il}(t)-\sigma_i\,dq_i(t)\ \text{ for }\ t\in[0,\omega]\ \ (i=1,\dots,n),$$
$$x_i(t_i)=x_i(\omega-t_i)-\alpha_i\ \ (i=1,\dots,n).$$

So, according to Theorem $1.1.1_1$, with regard for (4.1.19), (4.2.17), (4.2.20), and (4.2.21), we have

$$x_i(t)=x_{0i}(t)-\sigma_i\sum_{j=0}^{2}\int_0^{\omega}g_j(\sigma_i c_{ii})(t,\tau)\,ds_j(q_i)(\tau)+y_i(t)$$
$$\le x_{i0}(t)+y_i(t)\ \text{ for }\ t\in[0,\omega]\ \ (i=1,\dots,n), \tag{4.2.22}$$

where

$$y_i(t)\equiv\sigma_i\sum_{j=0}^{2}\sum_{l\ne i;\,l=1}^{n}\int_0^{\omega}g_j(\sigma_i c_{ii})(t,\tau)x_l(\tau)\,ds_j(c_{il})(\tau)$$

and x_{0i} is a solution of the problem

$$du_{ii}=\sigma_i\,u(t)\,dc_{ii}(t),\ \ u(t_i)=u(\omega-t_i)-\alpha_i\ \ (i=1,\dots,n).$$

On the other hand, it is not difficult to verify that

$$x_{0i}(t)=\frac{\sigma_i\alpha_i}{1-\lambda(c_{ii})(\omega)}\,\lambda(c_{ii})(t)\le 0\ \text{ for }\ t\in[0,\omega]\ \ (i=1,\dots,n).$$

Consequently, in view of (4.2.22),

$$x_i(t)\le y_i(t)\ \text{ for }\ t\in[0,\omega]\ \ (i=1,\dots,n). \tag{4.2.23}$$

In addition, by Theorem $1.1.1_1$, it follows from (4.2.23) that

$$\sigma_i(y_i(t)-y_i(s))=\int_s^t y_i(\tau)\,dc_{ii}(\tau)+\sum_{l\ne i;\,l=1}^{n}\int_s^t x_l(\tau)\,dc_{il}(\tau)$$
$$\le\sum_{l=1}^{n}\int_s^t y_l(\tau)\,dc_{il}(\tau)\ \text{ for }\ 0\le s\le t\le\omega\ \ (i=1,\dots,n)$$

and

$$y_i(0)=y_i(\omega)\ \ (i=1,\dots,n).$$

Therefore, the ω-periodic continuation on $\mathbb{R}$ of $(y_i)_{i=1}^n$ is a nontrivial nonnegative ω-periodic solution of system (4.1.20). The obtained contradiction proves that (4.2.13) follows from (4.1.24).

Finally, it is evident that from (4.1.24) follows (4.2.13), since the restriction on $[0,\omega]$ of an arbitrary ω-periodic solution of system (4.1.20) is a solution of problem (4.2.18), (4.2.19), as well. □

Lemma 4.2.7. *Let $\sigma_i\in\{-1,1\}$ $(i=1,\dots,n)$, $c_{il}\in\mathrm{BV}_\omega(\mathbb{R},\mathbb{R})$ $(i,l=1,\dots,n)$ and the functions c_{il} $(i\ne l)$ be nondecreasing on $[0,\omega]$. Let, moreover, conditions* (4.1.19) *and* (4.2.6) *hold and the module of every characteristic value of the matrix $S=(s_{il})_{i,l=1}^n$ be less then* 1*, where s_{il} $(i,l=1,\dots,n)$ are defined by* (4.1.25)*, and g_j $(j=0,1,2)$ are the operators defined by* (1.1.31)–(1.1.33)*. Then condition* (4.1.24) *holds.*

Proof. Let $(y_i)_{i=1}^n$ be an arbitrary nonnegative ω-periodic solution of system (4.1.20).

Put

$$q_i(t)=\sum_{l=1}^n\int\limits_0^t y_l(\tau)\,dc_{il}(\tau)-\sigma_i y_i(t)\ \text{ for } t\in\mathbb{R}\ (i=1,\dots,n).$$

Then $q_i\in \mathrm{BV}_\omega(\mathbb{R};\mathbb{R})$ $(i=1,\dots,n)$ and due to (4.1.20) they are nondecreasing on $[0,\omega]$.

It is clear that

$$dy_i(t)=\sigma_i\sum_{l=1}^n y_l(t)\,dc_{il}(t)-\sigma_i q_i(t)\ \text{ for } t\in\mathbb{R}\ (i=1,\dots,n).$$

Therefore, owing to Theorem $1.1.1_1$, we have

$$\begin{aligned}y_i(t)=y_{0i}(t)+\sigma_i\sum_{j=0}^2\sum_{l\neq i;\,l=1}^n\int\limits_0^\omega g_j(\sigma_i c_{ii})(t,\tau)y_l(t)\,ds_j(c_{il})(\tau)&\\ -\sigma_i\sum_{j=0}^2\int\limits_0^\omega g_j(\sigma_i c_{ii})(t,\tau)\,ds_j(q_i)(\tau)\ \text{ for } t\in[0,\omega]\ (i=1,\dots,n),&\quad(4.2.24)\end{aligned}$$

where y_{0i} is a ω-periodic solution of problem (4.2.16).

On the other hand, by condition (4.1.19), problem (4.2.16) has only the trivial ω-periodic solution. So, $y_{0i}(t)\equiv 0$. Besides, estimate (4.2.17) holds. Due to the above-said, from (4.2.24) follows

$$\begin{aligned}y_i(t)&\le\sigma_i\sum_{j=0}^2\sum_{l\neq i;\,l=1}^n\int\limits_0^\omega g_j(\sigma_i c_{ii})(t,\tau)y_l(t)\,ds_j(c_{il})(\tau)\\ &\le\sigma_i\sum_{j=0}^2\|y_l\|_\infty\int\limits_0^\omega g_j(\sigma_i c_{ii})(t,\tau)\,ds_j(c_{il})(\tau)\ \text{ for } t\in[0,\omega]\ (i=1,\dots,n).\end{aligned}$$

Consequently,

$$\big(\|y_i\|_\infty\big)_{i=1}^n\le S\,\big(\|y_i\|_\infty\big)_{i=1}^n.$$

Thus we get $\|y_i\|_\infty=0$ $(i=1,\dots,n)$, because the module of every characteristic value of the matrix S is less than 1. So, condition (4.1.24) holds. □

Lemma 4.2.8. *Let conditions* (4.1.19) *and* (4.1.26) *hold, where* $\sigma_i\in\{-1,1\}$, $\eta_{ii}\in\mathbb{R}$, $\eta_{il}\in\mathbb{R}_+$ $(l\neq i;\ i,l=1,\dots,n)$; $\alpha_i\in\mathrm{BV}_\omega(\mathbb{R},\mathbb{R})$ $(\alpha_i(\omega)\neq 0;\ i=1,\dots,n)$ *be nondecreasing on* $[0,\omega]$. *Then condition* (4.1.24) *holds if and only if conditions* (4.1.27) *and* (4.1.28) *are fulfilled, where the matrix* $\mathcal{H}=(\eta_{il})_{i,l=1}^n$ *is defined by* (4.1.29).

Proof. First, we show the necessity. Let (4.1.24) hold. Then, according to Lemma 4.2.5,

$$\begin{aligned}\eta_{ii}s_c(\alpha_i)(\omega)-\sigma_i\sum_{0<\tau\le\omega}\ln\big(1-\sigma_i\eta_{ii}\,d_1\alpha_i(\tau)\big)&\\ +\sigma_i\sum_{0\le\tau<\omega}\ln\big(1+\sigma_i\eta_{ii}\,d_2\alpha_i(\tau)\big)<0\ \ (i=1,\dots,n).&\quad(4.2.25)\end{aligned}$$

Let us show (4.1.27). Assume the contrary, i.e., $\eta_{kk}\ge 0$ for some $k\in\{1,\dots,n\}$. Then from (4.2.25) it follows that

$$\sigma_k\sum_{0<\tau\le\omega}\ln\big(1-\sigma_k\eta_{kk}\,d_1\alpha_k(\tau)\big)>\sigma_k\sum_{0\le\tau<\omega}\ln\big(1+\sigma_k\eta_{kk}\,d_2\alpha_k(\tau)\big),\quad(4.2.26)$$

since α_{kk} is nondecreasing on $[0,\omega]$ and $\eta_{kk}s_c(\alpha_k)(\omega)\geq 0$. On the other hand, it is easy to show the inequalities

$$\sigma_k\ln\big(1-\sigma_k\eta_{kk}\,d_1\alpha_k(\tau)\big)\leq 0\ \text{ and }\ \sigma_k\ln\big(1+\sigma_k\eta_k\,d_2\alpha_k(\tau)\big)\geq 0\ \text{ for }\ t\in[0,\omega].$$

But this contradicts (4.2.26). Thus (4.1.27) is proved.

Let us show (4.1.28). Assume the contrary, i.e.,

$$r_0=r(S)\geq 1.$$

By Theorem XIII.3.3 from [36], there exists a nonnegative eigenvector $(y_i)_{i=1}^n$ corresponding to the characteristic value r_0. It is clear that

$$0=\eta_{ii}y_i+\frac{1}{r_0}\sum_{l\neq i;\,l=1}^{n}\eta_{il}y_l\leq\sum_{l=1}^{n}\eta_{il}y_l\ \ (i=1,\dots,n).$$

Therefore, $(y_i)_{i=1}^n$ is a nontrivial nonnegative ω-periodic solution of system (4.1.20), since α_i $(i=1,\dots,n)$ are nondecreasing on $[0,\omega]$. But this contradicts (4.1.24). Consequently, the necessity is proved.

Let us proof the sufficiency. Due to (4.1.27), we have

$$(-1)^j\sigma_i\ln(1+(-1)^j\sigma_i\eta_{ii}\,d_j\alpha_i(\tau))\leq 0\ \text{ for }\ t\in[0,\omega]\ \ (j=1,2;\ i=1,\dots,n).\tag{4.2.27}$$

Moreover, by inequality $\alpha_i(\omega)\neq 0$, it is not difficult to see that if $s_c(\alpha_i)(\omega)=0$, then there exist $j\in\{1,2\}$ and $\tau\in[0,\omega]$ such that the inequality (4.2.27) is strict. Consequently, (4.2.25) holds.

Let now $t\in[0,\omega]$ and $i,l\in\{1,\dots,\}$ be fixed. Put

$$\lambda_i(t)=\exp\big(\sigma_i\eta_{ii}s_c(\alpha_i)(t)\big)\overline{\lambda}_i(t),$$
$$\overline{\lambda}_i(t)=\prod_{0\leq\tau<t}\big(1+\sigma_i\eta_{ii}d_2\alpha_i(\tau)\big)\prod_{0<\tau\leq t}\big(1-\sigma_i\eta_{ii}d_1\alpha_i(\tau)\big)^{-1}$$

and

$$\mathcal{I}_j(t)=\sigma_i\int_0^{\omega}g_j(\sigma_i\eta_{ii}\alpha_i)(t,\tau)\,ds_j(c_{il})\ \text{ for }\ t\in[0,\omega]\ \ (j=0,1,2),$$

where g_j $(j=0,1,2)$ are the operators defined by (1.1.31)–(1.1.33).

By virtue of the equalities

$$d_j\overline{\lambda}_i^{\,-1}(t)=\frac{\sigma_i\eta_{ii}\,d_j\alpha_i(t)}{1+(-1)^j\sigma_i\eta_{ii}\,d_j\alpha_i(\tau)}\,\overline{\lambda}_i^{\,-1}(t)\ \text{ for }\ t\in[0,\omega]\ \ (j=0,1,2),$$

we conclude

$$\begin{aligned}\int_a^b\lambda_i^{-1}(t)\,ds_c(\alpha_i)(\tau)=-\frac{1}{\sigma_i\eta_{ii}}\bigg(&\lambda_i^{-1}(b)-\lambda_i^{-1}(a)-\sum_{a<\tau\leq b}\lambda_i^{-1}(\tau)\overline{\lambda}_i(\tau)\,d_1\overline{\lambda}_i^{\,-1}(\tau)\\&-\sum_{a\leq\tau<b}\lambda_i^{-1}(\tau)\overline{\lambda}_i(\tau)\,d_2\overline{\lambda}_i^{\,-1}(\tau)\bigg)\ \text{ for }\ 0\leq a<b\leq\omega,\end{aligned}$$

$$\mathcal{I}_0(t)=\frac{\sigma_i\eta_{il}\lambda_i(t)}{1-\lambda_i(\omega)}\bigg(\int_0^t\lambda_i^{-1}(t)\,ds_c(\alpha_i)(\tau)+\lambda_i(\omega)\int_t^{\omega}\lambda_i^{-1}(t)\,ds_c(\alpha_i)(\tau)\bigg),$$

$$\mathcal{I}_1(t)=-\frac{\eta_{il}\lambda_i(t)}{\eta_{ii}(1-\lambda_i(\omega))}\bigg(\sum_{0<\tau\leq t}\lambda_i^{-1}(\tau)\overline{\lambda}_i(\tau)\,d_1\overline{\lambda}_i^{\,-1}(\tau)+\lambda_i(\omega)\sum_{t<\tau\leq\omega}\lambda_i^{-1}(\tau)\overline{\lambda}_i(\tau)\,d_1\overline{\lambda}_i^{\,-1}(\tau)\bigg),$$

$$\mathcal{I}_2(t)=-\frac{\eta_{il}\lambda_i(t)}{\eta_{ii}(1-\lambda_i(\omega))}\bigg(\sum_{0\leq\tau<t}\lambda_i^{-1}(\tau)\overline{\lambda}_i(\tau)\,d_2\overline{\lambda}_i^{\,-1}(\tau)+\lambda_i(\omega)\sum_{t\leq\tau<\omega}\lambda_i^{-1}(\tau)\overline{\lambda}_i(\tau)\,d_2\overline{\lambda}_i^{\,-1}(\tau)\bigg)$$
$$\text{for }\ t\in[0,\omega].$$

These equalities imply that

$$\sigma_i \sum_{j=0}^{2} \int_0^{\omega} g_j(\sigma_i \eta_{ii} \alpha_i)(t,\tau)\, ds_j(c_{il}) = -\frac{\eta_{il}}{\eta_{ii}} = s_{il} \ \text{ for } \ t \in [0,\omega] \ \ (i \neq l;\ i,l = 1,\dots,n).$$

Therefore, according to Lemma 4.2.7, condition (4.1.24) holds. □

Lemma 4.2.9. *Let $\sigma_1 = \dots \sigma_n = \sigma_0$, $\sigma_0 \in \{-1,1\}$, a matrix-function $C = (c_{il})_{i,l=1}^n \in \mathrm{BV}_\omega(\mathbb{R},\mathbb{R}^{n\times n})$ be quasi-nondecreasing on $[0,\omega]$. Then condition* (4.1.24) *holds if and only if the module of every multiplicator of system* (4.1.30), *where $C_{\sigma_0}(t) \equiv \sigma_0 C(\sigma_0 t + \frac{1}{2}(1-\sigma_0)\omega)$, is less than* 1.

Proof. Let $t_0 = \frac{1-\sigma_0}{2}\,\omega$ and let $Y(t,\varepsilon)$, $Y_(t_0,\varepsilon) = I_n$, be the fundamental matrix of the system

$$dy = dC(t,\varepsilon)\cdot y, \tag{4.2.28}$$

where $C(t,\varepsilon) \equiv \varepsilon C(t) + (1-\varepsilon)\operatorname{diag}(c_{11}(t),\dots,c_{nn}(t))$, and let $r(\varepsilon)$ be the spectral radius of the matrix $Y_\varepsilon(\omega)$ for every $\varepsilon \in [0,1]$. In view of Lemma 2.2.4,

$$Y_(t,\varepsilon) \geq O_{n\times n} \ \text{ for } \ t \in [0,\omega]$$

and $r : [0,1] \to]0,+\infty[$ is the continuous function.

First, consider the case $\sigma_0 = 1$. Then $t_0 = 0$ and $C_{\sigma_0}(t) = C_1(t) = C(t) = C(t,1)$ for $t \in [0,\omega]$. So, $Y(t) \equiv Y(t,1)$, where Y, $Y(0) = I_n$, is the fundamental matrix of system (4.1.30). In addition, the monodromy matrix of the system has the form $M = Y^{-1}(0)Y(\omega) = Y_1(\omega,1)$. Therefore, the condition imposed on multiplicators means that

$$r(1) < 1. \tag{4.2.29}$$

Let us show the sufficiency. Let (4.2.29) hold. Consider an arbitrary nonnegative ω-periodic solution $y = (y_i)_{i=1}^n$ of system (4.1.20). By Lemma 2.2.5, we have

$$y(t) \leq Y(t,1)y(0) \ \text{ for } \ t \in [0,\omega]$$

and

$$y(0) \leq Y(\omega,1)y(0).$$

By virtue of (4.2.29), it follows from the latter two estimates that $y(t) \equiv 0$. Therefore, condition (4.1.24) holds.

Now we show that (4.1.24) implies (4.2.29). Assume the contrary, i.e., (4.1.24) holds, but

$$r(1) \geq 1.$$

According to Lemma 4.2.5. condition (4.2.6) holds and

$$\lambda_i(\omega) < 1 \ \ (i = 1,\dots,n),$$

where $\lambda_i(t) \equiv \gamma_{c_{ii}}(t,t_0)$ is defined by (1.1.9), From this

$$r(0) < 1,$$

since

$$Y(\omega,0) = \operatorname{diag}\big(\lambda_1(\omega),\dots,\lambda_n(\omega)\big).$$

Therefore, there exists $\varepsilon \in]0,1[$ such that

$$r(\varepsilon) = 1.$$

Let $c_\varepsilon \in \mathbb{R}_+^n$ be an eigenvector corresponding to the characteristic value 1, i.e., $Y(\omega,\varepsilon)\, c_\varepsilon = c_\varepsilon$. Then the vector-function

$$y(t) \equiv Y(t,\varepsilon)c_\varepsilon$$

is a nontrivial nonnegative solution of system (4.2.28). Obviously, it is ω-periodic and satisfies system (4.1.20), as well, since $0 \le \varepsilon < 1$. But this contradicts (4.1.24).

Finally, we note that if $(y_i)_{i=1}^n$ is an arbitrary solution of system (4.1.20), then $(z_i)_{i=1}^n$, where

$$z_i(t) \equiv y_i(\omega - t) \ \ (i = 1, \dots, n),$$

is a solution of the system

$$-\sigma_i dz_i(t) \le \sum_{l=1}^{n} z_l(t)\, d(-c_{il}(\omega - t)) \ \text{ for } \ t \in \mathbb{R} \ \ (i = 1, \dots, n)$$

and, conversely, if $(z_i)_{i=1}^n$ is an arbitrary solution of the last system, then the $(y_i)_{i=1}^n$, where

$$y_i(t) \equiv z_i(\omega - t) \ \ (i = 1, \dots, n),$$

is a solution of system (4.1.20). In addition, $(y_i)_{i=1}^n$ is ω-periodic if and only if $(z_i)_{i=1}^n$ has the same property. So, the case $\sigma_0 = -1$ is reduced to the case $\sigma_0 = -1$. □

4.2.2 Proof of the results

By Lemma 4.2.1, Theorem 4.1.1 immediately follows from Theorem 1.1.1, and Theorems 4.1.2, 4.1.3 and Corollaries 4.1.1–4.1.3 immediately follow from Theorems 1.1.2–1.1.3 and Corollaries 4.1.1–4.1.3, respectively, if we assume that the linear functional ℓ appearing there is of the form $\ell(x) \equiv x(0) - x(\omega)$. Note that condition (4.1.6) has form (4.1.8) when the fundamental matrix of system $(4.1.1_0)$ is given by (4.1.9) in Corollary 4.1.1.

Proof of Theorem 4.1.4. By Theorem 1.1.1 and Lemma 4.2.3, problem (4.1.1), (4.1.2) is uniquely solvable, and problem $(4.1.1_0)$, (4.1.2) has a unique Green matrix $\mathcal{G}_\omega$. Therefore, for the proof it suffices to verify that the vector-function given by (4.1.14) is the ω-periodic solution of system (4.1.1).

Assume

$$\varphi(t) = \mathcal{A}(-A, f)(t) \ \text{ for } \ t \in \mathbb{R}.$$

Let us show that the vector-function x defined by (4.1.14) satisfies condition (4.1.2). By Lemma 4.2.4, it is evident that $\mathcal{A}(A, \varphi) \in \mathrm{BV}_\omega(\mathbb{R}, \mathbb{R}^n)$ and, therefore,

$$\mathcal{A}(A, \varphi)(t + \omega) = \mathcal{A}(A, \varphi)(t) + c \ \text{ for } \ t \in \mathbb{R}, \tag{4.2.30}$$

where c is some constant n-vector. Taking into account (4.2.30) and (4.1.14), we have

$$x(t+\omega) = \int_{t+\omega}^{t+2\omega} \mathcal{G}_\omega(t+\omega, \tau)\, d\mathcal{A}(A, \varphi)(\tau) = \int_{t}^{t+\omega} \mathcal{G}_\omega(t+\omega, \tau+\omega)\, d\mathcal{A}(A, \varphi)(\tau+\omega) = x(t) \ \text{ for } \ t \in \mathbb{R}.$$

Let us verify that the vector-function x satisfies system (4.1.1). By equality (4.2.2),

$$\mathcal{G}_\omega(t, \tau) = Y(t) C_\omega Y^{-1}(\tau) \ \text{ for } \ t, \tau \in \mathbb{R},$$

where Y is a fundamental matrix of system $(4.1.1_0)$, and

$$C_\omega = (Y^{-1}(\omega) Y(0) - I_n)^{-1}.$$

Thus, using the general integration-by-parts formula, we find that

$$x(t)-x(s)=\int\limits_s^t dx(\tau)=\int\limits_s^t d\Big(\int\limits_\tau^{\tau+\omega}\mathcal{G}_\omega(\tau,\eta)\,d\mathcal{A}(A,\varphi)(\eta)\Big)$$
$$=\int\limits_s^t d\Big(Y(\tau)C_\omega\int\limits_\tau^{\tau+\omega}Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta)\Big)=\int\limits_s^t dY(\tau)\cdot C_\omega\int\limits_\tau^{\tau+\omega}Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta)$$
$$+\int\limits_s^t Y(\tau)C_\omega\,d\Big(\int\limits_\tau^{\tau+\omega}Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta)\Big)-\sum_{s<\eta\le t}d_1Y(\tau)\cdot C_\omega\,d_1\Big(\int\limits_\tau^{\tau+\omega}Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta)\Big)$$
$$+\sum_{s\le\eta<t}d_2Y(\tau)\cdot C_\omega\,d_2\Big(\int\limits_\tau^{\tau+\omega}Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta)\Big)\ \text{ for }\ s<t,\ \ s,t\in\mathbb{R}. \tag{4.2.31}$$

On the other hand, due to (4.2.1),

$$Y^{-1}(t+\omega)-Y^{-1}(t)\equiv C_\omega^{-1}Y^{-1}(t). \tag{4.2.32}$$

By (4.2.30), we conclude that

$$\int\limits_\tau^{\tau+\omega}Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta)=\int\limits_\tau^{\omega}Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta)+\int\limits_\omega^{\tau+\omega}Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta)$$
$$=\int\limits_\tau^{\omega}Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta)+\int\limits_0^{\tau}Y^{-1}(\eta+\omega)\,d\mathcal{A}(A,\varphi)(\eta+\omega)$$
$$=\int\limits_0^{\omega}Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta)+\int\limits_0^{\tau}(Y^{-1}(\eta+\omega)-Y^{-1}(\eta))\,d\mathcal{A}(A,\varphi)(\eta)\ \text{ for }\ \tau\in\mathbb{R}.$$

From this, taking into account (4.2.32), we get

$$\int\limits_\tau^{\tau+\omega}Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta)\equiv\int\limits_0^{\omega}Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta)+C_\omega^{-1}\int\limits_0^{\tau}Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta).$$

Due to the last equality and the general integration-by-parts formula, taking into account the equalities

$$dY(t)=dA(t)\cdot Y(t)\ \text{ and }\ d_jY(t)=d_jA(t)\cdot Y(t)\ \text{ for }\ t\in\mathbb{R}\ \ (j=1,2),$$

from (4.2.31) it follows that

$$x(t)-x(s)=\int\limits_s^t dA(\tau)\cdot Y(\tau)C_\omega\int\limits_\tau^{\tau+\omega}Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta)+F(s,t)$$
$$=\int\limits_s^t dA(\tau)\cdot x(\tau)+F(s,t)\ \text{ for }\ s<t,\ \ s,t\in\mathbb{R}, \tag{4.2.33}$$

where

$$F(s,t)=\mathcal{A}(A,\varphi)(t)-\mathcal{A}(A,\varphi)(s)$$
$$-\sum_{s<\tau\le t}d_1A(\tau)\cdot d_1\mathcal{A}(A,\varphi)(\tau)+\sum_{s\le\tau<t}d_2A(\tau)\cdot d_2\mathcal{A}(A,\varphi)(\tau)\ \text{ for }\ s,t\in\mathbb{R},\ \ s<t.$$

Moreover, taking into account condition (4.1.13), according to the definition of the operator $\mathcal{A}$ and the function φ, we conclude that

$$d_1\varphi(\tau) = d_1 f(\tau) - \sum_{s<\tau\le t} d_1A(\tau)\cdot(I_n + d_1A(\tau))^{-1}\, d_1 f(\tau) \ \text{ for } \ \tau\in\mathbb{R},$$
$$d_2\varphi(\tau) = d_2 f(\tau) + \sum_{s\le\tau< t} d_2A(\tau)\cdot(I_n - d_2A(\tau))^{-1}\, d_2 f(\tau) \ \text{ for } \ \tau\in\mathbb{R}.$$

Using the last equalities, we can easily show that

$$\begin{aligned}F(s,t) = \varphi(t)-\varphi(s) &+ \sum_{s<\tau\le t} d_1A(\tau)\cdot(I_n - d_1A(\tau))^{-1}\, d_1\varphi(\tau)\\ &- \sum_{s\le\tau<t} d_2A(\tau)\cdot(I_n + d_2A(\tau))^{-1}\, d_2\varphi(\tau) - \sum_{s<\tau\le t} (d_1A(\tau))^2\cdot(I_n - d_1A(\tau))^{-1}\, d_1\varphi(\tau)\\ &- \sum_{s\le\tau<t} (d_2A(\tau))^2\cdot(I_n + d_2A(\tau))^{-1}\, d_2\varphi(\tau)\\ = \varphi(t)-\varphi(s) &+ \sum_{s<\tau\le t} d_1A(\tau)\cdot d_1\varphi(\tau) - \sum_{s\le\tau<t} d_2A(\tau)\cdot d_2\varphi(\tau) = f(t)-f(s) \ \text{ for } \ s,t\in\mathbb{R}, \ \ s<t.\end{aligned}$$

Consequently, due to (4.2.33), the vector-function x satisfies system (4.1.1). □

Proof of Theorem 4.1.5. According to Theorem 4.1.1, for the proof of the theorem it suffices to show that the homogeneous system $(4.1.1_0)$ has only the trivial ω-periodic solution. Let $x=(x_i)_{i=1}^n$ be an arbitrary solution of the latter problem. Assume

$$u_j(t) = \sum_{i=n_{j-1}+1}^{n_j} x_i^2(t) \ \text{ for } \ t\in[0,\omega] \ \ (j=1,\dots,m).$$

As in the proof of Theorem 3.1.4, we find that

$$\sigma_1(u_1(t)-u_1(s)) \ge \int_s^t u_1(\tau)\, d\beta_1(\tau) \ \text{ for } \ 0\le s\le t\le\omega$$

and

$$u_1(t) \le u_1(t_1)\gamma_{\beta_1}(t,t_1) \ \text{ for } \ t\in[0,\omega]. \tag{4.2.34}$$

Due to (4.1.5), we have $u_1(0)=u_1(\omega)$. Thus, from (4.2.34) it follows that

$$u_1(\omega-t_1) \le u_1(t_1)\,\gamma_{\beta_1}(\omega-t_1,t_1) = u_1(\omega-t_1)\,\gamma_{\beta_1}(\omega-t_1,t_1).$$

Therefore, due to (4.1.18),

$$u_(t_1) = u_1(0) = u_1(\omega) = 0,$$

so, by (4.2.34), we have

$$u_1(t)\equiv 0.$$

Using this identity and also (4.1.15)–(4.1.18), by induction we prove $u_j(t)\equiv 0$ $(j=2,\dots,m)$. Consequently, $x_i(t)=0$ for $t\in[0,\omega]$ $(i=1,\dots,n)$. □

Proof of Corollary 4.1.4. It is evident that

$$\sum_{i,k=1}^n p_{ik}(t)x_ix_k \equiv \frac{1}{2}\sum_{i,k=1}^n (p_{ik}(t)+p_{ki}(t))x_ix_k.$$

From this, by Lemma 3.1.1, we have

$$\lambda_0(P^*(t))\sum_{i=1}^n x_i^2 \le \sum_{i,k=1}^n p_{ik}(t)x_ix_k \le \lambda^0(P^*(t))\sum_{i=1}^n x_i^2 \ \text{ for } \mu(g)\text{-almost all } \ t\in[0,\omega], \ \ (x_i)_{i=1}^n\in\mathbb{R}^n.$$

Therefore, the corollary immediately follows from Theorem $4.1.5_1$. □

Proof of Theorem 4.1.6. By Lemma 4.2.6, condition (4.1.24) holds, where t_i $(i=1,\dots,n)$ and ℓ_0 are given by (4.2.14) and (4.2.15). Assume

$$\ell_i(y_1,\dots,y_n) = y(\omega - t_i) \ \ (i=1,\dots,n).$$

Then condition (4.1.5) has the form

$$x_i(t_i) = \ell_i(x_1,\dots,x_n) \ \ (i=1,\dots,n). \tag{4.2.35}$$

On the other hand, in view of conditions (4.2.14), (4.2.15), (4.1.21)–(4.1.23), the conditions of Theorem 2.2.1 and condition (4.2.13) guarantee the uniquely solvability of problem (4.1.1), (4.2.35). Therefore, problem (4.1.1), (4.1.5) has the unique solution, as well. Hence, due to Lemma 4.2.1, we conclude that system (4.1.1) has the unique ω-periodic solution. □

Corollaries 4.1.5, 4.1.6 and 4.1.7 follow immediately from Theorem 4.1.6 due to Lemmas 4.2.7, 4.2.8 and 4.2.9, respectively.

4.2.3 Nonnegativity of solutions of ω-periodic problem

In this subsection, we consider the question on the existence of nonnegative ω-periodic solutions of system (4.1.1). We realize the results of Section 2.3 for ω-periodic problem under consideration.

Theorem 4.2.1. *Let the matrix- and vector-functions* $A=(a_{ik})_{i,k=1}^n \in \mathrm{BV}_\omega(\mathbb{R};\mathbb{R}^{n\times n})$ *and* $f=(f_i)_{i=1}^n \in \mathrm{BV}_\omega(\mathbb{R};\mathbb{R}^n)$ *be such that the functions* $\sigma_i a_{il}(t)$ $(i\neq l;\ i,l=1,\dots,n)$ *are nondecreasing on* $[0,\omega]$, *the conditions*

$$\sigma_i(a_{il}(t)-a_{il}(s)) \leq (c_{il}(t)-c_{il}(s)) \ \ \text{for} \ \ \sigma_i(t-s)>0 \ \ (i,l=1,\dots,n) \tag{4.2.36}$$

and

$$\sigma_i f_i(t) \ \ \text{are nondecreasing on} \ \ [0,\omega] \ \ (i=1,\dots,n) \tag{4.2.37}$$

hold on $[0,\omega]$, *where* $\sigma_i\in\{-1,1\}$ $(i=1,\dots,n)$, *and*

$$C=(c_{il})_{i,l=1}^n \in \mathbb{U}_\omega^{\sigma_1,\dots,\sigma_n}.$$

Then system (4.1.1) *has one and only one* ω*-periodic solution and it is nonnegative.*

Due to Lemmas 4.2.7 and 4.2.8, from Theorem 4.2.1 follows the following

Corollary 4.2.1. *Let the matrix- and vector-functions* $A=(a_{ik})_{i,k=1}^n \in \mathrm{BV}_\omega(\mathbb{R};\mathbb{R}^{n\times n})$ *and* $f=(f_i)_{i=1}^n \in \mathrm{BV}_\omega(\mathbb{R};\mathbb{R}^n)$ *be such that the functions* $\sigma_i a_{il}(t)$ $(i\neq l;\ i,l=1,\dots,n)$ *are nondecreasing on* $[0,\omega]$, *conditions* (4.1.19), (4.2.6), (4.2.37) *and*

$$\sigma_i(a_{il}(t)-a_{il}(s)) \leq \int\limits_s^t h_{il}(\tau)\,d\alpha_i(\tau) \ \ \text{for} \ \ \sigma_i(t-s)>0 \ \ (i,l=1,\dots,n)$$

hold on $[0,\omega]$, *where* α_i $(i=1,\dots,n)$ *are functions nondecreasing on* $[0,\omega]$ *and having not more than a finite number of discontinuity points,* $h_{ii}\in L_\omega^\mu(\mathbb{R},\mathbb{R};\alpha_i)$, $h_{il}\in L_\omega^\mu(\mathbb{R},\mathbb{R}_+;\alpha_l)$ $(i\neq l;\ l=1,\dots,n)$, $1\leq\mu\leq+\infty$. *Let, moreover,*

$$r(S)<1,$$

where the matrix $S=\big(s_{il}\big)_{i,l=1}^n$ *is defined by* (4.1.25) *and* g_j $(j=0,1,2)$ *are the operators defined by* (1.1.31)–(1.1.33), *respectively. Then the conclusion of Theorem 4.2.1 is true.*

Corollary 4.2.2. *Let the matrix- and vector-functions* $A = (a_{ik})_{i,k=1}^n \in \mathrm{BV}_\omega(\mathbb{R};\mathbb{R}^{n\times n})$ *and* $f = (f_i)_{i=1}^n \in \mathrm{BV}_\omega(\mathbb{R};\mathbb{R}^n)$ *be such that the functions* $\sigma_i a_{il}(t)$ $(i \neq l;\ i,l = 1,\dots,n)$ *are nondecreasing on* $[0,\omega]$*, conditions* (4.1.19), (4.2.6), (4.2.37) *and*

$$\sigma_i(a_{il}(t) - a_{il}(s)) \leq \eta_{il}(\alpha_i(t) - \alpha_i(s)) \ \ \text{for} \ \ \sigma_i(t-s) > 0 \ \ (i,l = 1,\dots,n)$$

hold on $[0,\omega]$*, where* $\sigma_i \in \{-1,1\}$*,* $\eta_{il} \in \mathbb{R}_+$ $(i \neq l;\ i,l = 1,\dots,n)$*,* α_i $(\alpha_i(\omega) \neq 0;\ i = 1,\dots,n)$ *are nondecreasing on* $[0,\omega]$*. Let, moreover,*

$$\eta_{ii} < 0 \ \ (i = 1,\dots,n)$$

and

$$r(\mathcal{H}) < 1,$$

where $\mathcal{H} = (\eta_{il})_{i,l=1}^n$*,* $h_{ii} = 0$*,* $h_{il} = -\frac{\eta_{il}}{\eta_{ii}}$ $(i \neq l;\ i,l = 1,\dots,n)$*. Then the conclusion of Theorem 4.2.1 is true.*

Corollary 4.2.3. *Let the matrix- and vector-functions* $A = (a_{ik})_{i,k=1}^n \in \mathrm{BV}_\omega(\mathbb{R};\mathbb{R}^{n\times n})$ *and* $f = (f_i)_{i=1}^n \in \mathrm{BV}_\omega(\mathbb{R};\mathbb{R}^n)$ *be such that the functions* $\sigma_i a_{il}(t)$ $(i \neq l;\ i,l = 1,\dots,n)$ *are nondecreasing on* $[0,\omega]$*, conditions* (4.1.19), (4.2.6), (4.2.36) *and* (4.2.37) *hold on* $[0,\omega]$*, where* $\sigma_1 = \sigma_2 = \dots = \sigma_n = \sigma_0$*,* $\sigma_0 \in \{-1;1\}$*, a matrix-function* $C = (c_{il})_{i,l=1}^n \in \mathrm{BV}_\omega(\mathbb{R},\mathbb{R}^{n\times n})$ *is quasi-nondecreasing on* $[0,\omega]$*. Let, moreover, the module of every multiplicator of the system* (4.1.30)*, where* $C_{\sigma_0}(t) = \sigma_0 C(\sigma_0 t + \frac{1-\sigma_0}{2}\,\omega)$ *be less than* 1*. Then the conclusion of Theorem 4.2.1 is true.*

Remark 4.2.1. The fulfilment only of the conditions

$$\sigma_i a_{il}(t) \ \text{ are nondecreasing on } \ [0,\omega] \ \ (i \neq l;\ i,l = 1,\dots,n), \tag{4.2.38}$$

(4.2.37) and the existence of the unique ω-periodic solution of system (4.1.1) does not guarantees the positiveness of the solution. For the completeness, we give the corresponding example from [47].

Let $n = 2$, $a_{11}(t) = a_{22}(t) \equiv 0$, $a_{12}(t) = -a_{21}(t) \equiv t$, $f_1(t) \equiv t$ and $f_2(t) \equiv \frac{1}{2}(\cos 2t - 1)$. Then conditions (4.2.37) and (4.2.38) hold for $\sigma_1 = 1$, $\sigma_2 = -1$. On the other hand, the corresponding system has the unique solution

$$x_1(t) \equiv -\frac{1}{2}\,(1 - \cos t)^2, \quad x_2(t) \equiv = \frac{1}{3}\,(\sin 2t - 2\sin t)$$

with period π. It is evident that the solution is not nonnegative.

4.2.4 On a method for constructing the periodic solutions

In this subsection, we give a method for constructing the solutions of problems (4.1.1), (4.1.2).

We use the results of Section 2.4.

In case the conditions of Theorem 4.1.6 are fulfilled, for the construction of the ω-periodic solution of system (4.1.1) we can use the algorithm described in Section 2.4 for the construction of the solution of the multi-point boundary value problem.

Let

$$t_i = \frac{1-\sigma_i}{2}\,\omega \ \ (i = 1,\dots,n). \tag{4.2.39}$$

As the zero approximation to the solution of problem (4.1.1), (4.1.2), we choose an arbitrary function $(x_{0i})_{i=1}^n \in \mathrm{BV}_\omega(\mathbb{R};\mathbb{R}^n)$. If the $(m-1)$-th approximation $(x_{m-1\,i})_{i=1}^n$ is constructed, then by the m-th approximation we take $(x_{mi})_{i=1}^n \in \mathrm{BV}_\omega(\mathbb{R};\mathbb{R}^n)$, whose i-th components are defined by

$$\begin{aligned} &x_{mi}(t_i) = x_{m-1\,i}(\omega - t_i),\\ &x_{mi}(t) = \gamma_i(t,t_i)x_{m-1i}(\omega - t_i) + \psi_i(x_{m-1\,1},\dots,x_{m-1\,n},f_i)(t) \ \ \text{for} \ \ t \in [0,\omega] \ \ (i=1,\dots,n), \end{aligned} \tag{4.2.40}$$

where the operators $\psi_i : \mathrm{BV}_\omega(\mathbb{R}, \mathbb{R}^{n+1}) \to \mathrm{BV}_\omega(\mathbb{R}; \mathbb{R})$ $(i = 1, \dots, n)$ are defined as

$$\begin{gathered}\psi_i(y_1, \dots, y_{n+1})(t) = g_i(y_1, \dots, y_{n+1})(t) - \gamma_i(t, t_i) \int_{t_i}^{t} g_i(y_1, \dots, y_{n+1})(s)\, d\gamma_i^{-1}(s, t_i),\\ g_i(y_1, \dots, y_{n+1})(t) = \sum_{l=1}^{n} \int_{t_i}^{t} y_l(s)\, d(a_{il}(s) - \delta_{il}\widetilde{a}_i(s))\\ + y_{n+1}(t) - y_{n+1}(t_i) \ \text{ for } \ t \in [0, \omega] \ \ (i = 1, \dots, n);\\ \gamma_i(t, t_i) \equiv \gamma_{\widetilde{a}_i}(t, t_i), \ \ \widetilde{a}_i(t) \equiv s_c(a_{ii})(t) \ \ (i = 1, \dots, n),\end{gathered} \tag{4.2.41}$$

and the function $\gamma_{\widetilde{a}_i}(t, t_i)$ is defined by (1.1.9).

Theorem 4.2.2. *Let the conditions of Theorem 4.1.6 hold. Then system* (4.1.1) *has one and only one ω-periodic solution $x = (x_i)_{i=1}^n \in \mathrm{BV}_\omega(\mathbb{R}; \mathbb{R}^n)$ and there exist $\rho_0 > 0$ and $\delta \in]0, 1[$ such that*

$$\sum_{i=1}^{n} |x_i(t) - x_{mi}(t)| \le \rho_0 \delta^m \ \ \textit{for} \ \ t \in [0, \omega] \ \ (m = 1, 2 \dots), \tag{4.2.42}$$

where the vector-functions $(x_{mi})_{i=1}^n$ $(m = 1, 2, \dots)$ are defined by (4.2.40), (4.2.41).

Corollary 4.2.4. *Let the conditions from Corollaries 4.1.5–4.1.7 be fulfilled. Then system* (4.1.1) *has one and only one ω-periodic solution $x = (x_i)_{i=1}^n \in \mathrm{BV}_\omega(\mathbb{R}; \mathbb{R}^n)$ and estimate* (4.2.42) *holds, where $\rho_0 > 0$ and $\delta \in]0, 1[$ are the constants independent of m, and the vector-functions $(x_{mi})_{i=1}^n$ $(m = 1, 2, \dots)$ are defined by* (4.2.40), (4.2.41).

Remark 4.2.2. Using Lemma 4.2.6, we can show that the above process of constructing the ω-periodic solution of system (4.1.1) is stable in the sense given above (see Remark 2.4.1 and its proof in Subsection 2.4.3).

Remark 4.2.3. In view of (4.2.40) and (4.2.41), according to the variation-of-constant formula (1.1.12), the function x_{im} is a solution of the Cauchy problem

$$dx_{mi}(t) = x_{mi}(t)\, d\widetilde{a}_i(t) + \sum_{l=1}^{n} x_{m-1\,l}(t)\, d(a_{il}(t) - \delta_{il}\widetilde{a}_i(t)) + df_i(t), \tag{4.2.43}$$

$$x_{mi}(t_i) = x_{m-1\,i}(\omega - t_i) \tag{4.2.44}$$

for $i \in \{1, \dots, n\}$ and every natural m.

Here, the fact that the points t_i $(i = 1, \dots, n)$ are defined by (4.2.39) is of special importance. If, for example, for every $i \in \{1, \dots, n\}$ and every natural m, we replace condition (4.2.44) by the condition

$$x_{mi}(0) = x_{m-1\,i}(\omega), \tag{4.2.45}$$

then the process may be nonconvergent. In this connection, consider the example [47]

$$dx_i(t) = x_i(\omega - t_i)\, dt \ \ (i = 1, \dots, n).$$

It is evident that the conditions of Theorem 4.1.6 are fulfilled for $A(t) \equiv \mathrm{diag}(t, \dots, t)$, $f(t) \equiv 0_n$, $\sigma_i = -1$ $(i = 1, \dots, n)$ and $C(t) \equiv \mathrm{diag}(t, \dots, t)$. The system has only the trivial ω-periodic solution. Due to (4.2.39), $t_i = \omega$ $(i = 1, \dots, n)$. Let $x_{0i}(t) \equiv 1$ $(i = 1, \dots, n)$. Then if x_{mi} is the solution of problem (4.2.43), (4.2.44), we have

$$\begin{gathered}x_{mi}(t) = \exp(t - \omega)x_{m-1\,i}(0) = \exp(t - m\omega) \ \text{ for } \ t \in [0, \omega] \ \ (i = 1, \dots, n; \ m = 1, 2, \dots),\\ \lim_{m \to +\infty} x_{mi}(t) = 0 \ \text{ uniformly on } \ [0, \omega] \ \ (i = 1, \dots, n).\end{gathered}$$

If we replace condition (4.2.44) by condition (4.2.45), then

$$\begin{gathered}x_{mi}(t) = \exp(t)x_{m-1\,i}(\omega) = \exp(t + (m - 1)\omega) \ \text{ for } \ t \in [0, \omega] \ \ (i = 1, \dots, n; \ m = 1, 2, \dots),\\ \lim_{m \to +\infty} x_{mi}(t) = +\infty \ \text{ for } \ t \in [0, \omega] \ \ (i = 1, \dots, n).\end{gathered}$$

Chapter 5

Systems of linear impulsive differential equations

5.1 General linear boundary value problems

5.1.1 Unique solvability

In this chapter, some results of Chapter 1 for the general linear boundary value problem we realize for the following impulsive differential systems

$$\frac{dx}{dt} = P(t)x + q(t) \ \text{ for a.a. } \ t \in I \setminus T, \tag{5.1.1}$$

$$x(\tau_l+) - x(\tau_l-) = G(\tau_l)x(\tau_l) + u(\tau_l) \ \ (l = 1, 2, \dots); \tag{5.1.2}$$

$$\ell(x) = c_0, \tag{5.1.3}$$

where $P \in L(I; \mathbb{R}^{n\times n})$, $q \in L(I; \mathbb{R}^n)$, $G \in B(T; \mathbb{R}^{n\times n})$, $u \in B(T; \mathbb{R}^n)$, $T = \{\tau_1, \tau_2, \dots\}$, $\tau_l \in I$ $(l = 1, 2, \dots)$, $\tau_l \neq \tau_k$ if $l \neq k$ $(l, k = 1, 2, \dots)$, $\ell : \mathrm{BV}_\infty(I; \mathbb{R}^n) \to \mathbb{R}^n$ is a linear bounded vector-functional, and $c_0 \in \mathbb{R}^n$.

Everywhere we assume that $I = [a, b]$.

Definition 5.1.1. Under a solution of the impulsive differential system (5.1.1), (5.1.2) we understand a continuous from the left vector-function $x \in \mathrm{BVAC}_{loc}(I, T; \mathbb{R}^n)$ satisfying both the system

$$x'(t) = P(t)x(t) + q(t) \ \text{ for a.a. } \ t \in I \setminus T$$

and relation (5.1.2) for every $l \in \{1, 2, \dots\}$.

Quite a number of issues of the theory of linear systems of differential equations with impulsive effect have been studied sufficiently well (for survey of the results on impulsive systems see the references in Introduction). But the above-mentioned works do not contain the results analogous to those obtained in [46, 47] for ordinary differential equations. Using the theory of generalized ordinary differential equations, we extend these results to the systems of impulsive differential equations.

We assume that the condition

$$\det(I_n + G(\tau_l)) \neq 0 \ \ (l = 1, 2, \dots) \tag{5.1.4}$$

holds.

To establish the results dealing with the boundary value problems for the impulsive differential system (5.1.1), (5.1.2), we use the following conception.

Remark 5.1.1. A vector-function x is a solution of the impulsive system (5.1.1), (5.1.2) if and only if it is a solution of the system

$$dx = dA(t) \cdot x + df(t),$$

where

$$\begin{aligned} A(t) &\equiv \int_a^t P(\tau)\,d\tau + \sum_{\tau_l\in[a,t[} G(\tau_l) \ \text{ for } \ t\in I,\\ f(t) &\equiv \int_a^t q(\tau)\,d\tau + \sum_{\tau_l\in[a,t[} u(\tau_l) \ \text{ for } \ t\in I. \end{aligned} \tag{5.1.5}$$

It is evident that these matrix- and vector-functions A and f have the following properties:

$$\begin{gathered} d_1A(t)=O_{n\times n}, \quad d_1f(t)=0_n \ \text{ for } \ t\in I,\\ d_2A(t)=O_{n\times n}, \quad d_2f(t)=0_n \ \text{ for } \ t\in I\setminus T,\\ d_2A(\tau_l)=G(\tau_l), \quad d_2f(\tau_l)=u(\tau_l) \ \ (l=1,2,\dots); \end{gathered} \tag{5.1.6}$$

$$S_c(A)(t)-S_c(A)(s)=\int_s^t P(\tau)\,d\tau, \ \ s_c(f)(t)-s_c(f)(s)=\int_s^t q(\tau)\,d\tau \ \text{ for } \ s,t\in I\setminus T,$$

$$S_1(A)(t)=O_{n\times n}, \ \ s_1(f)(t)=0_n \ \text{ for } \ t\in I\setminus T,$$

$$S_2(A)(t)=S_2(A)(s)+\sum_{s\le\tau_l<t} G(\tau_l), \ \ s_2(f)(t)=s_2(f)(s)+\sum_{s\le\tau_l<t} u(\tau_l) \ \text{ for } \ s,t\in I; \ s<t$$

(in particular, they are continuous from the left everywhere).

So, condition (5.1.4) is equivalent to condition (1.1.8). Moreover, due to the conditions imposed on P, G, q and u, we have $A\in \mathrm{BV}(I;\mathbb{R}^{n\times n})$ and $f\in \mathrm{BV}(I;\mathbb{R}^n)$. Therefore, system (5.1.1) is a particular case of system (1.1.1).

We say that the pair (X,Y) consisting of the matrix-functions $X\in L(I;\mathbb{R}^{n\times n})$ and $Y\in B(T;\mathbb{R}^{n\times n})$ satisfies the Lappo–Danilevskiĭ condition at the point a if

$$X(t)\int_a^t X(\tau)\,d\tau = \int_a^t X(\tau)\,d\tau\cdot X(t),$$

$$\int_a^t X(\tau)\,d\tau\cdot \sum_{\tau_l\in[a,t[} Y(\tau_l) = \sum_{\tau_l\in[a,t[} Y(\tau_l)\cdot \int_a^t X(\tau)\,d\tau \ \text{ for } \ t\in I.$$

Remark 5.1.2. By Definition 5.1.1, under a solution of the impulsive system (5.1.1), (5.1.2) we understand the continuous from the left vector-function. If under a solution we understand the continuous from the right vector-function, then we have to require the condition

$$\det(I_n-G(\tau_l))\neq 0 \ \ (l=1,2,\dots)$$

instead of (5.1.4). In this case, the matrix $A(t)$ and vector $f(t)$ will be defined such that

$$\begin{gathered} d_1A(t)=O_{n\times n}, \quad d_1f(t)=0_n \ \text{ for } \ t\in I\setminus T,\\ d_1A(\tau_l)=G(\tau_l), \quad d_1f(\tau_l)=u(\tau_l) \ \ (l=1,2,\dots),\\ d_2A(t)=O_{n\times n}, \quad d_2f(t)=0_n \ \text{ for } \ t\in I \end{gathered}$$

instead of (5.1.6). In particular, $A(t)$ and $f(t)$ can be defined similarly as in (5.1.5) modifying the second component. The results corresponding to this case are analogous to the results corresponding to the first case given in Sections 5.1–5.3 below, if we replace the expressions of type $I_n+G(\tau_l)$ by $I_n-G(\tau_l)$, the intervals $[s,t[$ by $]s,t]$, and the right limits by the left ones.

We will need the forms of operators defined by means of $(1.1.35_1)$, $(1.1.35_2)$ and $(1.1.36_1)$, $(1.1.36_2)$. First of all, we note that the operators defined by $(1.1.35_1)$ ($(1.1.35_2)$) and $(1.1.36_1)$ ($(1.1.36_2)$) coincide among themselves if X is a continuous from the left (from the right) matrix-function.

For every matrix-function $X \in L(I;\mathbb{R}^{n\times n})$ and a sequence of constant matrices $Y_k \in \mathbb{R}^{n\times n}$ $(k = 0,1,\dots)$ we put

$$\big[\big(X,\{Y_k\}_{k=1}^{\infty}\big)\big]_0(t) = I_n \ \text{ for } \ a \le t \le b;$$

$$\big[\big(X,\{Y_k\}_{k=1}^{\infty}\big)\big]_i(a) = O_{n\times n} \ \ (i = 1,2,\dots),$$

$$\big[\big(X,\{Y_k\}_{k=1}^{\infty}\big)\big]_{i+1}(t) = \int\limits_a^t X(\tau)\cdot\big[\big(X,\{Y_k\}_{k=1}^{\infty}\big)\big]_i(\tau)\,d\tau + \\ + \sum_{\tau_k\in[a,t[} Y_k\cdot\big[\big(X,\{Y_k\}_{k=1}^{\infty}\big)\big]_i(\tau_k) \ \text{ for } \ a < t \le b \ \ (i = 1,2,\dots).$$

Note that in this case for the operators V_i $(i = 1,2,\dots)$ defined by $(1.1.37_1)$, we have

$$V_i(X,\{Y_k\}_{k=1}^{\infty})(t) = \big[\big(|X|,\{|Y_k|\}_{k=1}^{\infty}\big)\big]_i(t) \ \text{ for } \ a \le t \le b \ \ (i = 1,2,\dots).$$

Using the above-described properties the matrix- and vector-functions A and f corresponding to the impulsive system (5.1.1), (5.1.2), we obtain the results for the solvability of the impulsive boundary value problem (5.1.1)–(5.1.3). We do not cite here these results. They can be found in [18].

Our aim is to establish the necessary and sufficient conditions for the convergence of the difference schemes corresponding to linear impulsive boundary value problems. To this end, below we present only the results concerning the well-posedness of general linear boundary value problems.

As to the existence of nonnegative solutions of multi-point boundary value problems constructed by a method of solutions of the latest problem for the impulsive case and other results, they immediately follow from the case of corresponding results for the generalized ordinary differential equations.

5.1.2 The well-posedness of the general linear boundary value problems

Let x_0 be a unique solution of problem (5.1.1)–(5.1.3).

Here, as above, we will assume that $I = [a,b]$.

Along with the impulsive general boundary initial problem (5.1.1)–(5.1.3), consider the sequence of the problems

$$\frac{dx}{dt} = P_m(t)x + q_m(t) \ \text{ for a.a. } \ t \in I\setminus\{\tau_l\}_{l=1}^{\infty}, \tag{5.1.1$_m$}$$

$$x(\tau_l+) - x(\tau_{l-}) = G_m(\tau_l)x(\tau_l) + u_m(\tau_l) \ \ (l = 1,2,\dots); \tag{5.1.2$_m$}$$

$$\ell_m(x) = c_m \tag{5.1.3$_m$}$$

$(m = 1,2,\dots)$, where $P_m \in L(I;\mathbb{R}^{n\times n})$ $(m = 1,2,\dots)$, $q_m \in L(I;\mathbb{R}^n)$ $(m = 1,2,\dots)$, $G_m \in B(T;\mathbb{R}^{n\times n})$ $(m = 1,2,\dots)$, $u_m \in B(T;\mathbb{R}^n)$, $T = \{\tau_1,\tau_2,\dots\}$, $\ell_m : \mathrm{BV}_\infty(I;\mathbb{R}^n) \to \mathbb{R}^n$ $(m = 1,2,\dots)$ are linear bounded vector-functionals, and $c_m \in \mathbb{R}^n$ $(m = 1,2,\dots)$.

We assume that $P_m = (p_{mij})_{i,j=1}^n$ $(m = 0,1,\dots)$, $q_m = (q_{mi})_{i=1}^n$ $(m = 0,1,\dots)$; $G_m = (g_{mij})_{i,j=1}^n$ $(m = 0,1,\dots)$, $u_m = (u_{mi})_{i=1}^n$ $(m = 0,1,\dots)$.

Here, under the matrix- and vector-functions P_0, q_0, G_0, u_0 and functional ℓ_0 we understand P, q, G, u and ℓ, respectively.

We establish the necessary and sufficient and effective sufficient conditions for the boundary value problem $(5.1.1_m)$–$(5.1.3_m)$ to have a unique solution x_m for any sufficiently large m and

$$\lim_{m\to+\infty}\|x_m - x_0\|_\infty = 0. \tag{5.1.7}$$

Remark 5.1.3. If we consider the case where for every natural m, the impulses points depend on m in the impulsive system $(5.1.1_m)$, $(5.1.2_m)$, in particular, the linear algebraic system $(5.1.2_m)$ has the form

$$x(\tau_{lm}+) - x(\tau_{lm}-) = G_m(\tau_{lm})x(\tau_{lm}) + u_m(\tau_{lm}) \ \ (l = 1,2,\dots),$$

where $\tau_{lm} \in I$ $(l = 1,2,\dots)$, then the last general case will be reduced to case $(5.1.2_m)$ by using the following concept.

Let $T = T_0 \cup T_1 \cup T_2 \cup \dots$, where $T_m = \{\tau_{1m}, \tau_{2m}, \dots\}$ $(m = 0, 1, \dots)$, and $\tau_{10} = \tau_l$ $(l = 1, 2, \dots)$. The set T is countable. Therefore, $T = \{\tau_1^*, \tau_2^*, \dots\}$, where $\tau_l^* \in I$ $(l = 1, 2, \dots)$. For every $m \in \widetilde{\mathbb{N}}$ and $l \in \mathbb{N}$, we set $G_m^*(\tau_l^*) = G_m^*(\tau_{lm})$ and $u_m^*(\tau_l^*) = u_m^*(\tau_{lk})$ if $\tau_l^* \in T_m$, where $l_m \in \mathbb{N}$ is such that $\tau_l^* = \tau_{l_k m}$, and $G_m^*(\tau_l^*) = O_{n\times n}$ and $u_m^*(\tau_l^*) = 0$ if $\tau_l^* \notin T_m$. So, the last general case is equivalent to the impulsive system $(5.1.1_m), (5.1.2_m)$, where $\tau_l = \tau_l^*$ $(l = 1, 2, \dots)$, $G_m(\tau_l) = G_m^*(\tau_l^*)$ $(l = 1, 2, \dots)$ and $u_m(\tau_l) = u_m^*(\tau_l^*)$ $(l = 1, 2, \dots)$.

Below, we assume that $T = \{\tau_1, \tau_2, \dots\}$.

Along with systems $(5.1.1), (5.1.2)$ and $(5.1.1_m), (5.1.2_m)$, we consider the corresponding homogeneous systems

$$\frac{dx}{dt} = P_0(t)x \ \text{ for a.a. } \ t \in I \setminus T, \tag{$5.1.1_0$}$$

$$x(\tau_l+) - x(\tau_l-) = G_0(\tau_l)x(\tau_l) \ \ (l = 1, 2, \dots) \tag{$5.1.2_0$}$$

and

$$\frac{dx}{dt} = P_m(t)x \ \text{ for a.a. } \ t \in I \setminus T, \tag{$5.1.1_{m0}$}$$

$$x(\tau_l+) - x(\tau_l-) = G_m(\tau_l)x(\tau_l) \ \ (l = 1, 2, \dots) \tag{$5.1.2_{m0}$}$$

$(m = 1, 2, \dots)$.

Definition 5.1.2. We say that the sequence $(P_m, q_m; G_m, u_m; \ell_m)$ $(m = 1, 2, \dots)$ belongs to the set $\mathcal{S}(P_0, q_0; G_0, u_0; \ell_0)$ if for every $c_0 \in \mathbb{R}^n$ and a sequence $c_m \in \mathbb{R}^n$ $(m = 1, 2, \dots)$ satisfying condition

$$\lim_{m\to+\infty} c_m = c_0,$$

problem $(5.1.1_m)$–$(5.1.3_m)$ has a unique solution x_m for any sufficiently large m, and condition (5.1.7) holds.

As above, the impulsive systems $(5.1.1), (5.1.2)$ and $(5.1.1_m), (5.1.2_m)$ $(m = 1, 2, \dots)$ are the particular cases, respectively, of the general systems (1.2.1) and $(1.2.1_m)$ $(m = 1, 2, \dots)$ if we set

$$\begin{aligned} A_m(t) &= \int_a^t P_m(\tau)\,d\tau + \sum_{\tau_l \in [a,t[} G_m(\tau_l) \ \text{ for } \ t \in I \ \ (m = 0, 1, \dots), \\ f_m(t) &= \int_a^t q_m(\tau)\,d\tau + \sum_{\tau_l \in [a,t[} u_m(\tau_l) \ \text{ for } \ t \in I \ \ (m = 0, 1, \dots). \end{aligned} \tag{5.1.8}$$

To realize and formulate the well-posed results of Section 1.2, we use the following forms of the operators $\mathcal{B}(X, Y)$ and $\mathcal{I}(X, Y)$ (see (0.0.3) and (0.0.4)) for the impulsive case, in particular, when the matrix-functions X and Y are continuous from the left on I. Using the integration-by-parts formulas (0.0.10), (0.0.12) and the definition of the Kurzweil integral, we find that

$$\mathcal{B}(X, Y)(t) \equiv \int_a^t X(\tau)Y'(\tau)\,d\tau + \sum_{\tau_l \in [a,t[} X(\tau_l+)\,d_2Y(\tau_l) \tag{5.1.9}$$

if $X \in \mathrm{BV}(I; \mathbb{R}^{n\times j})$ and $Y \in \mathrm{BVAC}_{loc}(I, T; \mathbb{R}^{j\times m})$, and

$$\mathcal{I}(X, Y)(t) \equiv \int_a^t \big(X'(\tau) + X(\tau)Y'(\tau)\big)X^{-1}(\tau)\,d\tau + \sum_{\tau_l \in [a,t[} \big(d_2X(\tau_l) + X(\tau_l+)\,d_2Y(\tau_l)\big)\,X^{-1}(\tau_l) \tag{5.1.10}$$

if $X, Y \in \mathrm{BVAC}_{loc}(I, T; \mathbb{R}^{n\times n})$, $\det X(t) \neq 0$. In addition, if

$$Q(t) \equiv \int_s^t Y(\tau)\,d\tau + \sum_{\tau_l \in [a,t[} Z(\tau_l),$$

where $Y \in L_{loc}(I;\mathbb{R}^{n\times m})$ and $Z \in B_{loc}(T;\mathbb{R}^{n\times m})$, we set

$$\mathcal{B}_\iota(X;Y,Z)(t) \equiv \mathcal{B}(X,Q)(t) \ \text{ and } \ \mathcal{I}_\iota(X;Y,Z)(t) \equiv \mathcal{I}(X,Q)(t).$$

Consequently,

$$\mathcal{B}_\iota(X;Y,Z)(t) \equiv \int\limits_a^t X(\tau)Y(\tau)\,d\tau + \sum_{\tau_l\in[a,t[} X(\tau_l+)\,Z(\tau_l), \tag{5.1.11}$$

Note that if $X(t) \equiv I_n$, then

$$\mathcal{B}_\iota(I_n;Y,Z)(t) \equiv \int\limits_a^t Y(\tau)\,d\tau + \sum_{\tau_l\in[a,t[} Z(\tau_l), \quad \mathcal{I}_\iota(I_n;Y,Z)(t) \equiv \int\limits_a^t Y(\tau)\,d\tau + \sum_{\tau_l\in[a,t[} d_2 Z(\tau_l).$$

It is clear that, by (5.1.8),

$$A_m(t) \equiv \mathcal{B}_\iota(I_n;P_m,G_m)(t), \quad f_m(t) \equiv \mathcal{B}_\iota(I_n;q_m,u_m)(t) \ \ (m=0,1,\dots).$$

Theorem 5.1.1. *Let the conditions*

$$\lim_{m\to+\infty} \ell_m(x) = \ell(x) \ \textit{ for } \ x \in \mathrm{BV}(I;\mathbb{R}^n), \tag{5.1.12}$$

$$\limsup_{m\to+\infty} |||\ell_m||| < +\infty \tag{5.1.13}$$

hold. Then

$$\big((P_m,q_m;G_m,u_m;\ell_m)\big)_{m=1}^{\infty} \in \mathcal{S}(P_0,q_0;G_0,u_0;\ell_0) \tag{5.1.14}$$

if and only if there exists a sequence of matrix-functions $H_m \in \mathrm{BVAC}_{loc}(I,T;\mathbb{R}^{n\times n})$ $(m=0,1,\dots)$ *such that the conditions*

$$\limsup_{m\to+\infty} \bigvee_a^b \big(H_m + \mathcal{B}_\iota(H_m;P_m,G_m)\big) < +\infty, \tag{5.1.15}$$

and

$$\inf\big\{|\det(H_0(t))| : \ t \in I\big\} > 0, \tag{5.1.16}$$

hold, and the conditions

$$\lim_{m\to+\infty} H_m(t) = H_0(t), \tag{5.1.17}$$

$$\lim_{m\to+\infty} \mathcal{B}_\iota(H_m;P_m,G_m)(t) = \mathcal{B}_\iota(H_0;P_0,G_0)(t) \tag{5.1.18}$$

and

$$\lim_{m\to+\infty} \mathcal{B}_\iota(H_m;q_m,u_m)(t) = \mathcal{B}_\iota(H_0;q_0,u_0)(t)$$

hold uniformly on I*.*

Note that in Theorem 5.1.1, due to (5.1.9), (5.1.10) and (5.1.11), we have

$$\mathcal{B}_\iota(H_m;q_m,u_m)(t) \equiv \int\limits_a^t H_m(\tau)\,q_m(\tau)\,d\tau + \sum_{\tau_l\in[a,t[} H_m(\tau_l+)\,u_m(\tau_l) \ \ (m=0,1,\dots) \tag{5.1.19}$$

and

$$\begin{aligned}\mathcal{I}_\iota(H_m;P_m,G_m)(t) \equiv &\int\limits_a^t \big(H_m'(\tau) + H_m(\tau)P_m(\tau)\big)H_m^{-1}(\tau)\,d\tau \\ &+ \sum_{\tau_l\in[a,t[} \big(d_2H_m(\tau_l) + H_m(\tau_l+)G_m(\tau_l)\big)\,H_m^{-1}(\tau_l) \ \ (m=0,1,\dots).\end{aligned} \tag{5.1.20}$$

Theorem 5.1.2. *Let conditions* (5.1.12), (5.1.13) *and*

$$\det(I_n+G_m(\tau_l))\neq 0 \ \ (l=1,2,\dots;\ m=0,1,\dots)$$

hold. Then inclusion (5.1.14) *holds if and only if the conditions*

$$\lim_{m\to+\infty} X_m^{-1}(t)=X_0^{-1}(t)$$

and

$$\lim_{m\to+\infty}\bigg(\int\limits_a^t X_m^{-1}(\tau)q_m(\tau)\,d\tau+\sum_{\tau_l\in[a,t[} X_m^{-1}(\tau_l+)\,u_m(\tau_l)\bigg)
=\int\limits_a^t X_0^{-1}(\tau)q_0(\tau)\,d\tau+\sum_{\tau_l\in[a,t[} X_0^{-1}(\tau_l-)\,u_0(\tau_l)$$

hold uniformly on I, where X_m is the fundamental matrix of the homogeneous system $(5.1.1_{m0})$, $(5.1.2_{m0})$ *for any $m\in\widetilde{\mathbb{N}}$.*

Theorem 5.1.3. *Let $P_0^*\in L(I;\mathbb{R}^{n\times n})$, $q_0^*\in L(I;\mathbb{R}^n)$, $G_0^*\in B(T;\mathbb{R}^{n\times n})$, $u_0^*\in B(T;\mathbb{R}^n)$, $c_0^*\in\mathbb{R}^n$, and a $\ell_0^*:\mathrm{BV}_\infty(I;\mathbb{R}^{n\times n})\to\mathbb{R}^n$ be a linear bounded vector-functional such that*

$$\det(I_n+G_0^*(\tau_l))\neq\ 0\ \ (l=1,2,\dots)$$

and the boundary value problem

$$\frac{dx}{dt}=P_0^*(t)x+q_0^*(t)\ \ \text{for a.a. } t\in I\setminus T, \tag{5.1.1*}$$

$$x(\tau_l+)-x(\tau_l-)=G_0^*(\tau_l)x(\tau_l)+u_0^*(\tau_l)\ \ (l=1,2,\dots); \tag{5.1.2*}$$

$$\ell_0^*(x)=c_0^* \tag{5.1.3*}$$

has a unique solution x_0^*. Let, moreover, there exist the sequences of matrix- and vector-functions $H_m\in\mathrm{BVAC}_{loc}(I,T;\mathbb{R}^{n\times n})$ $(m=1,2,\dots)$ and $h_m\in\mathrm{BVAC}_{loc}(I,T;\mathbb{R}^n)$ $(m=1,2,\dots)$ such that

$$\inf\big\{|\det(H_m(t))|:\ t\in I\big\}>0\ \ \text{for every sufficiently large } m,$$

the conditions

$$\lim_{m\to+\infty}(c_m+\ell_m^*(h_m))=c_0^*,\quad \lim_{m\to+\infty}\ell_m^*(y)=\ell_0^*(y)\ \ \text{for } y\in\mathrm{BV}(I;\mathbb{R}^n),$$

$$\limsup_{m\to+\infty}|||\ell_m^*|||<+\infty\quad\text{and}\quad \limsup_{m\to+\infty}\bigvee_a^b\mathcal{I}_\iota(H_m;P_m,G_m)<+\infty$$

hold and the conditions

$$\lim_{m\to+\infty}\mathcal{I}_\iota(H_m;P_m,G_m)(t)=\int\limits_a^t P_0^*(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}G_0^*(\tau_l),$$

$$\lim_{m\to+\infty}\bigg(h_m(t)-h_m(a)+\mathcal{B}_\iota(H_m;q_m,u_m)(t)-\int\limits_a^t d\mathcal{I}_\iota(H_m;P_m,G_m)(s)\cdot h_m(s)\bigg)
=\int\limits_a^t q_0^*(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}u_0^*(\tau_l)$$

hold uniformly on I, where $\ell_m^*(y)=\ell_m(H_m^{-1}y)$ $(m=1,2,\dots)$, and the operators $\mathcal{B}_\iota$ and $\mathcal{I}_\iota$ are defined by (5.1.19) *and* (5.1.20), *respectively. Then problem* $(5.1.1_m)$–$(5.1.3_m)$ *has the unique solution x_m for any sufficiently large m and*

$$\lim_{m\to+\infty}\|H_m\,x_m+h_m-x_0^*\|_\infty=0.$$

Remark 5.1.4. In Theorem 5.1.3, the vector-function $x^*_m(t) \equiv H_m(t)x_m(t) + h_m(t)$ is a solution of the problem

$$\frac{dx}{dt} = P^*_m(t)x + q^*_m(t) \ \text{ for a.a. } \ t \in [a,b] \setminus T,$$
$$x(\tau_l+) - x(\tau_l-) = G^*_m(\tau_l)x(\tau_l) + u^*_m(\tau_l) \ \ (l = 1,2,\dots);$$
$$\ell^*_m(x) = c^*_m$$

for every sufficiently large m, where

$$\begin{aligned}
P^*_m(t) &\equiv \big(H'_m(t) + H_m(t)P_m(t)\big)H_m^{-1}(t),\\
G^*_m(\tau_l) &= \big(d_2H_m(\tau_l) + H_m(\tau_l+)G_m(\tau_l)\big)\,H_m^{-1}(\tau_l) \ \ (m = 0,1,\dots;\ l = 1,2,\dots);\\
q^*_m(t) &\equiv h'_m(t) + H_m(t)\,q_m(t) - P^*_m(t)h_m(t) \ \ (m = 1,2,\dots),\\
u^*_m(\tau_l) &= d_2h_m(\tau_l) + H_m(\tau_l+)\,u_m(\tau_l) - G^*_m(\tau_l)h_m(\tau_l) \ \ (m = 1,2,\dots;\ l = 1,2,\dots).
\end{aligned}$$

Corollary 5.1.1. *Let conditions* (5.1.12), (5.1.13), (5.1.15), (5.1.16) *and*

$$\lim_{m\to+\infty}(c_m - \varphi_m(a)) = c_0$$

hold, and conditions (5.1.17), (5.1.18) *and*

$$\lim_{m\to+\infty}\bigg(\mathcal{B}_\iota(H_m;q_m-\varphi'_m,u_m)(t) + \int_a^t d\mathcal{I}_\iota(H_m;P_m,G_m)\cdot\varphi_m(\tau)\bigg) = \mathcal{B}_\iota(H_0;q_0,u_0)(t)$$

hold uniformly on I, where $H_m \in \mathrm{BVAC}_{loc}(I,T;\mathbb{R}^{n\times n})$ $(m = 0,1,\dots)$, $\varphi_m \in \mathrm{BVAC}_{loc}(I,T;\mathbb{R}^n)$ $(m = 1,2,\dots)$, and the operators $\mathcal{B}_\iota$ and $\mathcal{I}_\iota$ are defined by (5.1.19) *and* (5.1.20), *respectively. Then problem* $(5.1.1_m)$–$(5.1.3_m)$ *has the unique solution x_m for any sufficiently large m and*

$$\lim_{m\to+\infty}\|x_m - \varphi_m - x_0\|_\infty = 0.$$

Remark 5.1.5. Note that the condition

$$\limsup_{m\to+\infty}\bigg(\int_a^b \big\|H'_m(t) + H_m(t)P_m(t)\big\|\,dt + \sum_{l=1}^{+\infty}\big\|d_2H_m(\tau_l) + H_m(\tau_l+)G_m(\tau_l)\big\|\bigg) < +\infty$$

guarantees the fulfilment of condition (5.1.15).

Now we give some effective sufficient conditions guaranteeing inclusion (5.1.14).

Theorem 5.1.4. *Let conditions* (5.1.12), (5.1.13) *and*

$$\limsup_{m\to+\infty}\bigg(\int_a^b \|P_m(t)\|\,dt + \sum_{l=1}^{\infty}\|G_m(\tau_l)\|\bigg) < +\infty$$

hold, and the conditions

$$\lim_{m\to+\infty}\bigg(\int_a^t P_m(\tau)\,d\tau + \sum_{\tau_l\in[a,t[} G_m(\tau_l)\bigg) = \int_a^t P_0(\tau)\,d\tau + \sum_{\tau_l\in[a,t[} G_0(\tau_l)$$

and

$$\lim_{m\to+\infty}\bigg(\int_a^t q_m(\tau)\,d\tau + \sum_{\tau_l\in[a,t[} u_m(\tau_l)\bigg) = \int_a^t q_0(\tau)\,d\tau + \sum_{\tau_l\in[a,t[} u_0(\tau_l)$$

hold uniformly on I. Then inclusion (5.1.14) *holds.*

Corollary 5.1.2. *Let conditions* (5.1.12), (5.1.13), (5.1.15) *and* (5.1.16) *hold, and conditions* (5.1.17)

$$\lim_{m\to+\infty}\int_a^t H_m(\tau)P_m(\tau)\,d\tau=\int_a^t H_0(\tau)P_0(\tau)\,d\tau$$

and

$$\lim_{m\to+\infty}\int_a^t H_m(\tau)\,q_m(\tau)\,d\tau=\int_a^t H_0(\tau)\,q_0(\tau)\,d\tau$$

hold uniformly on I, and

$$\lim_{m\to+\infty}G_m(\tau_l)=G_0(\tau_l)\quad\text{and}\quad\lim_{m\to+\infty}u_m(\tau_l)=u_0(\tau_l)$$

hold uniformly on T, where $H_m\in\mathrm{BVAC}_{loc}(I,T;\mathbb{R}^{n\times n})$ $(m=0,1,\dots)$. Let, moreover, either

$$\limsup_{m\to+\infty}\sum_{l=1}^{\infty}\big(\|G_m(\tau_l)\|+\|u_m(\tau_l)\|\big)<+\infty,\quad\text{or}\quad\limsup_{m\to+\infty}\sum_{l=1}^{\infty}\|H_m(\tau_l+)-H_m(\tau_l)\|<+\infty.$$

Then inclusion (5.1.14) *holds.*

Corollary 5.1.3. *Let conditions* (5.1.12), (5.1.13), (5.1.15) *and* (5.1.16) *hold, and conditions* (5.1.17)

$$\lim_{m\to+\infty}\bigg(\int_a^t H_m(\tau)P_m(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}H_m(\tau_l+)\,G_m(\tau_l)\bigg)=\int_a^t P_*(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}G_*(\tau_l)$$

and

$$\lim_{m\to+\infty}\bigg(\int_a^t H_m(\tau)q_m(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}H_m(\tau_l+)\,u_m(\tau_l)\bigg)=\int_a^t q_*(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}u_*(\tau_l)$$

hold uniformly on I, where $H_m\in\mathrm{BVAC}_{loc}(I,T;\mathbb{R}^{n\times n})$ $(m=1,2,\dots)$, $P_*\in L(I;\mathbb{R}^{n\times n})$, $q_*\in L(I;\mathbb{R}^n)$, $G_*\in B(T;\mathbb{R}^{n\times n})$, $u_*\in B(T;\mathbb{R}^n)$. Let, moreover, the system

$$\frac{dx}{dt}=(P_0(t)-P_*(t))x+(q_0(t)-q_*(t))\ \text{ for a.a. } t\in I\setminus T,$$
$$x(\tau_l+)-x(\tau_l-)=(G_0(\tau_l)-G_*(\tau_l))x(\tau_l)+(u_0(t)-u_*(\tau_l))\ \ (l=1,2,\dots)$$

have a unique solution satisfying condition (5.1.3)*. Then*

$$\big((P_m,q_m;G_m,u_m;\ell_m)\big)_{m=1}^{\infty}\in\mathcal{S}(P_0-P_*,q_0-q_*;G_0-G_*,u_0-u_*;\ell_0).$$

Corollary 5.1.4. *Let conditions* (5.1.12), (5.1.13) *hold and let there exist a natural number μ and matrix-functions $B_j\in\mathrm{BVAC}_{loc}(I,T;\mathbb{R}^{n\times n})$ $(j=0,\dots,\mu-1)$ such that the conditions*

$$\limsup_{m\to+\infty}\bigvee_a^b\big(H_{m\,\mu-1}+\mathcal{B}_\iota(H_{m\,\mu-1};P_m,G_m)\big)<+\infty$$

holds, and the conditions

$$\lim_{m\to+\infty}\mathcal{B}_\iota(I_n;P_m,G_m)(t)=B_0(t)-B_0(a),$$
$$\lim_{m\to+\infty}\big(H_{m\,j-1}(t)+\mathcal{B}_\iota(H_{m\,j-1};P_m,G_m)(t)\big)=I_n+B_j(t)-B_j(a)\ \ (j=1,\dots,\mu-1),$$
$$\lim_{m\to+\infty}\big(H_{m\,\mu-1}(t)+\mathcal{B}_\iota(H_{m\,\mu-1};P_m,G_m)(t)\big)=I_n+\int_{t_0}^t P_0(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}G_0(\tau_l)$$

and

$$\lim_{m\to+\infty}\mathcal{B}_\iota(H_{m\,\mu-1};q_m,u_m)(t)=\int_a^t q_0(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}u_0(\tau_l)$$

hold uniformly on I, where

$$H_{m0}(t)\equiv I_n,\ \ H_{mj}(t)\equiv-\big(H_{m\,j-1}(\tau)(t)+\mathcal{B}_\iota(H_{m\,j-1};P_m,G_m)(t)-B_j(t)+B_j(a)\big)H_{m\,j-1}(t)$$
$$(j=1,\dots,\mu-1;\ m=1,2,\dots).$$

Then inclusion (5.1.14) *holds.*

If $\mu=1$, then Corollary 5.1.4 coincides with Theorem 5.1.4.
If $\mu=2$, then Corollary 5.1.4 has the following form.

Corollary 5.1.4$_1$. *Let conditions* (5.1.12), (5.1.13) *and* (5.1.15) *hold, and the conditions*

$$\lim_{m\to+\infty}\bigg(\int_a^t P_m(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}G_m(\tau_l)\bigg)=B(t)-B(a),$$

$$\lim_{m\to+\infty}\bigg(\int_a^t H_m(\tau)P_m(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}\big(B(\tau_l+)-G_m(\tau_l+)\big)\,G_m(\tau_l)\bigg)=\int_a^t P_0(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}G_0(\tau_l)$$

and

$$\lim_{m\to+\infty}\bigg(\int_a^t H_m(\tau)\,q_m(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}\big(B(\tau_l+)-G_m(\tau_l+)\big)\,u_m(\tau_l)\bigg)=\int_{t_0}^t q_0(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}u_0(\tau_l)$$

hold uniformly on I, where $B\in\mathrm{BVAC}_{loc}(I,T;\mathbb{R}^{n\times n})$ and

$$H_m(t)\equiv I_n-\int_a^t P_m(\tau)\,d\tau-\sum_{\tau_l\in[a,t[}G_m(\tau_l)+B(t)-B(a)\ \ (m=1,2,\dots).$$

Then inclusion (5.1.14) *holds.*

Corollary 5.1.5. *Let conditions* (5.1.12) *and* (5.1.13) *hold. Then inclusion* (5.1.14) *holds if and only if there exist matrix-functions $Q_m\in L(I;\mathbb{R}^{n\times n})$ and $W_m\in B(T;\mathbb{R}^{n\times n})$ $(m=0,1,\dots)$ such that*

$$\limsup_{m\to+\infty}\bigg(\int_a^b\|P_m(t)-Q_m(t)\|\,dt+\sum_{l=1}^{\infty}\|G_m(\tau_l)-W_m(\tau_l)\|\bigg)<+\infty \tag{5.1.21}$$

and

$$\det(I_n+W_m(\tau_l))\neq0\ \ (m=0,1,\dots;\ l=1,2,\dots), \tag{5.1.22}$$

and the conditions

$$\lim_{m\to+\infty}Z_m^{-1}(t)=Z_0^{-1}(t), \tag{5.1.23}$$

$$\lim_{m\to+\infty}\mathcal{B}_\iota(Z_m^{-1};P_m,G_m)(t)=\mathcal{B}_\iota(Z_0^{-1};P_0,G_0)(t) \tag{5.1.24}$$

and

$$\lim_{m\to+\infty}\mathcal{B}_\iota(Z_m^{-1};q_m,u_m)(t)=\mathcal{B}_\iota(Z_0^{-1};q_0,u_0)(t) \tag{5.1.25}$$

hold uniformly on I, where Z_m ($Z_m(a)=I_n$) is a fundamental matrix of the homogeneous system

$$\frac{dx}{dt}=Q_m(t) \ \text{ for a.a. } \ t\in I\setminus T, \tag{5.1.26}$$

$$x(\tau_l+)-x(\tau_l-)=W_m(\tau_l)x(\tau_l) \ \ (l=1,2,\dots) \tag{5.1.27}$$

for every $m\in\widetilde{\mathbb{N}}$.

Corollary 5.1.6. *Let conditions* (5.1.12) *and* (5.1.13) *hold and let there exist sequences of matrix-functions $Q_m\in L(I;\mathbb{R}^{n\times n})$ $(m=0,1,\dots)$ and $W_m\in B(T;\mathbb{R}^{n\times n})$ $(m=0,1,\dots)$ such that the pairs (Q_m,W_m) $(m=1,2,\dots)$ satisfy the Lappo–Danilevskiĭ condition at the point a, conditions* (5.1.21) *and*

$$\det(I_n+W_0(\tau_l))\neq 0 \ \ (l=1,2,\dots) \tag{5.1.28}$$

hold, and the conditions

$$\lim_{m\to+\infty}\int_a^t Q_m(\tau)\,d\tau=\int_a^t Q_0(\tau)\,d\tau, \tag{5.1.29}$$

$$\lim_{m\to+\infty}\sum_{\tau_l\in[a,t[} W_m(\tau_l)=\sum_{\tau_l\in[a,t[} W_0(\tau_l), \tag{5.1.30}$$

$$\begin{aligned}\lim_{m\to+\infty}\bigg(\int_a^t Z_m^{-1}(\tau)P_m(\tau)\,d\tau+\sum_{\tau_l\in[a,t[} Z_m^{-1}(\tau_l)(I_n+W_m(\tau_l))^{-1}G_m(\tau_l)\bigg)\\ =\int_a^t Z_0^{-1}(\tau)P_0(\tau)\,d\tau+\sum_{\tau_l\in[a,t[} Z_0^{-1}(\tau_l)(I_n+W_0(\tau_l))^{-1}G_0(\tau_l)\end{aligned} \tag{5.1.31}$$

and

$$\begin{aligned}\lim_{m\to+\infty}\bigg(\int_a^t Z_m^{-1}(\tau)q_m(\tau)\,d\tau+\sum_{\tau_l\in[a,t[} Z_m^{-1}(\tau_l)(I_n+W_m(\tau_l))^{-1}u_m(\tau_l)\bigg)\\ =\int_a^t Z_0^{-1}(\tau)q_0(\tau)\,d\tau+\sum_{\tau_l\in[a,t[} Z_0^{-1}(\tau_l)(I_n+W_0(\tau_l))^{-1}u_0(\tau_l)\end{aligned} \tag{5.1.32}$$

hold uniformly on I, where Z_m ($Z_m(a)=I_n$) is a fundamental matrix of the homogeneous system (5.1.26), (5.1.27) *for any sufficiently large m. Then inclusion* (5.1.14) *holds.*

Remark 5.1.6. In Corollary 5.1.6, due to (5.1.30), it follows from (5.1.28) that condition (5.1.22) holds for every sufficiently large m and, therefore, conditions (5.1.31) and (5.1.32) of the corollary are correct.

Remark 5.1.7. In Corollaries 5.1.5 and 5.1.6, if we assume that $W_m(\tau_l)=O_{n\times n}$ $(m=0,1,\dots;$ $l=1,2,\dots)$, then conditions (5.1.22) and (5.1.28) are valid, obviously. Moreover, due to the definition of the operator $\mathcal{B}_\iota$, each of conditions (5.1.24) and (5.1.31) has the form

$$\lim_{m\to+\infty}\bigg(\int_a^t Z_m^{-1}(\tau)P_m(\tau)\,d\tau+\sum_{\tau_l\in[a,t[} Z_m^{-1}(\tau_l)G_m(\tau_l)\bigg)=\int_a^t Z_0^{-1}(\tau)P_0(\tau)\,d\tau+\sum_{\tau_l\in[a,t[} Z_0^{-1}(\tau_l)G_0(\tau_l)$$

and each of conditions (5.1.25) and (5.1.32) has the form

$$\lim_{m\to+\infty}\bigg(\int_a^t Z_m^{-1}(\tau)q_m(\tau)\,d\tau+\sum_{\tau_l\in[a,t[} Z_m^{-1}(\tau_l)u_m(\tau_l)\bigg)=\int_a^t Z_0^{-1}(\tau)q_0(\tau)\,d\tau+\sum_{\tau_l\in[a,t[} Z_0^{-1}(\tau_l)u_0(\tau_l).$$

Remark 5.1.8. If a pair (P, G) satisfies the Lappo–Danilevskiĭ condition at the point s and $\det(I_n + G(\tau_l)) \neq 0$ for $\tau_l < s$, then, due to (1.2.54), the fundamental matrix Z $(Z(s) = I_n)$ of the homogeneous system

$$\frac{dx}{dt} = P(t) \ \text{ for a.a. } \ t \in I \setminus T,$$
$$x(\tau_l+) - x(\tau_l-) = G(\tau_l)x(\tau_l) \ \ (l = 1, 2, \dots)$$

has the form

$$Z(t) = \begin{cases} \exp\Big(\int\limits_s^t P(\tau)\,d\tau\Big) \prod\limits_{s \le \tau_l < t} (I_n + G(\tau_l)) & \text{for } \ t > s, \\ \exp\Big(\int\limits_t^s P(\tau)\,d\tau\Big) \prod\limits_{t \le \tau_l < s} (I_n + G(\tau_l))^{-1} & \text{for } \ t < s, \\ I_n & \text{for } \ t = s. \end{cases} \tag{5.1.33}$$

Corollary 5.1.7. *Let conditions* (5.1.12), (5.1.13) *and*

$$\limsup_{m \to +\infty} \sum_{l=1}^{\infty} \|G_m(\tau_l)\| < +\infty$$

hold. Let, moreover, the matrix-functions P_m $(m = 0, 1, \dots)$ satisfy the Lappo–Danilevskiĭ condition at the point a and the conditions

$$\lim_{m \to +\infty} \int\limits_a^t P_m(\tau)\,d\tau = \int\limits_a^t P_0(\tau)\,d\tau,$$
$$\lim_{m \to +\infty} \sum_{\tau_l \in [a,t[} G_m(\tau_l) = \sum_{\tau_l \in [a,t[} G_0(\tau_l),$$
$$\lim_{m \to +\infty} \int\limits_a^t \exp\Big(-\int\limits_a^\tau P_m(s)\,ds\Big) P_m(\tau)\,d\tau = \int\limits_a^t \exp\Big(-\int\limits_a^\tau P_0(s)\,ds\Big) P_0(\tau)\,d\tau,$$
$$\lim_{m \to +\infty} \int\limits_a^t \exp\Big(-\int\limits_a^\tau P_m(s)\,ds\Big) q_m(\tau)\,d\tau = \int\limits_a^t \exp\Big(-\int\limits_a^\tau P_0(s)\,ds\Big) q_0(\tau)\,d\tau$$

and

$$\lim_{m \to +\infty} \sum_{\tau_l \in [a,t[} \exp\Big(-\int\limits_a^{\tau_l} P_m(s)\,ds\Big) u_m(\tau_l) = \sum_{\tau_l \in [a,t[} \exp\Big(-\int\limits_a^{\tau_l} P_0(s)\,ds\Big) u_0(\tau_l)$$

hold uniformly on I. Then inclusion (5.1.14) *holds.*

Corollary 5.1.8. *Let $P_m = (p_{mij})_{i,j=1}^n \in L(I; \mathbb{R}^{n\times n})$, $q_m = (q_{mi})_{i=1}^n \in L(I; \mathbb{R}^n)$, $G_m = (g_{mij})_{i,j=1}^n \in B(T; \mathbb{R}^{n\times n})$ and $u_m = (g_{mi})_{i=1}^n \in B(T; \mathbb{R}^n)$ $(m = 0, 1, \dots)$ and let the conditions* (5.1.12), (5.1.13),

$$\limsup_{m \to +\infty} \sum_{i,j=1;\, i \neq j}^{n} \Big(\int\limits_a^b |p_{mij}(t)|\,dt + \sum_{l=1}^{\infty} |g_{mij}(\tau_l)|\Big) < +\infty$$

and

$$1 + g_{0ii}(\tau_l) \neq 0 \ \ (i = 1, \dots, n; \ l = 1, 2, \dots)$$

hold. Let, moreover, the conditions

$$\lim_{m\to+\infty}\bigg(\int\limits_a^t p_{mii}(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}g_{mii}(\tau_l)\bigg)=\int\limits_a^t p_{0ii}(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}g_{0ii}(\tau_l)\ \ (i=1,\dots,n),$$

$$\lim_{m\to+\infty}\bigg(\int\limits_a^t z_{mii}^{-1}(\tau)p_{mij}(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}z_{mii}^{-1}(\tau_l)(1+g_{mii}(\tau_l))^{-1}g_{mij}(\tau_l)\bigg)$$
$$=\int\limits_a^t z_{0ii}^{-1}(\tau)p_{0ij}(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}z_{0ii}^{-1}(\tau_l)(1+g_{0ii}(\tau_l))^{-1}g_{0ij}(\tau_l)\ \ (i\neq j;\ i,j=1,\dots,n)$$

and

$$\lim_{m\to+\infty}\bigg(\int\limits_a^t z_{mii}^{-1}(\tau)q_{mi}(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}z_{mii}^{-1}(\tau_l)(1+g_{mii}(\tau_l))^{-1}u_{mi}(\tau_l)\bigg)$$
$$=\int\limits_a^t z_{0ii}^{-1}(\tau)q_{0i}(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}z_{0ii}^{-1}(\tau_l)(1+g_{0ii}(\tau_l))^{-1}u_{0i}(\tau_l)\ \ (i=1,\dots,n)$$

hold uniformly on I, where

$$z_{mii}(t)\equiv\exp\bigg(\int\limits_a^t p_{mii}(\tau)\,d\tau\bigg)\prod_{a\le\tau_l<t}(I_n+g_{mii}(\tau_l))\ \ (i=1,\dots,n)$$

for any sufficiently large m. Then inclusion (5.1.14) *holds.*

Remark 5.1.9. For Corollary 5.1.8, the remark analogous to Remark 1.2.3 is true, i.e.,

$$1+g_{mii}(\tau_l)\neq 0\ \ (i=1,\dots,n;\ l=1,2,\dots)$$

for every sufficiently large m and, therefore, all conditions of the corollary are correct.

Remark 5.1.10. In Theorem 5.1.1 and Corollaries 5.1.1, 5.1.2, without loss of generality, we can assume that $H_0(t)\equiv I_n$.

5.1.3 Nonnegativity of solutions of the Cauchy–Nicoletti type multi-point boundary value problems

In this subsection, for the impulsive case we realize some propositions on the existence of nonnegative solutions of multi-point boundary value problems.

We investigate the question on the existence of nonnegative solutions of the impulsive system (5.1.1), (5.1.2) satisfying the following boundary value conditions:

$$x_i(t_i)=\ell_i(x_1,\dots,x_n)+c_{0i}\ \ (i=1,\dots,n),\tag{5.1.34}$$

or

$$x_i(t_i)=c_{0i}\ \ (i=1,\dots,n),\tag{5.1.35}$$

where $\ell_i:\mathrm{BV}_\infty(I;\mathbb{R}^n)\to\mathbb{R}$ $(i=1,\dots,n)$ are linear bounded functionals; $c_{0i}\in\mathbb{R}$, and x_i is the i-th component of the vector-function x for every $i\in\{1,\dots,n\}$.

We assume that $I=[a,b]$.

Definition 5.1.3. We say that the triple (Q,H,ℓ_0) consisting of matrix-functions $Q=(q_{ik})_{i,k=1}^n \in L(I;\mathbb{R}^{n\times n})$ and $H=(h_{ik})_{i,k=1}^n \in B(T;\mathbb{R}^{n\times n})$ and a positive homogeneous nondecreasing bounded vector-functional $\ell_0=(\ell_{0i})_{i=1}^n : \mathrm{BV}_\infty(I;\mathbb{R}_+^n)\to\mathbb{R}_+^n$ belongs to the set $\mathbb{U}(t_1,\dots,t_n;\tau_1,\tau_2,\dots)$ if $q_{ik}(t)\geq 0$ $(i\neq k;\ i,k=1,\dots,n)$ for a.e. $t\in I$, $h_{ik}(\tau_l)\geq 0$ $(i\neq k;\ i,k=1,\dots,n;\ l=1,2,\dots)$, and the system

$$x_i'(t)\operatorname{sgn}(t-t_i)\leq \sum_{k=1}^n q_{ik}(t)x_k(t) \ \text{ for } \ t\in I \ \ (i=1,\dots,n),$$
$$x_i(\tau_l+)-x_i(\tau_l-)\leq \sum_{k=1}^n h_{ik}(\tau_l)x_k(\tau_l) \ \ (i=1,\dots,n;\ l=1,2,\dots)$$

has no nontrivial nonnegative solution satisfying the condition

$$x_i(t_i)\leq \ell_{0i}(x_1,\dots,x_n) \ \ (i=1,\dots,n).$$

Below, we give the general results on the existence of the nonnegative solution of system (5.1.1), (5.1.2) satisfying conditions (5.1.34), or (5.1.35). The particular cases follow from the corresponding results of Section 2.3 and the results obtained in [18].

Theorem 5.1.5. *Let there exist matrix-functions* $Q=(q_{il})_{i,l=1}^n \in L(I;\mathbb{R}^{n\times n})$ *and* $H=(h_{ik})_{i,k=1}^n \in B(T;\mathbb{R}^{n\times n})$ *and a positive homogeneous nondecreasing bounded vector-functional* $\ell_0=(\ell_{0i})_{i=1}^n$: $\mathrm{BV}_\infty(I;\mathbb{R}_+^n)\to\mathbb{R}_+^n$ *satisfying the condition*

$$(Q,H;\ell_0)\in\mathbb{U}(t_1,\dots,t_n;\ \tau_1,\tau_2,\dots)$$

such that

$$p_{ii}(t)\operatorname{sgn}(t-t_i)\leq q_{ii}(t),\ \ 0\leq p_{ik}(t)\leq q_{ik}(t) \ \textit{ for } \ t\in I \ \ (i\neq k;\ i,k=1,\dots,n),$$
$$g_{ii}(\tau_l)\operatorname{sgn}(\tau_l-t_i)\leq h_{ii}(\tau_l),\ \ 0\leq g_{ik}(\tau_l)\leq h_{ik}(\tau_l) \ \ (i\neq k;\ i,k=1,\dots,n;\ l=1,2,\dots),$$
$$q_i(t)\operatorname{sgn}(t-t_i)\geq 0 \ \textit{ for } \ t\in I,\ \ u_i(\tau_l)\operatorname{sgn}(\tau_l-t_i)\geq 0 \ \ (i=1,\dots,n;\ l=1,2,\dots),$$
$$c_{0i}\geq 0 \ \ (i=1,\dots,n)$$

and

$$0\leq \ell_i(x_1,\dots,x_n)\leq \ell_{0i}(x_1,\dots,x_n),\ \ x_l\in\mathrm{BV}(I;\mathbb{R}_+) \ \ (i,l=1,\dots,n).$$

Then problem (5.1.1)*,* (5.1.2)*;* (5.1.34) *has one and only one solution and it is nonnegative.*

The theorem immediately follows from Theorem 2.4.1.

5.1.4 On a method for constructing solutions of the Cauchy–Nicoletti type multi-point boundary value problems

In this subsection, we present a method for constructing solutions of the impulsive system (5.1.1), (5.1.2) satisfying one of the conditions (5.1.34), (5.1.35), or

$$x_i(t_i)=\mu_i x_i(\zeta_i)+c_{0i} \ \ (i=1,\dots,n),$$

where $c_{oi}\in\mathbb{R}$, $\mu_i\in\mathbb{R}$ and $\zeta_i\in I$, $\zeta_i\neq\tau_i$ $(i=1,\dots,n)$.

We use the designations given in Section 2.4 and realize them for the impulsive system under consideration.

As the zero approximation to the solution of problem (5.1.1), (5.1.2); (5.1.34), we choose an arbitrary function $(x_{0i})_{i=1}^n\in\mathrm{BV}(I;\mathbb{R}^n)$. If the $(m-1)$-th approximation $(x_{m-1\,i})_{i=1}^n$ is constructed, then by the m-th approximation we take $(x_{mi})_{i=1}^n$, i-th components of which are defined by

$$x_{mi}(t_i)=\ell_i(x_{m-1\,1},\dots,x_{m-1\,n})+c_{0i} \ \ (i=1,\dots,n),$$
$$x_{mi}(t)=\gamma_i(t,t_i)x_{mi}(t_i)+\omega_i(x_{m-1\,1},\dots,x_{m-1\,n},q_i,u_i)(t) \ \text{ for } \ t\in I \ \ (i=1,\dots,n),$$

where the operators $\omega_i : \mathrm{BV}(I;\mathbb{R}^{n+2}) \to \mathrm{BV}(I;\mathbb{R})$ $(i=1,\dots,n)$ are defined as

$$\omega_i(y_1,\dots,y_{n+2})(t) = g_i(y_1,\dots,y_{n+2})(t) + \int\limits_{t_i}^{t} p_{ii}(s)\, g_i(y_1,\dots,y_{n+2})(s) \exp\bigg(\int\limits_{s}^{t} p_{ii}(\tau)\,d\tau\bigg)\,ds,$$

$$g_i(y_1,\dots,y_{n+2})(t) = \sum_{l\neq i;\, l=1}^{n} \int\limits_{t_i}^{t} y_l(s) p_{il}(s)\,ds + \int\limits_{t_i}^{t} y_{n+1}(s)\,ds$$
$$+ \sum_{\tau_k\in[a,s[} \big(g_{il}(\tau_k) + y_{n+2}(\tau_k)\big)\Big|_{t_i}^{t} \ \text{ for } \ t\in I \ \ (i=1,\dots,n).$$

As above, we give general results on the method for constructing solutions of system (5.1.1), (5.1.2) satisfying condition (5.1.34), or (5.1.35). Particular cases follow from the corresponding results of Section 2.3 and the results obtained in [18].

Theorem 5.1.6. *Let there exist matrix-functions $Q=(q_{il})_{i,l=1}^n \in L(I;\mathbb{R}^{n\times n})$ and $H=(h_{ik})_{i,k=1}^n \in B(T;\mathbb{R}^{n\times n})$ and a positive homogeneous nondecreasing bounded vector-functional $\ell_0 = (\ell_{0i})_{i=1}^n : \mathrm{BV}_\infty(I;\mathbb{R}_+^n) \to \mathbb{R}_+^n$ satisfying the condition*

$$(Q,H;\ell_0) \in \mathbb{U}(t_1,\dots,t_n;\tau_1,\tau_2,\dots)$$

such that

$$p_{ii}(t)\,\mathrm{sgn}(t-t_i) \le q_{ii}(t), \ \ |p_{ik}(t)| \le q_{ik}(t) \ \text{ for } \ t\in I \ \ (i\neq k;\ i,k=1,\dots,n),$$
$$g_{ii}(\tau_l)\,\mathrm{sgn}(\tau_l-t_i) \le h_{ii}(\tau_l), \ \ |g_{ik}(\tau_l)| \le h_{ik}(\tau_l) \ \ (i\neq k;\ i,k=1,\dots,n;\ l=1,2,\dots)$$

and

$$|\ell_i(x_1,\dots,x_n)| \le \ell_{0i}(x_1,\dots,x_n) \ \text{ for } \ x_l\in \mathrm{BV}(I;\mathbb{R}_+) \ \ (i,l=1,\dots,n)$$

Then problem (5.1.1), (5.1.2); (5.1.34) has one and only one solution and there exist $\rho_0>0$ and $\delta\in\,]0,1[$ such that

$$\sum_{i=1}^n \|x_i - x_{mi}\|_\infty \le \rho_0\delta^m \ \ (m=1,2\dots),$$

where the vector-functions $(x_{mi})_{i=1}^n$ $(m=1,2,\dots)$ are defined by (2.4.5), (2.4.6).

5.2 Periodic problem

In this section, we consider the impulsive system

$$\frac{dx}{dt} = P(t)x + q(t) \ \text{ for a.a. } \ t\in\mathbb{R}\setminus T, \tag{5.2.1}$$

$$x(\tau_l+) - x(\tau_l-) = G(\tau_l)x(\tau_l) + u(\tau_l) \ \ (l=1,2,\dots) \tag{5.2.2}$$

with the $\omega>0$-periodic condition

$$x(t+\omega) = x(t) \ \text{ for } \ t\in\mathbb{R}, \tag{5.2.3}$$

where $P=(p_{ik})_{i,k=1}^n \in L_{loc}(\mathbb{R};\mathbb{R}^{n\times n})$, $q=(q_k)_{k=1}^n \in L_{loc}(\mathbb{R};\mathbb{R}^n)$, $G=(g_{ik})_{i,k=1}^n \in B_{loc}(T;\mathbb{R}^{n\times n})$, $u=(u_k)_{k=1}^n \in B_{loc}(T;\mathbb{R}^n)$, $T=\{\tau_1,\tau_2,\dots\}$, $\tau_l\in\mathbb{R}$ $(l=1,2,\dots)$, $\tau_l\neq\tau_k$ if $l\neq k$ $(l,k=1,2,\dots)$, and ω is a fixed positive number.

As we have noted in Section 5.1, a vector-function x is a solution of the impulsive system (5.2.1), (5.2.2) if and only if it is a solution of the generalized system (1.1.1), where

$$
\begin{aligned}
A(t) &= \int_0^t P(\tau)\,d\tau + \sum_{\tau_l \in [0,t[} G(\tau_l) \ \text{ for } \ t \in \mathbb{R},\\
f(t) &= \int_0^t q(\tau)\,d\tau + \sum_{\tau_l \in [0,t[} u(\tau_l) \ \text{ for } \ t \in \mathbb{R}.
\end{aligned}
\tag{5.2.4}
$$

Since P, G and q, u are ω-periodic matrix- and vector-functions, from (5.2.4) it follows that

$$A(t+\omega) \equiv A(t) + A(\omega) \quad \text{and} \quad f(t+\omega) \equiv f(t) + f(\omega).$$

We assume that

$$\det(I_n + G(\tau_l)) \neq 0 \ \ (l = 1, 2, \dots).$$

We realize only specific results corresponding to the ω-periodic problem, i.e., the above-established results obtained for generalized differential case.

Along with system (5.2.1), (5.2.2), we consider the corresponding homogeneous system

$$\frac{dx}{dt} = P(t)x \ \text{ for a.a. } \ t \in \mathbb{R} \setminus T, \tag{5.2.1$_0$}$$

$$x(\tau_l+) - x(\tau_l-) = G(\tau_l)x(\tau_l) \ \ (l = 1, 2, \dots). \tag{5.2.2$_0$}$$

Moreover, along with condition (5.2.3), we consider the condition

$$x(0) = x(\omega).$$

Definition 5.2.1. A matrix-function $\mathcal{G}_\omega : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n\times n}$ is said to be the Green matrix of problem $(5.2.1_0), (5.2.2_0); (5.2.3)$ if:

(a)
$$\mathcal{G}_\omega(t+\omega, \tau+\omega) = \mathcal{G}_\omega(t, \tau), \quad \mathcal{G}_\omega(t, t+\omega) - \mathcal{G}_\omega(t, t) = I_n \ \text{ for } \ t, \tau \in \mathbb{R};$$

(b) the matrix-function $\mathcal{G}_\omega(\,\cdot\,, \tau) : \mathbb{R} \to \mathbb{R}^{n\times n}$ is a fundamental matrix of system $(5.2.1_0), (5.2.2_0)$ for every $\tau \in \mathbb{R}$.

To use formulae (4.1.14) it is necessary to consider the expression $\mathcal{A}(A, \mathcal{A}(-A, f))$ for the case under consideration. Using (5.2.4) and the definition of the operator $\mathcal{A}$, we find that

$$
\begin{aligned}
\mathcal{A}(-A,f)(t) - \mathcal{A}(-A,f)(s) &= f(t) - f(s) + \sum_{\tau \in [s,t[} d_2A(\tau)(I_n - d_2A(\tau))^{-1} d_2 f(\tau)\\
&= \int_s^t q(\tau)\,d\tau + \sum_{\tau_l \in [s,t[} u(\tau_l) + \sum_{\tau_l \in [s,t[} G(\tau_l)(I_n - G(\tau_l))^{-1} u(\tau_l)\\
&= \int_s^t q(\tau)\,d\tau + \sum_{\tau_l \in [s,t[} (I_n - G(\tau_l))^{-1} u(\tau_l) \ \text{ for } \ s < t
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathcal{A}(A, \mathcal{A}(-A,f))(t) - \mathcal{A}(A, \mathcal{A}(-A,f))(s)\\
&\qquad = \mathcal{A}(-A,f)(t) - \mathcal{A}(-A,f)(s) - \sum_{\tau \in [s,t[} d_2A(\tau)(I_n + d_2A(\tau))^{-1} d_2\mathcal{A}(-A,f)(\tau)
\end{aligned}
$$

$$= \int\limits_s^t q(\tau)\,d\tau + \sum_{\tau_l\in[s,t[} (I_n - G(\tau_l))^{-1}u(\tau_l) - \sum_{\tau_l\in[s,t[} G(\tau_l)(I_n + G(\tau_l))^{-1}(I_n - G(\tau_l))^{-1}u(\tau_l)$$

$$= \int\limits_s^t q(\tau)\,d\tau + \sum_{\tau_l\in[s,t[} (I_n + G(\tau_l))^{-1}(I_n - G(\tau_l))^{-1}u(\tau_l) \ \text{ for } \ s<t, \ \ s,t\in\mathbb{R}.$$

Thus, we have the following

Theorem 5.2.1. *Let the condition*

$$\det(I_n \pm G(\tau_l)) \neq 0 \ \ (l=1,2,\dots)$$

hold, and system $(5.2.1_0), (5.2.2_0)$ *have only a trivial* ω*-periodic solution. Then system* $(5.2.1), (5.2.2)$ *also has the unique* ω*-periodic solution* x *and it is written in the form*

$$x(t) = \int\limits_t^{t+\omega} \mathcal{G}_\omega(t,\tau)q(\tau)\,d\tau + \sum_{\tau_l\in[t,t+\omega[} (I_n + G(\tau_l))^{-1}(I_n - G(\tau_l))^{-1}u(\tau_l) \ \ \textit{for} \ \ t\in\mathbb{R},$$

where $\mathcal{G}_\omega$ *is the Green matrix of problem* $(5.2.1_0), (5.2.2_0)$*;* $(5.2.3)$*.*

Definition 5.2.2. Let $\sigma_i \in \{-1,1\}$ $(i=1,\dots,n)$. We say that a pair (Q,H) consisting of the matrix-functions $Q=(q_{ik})_{i,k=1}^n \in L_\omega(\mathbb{R};\mathbb{R}^{n\times n})$ and $H=(h_{ik})_{i,k=1}^n \in B_\omega(T;\mathbb{R}^{n\times n})$ belongs to the set $\mathbb{U}_\omega^{\sigma_1,\dots,\sigma_n}$ if $q_{ik}(t)\geq 0$ for $t\in[0,\omega]$ and $h_{ik}(\tau_l)\geq 0$ $(i\neq k;\ i,k=1,\dots,n;\ l=1,2,\dots)$,

$$1+\sigma_i h_{ii}(\tau_l) > 0 \ \ (i=1,\dots,n;\ l=1,2,\dots) \tag{5.2.5}$$

and the system of impulsive inequalities

$$\sigma_i x_i'(t) \leq \sum_{k=1}^n q_{ik}(t)x_k(t) \ \text{ for } \ t\in\mathbb{R} \ \ (i=1,\dots,n),$$
$$x_i(\tau_l+) - x_i(\tau_l-) \leq \sum_{k=1}^n h_{ik}(\tau_l)x_k(\tau_l) \ \ (i=1,\dots,n;\ l=1,2,\dots)$$

has no nontrivial, nonnegative ω-periodic solution.

Theorem 5.2.2. *Let the conditions*

$$\sigma_i p_{ii}(t) \leq q_{ii}(t), \ \ |p_{ik}(t)| \leq q_{ik}(t) \ \ \textit{for} \ \ t\in\mathbb{R} \ \ (i\neq l;\ i,l=1,\dots,n); \tag{5.2.6}$$
$$|g_{ii}(\tau_l)| \leq |h_{ii}(\tau_l)|, \ \ |g_{il}(\tau_l)| \leq h_{il}(\tau_l) \ \ (j=1,2;\ i\neq l;\ i,l=1,\dots,n) \tag{5.2.7}$$

hold on $[0,\omega]$*, and*

$$(Q,H)\in\mathbb{U}_\omega^{\sigma_1,\dots,\sigma_n},$$

where $Q=(q_{ik})_{i,k=1}^n$ *and* $H=(h_{ik})_{i,k=1}^n$*. Then system* $(5.2.1), (5.2.1)$ *has the unique* ω*-periodic solution.*

Corollary 5.2.1. *Let conditions* $(5.2.5)$–$(5.2.7)$ *and*

$$\sigma_i\lambda_i(\omega) < 1 \ \ (i=1,\dots,n)$$

hold, where $\sigma_i\in\{-1,1\}$ $(i=1,\dots,n)$*,* $q_{ik}(t)\geq 0$ *for* $t\in[0,\omega]$ *and* $h_{ik}(\tau_l)\geq 0$ $(i\neq k;\ i,k=1,\dots,n;$ $l=1,2,\dots)$*, and*

$$\lambda_i(t) = \exp(\sigma_i q_{ii}(t)) \prod_{0\leq\tau_l<t} (1+h_{ii}(\tau_l)) \ \ \textit{for} \ \ t\in[0,\omega] \ \ (i=1,\dots,n).$$

Let, moreover,

$$r(\mathcal{S}) < 1,$$

where the matrix $\mathcal{S} = \big(s_{il}\big)_{i,l=1}^{n}$ *is defined by*

$$s_{ii} = 0, \quad s_{il} = \sup\bigg\{ \int\limits_0^{\omega} \sigma_i g_{0i}(t,\tau) q_{ik}(\tau)\,d\tau \\ + \sum_{\tau_l \in [0,\omega]} \sigma_i (I_n + \sigma_i h_{ii}(\tau_l))^{-1} g_{0i}(t,\tau_l) h_{ik}(\tau_l) :\ t \in [0,\omega] \bigg\} \ \ (i \neq l;\ i,l = 1,\dots,n),$$

where $g_{0i}(t,\tau) \equiv (1-\lambda_i(\omega))^{-1}\lambda_i(t)\lambda_i^{-1}(\tau)\xi_i(t,\tau)$, *and* $\xi_i(t,\tau) = \lambda_i(\omega)$ *if* $t \le \tau$ *and* $\xi_i(t,\tau) = 1$ *if* $\tau < t$. *Then the conclusion of Theorem 5.2.2 is true.*

Corollary 5.2.2. *Let conditions* (5.2.5)–(5.2.7) *hold, where*

$$p_{ik}(t) = \eta_{ik}\alpha_i'(t) \ \text{ for a.a. } \ t \in \mathbb{R} \ \ (i,k = 1,\dots,n),$$
$$q_{ik}(\tau_l) = \eta_{ik} d_2\alpha_i(\tau_l) \ \ (i,k = 1,\dots,n;\ l = 1,2,\dots),$$

$\sigma_i \in \{-1,1\}$, $\eta_{il} \in \mathbb{R}_+$ $(i \neq l;\, i,l = 1,\dots,n)$, α_i $(\alpha_i(\omega) \neq 0;\, i = 1,\dots,n)$ *are nondecreasing on* $[0,\omega]$ *functions. Let, moreover,*

$$\eta_{ii} < 0 \ \ (i = 1,\dots,n)$$

and

$$r(\mathcal{H}) < 1,$$

where $\mathcal{H} = (h_{il})_{i,l=1}^{n}$,

$$h_{ii} = 0, \quad h_{il} = -\frac{\eta_{il}}{\eta_{ii}} \ \ (i \neq l;\ i,l = 1,\dots,n).$$

Then the conclusion of Theorem 5.2.2 is true.

Corollary 5.2.3. *Let conditions* (5.2.5), (5.2.6), (5.2.7) *hold, where* $\sigma_1 = \sigma_2 = \cdots = \sigma_n = \sigma_0$, $\sigma_0 \in \{-1;1\}$, $Q = (q_{ik})_{i,k=1}^{n} \in L_\omega(\mathbb{R};\mathbb{R}^{n\times n})$ *and* $H = (h_{ik})_{i,k=1}^{n} \in B_\omega(T;\mathbb{R}^{n\times n})$ *are nondecreasing on* $[0,\omega]$ *matrix-functions. Let, moreover, the module of every multiplicator of the system*

$$\frac{dy}{dt} = Q_{\sigma_0}(t)y \ \text{ for a.a. } \ t \in \mathbb{R} \setminus T,$$
$$y(\tau_l+) - y(\tau_l-) = H_{\sigma_0}(\tau_l)y(\tau_l) \ \ (l = 1,2,\dots),$$

where $Q_{\sigma_0}(t) = \sigma_0 Q(\sigma_0 t + \frac{1-\sigma_0}{2}\,\omega)$ *and* $H_{\sigma_0}(\tau_l) = \sigma_0 H(\sigma_0\tau_l + \frac{1-\sigma_0}{2}\,\omega)$, *be less than* 1. *Then the conclusion of Theorem 5.2.2 is true.*

5.3 The numerical solvability of the general linear boundary value problem

5.3.1 Statement of the problem

In this section, we construct the difference schemes for the problem

$$\frac{dx}{dt} = P(t)x + q(t) \ \text{ for a.a. } \ t \in I \setminus T, \tag{5.3.1}$$
$$x(\tau_l+) - x(\tau_l-) = G(\tau_l)x(\tau_l) + u(\tau_l) \ \ (l = 1,2,\dots); \tag{5.3.2}$$

$$\ell(x) = c_0, \tag{5.3.3}$$

where $I = [a,b]$, $P \in L(I;\mathbb{R}^{n\times n})$, $q \in L(I;\mathbb{R}^n)$, $G \in B(T;\mathbb{R}^{n\times n})$, $u \in B(T;\mathbb{R}^n)$, $T = \{\tau_1, \tau_2, \dots\}$, $\tau_l \in I$ $(l = 1, 2, \dots)$, $\tau_l \neq \tau_k$ if $l \neq k$ $(l, k = 1, 2, \dots)$, $\ell : \mathrm{BV}_\infty(I;\mathbb{R}^n) \to \mathbb{R}^n$ is a linear bounded vector-functional and $c_0 \in \mathbb{R}^n$.

Throughout this section, we will assume that the vector-function $x_0 : I \to \mathbb{R}^n$ is the unique solution of problem (5.3.1), (5.3.2); (5.3.3).

Along with the problem, we consider the difference scheme

$$\Delta y(k-1) = \frac{1}{m}\big(G_{1m}(k)\,y(k) + G_{2m}(k-1)\,y(k-1) + g_{1m}(k) + g_{2m}(k-1)\big) \ \ (k=1,\dots,m), \tag{5.3.1$_m$}$$

$$\mathcal{L}_m(y) = \gamma_m, \tag{5.3.2$_m$}$$

where $m \in \mathbb{N}$ and G_{jm} and g_{jm} $(j = 1, 2)$ are, respectively, mappings of the set $\widetilde{\mathbb{N}}_m = \{0, \dots, m\}$ into $\mathbb{R}^{n\times n}$ and $\mathbb{R}^n$, $\gamma_m \in \mathbb{R}^n$. Furthermore, for a given $m \in \mathbb{N}_m$, $\mathcal{L}_m$ is a linear bounded functional of the space of vector-functions from $\widetilde{\mathbb{N}}_m$ into $\mathbb{R}^n$ and with values in $\mathbb{R}^n$.

In this section, we will present the effective necessary and sufficient (also, the effective sufficient) conditions for the convergence of the difference scheme $(5.3.1_m), (5.3.2_m)$ to x_0. Moreover, a criterion is obtained for the stability of the difference scheme $(5.3.1_m), (5.3.2_m)$.

It should be noted that no necessary and, the more so, no necessary and sufficient conditions were found in the earlier works.

Finally, we note that just as in [17], the 2-order $n \times n$-difference linear problem can be reduced to some 1-order $2n \times 2n$-difference linear problem of type $(5.3.1_m), (5.3.2_m)$ and, therefore, we can obtain the necessary and sufficient conditions for the convergence of the corresponding 2-order difference schemes. Analogously, we can consider the 3-order difference problem, and so on.

We assume that $G_{jm} \in \mathrm{E}(\widetilde{\mathbb{N}}_m;\mathbb{R}^{n\times n})$ $(j = 1, 2)$, $g_{jm} \in \mathrm{E}(\widetilde{\mathbb{N}}_m;\mathbb{R}^n)$ and $\mathcal{L}_m : \mathrm{E}(\widetilde{\mathbb{N}}_m;\mathbb{R}^n) \to \mathbb{R}^n$ is a given linear bounded vector-functional for $m \in \mathbb{N}$ and $j \in \{1, 2\}$. In addition, suppose

$$G_{1m}(0) = G_{2m}(m) = O_{n\times n} \ \text{ and } \ g_{1m}(0) = g_{2m}(m) = 0_n \ \text{ for } \ m \in \mathbb{N}.$$

Moreover, we assume that

$$\det(I_n + G(\tau_l)) \neq 0 \ \ (l = 1, 2, \dots).$$

5.3.2 The necessary and sufficient conditions for the convergence of the difference schemes. Formulation of the results

The proofs of the results of this chapter will be given below, in Subsection 5.3.3. We assume that $I = [a,b]$.

Definition 5.3.1. We say that a sequence $(G_{1m}, G_{2m}, g_{1m}, g_{2m}; \mathcal{L}_m)$ $(m = 1, 2, \dots)$ belongs to the set $\mathcal{CS}(P, q; G, u; \ell)$ if for every $c_0 \in \mathbb{R}^n$ and the sequence $\gamma_m \in \mathbb{R}^n$ $(m = 1, 2, \dots)$ satisfying the condition

$$\lim_{m\to+\infty} \gamma_m = c_0$$

the difference problem $(5.3.1_m), (5.3.2_m)$ has a unique solution $y_m \in \mathrm{E}(\widetilde{\mathbb{N}}_m;\mathbb{R}^n)$ for any sufficiently large m and

$$\lim_{m\to+\infty} \|y_m - p_m(x_0)\|_{\widetilde{\mathbb{N}}_m} = 0.$$

Theorem 5.3.1. *Let the conditions*

$$\lim_{m\to+\infty} \mathcal{L}_m(p_m(x)) = \ell(x) \ \ \textit{for} \ \ x \in \mathrm{BV}(I;\mathbb{R}^n), \tag{5.3.4}$$

$$\limsup_{m\to+\infty} |||\mathcal{L}_m||| < +\infty \tag{5.3.5}$$

hold. Then

$$\big((G_{1m},G_{2m},g_{1m},g_{2m};\mathcal{L}_m)\big)_{m=1}^{+\infty}\in\mathcal{CS}(P,q;G,u;\ell) \tag{5.3.6}$$

if and only if there exist a matrix-faction $H\in\mathrm{BVAC}_{loc}(I,T;\mathbb{R}^{n\times n})$ *and a sequence of matrix-functions* $H_{1m},H_{2m}\in\mathrm{E}(\widetilde{\mathbb{N}}_m;\mathbb{R}^{n\times n})$ $(m\in\mathbb{N})$ *such that the conditions*

$$\limsup_{m\to+\infty}\sum_{k=1}^{m}\bigg(\Big\|H_{2m}(k)-H_{1\,m}(k)+\frac{1}{m}\,H_{1m}(k)\,G_{1m}(k)\Big\| \\ +\Big\|H_{1m}(k)-H_{2m}(k-1)+\frac{1}{m}\,H_{1m}(k)\,G_{2m}(k-1)\Big\|\bigg)<+\infty, \tag{5.3.7}$$

$$\inf\big\{|\det(H(t))|:\ t\in I\big\}>0, \tag{5.3.8}$$

$$\lim_{m\to+\infty}\max_{k\in\widetilde{\mathbb{N}}_m}\Big\{\|H_{jm}(k)-H(\tau_{km})\|\Big\}=0\ \ (j=1,2) \tag{5.3.9}$$

hold, and the conditions

$$\lim_{m\to+\infty}\frac{1}{m}\sum_{k=1}^{\nu_m(t)}H_{1m}(k)\,\big(G_{1m}(k)+G_{2m}(k-1)\big)=\int_a^t H(\tau)P(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}H(\tau_l+)G(\tau_l), \tag{5.3.10}$$

$$\lim_{m\to+\infty}\frac{1}{m}\sum_{k=1}^{\nu_m(t)}H_{1m}(k)\,\big(g_{1m}(k)+g_{2m}(k-1)\big)=\int_a^t H(\tau)q(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}H(\tau_l+)u(\tau_l) \tag{5.3.11}$$

hold uniformly on I.

Remark 5.3.1. The limit equalities (5.3.10) and (5.3.11) hold uniformly on I if and only if the conditions

$$\lim_{m\to+\infty}\max_{i\in\mathbb{N}_m}\bigg\{\bigg|\frac{1}{m}\sum_{k=1}^{i}\sum_{j=0}^{1}H_{1m}(k)\,G_{j+1\,m}(k-j)-\int_a^{\tau_{im}}H(\tau)P(\tau)\,d\tau-\sum_{\tau_l\in[a,\tau_i[}H(\tau_l+)G(\tau_l)\bigg|\bigg\}=O_{n\times n},$$

$$\lim_{m\to+\infty}\max_{i\in\mathbb{N}_m}\bigg\{\bigg|\frac{1}{m}\sum_{k=1}^{i}\sum_{j=0}^{1}H_{1m}(k)\,g_{j+1\,m}(k-j)-\int_a^{\tau_{im}}H(\tau)q(\tau)\,d\tau-\sum_{\tau_l\in[a,\tau_i[}H(\tau_l+)u(\tau_l)\bigg|\bigg\}=0_n$$

hold, respectively.

Let X be the fundamental matrix of the system

$$\frac{dx}{dt}=P(t)x\ \text{ for a.a. }\ t\in I\setminus T,$$
$$x(\tau_l+)-x(\tau_l-)=G(\tau_l)x(\tau_l)\ \ (l=1,2,\dots)$$

such that $X(a)=I_n$ and let Y_m for any $m\in\mathbb{N}$ be the fundamental matrix of the system

$$\Delta y(k-1)=\frac{1}{m}\,\big(G_{1m}(k)\,y(k)+G_{2m}(k-1)\,y(k-1)\big)\ \ (k\in\mathbb{N}_m) \tag{5.3.12}$$

such that $Y_m(0)=I_n$.

Theorem 5.3.2. *Let conditions* (5.3.4), (5.3.5) *and*

$$\det\Big(I_n+(-1)^j\,\frac{1}{m}\,G_{jm}(k)\Big)\neq 0\ \ (j=1,2;\ k\in\mathbb{N}_m;\ m\in\mathbb{N}) \tag{5.3.13}$$

hold. Then inclusion (5.3.6) *holds if and only if the conditions*

$$\lim_{m\to+\infty}\max_{k\in\widetilde{\mathbb{N}}_m}\Big\{\|Y_m^{-1}(k)-X^{-1}(\tau_{km})\|\Big\}=0$$

and

$$\lim_{m\to+\infty}\max_{i\in\mathbb{N}_m}\bigg\{\bigg|\frac{1}{m}\sum_{k=1}^{i}\sum_{j=0}^{1}Y_m^{-1}(k)\,g_{j+1\,m}(k-j)\\ -\int\limits_a^{\tau_{im}}X^{-1}(\tau)q(\tau)\,d\tau-\sum_{\tau_l\in[a,\tau_i[}X^{-1}(\tau_l+)u(\tau_l)\bigg|\bigg\}=0_n$$

hold.

Remark 5.3.2.

(a) If a pair (P,G) satisfied the Lappo–Danilevskiĭ condition at the point s and $\det(I_n+G(\tau_l))\neq 0$ for $\tau_l<s$, then, due to (1.2.54), the fundamental matrix X $(X(s)=I_n)$ of the above-given homogeneous system has form (5.1.33).

(b) By (5.3.13), we conclude

$$Y_m(k)=\prod_{i=k}^{1}\Big(I_n-\frac{1}{m}\,G_{1m}(i)\Big)^{-1}\Big(I_n+\frac{1}{m}\,G_{2m}(i-1)\Big)\ \ (k\in\mathbb{N}_m) \tag{5.3.14}$$

for every natural m.

(c) In Theorem 5.3.2, condition (5.3.8) holds automatically, since Y_m is the fundamental matrix of the homogeneous system (5.3.12) for every natural m.

Theorem 5.3.3. *Let conditions* (5.3.4), (5.3.5) *and*

$$\limsup_{m\to+\infty}\frac{1}{m}\sum_{k=1}^{m}\big(\|G_{1m}(k)\|+\|G_{2m}(k-1)\|\big)<+\infty$$

hold and let the conditions

$$\lim_{m\to+\infty}\frac{1}{m}\sum_{k=1}^{\nu_m(t)}\big(G_{1m}(k)+G_{2m}(k-1)\big)=\int\limits_a^t P(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}G(\tau_l) \tag{5.3.15}$$

and

$$\lim_{m\to+\infty}\frac{1}{m}\sum_{k=1}^{\nu_m(t)}\big(g_{1m}(k)+g_{2m}(k-1)\big)=\int\limits_a^t q(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}q(\tau_l) \tag{5.3.16}$$

hold uniformly on I. Then inclusion (5.3.6) *holds.*

Proposition 5.3.1. *Let conditions* (5.3.4), (5.3.5), (5.3.7)–(5.3.9) *and*

$$\lim_{m\to+\infty}\frac{1}{m}\max_{k\in\widetilde{\mathbb{N}}_m}\Big\{\|G_{jm}(k)\|+\|g_{jm}(k)\|\Big\}=0\ \ (j=1,2) \tag{5.3.17}$$

hold and let conditions (5.3.10) *and* (5.3.11) *hold uniformly on I, where $H\in\mathrm{AC}(I;\mathbb{R}^{n\times n})$, H_{1m}, $H_{2m}\in\mathrm{E}(\widetilde{\mathbb{N}}_m;\mathbb{R}^{n\times n})$ $(m\in\mathbb{N})$. Let, moreover, either*

$$\limsup_{m\to+\infty}\Big(\frac{1}{m}\sum_{k=0}^{m}\big(\|G_{jm}(k)\|+\|g_{jm}(k)\|\big)\Big)<+\infty\ \ (j=1,2),$$

or

$$\limsup_{m\to+\infty}\sum_{k=0}^{m}\Big(\|H_{2m}(k)-H_{1m}(k)\|+\|H_{1m}(k)-H_{2m}(k-1)\|\Big)<+\infty.$$

Then inclusion (5.3.6) *holds.*

Theorem 5.3.4. *Let conditions* (5.3.4), (5.3.5), (5.3.7)–(5.3.9) *and* (5.3.17) *hold and let conditions* (5.3.15), (5.3.16),

$$\lim_{m\to+\infty}\frac{1}{m}\sum_{k=1}^{\nu_m(t)} H_{1m}(k)\big(G_{1m}(k)+G_{2m}(k-1)\big)=\int_a^t P_*(\tau)\,d\tau+\sum_{\tau_l\in[a,t[} G_*(\tau_l)$$

and

$$\lim_{m\to+\infty}\frac{1}{m}\sum_{k=1}^{\nu_m(t)} H_{1m}(k)\big(g_{1m}(k)+g_{2m}(k-1)\big)=\int_a^t q_*(\tau)\,d\tau+\sum_{\tau_l\in[a,t[} u_*(\tau_l)$$

hold uniformly on I, *where* $P_*\in L(I;\mathbb{R}^{n\times n})$, $q_*\in L(I;\mathbb{R}^n)$, $G_*\in B(T;\mathbb{R}^{n\times n})$, $u_*\in B(T;\mathbb{R}^n)$, $H\in \mathrm{AC}(I;\mathbb{R}^{n\times n})$, H_{1m}, $H_{2m}\in \mathrm{E}(\mathbb{N}_m;\mathbb{R}^{n\times n})$ $(m\in\mathbb{N})$. *Let, moreover, the system*

$$\frac{dx}{dt}=(P(t)-P^*(t))x+q(t)-q_*(t)\ \ \textit{for a.a. } t\in I\setminus T,$$
$$x(\tau_l+)-x(\tau_l-)=\big(G(\tau_l)-G^*(\tau_l)\big)x(\tau_l)+\big(u(\tau_l)-u^*(\tau_l)\big)\ \ (l=1,2,\dots)$$

have a unique solution satisfying the boundary value condition (5.3.3). *Then*

$$\big((G_{1m},G_{2m},g_{1m},g_{2m};\mathcal{L}_m)\big)_{m=1}^{+\infty}\in\mathcal{CS}(P-P_*,q-q_*;G-G_*,u-u_*;\ell).$$

Corollary 5.3.1. *Let conditions* (5.3.4) *and* (5.3.5) *hold and let there exist a natural* μ *and matrix-functions* $B_j\in \mathrm{E}(\widetilde{\mathbb{N}}_m;\mathbb{R}^{n\times n})$, $B_j(0)=O_{n\times n}$ $(j=0,\dots,\mu-1)$ *such that*

$$\limsup_{m\to+\infty}\sum_{k=1}^{m}\bigg(\Big\|H_{2m\mu}(k)-H_{1m\mu}(k)+\frac{1}{m}\,H_{1m\mu}(k)\,G_{1m\mu}(k)\Big\|$$
$$+\Big\|H_{1m\mu}(k)-H_{2m\mu}(k-1)+\frac{1}{m}\,H_{1m\mu}(k)\,G_{2m\mu}(k-1)\Big\|\bigg)<+\infty,$$
$$\lim_{m\to+\infty}\max_{k\in\widetilde{\mathbb{N}}_m}\big\{\|H_{lm\mu}(k)-I_n\|\big\}=0\ \ (l=1,2,\dots),$$

and let the conditions

$$\lim_{m\to+\infty}\frac{1}{m}\sum_{k=1}^{\nu_m(t)}\big(G_{1mj}(k)+G_{2mj}(k-1)\big)=B_j(\nu_m(t))\ \ (j=0,\dots,\mu-1),$$
$$\lim_{m\to+\infty}\frac{1}{m}\sum_{k=1}^{\nu_m(t)}\big(G_{1m\mu}(k)+G_{2m\mu}(k-1)\big)=\int_a^t P(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}G(\tau_l),$$
$$\lim_{m\to+\infty}\frac{1}{m}\sum_{k=1}^{\nu_m(t)}\big(g_{1m\mu}(k)+g_{2m\mu}(k-1)\big)=\int_a^t q(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}u(\tau_l)$$

hold uniformly on I, *where*

$$G_{1m0}(k)\equiv G_{1m}(k),\quad G_{2m0}(k)\equiv G_{2m}(k),$$
$$G_{1m\,j+1}(k)\equiv H_{1mj}(k)G_{1m}(k),\quad G_{2m\,j+1}(k)\equiv H_{1mj}(k+1)G_{2m}(k),$$
$$g_{1m\,j+1}(k)\equiv H_{1mj}(k)g_{1m}(k),\quad g_{2m\,j+1}(k)\equiv H_{2mj}(k+1)g_{2m}(k),$$
$$H_{1m0}(k)=H_{2m0}(k)\equiv I_n,$$
$$H_{1m\,j+1}(k)\equiv\Big(\frac{1}{m}\,H_{1mj}(k)\,G_{1m}(k)+\mathcal{Q}_1(H_{1mj},G_{1m},G_{2m})(k)+B_{j+1}(k)\Big)H_{1mj}(k),$$
$$H_{2m\,j+1}(k)\equiv\big(\mathcal{Q}_2(H_{1mj},G_{1m},G_{2m})(k)+B_{j+1}(k)\big)H_{2mj}(k),$$

$$\mathcal{Q}_l(H_{1mj},G_{1m},G_{2m})(k)\equiv 2I_n-H_{nmj}(k)-\frac{1}{m}\sum_{i=1}^{k}H_{2mj}(i)\big(G_{1m}(i)+G_{2m}(i-1)\big)$$
$$(l=1,2;\ \ j=0,\dots,\mu-1;\ \ m=1,2,\dots).$$

Then inclusion (5.3.6) *holds.*

If $\mu=1$ and $B_{j0}(0)\equiv O_{n\times n}$ $(j=1,2)$, then Corollary 5.3.1 has the form of Theorem 5.3.3.
If $\mu=2$, then Corollary 5.3.1 has the following form.

Corollary 5.3.1$_1$. *Let conditions* (5.3.4), (5.3.5) *and* (5.3.7) *hold, and the conditions*

$$\lim_{m\to+\infty}\frac{1}{m}\sum_{k=1}^{\nu_m(t)}\big(G_{1m}(k)+G_{2m}(k-1)\big)=B(\nu_m(t)),$$
$$\lim_{m\to+\infty}\sum_{k=1}^{\nu_m(t)}H_m(k)\big(G_{1m}(k)+G_{2m}(k-1)\big)=\int_a^t P_0(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}G_0(\tau_l)$$

and

$$\lim_{m\to+\infty}\sum_{k=1}^{\nu_m(t)}H_m(k)\big(g_{1m}(k)+g_{2m}(k-1)\big)=\int_{t_0}^t q_0(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}u_0(\tau_l)$$

hold uniformly on I, where $B\in\mathrm{E}(\widetilde{\mathbb{N}}_m;\mathbb{R}^{n\times n})$, $B(0)=O_{n\times n}$ and

$$H_m(k)\equiv I_n-\frac{1}{m}\sum_{i=1}^{k-1}\big(G_{1m}(i)+G_{2m}(i-1)\big)+B(k)\ \ (m=1,2,\dots).$$

Then inclusion (5.3.6) *holds.*

Corollary 5.3.2. *Let conditions* (5.3.4) *and* (5.3.5) *hold. Then inclusion* (5.3.6) *holds if and only if there exist matrix-functions $Q_m\in L(I;\mathbb{R}^{n\times n})$ $(m=0,1,\dots)$ and $W_m\in B(T;\mathbb{R}^{n\times n})$ $(m=0,1,\dots)$ such that the conditions* (5.1.22) *and*

$$\limsup_{m\to+\infty}\bigg(\int_a^b\|Q_m(t)\|\,dt+\sum_{k=1}^{\infty}\Big\|\frac{1}{m}\sum_{i=0}^{k-1}\big(G_{1m}(i)+G_{2m}(i)\big)-W_m(\tau_k)\Big\|\bigg)<+\infty \tag{5.3.18}$$

hold, and the conditions (5.1.23),

$$\lim_{m\to+\infty}\frac{1}{m}\sum_{l:\,\tau_l\in[a,t[}\Big(Z_m^{-1}(\tau_l-)G_{1m}(l)+Z_m^{-1}(\tau_{l-1}+)G_{2m}(l-1)\Big)$$
$$=\int_a^t Z_0^{-1}(\tau)P_0(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}Z_0^{-1}(\tau_l+)G_0(\tau_l) \tag{5.3.19}$$

and

$$\lim_{m\to+\infty}\frac{1}{m}\sum_{l:\,\tau_l\in[a,t[}\Big(Z_m^{-1}(\tau_l-)g_{1m}(l)+Z_m^{-1}(\tau_{l-1}+)g_{2m}(l-1)\Big)$$
$$=\int_a^t Z_0^{-1}(\tau)p_0(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}Z_0^{-1}(\tau_l+)g_0(\tau_l) \tag{5.3.20}$$

hold uniformly on I, where Z_m $(Z_m(a)=I_n)$ is a fundamental matrix of the homogeneous system

$$\frac{dx}{dt}=Q_m(t) \ \text{ for a.a. } \ t\in I\setminus T, \tag{5.3.21}$$

$$x(\tau_l+)-x(\tau_l-)=W_m(\tau_l)x(\tau_l) \ \ (l=1,2,\dots) \tag{5.3.22}$$

for any sufficiently large m.

Corollary 5.3.3. *Let conditions* (5.3.4) *and* (5.3.5) *hold and let there exist sequences of matrix-functions $Q_m\in L(I;\mathbb{R}^{n\times n})$ $(m=0,1,\dots)$ and $W_m\in B(T;\mathbb{R}^{n\times n})$ $(m=0,1,\dots;\ l=1,2,\dots)$ such that the pairs (Q_m,W_m) $(m=1,2,\dots)$ satisfy the Lappo–Danilevskiĭ condition at the point a, conditions* (5.1.28), (5.3.18) *hold, and conditions* (5.1.23), (5.1.29), (5.3.19), (5.3.20) *hold uniformly on $[a,b]$, where Z_m $(Z_m(a)=I_n)$ is a fundamental matrix of the homogeneous system* (5.3.21), (5.3.22) *for any sufficiently large m. Then inclusion* (5.3.6) *holds.*

Corollary 5.3.4. *Let $G_{km}=(g_{kmij})_{i,j=1}^n\in E(\widetilde{\mathbb{N}}_m;\mathbb{R}^{n\times n})$ and $g_{km}=(g_{kmi})_{i=1}^n\in E(\widetilde{\mathbb{N}}_m;\mathbb{R}^n)$ $(k=1,2;\ m=0,1,\dots)$ and let conditions* (5.3.4), (5.3.5),

$$\limsup_{m\to+\infty}\frac{1}{m}\sum_{i,j=1;\,i\neq j}^{n}\Big(\sum_{l=1}^{\infty}\big(|g_{1mij}(\tau_l)|+|g_{2mij}(\tau_l)|\big)\Big)<+\infty$$

and

$$1+g_{0ii}(\tau_l)\neq 0 \ \ (i=1,\dots,n;\ l=1,2,\dots)$$

hold. Let, moreover, the conditions

$$\lim_{m\to+\infty}\frac{1}{m}\sum_{k=0}^{\nu_m(t)}(g_{1mii}(k)+g_{2mii}(k))=\int_a^t p_{0ii}(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}g_{0ii}(\tau_l) \ \ (i=1,\dots,n),$$

$$\lim_{m\to+\infty}\frac{1}{m}\Big(\sum_{l:\,\tau_l\in\,]a,t]}z_{mii}^{-1}(\tau_l)h_{1mij}(l)-\sum_{l:\,\tau_l\in[a,t[}z_{mii}^{-1}(\tau_l)h_{2mij}(l)\Big)$$
$$=\int_a^t z_{0ii}^{-1}(\tau)p_{0ij}(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}z_{0ii}^{-1}(\tau_l)(1+g_{0ii}(\tau_l))^{-1}g_{0ij}(\tau_l) \ \ (i\neq j;\ i,j=1,\dots,n)$$

and

$$\lim_{m\to+\infty}\frac{1}{m}\Big(\sum_{l:\,\tau_l\in\,]a,t]}z_{mii}^{-1}(\tau_l)h_{1mi}(l)-\sum_{l:\,\tau_l\in[a,t[}z_{mii}^{-1}(\tau_l)h_{2mi}(l)\Big)$$
$$=\int_a^t z_{0ii}^{-1}(\tau)q_{0i}(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}z_{0ii}^{-1}(\tau_l)(1+g_{0ii}(\tau_l))^{-1}u_{0i}(\tau_l) \ \ (i=1,\dots,n)$$

hold uniformly on I, where

$$h_{kmij}(l)\equiv\Big(1+(-1)^k\,\frac{1}{m}\,g_{kmii}(l)\Big)^{-1}g_{kmij}(l),\quad h_{kmi}(l)\equiv\Big(1+(-1)^k\,\frac{1}{m}\,g_{kmii}(l)\Big)^{-1}g_{kmi}(l)$$
$$(k=1,2;\ i,j=1,\dots,n),$$

$$z_{mii}(\tau_l)\equiv\prod_{k=0}^{l-1}(1+g_{mii}(\tau_k)) \ \ (i=1,\dots,n)$$

for any sufficiently large m. Then inclusion (5.3.6) *holds.*

Remark 5.3.3. For Corollary 5.3.4, the remark analogous to Remark 1.2.3, is true, i.e.,

$$1+g_{mii}(\tau_l)\neq 0 \ \ (i=1,\dots,n;\ l=1,2,\dots)$$

for every sufficiently large m and, therefore, all conditions of the corollary are correct.

Remark 5.3.4. In Theorems 5.3.1, 5.3.4, Proposition 5.3.1 and Corollary 5.3.1, if condition (5.3.13) holds, we may assume that $H_m(t)\equiv Y_m^{-1}(t)$, where Y_m is the fundamental matrix of the homogeneous system (5.3.12) defined by (5.3.14) for every natural m. Moreover, condition (5.3.1) and analogous conditions hold automatically everywhere in the above results, as well.

5.3.3 Auxiliary propositions and proofs of the results

The proofs of the results are based on the following concept. We rewrite problems (5.3.1), (5.3.2); (5.3.3) and $(5.3.1_m), (5.3.2_m)$ $(m\in\mathbb{N})$ as the linear boundary value problem for the systems of generalized ordinary differential equations. So, the impulsive system (5.3.1), (5.3.2), as well as the discrete systems $(5.3.1_m)$ $(m\in\mathbb{N})$ are, really, the same type equations. Therefore, the convergence of difference scheme $(5.3.1_m), (5.3.2_m)$ $(m\in\mathbb{N})$ to the solution of problem (5.3.1), (5.3.2); (5.3.3) is equivalent to the well-possed question for the boundary value problem for the last systems. So, using the results of Section 1.2, we established the present results.

As above, in Subsection 5.1.1, we rewrite the boundary value problem (5.3.1), (5.3.2); (5.3.3) as the boundary value problem

$$dx=dA_0(t)\cdot x+df_0(t),$$
$$\ell_0(x)=c_0,$$

where $A_0\in\mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_0\in\mathrm{BV}(I;\mathbb{R}^n)$, $\ell_0:\mathrm{BV}_\infty(I;\mathbb{R}^n)\to\mathbb{R}^n$ is a linear bounded vector-functional and $c_0\in\mathbb{R}^n$ is a constant vector.

Consider now the difference boundary value problem $(5.3.1_m), (5.3.2_m)$, where $m\in\mathbb{N}$.

For every natural m, we define the matrix- and vector-functions $A_m\in\mathrm{BV}(I;\mathbb{R}^{n\times n})$ and $f_m\in\mathrm{BV}(I;\mathbb{R}^n)$ and the bounded vector-functional $\ell_m:\mathrm{BV}_\infty(I;\mathbb{R}^n)\to\mathbb{R}^n$, respectively, by the equalities

$$A_m(a)=A_m(\tau_{0m})=O_{n\times n},\quad A_m(\tau_{km})=\frac{1}{m}\Big(\sum_{i=0}^{k}G_{1m}(i)+\sum_{i=1}^{k}G_{2m}(i-1)\Big),\tag{5.3.23}$$

$$A_m(t)=\frac{1}{m}\Big(\sum_{i=0}^{k-1}G_{1m}(i)+\sum_{i=1}^{k}G_{2m}(i-1)\Big)\ \text{ for }\ t\in\,]\tau_{k-1\,m},\tau_{k\,m}[\ \ (k\in\mathbb{N}_m);$$

$$f_m(a)=f(\tau_{0m})=0_n,\quad f_m(\tau_{km})=\frac{1}{m}\Big(\sum_{i=0}^{k}g_{1m}(i)+\sum_{i=1}^{k}g_{2m}(i-1)\Big),\tag{5.3.24}$$

$$f_m(t)=\frac{1}{m}\Big(\sum_{i=0}^{k-1}g_{1m}(i)+\sum_{i=1}^{k}g_{2m}(i-1)\Big)\ \text{ for }\ t\in\,]\tau_{k-1\,m},\tau_{k\,m}[\ \ (k\in\mathbb{N}_m);$$

$$\ell_m(x)=\mathcal{L}_m(p_m(x))\ \text{ for }\ x\in\mathrm{BV}(I;\mathbb{R}^n),\quad c_m=\gamma_m.\tag{5.3.25}$$

It is not difficult to verify that the defined matrix- and vector-functions have the following properties:

$$d_1A_m(\tau_{k\,m})=\frac{1}{m}\,G_{1m}(k),\quad d_2A_m(\tau_{km})=\frac{1}{m}\,G_{2m}(k)\ \ (k=1,\dots,m),\tag{5.3.26}$$
$$d_jA_m(t)=O_{n\times n}\ \text{ for }\ t\in I\setminus\big\{\tau_{1m},\dots,\tau_{km}\big\}\ \ (j=1,2);$$

$$d_1f_m(\tau_{k\,m})=\frac{1}{m}\,g_{1m}(k),\quad d_2f_m(\tau_{km})=\frac{1}{m}\,g_{2m}(k)\ \ (k=1,\dots,m),\tag{5.3.27}$$
$$d_jf_m(t)=0_n\ \text{ for }\ t\in I\setminus\{\tau_{1m},\dots,\tau_{km}\}\ \ (j=1,2)$$

for every $m\in\mathbb{N}$.

Lemma 5.3.1. *Let m be an arbitrary natural number. Then the vector-function $y\in\mathrm{E}(\widetilde{\mathbb{N}}_m;\mathbb{R}^n)$ is a solution of the difference problem $(5.3.1_m), (5.3.2_m)$ if and only if the vector-function $x=q_m(y)\in\mathrm{BV}(I;\mathbb{R}^n)$ is a solution of the generalized problem*

$$dx=dA_m(t)\cdot x+df_m(t),$$

$$\ell_m(x) = c_m,$$

where the matrix- and vector-functions $A_m \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$ *and* $f_m \in \mathrm{BV}(I;\mathbb{R}^n)$ *and the bounded vector-functional* ℓ_m *are defined by* (5.3.23)–(5.3.25)*, respectively.*

Proof. Let $y \in \mathrm{E}(\widetilde{\mathbb{N}}_m;\mathbb{R}^n)$ be a solution of the difference system $(5.3.1_m)$ $(m\in\mathbb{N})$. Then by (0.0.12), (0.0.13) and the equality $x(\tau_{km}) = q_m(y)(\tau_{km}) = y(k)$ $(k\in\widetilde{\mathbb{N}}_m)$, we get

$$\begin{aligned}
\int\limits_{\tau_{k-1\,m}}^{\tau_{km}} dA_m(\tau)x_m(\tau) + f(\tau_{km}) - f(\tau_{k-1\,m})& \\
&= \frac{1}{m}\,G_{1m}(k)x_m(\tau_{km}) + \frac{1}{m}\,G_{2m}(k-1)x_m(\tau_{k-1\,m}) + \frac{1}{m}\,g_{1m}(k) + \frac{1}{m}\,g_{2m}(k-1)\\
&= \frac{1}{m}\,G_{1m}(k)y(k) + \frac{1}{m}\,G_{2m}(k-1)y(k-1) + \frac{1}{m}\,g_{1m}(k) + \frac{1}{m}\,g_{2m}(k-1)\\
&= \Delta y(k-1) = x_m(\tau_{km}) - x_m(\tau_{k-1\,m})
\end{aligned}$$

and

$$\begin{aligned}
d_1x_m(\tau_{km}) &= x_m(\tau_{km}) - x_m(\tau_{km}-) = \frac{1}{m}\,G_{1m}(k)y(k) + \frac{1}{m}\,g_{1m}(k)\\
&= d_1A_m(\tau_{km}) + d_1f_m(\tau_{km}) \ \ (k\in\mathbb{N}_m);\\
d_2x_m(\tau_{k-1\,m}) &= x_m(\tau_{k-1\,m}+) - x_m(\tau_{k-1\,m}) = y(k) - y(k-1) - \frac{1}{m}\,G_{1m}(k)y(k) - \frac{1}{m}\,g_{1m}(k)\\
&= \frac{1}{m}\,G_{2m}(k-1)y(k-1) + \frac{1}{m}\,g_{2m}(k-1) = d_2A_m(\tau_{k-1\,m}) + d_2f_m(\tau_{k-1\,m})
\end{aligned}$$

for every $m\in\mathbb{N}$ and $k\in\mathbb{N}_m$. □

Analogously, we show that if the vector-function $x\in\mathrm{BV}(I;\mathbb{R}^n)$ is a solution of the generalized problem defined above, then the vector-function $y(k) = p_m(x)(k)$ $(k=1,\dots,m)$ will be a solution of the difference problem $(5.3.1_m),(5.3.2_m)$ for every natural m.

So, we show that the convergence of the difference scheme $(5.3.1_m),(5.3.2_m)$ $(m\in\mathbb{N})$ is equivalent to the well-posed question for the corresponding linear generalized boundary value problem given at the beginning of the subsection.

Moreover, in view of Definition 1.2.1, the following lemma is true.

Lemma 5.3.2. *Inclusion* (5.3.6) *holds if and only if the inclusion*

$$\big((A_m,f_m;\ell_m)\big)_{m=1}^{+\infty} \in \mathcal{S}(A,f;\ell)$$

holds, where the $n\times n$*-matrix-functions* A, A_m, n*-vector-functions* f, f_m *and* n*-vector-functionals* ℓ, ℓ_m $(m=1,2,\dots)$ *are defined as above by* (5.3.23)–(5.3.25)*, respectively.*

Remark 5.3.5. In view of (5.3.23) and (5.3.24), we have $A_m(t) = const$ and $f_m(t) = const$ for $t\in\,]\tau_{k-1\,m},\tau_{km}[$ $(k=1,\dots,m;\ m=1,2,\dots)$, i.e., they are the break matrix- and vector-functions. So, all the solutions of system $(5.3.1_m)$ $(m=1,2,\dots)$ have the same property. Such property have also matrix-functions H_m $(m=1,2,\dots)$ in the results of Section 1.2. So, they are also break functions. Therefore,

$$H_m(\tau_{k-1\,m}+) = H_m(\tau_{km}-) = const \ \ (k=1,\dots,m;\ m=1,2,\dots). \tag{5.3.28}$$

Below we realize some results from Chapter 2. To this end, we use the following

Lemma 5.3.3. *Let the matrix-functions* $A_m\in\mathrm{BV}(I;\mathbb{R}^{n\times n})$ $(m=1,2,\dots)$ *and the vector-functions* $f_m\in\mathrm{BV}(I;\mathbb{R}^n)$ $(m=1,2,\dots)$ *be defined by* (5.3.23) *and* (5.3.24)*, respectively, and* $Q_m\in\mathrm{BV}(I;\mathbb{R}^{n\times n})$ $(m=1,2,\dots)$*. Then there exist discrete matrix-functions* $Q_{1m},Q_{2m}\in\mathrm{E}(\widetilde{\mathbb{N}}_m;\mathbb{R}^{n\times n})$ $(m=1,2,\dots)$ *such that* $Q_{1m}(k)\equiv Q_{2m}(k-1)$ *and*

$$\mathcal{B}(Q_m,A_m)(t) \equiv \frac{1}{m}\sum_{k=1}^{\nu_m(t)} \Big(Q_{1m}(k)\,G_{1m}(k) + Q_{2m}(k-1)\,G_{2m}(k-1)\Big) \ \ (m=1,2,\dots) \tag{5.3.29}$$

and

$$\mathcal{B}(Q_m,f_m)(t)\equiv\frac{1}{m}\sum_{k=1}^{\nu_m(t)}\Big(Q_{1m}(k)\,g_{1m}(k)+Q_{2m}(k-1)\,g_{2m}(k-1)\Big)\ \ (m=1,2,\dots). \tag{5.3.30}$$

Proof. By the definition of the operator $\mathcal{B}(H,A)$, the integration-by-parts formula and equalities (0.0.12), we have

$$\begin{aligned}\mathcal{B}(Q_m,A_m)(t)&=\int_a^t Q_m(\tau)\,dA_m(\tau)-\sum_{a<\tau\le t}d_1Q_m(\tau)\,d_1A_m(\tau)+\sum_{0\le\tau<t}d_2Q(\tau)\,d_2A_m(\tau)\\&=\sum_{a<\tau_{km}\le t}Q_m(\tau_{km}-)\,d_1A_m(\tau_{km})+\sum_{a\le\tau_{km}<t}Q_m(\tau_{km}+)\,d_2A_m(\tau_{km})\\&=\sum_{k=1}^{\nu_m(t)}Q_m(\tau_{km}-)\,d_1A_m(\tau_{km})+\sum_{k=0}^{\nu_m(t)-1}Q_m(\tau_{km}+)\,d_2A_m(\tau_{km})\end{aligned}$$

$$=\sum_{k=1}^{\nu_m(t)}\Big(Q_m(\tau_{km}-)\,d_1A_m(\tau_{km})+Q_m(\tau_{k-1\,m}+)\,d_2A_m(\tau_{k-1\,m})\Big)\ \text{ for }\ t\in I\ \ (m=1,2,\dots). \tag{5.3.31}$$

Owing to (5.3.26), from (5.3.31) we get presentation (5.3.29), where $Q_{1m}(k)\equiv Q_m(\tau_{km}-)$ and $Q_{2m}(k)\equiv Q_m(\tau_{km}+)$ $(m=1,2\dots)$. Analogously, using (5.3.27), we obtain presentation (5.3.30). Due to (5.3.23) and (5.3.24), the lemma is proved. □

Proof of Theorem 5.3.1. Let us show the sufficiency.

Let the matrix-functions $H_m\in\mathrm{BV}(I;\mathbb{R}^n)$ $(m=1,2,\dots)$ be defined by the equalities

$$H_m(t)=H_{1m}(k)\ \text{ for }\ \tau_{k-1\,m}<t<\tau_{km},\ \ H_m(\tau_{km})=H_{2m}(k)\ \ (k=1,\dots,m;\ m=1,2,\dots).$$

It is evident that H_m $(m=1,2,\dots)$ are the break matrix-functions and they are constant on the intervals $]\tau_{k-1\,m},\tau_{km}[$. Hence equalities (5.3.28) hold and

$$d_1H_m(\tau_{k\,m})=H_{2m}(k)-H_{1m}(k),\ \ d_2H_m(\tau_{km})=H_{1m}(k+1)-H_{2m}(k)\ \ (k=1,\dots,m;\ m=1,2,\dots).$$

By Lemma 5.3.3, Remark 5.3.5 and equalities (5.3.28), we get

$$\mathcal{B}(H_m,A_m)(t)\equiv\frac{1}{m}\sum_{k=1}^{\nu_m(t)}H_{1m}(k)\big(G_{1m}(k)+G_{2m}(k-1)\big)\ \ (m=1,2,\dots)$$

and

$$\mathcal{B}(H_m,f_m)(t)\equiv\frac{1}{m}\sum_{k=1}^{\nu_m(t)}H_{1m}(k)\big(g_{1m}(k)+g_{2m}(k-1)\big)\ \ (m=1,2,\dots).$$

On the other hand, condition (1.2.9) is equivalent to condition (5.3.7). So, conditions (5.3.8)–(5.3.11) guarantee the fulfilment of the condition of Theorem 1.2.1. The sufficiency is proved.

Let us show the necessity. Inclusion (5.3.6) is equivalent to inclusion (1.2.8), where A_m and f_m $(m=1,2,\dots)$ are defined as above. Due to Theorem 5.3.1, there exists the sequence $H_m\in\mathrm{BV}(I;\mathbb{R}^n)$ $(m=1,2,\dots)$ satisfying the conditions given in the theorem. Let

$$H_{1m}(k)\equiv H_m(\tau_{km}-),\quad H_{2m}(k)\equiv H_m(\tau_{km})\ \ (m=1,2,\dots).$$

According to Remark 5.3.5, equality (5.3.28) holds. Using Lemma 5.3.3, we easily show that the defined discrete matrix-functions H_{1m} and H_{2m} $(m=1,2,\dots)$ satisfy the condition of Theorem 5.3.1. □

Due to the above lemmas and remark, we conclude that Theorems 5.3.2 and 5.3.3 are the particular cases of Theorems 1.2.2 and 1.2.4, respectively. Moreover, Proposition 5.3.1, Theorem 5.3.4 and Corollary 5.3.1 are the particular cases of Corollaries 1.2.2, 1.2.3 and 1.2.4, respectively, etc.

5.4 The stability of difference schemes

5.4.1 Statement of the problem and formulation of the results

In this section, we consider the question on the stability of a solutions of the difference linear boundary value problem

$$\Delta y(k-1) = G_1(k)\,y(k) + G_2(k-1)y(k-1) + g_1(k) + g_2(k-1) \ \ (k \in \mathbb{N}_{m_0}), \tag{5.4.1}$$

$$\mathcal{L}(y) \equiv \sum_{k=0}^{m_0} H(k)y(k) = \gamma_0, \tag{5.4.2}$$

where $m_0 \geq 2$ is a fixed natural number, $G_j \in \mathrm{E}(\widetilde{\mathbb{N}}_{m_0};\mathbb{R}^{n\times n})$ $(j=1,2)$, $g_j \in \mathrm{E}(\widetilde{\mathbb{N}}_{m_0};\mathbb{R}^{n})$ $(j=1,2)$, $H \in \mathrm{E}(\widetilde{\mathbb{N}}_{m_0};\mathbb{R}^{n})$, and $\gamma_0 \in \mathbb{R}^n$.

Along with problem (5.4.1), (5.4.2), consider the sequence of the problems

$$\Delta y(k-1) = G_{1m}(k)y(k) + G_{2m}(k-1)y(k-1) + g_{1m}(k) + g_{2m}(k-1) \ \ (k \in \mathbb{N}_{m_0}), \tag{5.4.1$_m$}$$

$$\mathcal{L}_m(y) \equiv \sum_{k=0}^{m_0} H_m(k)y(k) = \gamma_m \tag{5.4.2$_m$}$$

$(m \in \mathbb{N})$, where $G_{jm} \in \mathrm{E}(\mathbb{N}_{m_0};\mathbb{R}^{n\times n})$ $(j=1,2)$, $g_{jm} \in \mathrm{E}(\mathbb{N}_{m_0};\mathbb{R}^{n})$ $(j=1,2)$, $H_m \in \mathrm{E}(N_{m_0};\mathbb{R}^{n})$, and $\gamma_m \in \mathbb{R}^n$ for every natural m.

We assume that

$$G_1(0) = G_{1m}(0) = O_{n\times n}, \quad g_1(0) = g_{1m}(0) = 0_n \ \ (m \in \mathbb{N}),$$
$$G_2(m_0) = G_{2m}(m_0) = O_{n\times n}, \quad g_2(m_0) = g_{2m}(m_0) = 0_n \ \ (m \in \mathbb{N})$$

and problem (5.4.1), (5.4.2) has the unique solution $y^0 \in \mathrm{E}(\widetilde{\mathbb{N}}_{m_0};\mathbb{R}^{n})$ (the necessary and sufficient conditions are given, e.g., in [18]).

Besides, we assume that $G_{10}(k) \equiv G_1(k)$ and $g_{10}(k) \equiv g_1(k)$, if necessary.

Definition 5.4.1. We say that a sequence $(G_{1m}, G_{2m}, g_{1m}, g_{2m}; \mathcal{L}_m)$ $(m = 1,2,\dots)$ belongs to the set $\mathcal{S}(G_1, G_2, g_1, g_2; \mathcal{L})$ if for every $\gamma_0 \in \mathbb{R}^n$ and the sequence $\gamma_m \in \mathbb{R}^n$ $(m=1,2,\dots)$ satisfying the condition

$$\lim_{m\to+\infty} \gamma_m = \gamma_0$$

the difference boundary value problem (5.4.1$_m$), (5.4.2$_m$) has a unique solution $y_m \in \mathrm{E}(\widetilde{\mathbb{N}}_{m_0};\mathbb{R}^{n})$ for any sufficiently large m and

$$\lim_{m\to+\infty} \|y_m - y_0\|_{\widetilde{\mathbb{N}}_{m_0}} = 0.$$

Theorem 5.4.1. *Let*

$$\det\big(I_n + (-1)^j G_j(k)\big) \neq 0 \ \ \textit{for} \ \ k \in \widetilde{\mathbb{N}}_{m_0} \ \ (j=1,2) \tag{5.4.3}$$

and

$$\lim_{m\to+\infty} H_m(k) = H(k) \ \ \textit{for} \ \ k \in \widetilde{\mathbb{N}}_{m_0}. \tag{5.4.4}$$

Then

$$\big((G_{1m}, G_{2m}, g_{1m}, g_{2m}; \mathcal{L}_m)\big)_{m=1}^{+\infty} \in \mathcal{S}(G_1, G_2, g_1, g_2; \mathcal{L}) \tag{5.4.5}$$

if and only if

$$\lim_{m\to+\infty} \big(G_{1m}(k) + G_{2m}(k-1)\big) = G_1(k) + G_2(k-1) \ \ \textit{for} \ \ k \in N_{m_0},$$
$$\lim_{m\to+\infty} \big(g_{1m}(k) + g_{2m}(k-1)\big) = g_1(k) + g_2(k-1) \ \ \textit{for} \ \ k \in N_{m_0}.$$

Proposition 5.4.1. *Let conditions* (5.4.3), (5.4.4),

$$\lim_{m\to+\infty} G_{jm}(k) = G_j(k) \ \ for \ \ k\in\mathbb{N}_{m_0} \ \ (j=1,2),$$
$$\lim_{m\to+\infty} g_{jm}(k) = g_j(k) \ \ for \ \ k\in\mathbb{N}_{m_0} \ \ (j=1,2)$$

hold. Then inclusion (5.4.5) *holds.*

Corollary 5.4.1. *Let conditions* (5.4.3) *and* (5.4.4) *hold and there exist a natural* μ *and matrix-functions* $B_l\in\mathrm{E}(\widetilde{\mathbb{N}}_{m_0};\mathbb{R}^{n\times n})$, $B_l(0)=O_{n\times n}$ $(l=0,\dots,\mu-1)$ *such that the conditions*

$$\limsup_{m\to+\infty}\Big(\big\|H_{2m\mu}(k)-H_{1m\mu}(k)+H_{1m\mu}(k)\,G_{1m\mu}(i)\big\|$$
$$+\big\|H_{1m\mu}(k)-H_{2m\mu}(k-1)+H_{1m\mu}(k)\,G_{2m\mu}(k-1)\big\|\Big)<+\infty \ \ for \ \ k\in\mathbb{N}_{m_0},$$
$$\lim_{m\to+\infty} H_{jm\mu}(k)=I_n \ \ for \ \ k\in\widetilde{\mathbb{N}}_{m_0} \ \ (j=1,2),$$
$$\lim_{m\to+\infty}\big(G_{1ml}(k)+G_{2ml}(k-1)\big)=B_l(k) \quad for \ \ k\in\mathbb{N}_{m_0} \ \ (l=0,\dots\mu-1),$$
$$\lim_{m\to+\infty}\big(G_{1m\mu}(k)+G_{2m\mu}(k-1)\big)=G_1(k)+G_2(k-1) \ \ for \ \ k\in\mathbb{N}_{m_0}$$

and

$$\lim_{m\to+\infty}\big(g_{1m\mu}(k)+g_{2m\mu}(k-1)\big)=g_1(k)+g_2(k-1) \ \ for \ \ k\in\mathbb{N}_{m_0}$$

hold, where

$$G_{1m0}(k)\equiv G_{1m}(k), \quad G_{2m0}(k)\equiv G_{2m}(k),$$
$$G_{1m\,l+1}(k)\equiv H_{1ml}(k)G_{1m}(k), \quad G_{2m\,l+1}(k)\equiv H_{1ml}(k+1)G_{2m}(k),$$
$$g_{1m\,l+1}(k)\equiv H_{ml}(k)g_{1m}(k), \quad g_{2m\,l+1}(k)\equiv H_{ml}(k+1)g_{2m}(k),$$
$$H_{1m0}(k)=H_{2m0}(k)\equiv I_n,$$
$$H_{1m\,l+1}(k)\equiv\big(H_{1ml}(k)G_{1m}(k)+\mathcal{Q}_1(H_{1ml},G_{1m},G_{2m})(k)+B_{l+1}(k)\big)H_{1ml}(k),$$
$$H_{2m\,l+1}(k)\equiv\big(\mathcal{Q}_2(H_{1ml},G_{1m},G_{2m})(k)+B_{l+1}(k)\big)H_{2ml}(k),$$
$$\mathcal{Q}_j(H_{1ml},G_{1m},G_{2m})(k)\equiv 2I_n-H_{jml}(k)-\sum_{i=1}^{k}H_{1ml}(i)\,\big(G_{1m}(i)+G_{2m}(i-1)\big)$$
$$(j=1,2;\ l=0,\dots,\mu-1;\ m=1,2,\dots).$$

Then inclusion (5.4.5) *holds.*

If $\mu=1$ and $B_0(k)\equiv O_{n\times n}$, then Corollary 5.1.1 coincides with the sufficient part of Theorem 5.4.1.

If $\mu=2$, then Corollary 5.4.1 has the following form.

Corollary 5.4.1$_1$. *Let conditions* (5.4.3), (5.4.4),

$$\lim_{m\to+\infty}(G_{1m}(k)+G_{2m}(k-1))=B(k) \ \ for \ \ k\in\mathbb{N}_{m_0},$$
$$\lim_{m\to+\infty}\big(H_m(k)\,(G_{1m}(k)+G_{2m}(k-1))\big)=G_1(k)+G_2(k-1) \ \ for \ \ k\in\mathbb{N}_{m_0}$$

and

$$\lim_{m\to+\infty}\big(H_m(k)\,(g_{1m}(k)+g_{2m}(k-1))\big)=g_1(k)+g_2(k-1) \ \ for \ \ k\in\mathbb{N}_{m_0}$$

hold, where $B\in\mathrm{E}(\widetilde{\mathbb{N}}_{m_0};\mathbb{R}^{n\times n})$, $B(0)=O_{n\times n}$ *and*

$$H_m(k)\equiv I_n-\sum_{i=1}^{k-1}\big(G_{1m}(i)+G_{2m}(i-1)\big)+B(k) \ \ (m=1,2,\dots).$$

Then inclusion (5.4.5) *holds.*

Corollary 5.4.2. *Let conditions* (5.4.3) *and* (5.4.4) *hold. Then inclusion* (5.4.5) *holds if and only if there exist matrix-functions* $W_m \in \mathrm{E}(\widetilde{\mathbb{N}}_{m_0};\mathbb{R}^{n\times n})$, $W_m(0) = O_{n\times n}$ $(m = 0,1,\dots)$ *such that the conditions*

$$\det(I_n + W_m(k)) \neq 0 \ \ (m = 0,1,\dots),$$

$$\lim_{m\to+\infty} Z_m^{-1}(k) = Z_0^{-1}(k),$$

$$\lim_{m\to+\infty}\Big(Z_m^{-1}(k)G_{1m}(k) + Z_m^{-1}(k-1)G_{2m}(k-1)\Big) = Z_0^{-1}(k)G_1(k) + Z_0^{-1}(k-1)G_2(k-1) \quad (5.4.6)$$

and

$$\lim_{m\to+\infty}\Big(Z_m^{-1}(k)g_{1m}(k) + Z_m^{-1}(k-1)g_{2m}(k-1)\Big) = Z_0^{-1}(k)g_1(k) + Z_0^{-1}(k-1)g_2(k-1) \quad (5.4.7)$$

hold for $k \in \widetilde{\mathbb{N}}_{m_0}$, *where*

$$Z_m(k) \equiv \prod_{i=0}^{k}(I_n + W_m(k-i)) \ \ (m = 0,1,\dots).$$

Corollary 5.4.3. *Let conditions* (5.4.3) *and* (5.4.4) *hold and let there exist sequences of matrix-functions* $W_m \in \mathrm{E}(\widetilde{\mathbb{N}}_{m_0};\mathbb{R}^{n\times n})$, $W_m(0) = O_{n\times n}$ $(m = 0,1,\dots)$ *such that conditions* (5.4.6), (5.4.7),

$$\det(I_n + W_0(k)) \neq 0$$

and

$$\lim_{m\to+\infty} W_m(k) = W_0(k)$$

hold for $k \in \widetilde{\mathbb{N}}_{m_0}$, *where the matrix-functions* Z_m $(m = 0,1,\dots)$ *are defined as in Corollary 5.3.2. Then inclusion* (5.4.5) *holds.*

Corollary 5.4.4. *Let* $G_{km} = (g_{kmij})_{i,j=1}^{n} \in B(T;\mathbb{R}^{n\times n})$ *and* $g_{km} = (g_{kmi})_{i=1}^{n} \in B(T;\mathbb{R}^n)$ $(k = 1,2;$ $m = 0,1,\dots)$ *and let conditions* (5.4.3), (5.4.4),

$$\limsup_{m\to+\infty}\sum_{i,j=1;\, i\neq j}^{n}\Big(\big(|g_{1mij}(k)| + |g_{2mij}(k)|\big)\Big) < +\infty \ \ \textit{for}\ \ k \in \widetilde{\mathbb{N}}_{m_0}$$

and

$$1 + g_{0ii}(k) \neq 0 \ \ \textit{for}\ \ k \in N_{m_0} \ \ (i = 1,\dots,n)$$

hold. Let, moreover, the conditions

$$\lim_{m\to+\infty}\big(g_{1mii}(k) + g_{2mii}(k)\big) = g_{1ii}(k) + g_{2ii}(k) \ \ (i = 1,\dots,n),$$

$$\lim_{m\to+\infty}\big(z_{mii}^{-1}(k)(h_{1mij}(k) - h_{2mij}(k))\big) = z_{0ii}^{-1}(k)\big(h_{10ij}(k) - h_{20ij}(k)\big) \ \ (i \neq j;\ i,j = 1,\dots,n)$$

and

$$\lim_{m\to+\infty}\big(z_{mii}^{-1}(k)(h_{1mi}(k) - h_{2mi}(k))\big) = z_{0ii}^{-1}(k)\big(h_{10i}(k) - h_{20i}(k)\big) \ \ (i = 1,\dots,n)$$

hold for $k \in \widetilde{\mathbb{N}}_{m_0}$, *where*

$$h_{lmij}(k) \equiv \big(1 + (-1)^l g_{lmii}(k)\big)^{-1} g_{lmij}(k), \quad h_{lmi}(k) \equiv \big(1 + (-1)^l g_{lmii}(k)\big)^{-1} g_{lmi}(k)$$
$$(l = 1,2;\ i,j = 1,\dots,n),$$

and

$$z_{mii}(k) \equiv \prod_{l=0}^{k-1}(1 + g_{mii}(l)) \ \ (i = 1,\dots,n)$$

for any sufficiently large m. *Then inclusion* (5.4.5) *holds.*

Remark 5.4.1. In this section, we consider the case where the limit problem (5.4.1), (5.4.2) and the approximate problems $(5.4.1_m), (5.4.2_m)$ $(m = 1, 2, \dots)$ are given on the same (constant) sets $\widetilde{\mathbb{N}}_{m_0}$. The method considered below enables one to investigate the same problem for the general case, i.e., when the approximate problems $(5.4.1_m), (5.4.2_m)$ are given on different sets $\widetilde{\mathbb{N}}_m$ $(m = 1, 2, \dots)$ differing from $\widetilde{\mathbb{N}}_{m_0}$.

5.4.2 Proofs of the results

As in the previous section, the proofs of the results are based on the following concept. We rewrite problems (5.4.1), (5.4.2) and $(5.4.1_m), (5.4.2_m)$ $(m \in \mathbb{N})$ as the linear boundary value problem for the systems of generalized ordinary differential equations.

We assume that

$$\mathcal{L}_0(y) \equiv \mathcal{L}(y), \ \ G_{j0}(k) \equiv G_j(k) \ \text{ and } \ g_{j0}(k) \equiv g_j(k) \ \ (j = 1, 2).$$

For every $m \in \widetilde{\mathbb{N}}$, we define the matrix- and vector-functions $A_m \in \mathrm{BV}([0, m_0]; \mathbb{R}^{n\times n})$ and $f_m \in \mathrm{BV}([0, m_0]; \mathbb{R}^n)$ and the bounded vector-functional $\ell_m : \mathrm{BV}_\infty([0, m_0]; \mathbb{R}^n) \to \mathbb{R}^n$, respectively, by the equalities

$$A_m(0) = O_{n\times n}, \quad A_m(k) = \sum_{i=0}^{k} G_{1m}(i) + \sum_{i=1}^{k} G_{2m}(i-1),$$

$$A_m(t) = \sum_{i=0}^{k-1} G_{1m}(i) + \sum_{i=1}^{k} G_{2m}(i-1) \ \text{ for } \ t \in]k-1, k[\ \ (k \in \mathbb{N}_{m_0}); \tag{5.4.8}$$

$$f_m(0) = 0_n, \quad f_m(k) = \sum_{i=0}^{k} g_{1m}(i) + \sum_{i=1}^{k} g_{2m}(i-1),$$

$$f_m(t) = \sum_{i=0}^{k-1} g_{1m}(i) + \sum_{i=1}^{k} g_{2m}(i-1) \ \text{ for } \ t \in]k-1, k[\ \ (k \in \mathbb{N}_{m_0}); \tag{5.4.9}$$

$$\ell_m(x) = \mathcal{L}_m(p_m(x)) \ \text{ for } \ x \in \mathrm{BV}([0, m_0]; \mathbb{R}^n), \ \ c_m = \gamma_m. \tag{5.4.10}$$

Analogously, as in the pervious section, the following lemmas are true.

Lemma 5.4.1. *Let $m \in \widetilde{\mathbb{N}}$ be arbitrary. Then the vector-function $y \in \mathrm{E}(\widetilde{\mathbb{N}}_m; \mathbb{R}^n)$ is a solution of the difference problem $(5.4.1_m), (5.4.2_m)$ if and only if the vector-function $x = q_m(y) \in \mathrm{BV}([0, m_0]; \mathbb{R}^n)$ is a solution of the generalized problem*

$$dx = dA_m(t) \cdot x + df_m(t),$$
$$\ell_m(x) = c_m,$$

where the matrix- and vector-functions $A_m \in \mathrm{BV}([0, m_0]; \mathbb{R}^{n\times n})$ and $f_m \in \mathrm{BV}([0, m_0]; \mathbb{R}^n)$ and the vector-functional ℓ_m are defined by (5.4.8)–(5.4.10)*, respectively.*

Lemma 5.4.2. *Inclusion* (5.4.5) *holds if and only if the inclusion*

$$\big((A_m, f_m; \ell_m)\big)_{m=1}^{+\infty} \in \mathcal{S}(A, f; \ell)$$

holds, where the matrix-functions A, A_m, vector-functions f, f_m and vector-functionals ℓ, ℓ_m $(m = 1, 2, \dots)$ are defined as above by (5.4.8)–(5.4.10)*, respectively.*

So, the discrete systems (5.4.1), (5.4.1) and $(5.4.1_m), (5.4.2_m)$ $(m \in \mathbb{N})$ are the particular cases of the generalized linear boundary value problems.

Therefore, the convergence of solutions of the difference problems $(5.4.1_m), (5.4.2_m)$ $(m \in \mathbb{N})$ to the solution of problem (5.4.1), (5.4.2) is equivalent to the well-posed question for the boundary value problem for the latter systems.

Due to the lemmas, we conclude that Theorem 5.4.1, Proposition 5.4.1 and Corollary 5.4.1 are the particular cases of Theorem 1.2.1, Corollary 1.2.2 and Corollary 1.2.4, respectively, etc.

Chapter 6

The well-posedness and the numerical solvability of the general linear boundary value problems for systems of ordinary differential equations

In this chapter, we realize the results of Sections 5.1 and 5.3 for the general linear boundary value problem for the following differential systems of ordinary differential equations.

Below:

(a) in Section 6.1, we give the conditions guaranteeing the approximation of the solution of the considered problem by solutions of the nearly problems of the same type, i.e., by absolutely continuous vector-functions;

(b) in Section 6.2, we give the conditions guaranteeing the approximation of the solution of the considered problem by solutions of the nearly difference problems, i.e., by piecewise constants vector-functions;

(c) in Section 6.3, we give the conditions guaranteeing the approximation of the solution of the considered problem by solutions of the nearly impulsive problems, i.e., by piecewise continuous vector-functions.

6.1 The necessary and sufficient conditions for the well-posedness

Consider the problem

$$\frac{dx}{dt} = P_0(t)x + q_0(t) \ \text{ for a.a. } \ t \in I, \tag{6.1.1}$$

$$\ell_0(x) = c_0, \tag{6.1.2}$$

where $I = [a, b]$, $P_0 \in L(I; \mathbb{R}^{n\times n})$, $q_0 \in L(I; \mathbb{R}^n)$, $\ell_0 : C(I; \mathbb{R}^n) \to \mathbb{R}^n$ is a linear vector-functional, bounded with respect to the norm $\|\cdot\|_c$, and $c_0 \in \mathbb{R}^n$.

Note that by the Hahn–Banach theorem there exists a linear bounded vector-functional $\ell_* \in \mathrm{BV}_\infty(I; \mathbb{R}^n) \to \mathbb{R}^n$ such that

$$\ell_*(x) = \ell_0(x) \ \text{ for } \ x \in C(I, \mathbb{R}^n)$$

and the norm of ℓ_* on $\mathrm{BV}_\infty(I;\mathbb{R}^n)$ equals to the norm of ℓ_0 on $C(I;\mathbb{R}^n)$, i.e., $|||\ell_*||| = |||\ell_0|||$. So we can assume, without loss of generality, that the vector-functional ℓ_0 is given on $\mathrm{BV}_\infty(I;\mathbb{R}^n)$.

Along with the boundary initial problem (6.1.1), (6.1.2), consider the sequence of problems

$$\frac{dx}{dt} = P_m(t)x + q_m(t) \ \text{ for a.a. } \ t \in I, \tag{6.1.1$_m$}$$

$$\ell_m(x) = c_m, \tag{6.1.2$_m$}$$

where $P_m \in L(I;\mathbb{R}^{n\times n})$ $(m = 1,2,\dots)$, $q_m \in L(I;\mathbb{R}^n)$ $(m = 1,2,\dots)$, $\ell_m : C(I;\mathbb{R}^{n\times n}) \to \mathbb{R}^n$ $(m = 1,2,\dots)$ are linear bounded vector-functionals, and $c_m \in \mathbb{R}^n$ $(m = 1,2,\dots)$.

We assume that $P_m = (p_{mij})_{i,j=1}^n$ $(m = 0,1,\dots)$, $q_m = (q_{mi})_{i=1}^n$ $(m = 0,1,\dots)$.

In this section, we establish the necessary and sufficient as well as effective sufficient conditions for the boundary value problem (6.1.1$_m$), (6.1.2$_m$) to have a unique solution x_m for any sufficiently large m and

$$\lim_{m\to+\infty} \|x_m - x_0\|_c = 0. \tag{6.1.3}$$

Along with systems (6.1.1) and (6.1.1$_m$), we consider the corresponding homogeneous systems

$$\frac{dx}{dt} = P_0(t)x \ \text{ for a.a. } \ t \in I \tag{6.1.1$_0$}$$

and

$$\frac{dx}{dt} = P_m(t)x \ \text{ for a.a. } \ t \in I, \tag{6.1.1$_{m0}$}$$

$(m = 1,2,\dots)$.

Definition 6.1.1. We say that the sequence $(P_m, q_m; \ell_m)$ $(m=1,2,\dots)$ belongs to the set $\mathcal{S}(P_0, q_0; \ell_0)$ if for every $c_0 \in \mathbb{R}^n$ and a sequence $c_m \in \mathbb{R}^n$ $(m = 1,2,\dots)$ satisfying condition

$$\lim_{m\to+\infty} c_m = c_0$$

problem (6.1.1$_m$), (6.1.2$_m$) has a unique solution x_m for any sufficiently large m and condition (6.1.3) holds.

Systems (6.1.1) and (6.1.1$_m$) $(m = 1,2,\dots)$ are particular cases of impulsive systems (5.1.1), (5.1.2) and (5.1.1$_m$), (5.1.2$_m$) $(m = 1,2,\dots)$, respectively, where $G(\tau_l) = G_m(\tau_l) \equiv O_{n\times n}$, $u(\tau_l) = u_m(\tau_l) \equiv 0_n$ $(m = 1,2,\dots)$.

To realize and formulate the well-posed results of Section 5.3, we use the following forms of the operators $\mathcal{B}(X,Y)$ and $\mathcal{I}(X,Y)$ (see (0.0.3) and (0.0.4)) for the ordinary differential case, in particular, when the matrix-functions X and Y are continuous on I. Using the integration-by-parts formula (0.0.10), (0.0.12) and the definition of the Kurzweil integral, we find that

$$\mathcal{B}(X,Y)(t) \equiv \int_a^t X(\tau)Y'(\tau)\,d\tau$$

if $X \in \mathrm{BV}(I;\mathbb{R}^{n\times j})$ and $Y \in \mathrm{AC}(I;\mathbb{R}^{j\times m})$, and

$$\mathcal{I}(X,Y)(t) \equiv \int_a^t \big(X'(\tau) + X(\tau)Y'(\tau)\big)X^{-1}(\tau)\,d\tau$$

if $X, Y \in \mathrm{AC}(I;\mathbb{R}^{n\times n})$, $\det X(t) \neq 0$. In addition, if

$$Q(t) \equiv \int_s^t Y(\tau)\,d\tau,$$

where $Y \in L(I;\mathbb{R}^{n\times m})$, we set

$$\mathcal{B}_\iota(X;Y,O_{n\times n})(t) \equiv \mathcal{B}(X,Q)(t) \text{ and } \mathcal{I}_\iota(X;Y,O_{n\times n})(t) \equiv \mathcal{I}(X,Q)(t).$$

Consequently,

$$\mathcal{B}_\iota(X;Y,O_{n\times n})(t) \equiv \int_a^t X(\tau)Y(\tau)\,d\tau,$$

$$\mathcal{I}_\iota(X;Y,O_{n\times n})(t) \equiv \int_a^t \big(X'(\tau)+X(\tau)Y(\tau)\big)X^{-1}(\tau)\,d\tau.$$

Thus we obtain the following results.

Theorem 6.1.1. *Let the conditions*

$$\lim_{m\to+\infty} \ell_m(x)=\ell_0(x) \;\; \textit{for} \;\; x\in C(I;\mathbb{R}^n), \tag{6.1.4}$$

$$\limsup_{m\to+\infty} |||\ell_m||| < +\infty \tag{6.1.5}$$

hold. Then

$$\big((P_m,q_m;\ell_m)\big)_{m=1}^{\infty} \in \mathcal{S}(P_0,q_0;\ell_0) \tag{6.1.6}$$

if and only if there exists a sequence of matrix-functions $H_m \in \mathrm{AC}(I;\mathbb{R}^{n\times n})$ $(m=0,1,\dots)$ *such that the conditions*

$$\limsup_{m\to+\infty} \int_a^b \big\|H_m'(t)+H_m(t)P_m(t)\big\|\,dt < +\infty \tag{6.1.7}$$

and

$$\inf\big\{|\det(H_0(t))|:\ t\in I\big\} > 0, \tag{6.1.8}$$

hold, and the conditions

$$\lim_{m\to+\infty} H_m(t)=H_0(t), \tag{6.1.9}$$

$$\lim_{m\to+\infty} \int_a^t H_m(t)P_m(t)\,dt = \int_a^t H_0(t)P_0(t)\,dt \tag{6.1.10}$$

and

$$\lim_{m\to+\infty} \int_a^t H_m(t)q_m(t)\,dt = \int_a^t H_0(t)q_0(t)\,dt$$

hold uniformly on I*.*

Theorem 6.1.2. *Let conditions* (6.1.4) *and* (6.1.5) *hold. Then inclusion* (6.1.6) *holds if and only if the conditions*

$$\lim_{m\to+\infty} X_m^{-1}(t)=X_0^{-1}(t)$$

and

$$\lim_{m\to+\infty} \int_a^t X_m^{-1}(\tau)q_m(\tau)\,d\tau = \int_a^t X_0^{-1}(\tau)q_0(\tau)\,d\tau$$

hold uniformly on I*, where* X_m *is the fundamental matrix of the homogeneous system* $(6.1.1_{m0})$ *for every* $m\in\widetilde{\mathbb{N}}$*.*

Theorem 6.1.3. *Let $P_0^* \in L(I;\mathbb{R}^{n\times n})$, $q_0^* \in L(I;\mathbb{R}^n)$, $c_0^* \in \mathbb{R}^n$, and a $\ell_0^* : C(I;\mathbb{R}^{n\times n}) \to \mathbb{R}^n$ be a linear bounded vector-functional such that the boundary value problem*

$$\frac{dx}{dt} = P_0^*(t)x + q_0^*(t) \ \text{ for a.a. } t \in I, \tag{6.1.1*}$$

$$\ell_0^*(x) = c_0^* \tag{6.1.2*}$$

has a unique solution x_0^*. Let, moreover, there exist sequences of matrix- and vector-functions $H_m \in \mathrm{AC}(I;\mathbb{R}^{n\times n})$ $(m=1,2,\dots)$ and $h_m \in \mathrm{AC}(I;\mathbb{R}^n)$ $(m=1,2,\dots)$ such that

$$\inf\Big\{|\det(H_m(t))| : \ t \in I\Big\} > 0 \ \text{ for every sufficiently large } m,$$

the conditions

$$\lim_{m\to+\infty}(c_m + \ell_m^*(h_m)) = c_0^*, \quad \lim_{m\to+\infty}\ell_m^*(y) = \ell_0^*(y) \ \text{ for } \ y \in C(I;\mathbb{R}^n),$$

$$\limsup_{m\to+\infty}|||\ell_m^*||| < +\infty \quad \text{and} \quad \limsup_{m\to+\infty}\int_a^b \|P_m^*(t)\|\,dt < +\infty$$

hold, and the conditions

$$\lim_{m\to+\infty}\int_a^t P_m^*(\tau)\,d\tau = \int_a^t P_0^*(\tau)\,d\tau,$$

$$\lim_{m\to+\infty}\bigg(h_m(t) - h_m(a) + \int_a^t \big(H_m(\tau)q_m(\tau) - P_m^*(\tau)h_m(\tau)\big)\,d\tau\bigg) = \int_a^t q_0^*(\tau)\,d\tau$$

hold uniformly on I, where $P_m^*(t) \equiv (H_m'(t)+H_m(t)P_m(t))H_m^{-1}(t)$ $(m=1,2,\dots)$, $\ell_m^*(y) = \ell_m(H_m^{-1}y)$ $(m=1,2,\dots)$. Then problem $(6.1.1_m), (6.1.2_m)$ has the unique solution x_m for any sufficiently large m and

$$\lim_{m\to+\infty}\|H_m\,x_m + h_m - x_0^*\|_c = 0.$$

Remark 6.1.1. In Theorem 6.1.3, the vector-function $x_m^*(t) \equiv H_m(t)x_m(t) + h_m(t)$ is a solution of the problem

$$\frac{dx}{dt} = P_m^*(t)x + q_m^*(t) \ \text{ for a.a. } t \in I,$$

$$\ell_m^*(x) = c_m^*$$

for every sufficiently large m, where

$$q_m^*(t) \equiv h_m'(t) + H_m(t)\,q_m(t) - P_m^*(t)h_m(t) \ \ (m=1,2,\dots).$$

Corollary 6.1.1. *Let conditions* (6.1.4), (6.1.5), (6.1.7), (6.1.8) *and*

$$\lim_{m\to+\infty}(c_m - \varphi_m(a)) = c_0$$

hold, and conditions (6.1.9), (6.1.10) *and*

$$\lim_{m\to+\infty}\int_a^t \Big(H_m(\tau)(q_m(\tau) - \varphi_m'(\tau)) + P_m^*(\tau)\varphi_m(\tau)\Big)\,d\tau = \int_a^t H_0(\tau)q_0(\tau)\,d\tau$$

hold uniformly on I, where $H_m \in \mathrm{AC}(I;\mathbb{R}^{n\times n})$ $(m=0,1,\dots)$, $\varphi_m \in \mathrm{AC}(I;\mathbb{R}^n)$ $(m=1,2,\dots)$. Then problem $(6.1.1_m), (5.1.2_m)$ has the unique solution x_m for any sufficiently large m and

$$\lim_{m\to+\infty}\|x_m - \varphi_m - x_0\|_c = 0.$$

Now we give some effective sufficient conditions guaranteeing inclusion (6.1.6).

Theorem 6.1.4. *Let conditions* (6.1.4), (6.1.5) *and*

$$\limsup_{m\to+\infty}\int_a^b \|P_m(t)\|\,dt<+\infty$$

hold, and the conditions

$$\lim_{m\to+\infty}\int_a^t P_m(\tau)\,d\tau=\int_a^t P_0(\tau)\,d\tau \tag{6.1.11}$$

and

$$\lim_{m\to+\infty}\int_a^t q_m(\tau)\,d\tau=\int_m^t q_0(\tau)\,d\tau$$

hold uniformly on I. Then inclusion (6.1.6) *holds.*

Corollary 6.1.2. *Let conditions* (6.1.4), (6.1.5), (6.1.7) *and* (6.1.8) *hold, and conditions* (6.1.9)

$$\lim_{m\to+\infty}\int_a^t H_m(\tau)P_m(\tau)\,d\tau=\int_a^t P_*(\tau)\,d\tau$$

and

$$\lim_{m\to+\infty}\int_a^t H_m(\tau)q_m(\tau)\,d\tau=\int_a^t q_*(\tau)\,d\tau$$

hold uniformly on I, where $H_m\in \mathrm{AC}(I;\mathbb{R}^{n\times n})$ $(m=1,2,\dots)$, $P_*\in L(I;\mathbb{R}^{n\times n})$, $q_*\in L(I;\mathbb{R}^n)$. Let, moreover, the system

$$\frac{dx}{dt}=(P_0(t)-P_*(t))x+(q_0(t)-q_*(t) \ \ \text{for a.a. } t\in I$$

have a unique solution satisfying condition (6.1.2). *Then*

$$\big((P_m,q_m;\ell_m)\big)_{m=1}^{\infty}\in\mathcal{S}(P_0-P_*,q_0-q_*;\ell_0).$$

Corollary 6.1.3. *Let conditions* (6.1.4), (6.1.5) *hold and let there exist a natural number μ and matrix-functions $B_j\in\mathrm{AC}(I;\mathbb{R}^{n\times n})$ $(j=0,\dots,\mu-1)$ such that*

$$\limsup_{m\to+\infty}\int_a^b \big\|H'_{m\,\mu-1}(t)+H_{m\,\mu-1}(t)P_m(t)\big\|\,dt<+\infty,$$

and conditions

$$\lim_{m\to+\infty}\int_a^t P_m(\tau)\,d\tau=B_0(t)-B_0(a),$$

$$\lim_{m\to+\infty}\bigg(H_{m\,j-1}(t)+\int_a^t H_{m\,j-1}(\tau)P_m(\tau)\,d\tau\bigg)=I_n+B_j(t)-B_j(a)\ \ (j=1,\dots,\mu-1),$$

$$\lim_{m\to+\infty}\bigg(H_{m\,\mu-1}(t)+\int_a^t H_{m\,\mu-1}(\tau)P_m(\tau)\,d\tau\bigg)=I_n+\int_{t_0}^t P_0(\tau)\,d\tau,$$

$$\lim_{m\to+\infty}\int\limits_a^t H_{m\,\mu-1}(\tau)q_m(\tau)\,d\tau=\int\limits_a^t q_0(\tau)\,d\tau$$

hold uniformly on I, where

$$H_{m0}(t)\equiv I_n,\quad H_{mj}(t)\equiv-\bigg(H_{m\,j-1}(\tau)(t)+\int\limits_a^t H_{m\,j-1}(\tau)P_m(\tau)\,d\tau-B_j(t)+B_j(a)\bigg)H_{m\,j-1}(t)$$
$$(j=1,\dots,\mu-1;\ m=1,2,\dots).$$

Then inclusion (6.1.6) *holds.*

If $\mu=1$, then Corollary 6.1.3 coincides with Theorem 6.1.4.
If $\mu=2$, then Corollary 6.1.3 has the following form.

Corollary 6.1.3$_1$. *Let conditions* (6.1.4), (6.1.5) *and* (6.1.7) *hold, and the conditions*

$$\lim_{m\to+\infty}\int\limits_a^t P_m(\tau)\,d\tau=B(t)-B(a),$$
$$\lim_{m\to+\infty}\int\limits_a^t H_m(\tau)P_m(\tau)\,d\tau=\int\limits_a^t P_0(\tau)\,d\tau$$

and

$$\lim_{m\to+\infty}\int\limits_a^t H_m(\tau)\,q_m(\tau)\,d\tau=\int\limits_{t_0}^t q_0(\tau)\,d\tau$$

hold uniformly on I, where $B\in\mathrm{AC}(I;\mathbb{R}^{n\times n})$ and

$$H_m(t)\equiv I_n-\int\limits_a^t P_m(\tau)\,d\tau+B(t)-B(a)\ \ (m=1,2,\dots).$$

Then inclusion (6.1.6) *holds.*

In Corollary 6.1.3$_1$, if we choose

$$B(t)\equiv\int\limits_a^t P_0(\tau)\,d\tau,$$

then the corollary has the following simple form.

Corollary 6.1.3$_2$. *Let conditions* (6.1.4), (6.1.5) *and*

$$\limsup_{m\to+\infty}\int\limits_a^b \|(I_n-H_m(t))\ P_m(t)\|\,dt<+\infty$$

hold, and the conditions

$$\lim_{m\to+\infty}B_m(t)=O_{n\times n},$$
$$\lim_{m\to+\infty}\int\limits_a^t B_m'(\tau)\bigg(\int\limits_a^\tau P_m(s)\,ds\bigg)\,d\tau=O_{n\times n},$$

$$\lim_{m\to+\infty}\int_a^t (I_n - B_m(\tau))\,q_m(\tau)\,d\tau = \int_a^t q_0(\tau)\,d\tau$$

hold uniformly on I, where $B_m(t) \equiv \int_a^t (P_m(\tau)-P_0(\tau))\,d\tau$ $(m=1,2,\dots)$. Then inclusion (6.1.6) *holds.*

Remark 6.1.2. In this corollary, the last limit condition holds, in particular, if

$$\lim_{m\to+\infty}\int_a^t q_m(\tau)\,d\tau = \int_a^t q_0(\tau)\,d\tau \quad\text{and}\quad \lim_{m\to+\infty}\int_a^t B_m'(\tau)\bigg(\int_a^\tau q_m(s)\,ds\bigg)\,d\tau = 0_n$$

uniformly on I.

Corollary 6.1.4. *Let conditions* (6.1.4) *and* (6.1.5) *hold. Then inclusion* (6.1.6) *holds if and only if there exist matrix-functions $Q_m \in L(I;\mathbb{R}^{n\times n})$ $(m=0,1,\dots)$ such that*

$$\limsup_{m\to+\infty}\int_a^b \|P_m(t)-Q_m(t)\|\,dt < +\infty, \tag{6.1.12}$$

and the conditions

$$\lim_{m\to+\infty} Z_m^{-1}(t) = Z_0^{-1}(t), \tag{6.1.13}$$

$$\lim_{m\to+\infty}\int_a^t Z_m^{-1}(\tau)\,P_m(\tau)\,d\tau = \int_a^t Z_0^{-1}(\tau)\,P_0(\tau)\,d\tau \tag{6.1.14}$$

and

$$\lim_{m\to+\infty}\int_a^t Z_m^{-1}(\tau)\,q_m(\tau)\,d\tau = \int_a^t Z_0^{-1}(\tau)\,q_0(\tau)\,d\tau \tag{6.1.15}$$

hold uniformly on I, where Z_m $(Z_m(a)=I_n)$ is a fundamental matrix of the homogeneous system

$$\frac{dx}{dt} = Q_m(t) \ \text{ for a.a. } t\in I \tag{6.1.16}$$

for every $m\in\widetilde{\mathbb{N}}$.

Corollary 6.1.5. *Let conditions* (6.1.4) *and* (6.1.5) *hold and let there exist the sequence of matrix-functions $Q_m \in L(I;\mathbb{R}^{n\times n})$ $(m=0,1,\dots)$ such that Q_m $(m=1,2,\dots)$ satisfies the Lappo–Danilevskiĭ condition at the point a, condition* (6.1.12) *holds, and conditions* (6.1.14), (6.1.15) *and*

$$\lim_{m\to+\infty}\int_a^t Q_m(\tau)\,d\tau = \int_a^t Q_0(\tau)\,d\tau$$

hold uniformly on I, where Z_m $(Z_m(a)=I_n)$ is a fundamental matrix of the homogeneous system (6.1.16) *for every $m\in\widetilde{\mathbb{N}}$. Then inclusion* (6.1.6) *holds.*

Corollary 6.1.6. *Let conditions* (6.1.4), (6.1.5) *and* (6.1.11) *hold and let the matrix-functions P_m $(m=0,1,\dots)$ satisfy the Lappo–Danilevskiĭ condition at the point a and conditions* (6.1.11),

$$\lim_{m\to+\infty}\int_a^t \exp\bigg(-\int_a^\tau P_m(s)\,ds\bigg)P_m(\tau)\,d\tau = \int_a^t \exp\bigg(-\int_a^\tau P_0(s)\,ds\bigg)P_0(\tau)\,d\tau$$

and

$$\lim_{m\to+\infty}\int\limits_a^t \exp\bigg(-\int\limits_a^\tau P_m(s)\,ds\bigg)q_m(\tau)\,d\tau=\int\limits_a^t \exp\bigg(-\int\limits_a^\tau P_0(s)\,ds\bigg)q_0(\tau)\,d\tau$$

hold uniformly on I. Then inclusion (6.1.6) *holds.*

Corollary 6.1.7. *Let $P_m=(p_{mij})_{i,j=1}^n\in L(I;\mathbb{R}^{n\times n})$ $(m=0,1,\dots)$, $q_m=(q_{mi})_{i=1}^n\in L(I;\mathbb{R}^n)$ $(m=0,1,\dots)$, and let conditions* (6.1.4), (6.1.5) *and*

$$\limsup_{m\to+\infty}\sum_{i,j=1;\,i\neq j}^n\bigg(\int\limits_a^b |p_{mij}(t)|\,dt\bigg)<+\infty$$

hold. Let, moreover, the conditions

$$\lim_{m\to+\infty}\int\limits_a^t p_{mii}(\tau)\,d\tau=\int\limits_a^t p_{0ii}(\tau)\,d\tau\ \ (i=1,\dots,n),$$

$$\lim_{m\to+\infty}\int\limits_a^t \exp\bigg(-\int\limits_a^\tau p_{mii}(s)\,ds\bigg)p_{mij}(\tau)\,d\tau=\int\limits_a^t \exp\bigg(-\int\limits_a^\tau p_{0ii}(s)\,ds\bigg)p_{0ij}(\tau)\,d\tau$$
$$(i\neq j;\ i,j=1,\dots,n)$$

and

$$\lim_{m\to+\infty}\int\limits_a^t \exp\bigg(-\int\limits_a^\tau p_{mii}(s)\,ds\bigg)q_{mi}(\tau)\,d\tau=\int\limits_a^t \exp\bigg(-\int\limits_a^\tau p_{0ii}(s)\,ds\bigg)q_{0i}(\tau)\,d\tau\ \ (i=1,\dots,n)$$

hold uniformly on I. Then inclusion (6.1.6) *holds.*

Remark 6.1.3. In Theorem 6.1.1 and Corollary 6.1.1, we can assume, without loss of generality, that $H_0(t)\equiv I_n$.

Corollary $6.1.3_2$ follows from Corollary $1.2.4_2$. The other results immediately follow from the corresponding results concerning the impulsive problems.

6.2 The necessary and sufficient conditions for the convergence of difference schemes

In this section, we construct the difference schemes for our problem.

Throughout the section, we assume that the vector-function $x_0: I\to\mathbb{R}^n$ is a unique solution of the problem

$$\frac{dx}{dt}=P(t)x+q(t)\ \text{ for a.a. } t\in I, \tag{6.2.1}$$

$$\ell(x)=c_0, \tag{6.2.2}$$

where $I=[a,b]$, $P\in L(I;\mathbb{R}^{n\times n})$, $q\in L(I;\mathbb{R}^n)$, $\ell: C(I;\mathbb{R}^n)\to\mathbb{R}^n$ is a linear bounded vector-functional, and $c_0\in\mathbb{R}^n$.

Along with the problem, we consider the difference scheme

$$\Delta y(k-1)=\frac{1}{m}\Big(G_{1m}(k)y(k)+G_{2m}(k-1)y(k-1)+g_{1m}(k)+g_{2m}(k-1)\Big)\ \ (k=1,\dots,m), \tag{$6.2.1_m$}$$

$$\mathcal{L}_m(y) = \gamma_m, \tag{6.2.2$_m$}$$

where $m \in \mathbb{N}$ and G_{jm} and g_{jm} $(j = 1,2)$ are, respectively, the mappings of the set $\widetilde{\mathbb{N}}_m = \{1,\dots,m\}$ into $\mathbb{R}^{n\times n}$ and $\mathbb{R}^n$, $\gamma_m \in \mathbb{R}^n$. Furthermore, for a given $m \in \mathbb{N}_m$, $\mathcal{L}_m$ is a linear bounded mapping of the space of vector-functions from $\widetilde{\mathbb{N}}_m$ into $\mathbb{R}^n$ with values in $\mathbb{R}^n$.

In this section, we present the effective necessary and sufficient (moreover, the effective sufficient) conditions for the convergence of the difference schemes $(6.2.1_m), (6.2.2_m)$ to x_0.

The problem analogous to the one under consideration, for the initial problem is investigated in [23].

We assume that $G_{jm} \in \mathrm{E}(\widetilde{\mathbb{N}}_m;\mathbb{R}^{n\times n})$ $(j = 1,2)$, $g_{jm} \in \mathrm{E}(\widetilde{\mathbb{N}}_m;\mathbb{R}^n)$ and $\mathcal{L}_m : \mathrm{E}(\widetilde{\mathbb{N}}_m;\mathbb{R}^n) \to \mathbb{R}^n$ is a given linear bounded vector-functional for $m \in \mathbb{N}$ and $j \in \{1,2\}$; in addition, assume

$$G_{1m}(0) = G_{2m}(m) = O_{n\times n}, \quad g_{1m}(0) = g_{2m}(m) = 0_n \ \text{ for } \ m \in \mathbb{N}.$$

Definition 6.2.1. We say that a sequence $(G_{1m}, G_{2m}, g_{1m}, g_{2m}; \mathcal{L}_m)$ $(m = 1,2,\dots)$ belongs to the set $\mathcal{CS}(P,q;\ell)$ if for every $c_0 \in \mathbb{R}^n$ and the sequence $\gamma_m \in \mathbb{R}^n$ $(m = 1,2,\dots)$ satisfying the condition

$$\lim_{m\to+\infty} \gamma_m = c_0$$

the difference problem $(6.2.1_m), (6.2.2_m)$ has a unique solution $y_m \in \mathrm{E}(\widetilde{\mathbb{N}}_m;\mathbb{R}^n)$ for any sufficiently large m and

$$\lim_{m\to+\infty} \|y_m - p_m(x_0)\|_{\widetilde{\mathbb{N}}_m} = 0.$$

Theorem 6.2.1. *Let the conditions*

$$\lim_{m\to+\infty} \mathcal{L}_m(p_m(x)) = \ell(x) \ \textit{ for } \ x \in \mathrm{BV}(I;\mathbb{R}^n), \tag{6.2.3}$$

$$\limsup_{m\to+\infty} |||\mathcal{L}_m||| < +\infty \tag{6.2.4}$$

hold. Then

$$\big((G_{1m}, G_{2m}, g_{1m}, g_{2m}; \mathcal{L}_m)\big)_{m=1}^{+\infty} \in \mathcal{CS}(P,q;\ell) \tag{6.2.5}$$

if and only if there exist a matrix-faction $H \in \mathrm{AC}(I;\mathbb{R}^{n\times n})$ and a sequence of matrix-functions $H_{1m}, H_{2m} \in \mathrm{E}(\widetilde{\mathbb{N}}_m;\mathbb{R}^{n\times n})$ $(m \in \mathbb{N})$ such that the conditions

$$\begin{aligned}\limsup_{m\to+\infty} \sum_{k=1}^{m} \bigg(\Big\| H_{2m}(k) - H_{1\,m}(k) + \frac{1}{m}\, H_{1m}(k)\, G_{1m}(k) \Big\| \\ + \Big\| H_{1m}(k) - H_{2m}(k-1) + \frac{1}{m}\, H_{1m}(k)\, G_{2m}(k-1) \Big\| \bigg) < +\infty,\end{aligned} \tag{6.2.6}$$

$$\inf\big\{ |\det(H(t))| : \ t \in I \big\} > 0, \tag{6.2.7}$$

$$\lim_{m\to+\infty} \max_{k\in\widetilde{\mathbb{N}}_m} \Big\{ \|H_{jm}(k) - H(\tau_{km})\| \Big\} = 0 \ \ (j = 1,2) \tag{6.2.8}$$

hold, and the conditions

$$\lim_{m\to+\infty} \frac{1}{m} \sum_{k=1}^{\nu_m(t)} H_{1m}(k) \big(G_{1m}(k) + G_{2m}(k-1)\big) = \int_a^t H(\tau)P(\tau)\,d\tau, \tag{6.2.9}$$

$$\lim_{m\to+\infty} \frac{1}{m} \sum_{k=1}^{\nu_m(t)} H_{1m}(k) \big(g_{1m}(k) + g_{2m}(k-1)\big) = \int_a^t H(\tau)q(\tau)\,d\tau \tag{6.2.10}$$

hold uniformly on I.

Remark 6.2.1. The limit equalities (6.2.9) and (6.2.10) are fulfilled uniformly on I if and only if the conditions

$$\lim_{m\to+\infty}\max_{i\in\mathbb{N}_m}\bigg\{\bigg|\frac{1}{m}\sum_{k=1}^{i}\sum_{j=0}^{1}H_{1m}(k)\,G_{j+1\,m}(k-j)-\int\limits_a^{\tau_{im}}H(\tau)P(\tau)\,d\tau\bigg|\bigg\}=O_{n\times n},$$

$$\lim_{m\to+\infty}\max_{i\in\mathbb{N}_m}\bigg\{\bigg|\frac{1}{m}\sum_{k=1}^{i}\sum_{j=0}^{1}H_{1m}(k)\,g_{j+1\,m}(k-j)-\int\limits_a^{\tau_{im}}H(\tau)q(\tau)\,d\tau\bigg|\bigg\}=0_n$$

hold, respectively.

Let X be the fundamental matrix of the system

$$\frac{dx}{dt}=P(t)x$$

such that $X(a)=I_n$ and, for any $m\in\mathbb{N}$, let Y_m be the fundamental matrix of the system

$$\Delta y(k-1)=\frac{1}{m}\Big(G_{1m}(k)\,y(k)+G_{2m}(k-1)\,y(k-1)\Big)\ \ (k\in\mathbb{N}_m) \tag{6.2.11}$$

such that $Y_m(0)=I_n$.

Theorem 6.2.2. *Let conditions* (6.2.3), (6.2.4) *and*

$$\det\Big(I_n+(-1)^j\,\frac{1}{m}\,G_{jm}(k)\Big)\neq 0\ \ (j=1,2;\ k\in\mathbb{N}_m;\ m\in\mathbb{N}) \tag{6.2.12}$$

hold. Then inclusion (6.2.5) *holds if and only if the conditions*

$$\lim_{m\to+\infty}\max_{k\in\widetilde{\mathbb{N}}_m}\Big\{\big\|Y_m^{-1}(k)-X^{-1}(\tau_{km})\big\|\Big\}=0$$

and

$$\lim_{m\to+\infty}\max_{i\in\mathbb{N}_m}\bigg\{\bigg|\frac{1}{m}\sum_{k=1}^{i}\sum_{j=0}^{1}Y_m^{-1}(k)\,g_{j+1\,m}(k-j)-\int\limits_a^{\tau_{im}}X^{-1}(\tau)q(\tau)\,d\tau\bigg|\bigg\}=0_n \tag{6.2.13}$$

hold.

Remark 6.2.2.

(a) If P satisfies the Lappo–Danilevskiĭ condition at the point s, then the fundamental matrix X ($X(s)=I_n$) of the given above homogeneous system has the form

$$X(t)=\exp\bigg(\int\limits_s^t P(\tau)\,d\tau\bigg).$$

(b) By (6.2.12), we conclude that

$$Y_m(k)=\prod_{i=k}^{1}\Big(I_n-\frac{1}{m}\,G_{1m}(i)\Big)^{-1}\Big(I_n+\frac{1}{m}\,G_{2m}(i-1)\Big)\ \ (k\in\mathbb{N}_m) \tag{6.2.14}$$

for every natural m.

(c) In Theorem 6.2.2, condition (6.2.6) holds automatically, since Y_m is the fundamental matrix of the homogeneous system (6.2.11) for every natural m.

Now we present a method of constructing discrete real matrix- and vector-functions, respectively, G_{jm} and g_{jm} ($j = 1, 2;\ m \in \mathbb{N}$) for which the conditions of Theorem 6.2.2 hold.

Toward this end, we use the inductive method. Let $\mathcal{E}_m : \widetilde{\mathbb{N}}_m \to \mathbb{R}^{n\times n}$ and $\xi_m : \widetilde{\mathbb{N}}_m \to \mathbb{R}^n$ ($m \in \mathbb{N}$) be discrete matrix-and vector-functions, respectively, such that

$$\lim_{m\to+\infty} \|\mathcal{E}_m\|_{\widetilde{\mathbb{N}}_m} = 0 \ \text{ and } \ \lim_{m\to+\infty} m\|\xi_m\|_{\widetilde{\mathbb{N}}_m} = 0.$$

Let

$$P_{lm} = X(\tau_{lm}) + \mathcal{E}_m(l) \ \text{ for } \ l \in \widetilde{\mathbb{N}}_m \ \text{ and } \ m \in \mathbb{N}.$$

Let m be an arbitrary natural number and let $G_{1m}(1)$ and $G_{2m}(0)$ be such that

$$Y_m(1) = P_{1m}.$$

According to (6.2.14), we get

$$\Big(I_n - \frac{1}{m}\, G_{1m}(1)\Big)^{-1}\Big(I_n + \frac{1}{m}\, G_{2m}(0)\Big) = P_{1m}.$$

Therefore, $G_{1m}(1)$ and $G_{2m}(0)$ will be arbitrary matrices such that

$$G_{1m}(1) = m\big(I_n - P_{1m}^{-1}\big) - G_{2m}(0)\, P_{1m}^{-1}.$$

Thus the matrices $G_{1m}(k)$, $G_{2m}(k-1)$ and $Y_m(k)$ ($k = 1, \dots, l-1$) are constructed. To construct $G_{1m}(l)$ and $G_{2m}(l-1)$, we use the equalities

$$Y_m(l) = P_{lm}$$

and

$$Y_m(l) = \Big(I_n - \frac{1}{m}\, G_{1m}(l)\Big)^{-1}\Big(I_n + \frac{1}{m}\, G_{2m}(l-1)\Big)Y_m(l-1).$$

As above, we obtain the relation

$$G_{1m}(l) = m\big(I_n - P_{l-1\,m} P_{lm}^{-1}\big) - G_{2m}(l-1)\, P_{l-1\,m}\, P_{lm}^{-1}.$$

So, $G_{1m}(l)$ and $G_{2m}(l-1)$ will be arbitrary matrices satisfying the latter equality.

Construct the discrete vector-functions g_{1m} and g_{2m} ($m \in \mathbb{N}$). As $g_{1m}(l)$ and $g_{2m}(l-1)$ we choose arbitrary vectors satisfying the equalities

$$\frac{1}{m}\, Y_m^{-1}(l)\big(g_{1m}(1) + g_{2m}(l-1)\big) = q_{lm} \ \ (l \in \mathbb{N}_m),$$

where

$$q_{lm} = \xi_m(l) + \int\limits_a^{\tau_{lm}} X^{-1}(\tau) q(\tau)\, d\tau \ \ (l \in \mathbb{N}_m)$$

for every natural m. Therefore, we have

$$g_{1m}(l) + g_{2m}(l-1) = mY_m(l) q_{lm} \ \ (l \in \mathbb{N}_m,\ m \in \mathbb{N})$$

for the definition of the vector-functions g_{1m} and g_{2m} ($m \in \mathbb{N}$).

It is evident that the constructed vector-functions satisfy condition (6.2.13).

Realization of above-constructed discrete matrix- and vector-functions are illustrated by the following

Example 6.2.1. Let $X(t) \equiv \exp\big(\int\limits_a^t P(\tau)\, d\tau\big)$ be the fundamental matrix of the homogeneous system corresponding to system (6.2.1) and let $\mathcal{E}_m \equiv O_{n\times n}$ and $\xi_m \equiv 0_n$ for $m \in \mathbb{N}$. Then

$$P_{lm} = \exp\Bigg(\int\limits_a^{\tau_{lm}} P(\tau)\, d\tau\Bigg) \ \text{ for } \ l \in \widetilde{\mathbb{N}}_m \ \text{ and } \ m \in \mathbb{N}.$$

If we choose

$$G_{2m}(l-1)=P_{lm}P_{l-1\,m}^{-1}=\exp\bigg(\int\limits_{\tau_{l-1\,m}}^{\tau_{lm}}P(\tau)\,d\tau\bigg)\ \text{ for }\ l\in\mathbb{N}_m,\ \ m\in\mathbb{N},$$

then

$$G_{1m}(l)=(m-1)I_n-m\exp\bigg(-\int\limits_{\tau_{l-1\,m}}^{\tau_{lm}}P(\tau)\,d\tau\bigg)\ \text{ for }\ l\in\mathbb{N}_m,\ \ m\in\mathbb{N}.$$

For the definition of the discrete vector-functions g_{1m} and g_{2m}, we have the relations

$$g_{1m}(l)+g_{2m}(l-1)=m\int\limits_a^{\tau_{lm}}U(\tau_{lm},\tau)q(\tau)\,d\tau\ \text{ for }\ l\in\mathbb{N}_m,\ \ m\in\mathbb{N},$$

where $U(t,\tau)$ is the Cauchy matrix of system (6.2.1).

In particular, we can take

$$g_{1m}(l)=\alpha\,m\int\limits_a^{\tau_{lm}}U(\tau_{lm},\tau)q(\tau)\,d\tau,\quad g_{2m}(l-1)=(1-\alpha)\,m\int\limits_a^{\tau_{lm}}U(\tau_{lm},\tau)q(\tau)\,d\tau\ \ l\in\mathbb{N}_m,\ \ m\in\mathbb{N},$$

where α is some number.

Moreover, we can choose these discrete vector-functions in connection with the Cauchy formula for system (6.2.1).

Theorem 6.2.3. *Let conditions* (6.2.3), (6.2.4) *and*

$$\limsup_{m\to+\infty}\sum_{k=1}^{m}\Big(\frac{1}{m}\,\big(\|G_{1m}(k)\|+\|G_{2m}(k-1)\|\big)\Big)<+\infty$$

hold, and let the conditions

$$\lim_{m\to+\infty}\frac{1}{m}\sum_{k=1}^{\nu_m(t)}\big(G_{1m}(k)+G_{2m}(k-1)\big)=\int\limits_a^t P(\tau)\,d\tau \tag{6.2.15}$$

and

$$\lim_{m\to+\infty}\frac{1}{m}\sum_{k=1}^{\nu_m(t)}\big(g_{1m}(k)+g_{2m}(k-1)\big)=\int\limits_a^t q(\tau)\,d\tau \tag{6.2.16}$$

hold uniformly on I. Then inclusion (6.2.5) *holds.*

Proposition 6.2.1. *Let conditions* (6.2.3), (6.2.4), (6.2.6)–(6.2.8) *and*

$$\lim_{m\to+\infty}\frac{1}{m}\max_{k\in\widetilde{\mathbb{N}}_m}\Big\{\|G_{jm}(k)\|+\|g_{jm}(k)\|\Big\}=0\ \ (j=1,2) \tag{6.2.17}$$

hold and let conditions (6.2.9) *and* (6.2.10) *hold uniformly on I, where $H\in\mathrm{AC}(I;\mathbb{R}^{n\times n})$, $H_{1m},H_{2m}\in\mathrm{E}(\widetilde{\mathbb{N}}_m;\mathbb{R}^{n\times n})$ $(m\in\mathbb{N})$. Let, moreover, either*

$$\limsup_{m\to+\infty}\Big(\frac{1}{m}\sum_{k=0}^{m}\big(\|G_{jm}(k)\|+\|g_{jm}(k)\|\big)\Big)<+\infty\ \ (j=1,2),$$

or

$$\limsup_{m\to+\infty}\sum_{k=0}^{m}\Big(\|H_{2m}(k)-H_{1m}(k)\|+\|H_{1m}(k)-H_{2m}(k-1)\|\Big)<+\infty.$$

Then inclusion (6.2.5) *holds.*

Theorem 6.2.4. *Let conditions* (6.2.3), (6.2.4), (6.2.6)–(6.2.8) *and* (6.2.17) *hold and let conditions* (6.2.15), (6.2.16),

$$\lim_{m\to+\infty}\frac{1}{m}\sum_{k=1}^{\nu_m(t)}H_{1m}(k)\big(G_{1m}(k)+G_{2m}(k-1)\big)=\int_a^t P_*(\tau)\,d\tau$$

and

$$\lim_{m\to+\infty}\frac{1}{m}\sum_{k=1}^{\nu_m(t)}H_{1m}(k)\big(g_{1m}(k)+g_{2m}(k-1)\big)=\int_a^t q_*(\tau)\,d\tau$$

hold uniformly on I, where $P_*\in L(I;\mathbb{R}^{n\times n})$, $q_*\in L(I;\mathbb{R}^n)$, $H\in \mathrm{AC}(I;\mathbb{R}^{n\times n})$, $H_{1m},H_{2m}\in \mathrm{E}(\mathbb{N}_m;\mathbb{R}^{n\times n})$ $(m\in\mathbb{N})$. Let, moreover, the system

$$\frac{dx}{dt}=(P(t)-P^*(t))x+q(t)-q_*(t)\ \ \text{for a.a. } t\in I$$

have a unique solution satisfying the boundary value condition (6.2.2)*. Then*

$$\big((G_{1m},G_{2m},g_{1m},g_{2m};\mathcal{L}_m)\big)_{m=1}^{+\infty}\in\mathcal{CS}(P-P_*,q-q_*;\ell).$$

Corollary 6.2.1. *Let conditions* (6.2.3) *and* (6.2.4) *hold and let there exist a natural μ and matrix-functions $B_{jl}\in \mathrm{E}(\widetilde{\mathbb{N}}_m;\mathbb{R}^{n\times n})$, $B_{jl}(a)=O_{n\times n}$ $(j=1,2;\ l=0,\dots,\mu-1)$ such that*

$$\limsup_{m\to+\infty}\sum_{k=1}^{m}\bigg(\Big\|H_{2m\mu}(k)-H_{1m\mu}(k)+\frac{1}{m}\,H_{1m\mu}(k)\,G_{1m\mu}(k)\Big\|$$
$$+\Big\|H_{1m\mu}(k)-H_{2m\mu}(k-1)+\frac{1}{m}\,H_{1m\mu}(k)\,G_{2m\mu}(k-1)\Big\|\bigg)<+\infty,$$
$$\lim_{m\to+\infty}\max_{k\in\widetilde{\mathbb{N}}_m}\big\{\|H_{jm\mu}(k)-I_n\|\big\}=0\ \ (j=1,2),$$

and let the conditions

$$\lim_{m\to+\infty}\frac{1}{m}\sum_{k=1}^{\nu_m(t)}\big(G_{1m\mu}(k)+G_{2m\mu}(k-1)\big)=\int_a^t P(\tau)\,d\tau,$$
$$\lim_{m\to+\infty}\frac{1}{m}\sum_{k=1}^{\nu_m(t)}\big(g_{1m\mu}(k)+g_{2m\mu}(k-1)\big)=\int_a^t q(\tau)\,d\tau$$

hold uniformly on I, where

$$G_{1m0}(k)\equiv G_{1m}(k),\quad G_{2m0}(k)\equiv G_{2m}(k),$$
$$G_{1m\,l+1}(k)\equiv H_{1ml}(k)G_{1m}(k),\quad G_{2m\,l+1}(k)\equiv H_{1ml}(k+1)G_{2m}(k),$$
$$g_{1m\,l+1}(k)\equiv H_{ml}(k)g_{1m}(k),\quad g_{2m\,l+1}(k)\equiv H_{ml}(k+1)g_{2m}(k),$$
$$H_{1m0}(k)=H_{2m0}(k)\equiv I_n,$$

$$H_{1m\,l+1}(k) \equiv \left(\frac{1}{m}\,H_{1ml}(k)\,G_{1m}(k) + \mathcal{Q}_1(H_{1ml},G_{1m},G_{2m})(k) + B_{1\,l+1}(k)\right)H_{1ml}(k),$$
$$H_{2m\,l+1}(k) \equiv \big(\mathcal{Q}_2(H_{1ml},G_{1m},G_{2m})(k) + B_{2\,l+1}(k)\big)H_{2ml}(k),$$
$$\mathcal{Q}_j(H_{1ml},G_{1m},G_{2m})(k) \equiv 2I_n - H_{jml}(k) - \frac{1}{m}\sum_{i=1}^{k} H_{1ml}(i)\,\big(G_{1m}(i) + G_{2m}(i-1)\big)$$
$$(j=1,2;\ l=0,\dots,\mu-1;\ m=1,2,\dots).$$

Then inclusion (6.2.5) *holds.*

If $\mu = 1$ and $B_{j0}(t) \equiv O_{n\times n}$ $(j=1,2)$, then Corollary 1.2.1 has the form of Theorem 6.2.3.
If $\mu = 2$, then Corollary 6.2.1 has the following form.

Corollary 6.2.1$_1$. *Let conditions* (6.2.3), (6.2.4) *and* (6.2.6) *hold, and the conditions*

$$\lim_{m\to+\infty}\frac{1}{m}\sum_{k=1}^{\nu_m(t)}\big(G_{1m}(k)+G_{2m}(k-1)\big) = B(\nu_m(t)),$$
$$\lim_{m\to+\infty}\sum_{k=1}^{\nu_m(t)} H_m(k)\big(G_{1m}(k)+G_{2m}(k-1)\big) = \int_a^t P_0(\tau)\,d\tau$$

and

$$\lim_{m\to+\infty}\sum_{k=1}^{\nu_m(t)} H_m(k)\big(g_{1m}(k)+g_{2m}(k-1)\big) = \int_{t_0}^t q_0(\tau)\,d\tau$$

hold uniformly on I, where $B \in \mathrm{E}(\widetilde{\mathbb{N}}_m;\mathbb{R}^{n\times n})$, $B(0)=O_{n\times n}$ and

$$H_m(k) \equiv I_n - \frac{1}{m}\sum_{i=1}^{k-1}\big(G_{1m}(i)+G_{2m}(i-1)\big) + B(k)\ \ (m=1,2,\dots).$$

Then inclusion (6.2.5) *holds.*

Corollary 6.2.2. *Let conditions* (6.2.3) *and* (6.2.4) *hold. Then inclusion* (6.2.5) *holds if and only if there exist matrix-functions $Q_m \in L(I;\mathbb{R}^{n\times n})$ $(m=0,1,\dots)$ and $W_m \in B(T;\mathbb{R}^{n\times n})$ $(m=0,1,\dots)$ such that*

$$\limsup_{m\to+\infty}\left(\int_a^b \|Q_m(t)\|\,dt + \sum_{k=1}^{\infty}\left\|\frac{1}{m}\sum_{i=0}^{k-1}\big(G_{1m}(i)+G_{2m}(i)\big) - W_m(\tau_k)\right\|\right) < +\infty, \tag{6.2.18}$$
$$\det(I_n + W_m(\tau_l)) \neq 0\ \ (m=1,2,\dots;\ l=1,2,\dots), \tag{6.2.19}$$

and the conditions (6.1.13),

$$\lim_{m\to+\infty}\frac{1}{m}\sum_{l:\,\tau_l\in[a,t[}\Big(Z_m^{-1}(\tau_l-)G_{1m}(l) + Z_m^{-1}(\tau_{l-1}+)G_{2m}(l-1)\Big) = \int_a^t Z_0^{-1}(\tau)P_0(\tau)\,d\tau \tag{6.2.20}$$

and

$$\lim_{m\to+\infty}\frac{1}{m}\sum_{l:\,\tau_l\in[a,t[}\Big(Z_m^{-1}(\tau_l-)g_{1m}(l) + Z_m^{-1}(\tau_{l-1}+)g_{2m}(l-1)\Big) = \int_a^t Z_0^{-1}(\tau)P_0(\tau)\,d\tau \tag{6.2.21}$$

hold uniformly on I, where Z_m ($Z_m(a) = I_n$) is a fundamental matrix of the homogeneous system

$$\frac{dx}{dt} = Q_m(t) \ \text{ for a.a. } \ t \in I \setminus T, \tag{6.2.22}$$

$$x(\tau_l+) - x(\tau_l-) = W_m(\tau_l)x(\tau_l) \ \ (l = 1, 2, \dots) \tag{6.2.23}$$

for every $m \in \widetilde{\mathbb{N}}$.

Corollary 6.2.3. *Let conditions* (6.2.3) *and* (6.2.4) *hold and let there exist sequences of matrix-functions $Q_m \in L(I; \mathbb{R}^{n\times n})$ $(m = 0, 1, \dots)$ and $W_m \in B(T; \mathbb{R}^{n\times n})$ $(m = 0, 1, \dots;\ l = 1, 2, \dots)$ such that the pairs (Q_m, W_m) $(m = 1, 2, \dots)$ satisfy the Lappo–Danilevskiĭ condition at the point a, conditions* (6.2.18) *and*

$$\det(I_n + W_0(\tau_l)) \neq 0 \ \ (l = 1, 2, \dots)$$

hold, and conditions (6.2.20), (6.2.21),

$$\lim_{m\to+\infty} \int_a^t Q_m(\tau)\, d\tau = \int_a^t Q_0(\tau)\, d\tau \ \text{ and } \ \lim_{m\to+\infty} \sum_{\tau_l\in[a,t[} W_m(\tau_l) = \sum_{\tau_l\in[a,t[} W_0(\tau_l)$$

hold uniformly on I, where Z_m ($Z_m(a) = I_n$) is a fundamental matrix of the homogeneous system (6.2.22), (6.2.23) *for any sufficiently large m. Then inclusion* (6.2.5) *holds.*

Corollary 6.2.4. *Let $G_{km} = (g_{kmij})_{i,j=1}^n \in B(T; \mathbb{R}^{n\times n})$ $(k = 1, 2;\ m = 0, 1, \dots)$ and $g_{km} = (g_{kmi})_{i=1}^n \in B(T; \mathbb{R}^n)$ $(k = 1, 2;\ m = 0, 1, \dots)$ and let conditions* (6.2.3), (6.2.4) *and*

$$\limsup_{m\to+\infty} \frac{1}{m} \sum_{i,j=1;\, i\neq j}^n \Big(\sum_{l=1}^\infty \big(|g_{1mij}(\tau_l)| + |g_{2mij}(\tau_l)|\big)\Big) < +\infty$$

hold. Let, moreover, the conditions

$$\lim_{m\to+\infty} \frac{1}{m} \sum_{k=0}^{\nu_m(t)} (g_{1mii}(k) + g_{2mii}(k)) = \int_a^t p_{0ii}(\tau)\, d\tau \ \ (i = 1, \dots, n),$$

$$\lim_{m\to+\infty} \frac{1}{m} \Big(\sum_{l:\, \tau_l\in\,]a,t]} z_{mii}^{-1}(\tau_l) h_{1mij}(l) - \sum_{l:\, \tau_l\in[a,t[} z_{mii}^{-1}(\tau_l) h_{2mij}(l) \Big) = \int_a^t z_{0ii}^{-1}(\tau) p_{0ij}(\tau)\, d\tau$$

$$(i \neq j;\ i, j = 1, \dots, n)$$

and

$$\lim_{m\to+\infty} \frac{1}{m} \Bigg(\sum_{l:\, \tau_l\in\,]a,t]} z_{mii}^{-1}(\tau_l) h_{1mi}(l) - \sum_{l:\, \tau_l\in[a,t[} z_{mii}^{-1}(\tau_l) h_{2mi}(l) \Bigg) = \int_a^t z_{0ii}^{-1}(\tau) q_{0i}(\tau)\, d\tau \ \ (i = 1, \dots, n)$$

hold uniformly on I, where

$$h_{kmij}(l) \equiv \Big(1 + (-1)^k \frac{1}{m}\, g_{kmii}(l)\Big)^{-1} g_{kmij}(l), \quad h_{kmi}(l) \equiv \Big(1 + (-1)^k \frac{1}{m}\, g_{kmii}(l)\Big)^{-1} g_{kmi}(l)$$

$$(k = 1, 2;\ i, j = 1, \dots, n),$$

and

$$z_{mii}(\tau_l) \equiv \prod_{k=0}^{l-1} (1 + g_{mii}(\tau_k)) \ \ (i = 1, \dots, n)$$

for any sufficiently large m. Then inclusion (6.2.5) *holds.*

Remark 6.2.3. In Theorems 6.2.1, 6.2.4, Proposition 6.2.1 and Corollary 6.2.1, if condition (6.2.12) holds, we can assume that $H_m(t) \equiv Y_m^{-1}(t)$, where Y_m is the fundamental matrix of the homogeneous system (6.2.11) defined by (6.2.14) for every natural m. Moreover, condition (6.2.6) and analogous conditions hold automatically everywhere in the above results, as well.

6.3 The necessary and sufficient conditions for the convergence of discontinuous vector-functions

Let x_0 be a unique solution of the problem

$$\frac{dx}{dt} = P_0(t)x + q_0(t) \ \text{ for a.a. } \ t \in I, \tag{6.3.1}$$

$$\ell_0(x) = c_0 \tag{6.3.2}$$

where $I = [a,b]$, $P_0 \in L(I;\mathbb{R}^{n\times n})$, $q_0 \in L(I;\mathbb{R}^n)$, $\ell_0 : BV_\infty(I;\mathbb{R}^n) \to \mathbb{R}^n$ be a linear bounded vector-functionals, and $c_0 \in \mathbb{R}^n$.

Along with the problem, consider the impulsive boundary initial problems

$$\begin{gathered} \frac{dx}{dt} = P_m(t)x + q_m(t) \ \text{ for a.a. } \ t \in I \setminus \{\tau_l\}_{l=1}^{\infty}, \\ x(\tau_l+) - x(\tau_{l-}) = G_m(\tau_l)x(\tau_l) + u_m(\tau_l) \ \ (l = 1,2,\dots); \end{gathered} \tag{6.3.1$_m$}$$

$$\ell_m(x) = c_m \tag{6.3.2$_m$}$$

$(m = 1,2,\dots)$, where $P_m \in L(I;\mathbb{R}^{n\times n})$ $(m = 1,2,\dots)$, $q_m \in L(I;\mathbb{R}^n)$ $(m = 1,2,\dots)$, $G_m \in B(T;\mathbb{R}^{n\times n})$ $(m = 1,2,\dots)$, $u_m \in B(T;\mathbb{R}^n)$, $T = \{\tau_1,\tau_2,\dots\}$, $\ell_m : \mathrm{BV}_\infty(I;\mathbb{R}^n) \to \mathbb{R}^n$ $(m = 1,2,\dots)$ are linear bounded vector-functionals, and $c_m \in \mathbb{R}^n$ $(m = 1,2,\dots)$.

We assume that $P_m = (p_{mij})_{i,j=1}^n$ $(m = 0,1,\dots)$, $q_m = (q_{mi})_{i=1}^n$ $(m = 0,1,\dots)$; $G_m = (g_{mij})_{i,j=1}^n$ $(m = 1,2,\dots)$, $u_m = (u_{mi})_{i=1}^n$ $(m = 1,2,\dots)$.

In this section, we establish the necessary and sufficient and effective sufficient conditions for the boundary value problem $(6.3.1_m),(6.3.2_m)$ to have a unique solution x_m for any sufficiently large m and

$$\lim_{m\to+\infty} \|x_m - x_0\|_\infty = 0. \tag{6.3.3}$$

Along with systems (6.3.1) and $(6.3.1_m)$, we consider the corresponding homogeneous systems

$$\frac{dx}{dt} = P_0(t)x \ \text{ for a.a. } \ t \in I \setminus T \tag{6.3.1$_0$}$$

and

$$\begin{gathered} \frac{dx}{dt} = P_m(t)x \ \text{ for a.a. } \ t \in I \setminus \{\tau_l\}_{l=1}^{\infty}, \\ x(\tau_l+) - x(\tau_{l-}) = G_m(\tau_l)x(\tau_l) \ \ (l = 1,2,\dots) \end{gathered} \tag{6.3.1$_{m0}$}$$

$(m = 1,2,\dots)$.

Definition 6.3.1. We say that the sequence $(P_m,q_m;G_m,u_m;\ell_m)$ $(m = 1,2,\dots)$ belongs to the set $\mathcal{S}(P_0,q_0;\ell_0)$ if for every $c_0 \in \mathbb{R}^n$ and a sequence $c_m \in \mathbb{R}^n$ $(m = 1,2,\dots)$ satisfying the condition

$$\lim_{m\to+\infty} c_m = c_0$$

problem $(6.3.1_m),(6.3.2_m)$ has a unique solution x_m for any sufficiently large m and (6.3.3) holds.

As in Subsection 5.1.2, we use the following forms of the operators $\mathcal{B}(X,Y)$ and $\mathcal{I}(X,Y)$:

$$\mathcal{B}_\iota(X;Y,Z)(t) \equiv \int_a^t X(\tau)Y(\tau)\,d\tau + \sum_{\tau_l\in[a,t[} X(\tau_l+)Z(\tau_l),$$

$$\mathcal{I}_\iota(X;Y,Z)(t) \equiv \int_a^t \big(X'(\tau)+X(\tau)Y(\tau)\big)X^{-1}(\tau)\,d\tau + \sum_{\tau_l\in[a,t[} \big(d_2X(\tau_l)+X(\tau_l+)\,d_2Z(\tau_l)\big)X^{-1}(\tau_l)$$

for the corresponding $X \in \mathrm{BV}(I;\mathbb{R}^{n\times j})$, $Y \in \mathrm{BVAC}_{loc}(I,T;\mathbb{R}^{j\times m})$ and $Z \in B_{loc}(T;\mathbb{R}^{n\times m})$.

Theorem 6.3.1. *Let the conditions*

$$\lim_{m\to+\infty} \ell_m(x) = \ell_0(x) \ \ \textit{for} \ \ x \in \mathrm{BV}(I;\mathbb{R}^n), \tag{6.3.4}$$

$$\limsup_{m\to+\infty} |||\ell_m||| < +\infty \tag{6.3.5}$$

hold. Then

$$\big((P_m, q_m; G_m, u_m; \ell_m)\big)_{m=1}^{\infty} \in \mathcal{S}(P_0, q_0; \ell_0) \tag{6.3.6}$$

if and only if there exists a sequence of matrix-functions $H_m \in \mathrm{BVAC}_{loc}(I,T;\mathbb{R}^{n\times n})$ $(m = 0,1,\dots)$ *such that the condition*

$$\limsup_{m\to+\infty} \bigvee_a^b \big(H_m + \mathcal{B}_\iota(H_m; P_m, G_m)\big) < +\infty \tag{6.3.7}$$

holds, and the conditions

$$\lim_{m\to+\infty} H_m(t) = I_n, \tag{6.3.8}$$

$$\lim_{m\to+\infty} \mathcal{B}_\iota(H_m; P_m, G_m)(t) = \int_a^t P_0(\tau)\, d\tau \tag{6.3.9}$$

and

$$\lim_{m\to+\infty} \mathcal{B}_\iota(H_m; q_m, u_m)(t) = \int_a^t q_0(\tau)\, d\tau$$

hold uniformly on I.

Theorem 6.3.2. *Let conditions* (6.3.4), (6.3.5) *and*

$$\det(I_n + G_m(\tau_l)) \neq 0 \ \ (l = 1,2,\dots;\ m = 1,2,\dots)$$

hold. Then inclusion (6.3.6) *holds if and only if the conditions*

$$\lim_{m\to+\infty} X_m^{-1}(t) = X_0^{-1}(t)$$

and

$$\lim_{m\to+\infty} \bigg(\int_a^t X_m^{-1}(\tau) q_m(\tau)\, d\tau + \sum_{\tau_l\in[a,t[} X_m^{-1}(\tau_l+)\, u_m(\tau_l) \bigg) = \int_a^t X_0^{-1}(\tau) q_0(\tau)\, d\tau(\tau_l)$$

hold uniformly on I, *where* X_m *is the fundamental matrix of system* (6.3.1_{m0}) *for every* $m \in \widetilde{\mathbb{N}}$.

Theorem 6.3.3. *Let* $P_0^* \in L(I;\mathbb{R}^{n\times n})$, $q_0^* \in L(I;\mathbb{R}^n)$, $c_0^* \in \mathbb{R}^n$, *and a* $\ell_0^* : \mathrm{BV}_\infty(I;\mathbb{R}^{n\times n}) \to \mathbb{R}^n$ *be a linear bounded vector-functional such that the boundary value problem*

$$\frac{dx}{dt} = P_0^*(t)x + q_0^*(t) \ \ \textit{for a.a.} \ \ t \in I,$$
$$\ell_0^*(x) = c_0^*$$

has a unique solution x_0^*. *Let, moreover, there exist sequences of matrix- and vector-functions* $H_m \in \mathrm{BVAC}_{loc}(I,T;\mathbb{R}^{n\times n})$ $(m = 1,2,\dots)$ *and* $h_m \in \mathrm{BVAC}_{loc}(I,T;\mathbb{R}^n)$ $(m = 1,2,\dots)$ *such that*

$$\inf\big\{ |\det(H_m(t))| :\ t \in I \big\} > 0 \ \ \textit{for every sufficiently large}\ m,$$

the conditions

$$\lim_{m\to+\infty}(c_m+\ell^*_m(h_m))=c^*_0,\quad \lim_{m\to+\infty}\ell^*_m(y)=\ell^*_0(y)\ \textit{ for }\ y\in\mathrm{BV}(I;\mathbb{R}^n),$$

$$\limsup_{m\to+\infty}|||\ell^*_m|||<+\infty\quad\textit{and}\quad \limsup_{m\to+\infty}\bigvee_a^b(\mathcal{I}_\iota(H_m;P_m,G_m)<+\infty$$

hold, and the conditions

$$\lim_{m\to+\infty}\mathcal{I}_\iota(H_m;P_m,G_m)(t)=\int_a^t P^*_0(\tau)\,d\tau,$$

$$\lim_{m\to+\infty}\bigg(h_m(t)-h_m(a)+\mathcal{B}_\iota(H_m;q_m,u_m)(t)-\int_a^t d\mathcal{I}_\iota(H_m;P_m,G_m)(s)\cdot h_m(s)\bigg)=\int_a^t q^*_0(\tau)\,d\tau$$

hold uniformly on I, where $\ell^*_m(y)=\ell_m(H_m^{-1}y)$ $(m=1,2,\dots)$. Then problem $(6.3.1_m),(6.3.2_m)$ *has the unique solution x_m for any sufficiently large m and*

$$\lim_{m\to+\infty}\|H_m\,x_m+h_m-x^*_0\|_\infty=0.$$

Remark 6.3.1. In Theorem 6.3.3, the vector-function $x^*_m(t)\equiv H_m(t)x_m(t)+h_m(t)$ is a solution of the problem

$$\begin{gathered}\frac{dx}{dt}=P^*_m(t)x+q^*_m(t)\ \text{ for a.a. }\ t\in[a,b]\setminus T,\\ x(\tau_l+)-x(\tau_l-)=G^*_m(\tau_l)x(\tau_l)+u^*_m(\tau_l)\ \ (l=1,2,\dots);\\ \ell^*_m(x)=c^*_m\end{gathered}$$

for every sufficiently large m, where

$$\begin{aligned}P^*_m(t)&\equiv\big(H'_m(t)+H_m(t)P_m(t)\big)H_m^{-1}(t),\\ G^*_m(\tau_l)&=\big(d_2H_m(\tau_l)+H_m(\tau_l+)G_m(\tau_l)\big)H_m^{-1}(\tau_l)\ \ (m=1,2,\dots;\ l=1,2,\dots);\\ q^*_m(t)&\equiv h'_m(t)+H_m(t)\,q_m(t)-P^*_m(t)h_m(t)\ \ (m=1,2,\dots),\\ u^*_m(\tau_l)&=d_2h_m(\tau_l)+H_m(\tau_l+)u_m(\tau_l)-G^*_m(\tau_l)h_m(\tau_l)\ \ (m=1,2,\dots;\ l=1,2,\dots).\end{aligned}$$

Corollary 6.3.1. *Let conditions* (6.3.4), (6.3.5), (6.3.7) *and*

$$\lim_{m\to+\infty}(c_m-\varphi_m(a))=c_0$$

hold, and conditions (6.3.8), (6.3.9) *and*

$$\lim_{m\to+\infty}\bigg(\mathcal{B}_\iota(H_m;q_m-\varphi'_m,u_m)(t)+\int_a^t d\mathcal{I}_\iota(H_m;P_m,G_m)\cdot\varphi_m(\tau)\bigg)=\int_a^t q_0(\tau)\,d\tau$$

hold uniformly on I, where $H_m\in\mathrm{BVAC}_{loc}(I,T;\mathbb{R}^{n\times n})$ $(m=0,1,\dots)$, $\varphi_m\in\mathrm{BVAC}_{loc}(I,T;\mathbb{R}^n)$ $(m=1,2,\dots)$. Then problem $(6.3.1_m),(6.3.2_m)$ *has the unique solution x_m for any sufficiently large m and*

$$\lim_{m\to+\infty}\|x_m-\varphi_m-x_0\|_\infty=0.$$

Remark 6.3.2. Note that the condition

$$\limsup_{m\to+\infty}\bigg(\int_a^b\big\|H'_m(t)+H_m(t)P_m(t)\big\|\,dt+\sum_{l=1}^{+\infty}\big\|H_m(\tau_l+)-H_m(\tau_l)+H_m(\tau_l+)G_m(\tau_l)\big\|\bigg)<+\infty$$

guarantees the fulfilment of condition (6.3.7).

Now we give some effective sufficient conditions guaranteeing inclusion (6.3.6).

Theorem 6.3.4. *Let conditions* (6.3.4), (6.3.5) *and*

$$\limsup_{m\to+\infty}\bigg(\int_a^b \|P_m(t)\|\,dt+\sum_{l=1}^{\infty}\|G_m(\tau_l)\|\bigg)<+\infty$$

hold, and the conditions

$$\lim_{m\to+\infty}\bigg(\int_a^t P_m(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}G_m(\tau_l)\bigg)=\int_a^t P_0(\tau)\,d\tau$$

and

$$\lim_{m\to+\infty}\bigg(\int_a^t q_m(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}u_m(\tau_l)\bigg)=\int_m^t q_0(\tau)\,d\tau$$

hold uniformly on I. Then inclusion (6.3.6) *holds.*

Corollary 6.3.2. *Let conditions* (6.3.4), (6.3.5) *and* (6.3.7) *hold, and conditions* (6.3.8)

$$\lim_{m\to+\infty}\int_a^t H_m(\tau)P_m(\tau)\,d\tau=\int_a^t P_0(\tau)\,d\tau$$

and

$$\lim_{m\to+\infty}\int_a^t H_m(\tau)\,q_m(\tau)\,d\tau=\int_a^t q_0(\tau)\,d\tau$$

hold uniformly on I, and

$$\lim_{m\to+\infty}G_m(\tau_l)=O_{n\times n},\qquad \lim_{m\to+\infty}u_m(\tau_l)=0_n$$

hold uniformly on T, where $H_m\in\mathrm{BVAC}_{loc}(I,T;\mathbb{R}^{n\times n})$ $(m=0,1,\dots)$. Let, moreover, either

$$\limsup_{m\to+\infty}\sum_{l=1}^{\infty}\big(\|G_m(\tau_l)\|+\|u_m(\tau_l)\|\big)<+\infty,\quad or\quad \limsup_{m\to+\infty}\sum_{l=1}^{\infty}\|H_m(\tau_l+)-H_m(\tau_l)\|<+\infty.$$

Then inclusion (6.3.6) *holds.*

Corollary 6.3.3. *Let conditions* (6.3.4), (6.3.5) *and* (6.3.7) *hold, and conditions* (6.3.8),

$$\lim_{m\to+\infty}\bigg(\int_a^t H_m(\tau)P_m(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}H_m(\tau_l+)G_k(\tau_l)\bigg)=\int_a^t P_*(\tau)\,d\tau,$$

$$\lim_{m\to+\infty}\bigg(\int_a^t H_m(\tau)q_m(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}H_m(\tau_l+)u_m(\tau_l)\bigg)=\int_a^t q_*(\tau)\,d\tau$$

hold uniformly on I, where $H_m\in\mathrm{BVAC}_{loc}(I,T;\mathbb{R}^{n\times n})$ $(m=1,2,\dots)$, $P_*\in L(I;\mathbb{R}^{n\times n})$, $q_*\in L(I;\mathbb{R}^n)$, $G_*\in B(T;\mathbb{R}^{n\times n})$, $u_*\in B(T;\mathbb{R}^n)$. Let, moreover, the system

$$\frac{dx}{dt}=(P_0(t)-P_*(t))x+(q_0(t)-q_*(t))\ \text{ for a.a. } t\in I$$

have a unique solution satisfying condition (6.3.2). *Then*

$$\big((P_m,q_m;G_m,u_m;\ell_m)\big)_{m=1}^{\infty}\in\mathcal{S}(P_0-P_*,q_0-q_*;\ell_0).$$

Corollary 6.3.4. *Let conditions* (6.3.4), (6.3.5) *hold and let there exist a natural number* μ *and matrix-functions* $B_j \in \mathrm{BVAC}_{loc}(I,T;\mathbb{R}^{n\times n})$ $(j=0,\dots,\mu-1)$ *such that*

$$\limsup_{m\to+\infty}\bigvee_a^b \big(H_{m\,\mu-1}+\mathcal{B}_\iota(H_{m\,\mu-1};P_m,G_m)\big)<+\infty$$

holds, and the conditions

$$\lim_{m\to+\infty}\mathcal{B}_\iota(I_n;P_m,G_m)(t)=B_0(t)-B_0(a),$$

$$\lim_{m\to+\infty}\Big(H_{m\,j-1}(t)+\mathcal{B}_\iota(H_{m\,j-1};P_m,G_m)(t)\Big)=I_n+B_j(t)-B_j(a)\ \ (j=1,\dots,\mu-1),$$

$$\lim_{m\to+\infty}\Big(H_{m\,\mu-1}(t)+\mathcal{B}_\iota(H_{m\,\mu-1};P_m,G_m)(t)\Big)=I_n+\int_{t_0}^t P_0(\tau)\,d\tau,$$

$$\lim_{m\to+\infty}\mathcal{B}_\iota(H_{m\,\mu-1};q_m,u_m)(t)=\int_a^t q_0(\tau)\,d\tau$$

hold uniformly on I, *where*

$$H_{m0}(t)\equiv I_n,\quad H_{mj}(t)\equiv-\Big(H_{m\,j-1}(\tau)(t)+\mathcal{B}_\iota(H_{m\,j-1};P_m,G_m)(t)-B_j(t)+B_j(a)\Big)H_{m\,j-1}(t)$$
$$(j=1,\dots,\mu-1;\ m=1,2,\dots).$$

Then inclusion (6.3.6) *holds.*

If $\mu=1$, then Corollary 6.3.4 coincides with Theorem 6.3.4.
If $\mu=2$, then Corollary 6.3.4 has the following form.

Corollary 6.3.4$_1$. *Let conditions* (6.3.4), (6.3.5) *and* (6.3.7) *hold, and the conditions*

$$\lim_{m\to+\infty}\bigg(\int_a^t P_m(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}G_m(\tau_l)\bigg)=B(t)-B(a),$$

$$\lim_{m\to+\infty}\bigg(\int_a^t H_m(\tau)P_m(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}H_m(\tau_l+)G_m(\tau_l)\bigg)=\int_a^t P_0(\tau)\,d\tau,$$

$$\lim_{m\to+\infty}\bigg(\int_a^t H_m(\tau)\,q_m(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}H_m(\tau_l+)u_m(\tau_l)\bigg)=\int_{t_0}^t q_0(\tau)\,d\tau$$

hold uniformly on I, *where* $B\in\mathrm{BVAC}_{loc}(I,T;\mathbb{R}^{n\times n})$ *and*

$$H_m(t)\equiv I_n-\int_a^t P_m(\tau)\,d\tau-\sum_{\tau_l\in[a,t[}G_m(\tau_l)+B(t)-B(a)\ \ (m=1,2,\dots).$$

Then inclusion (6.3.6) *holds.*

Corollary 6.3.5. *Let conditions* (6.3.4) *and* (6.3.5) *hold. Then inclusion* (6.3.6) *holds if and only if there exist matrix-functions* $Q_m\in L(I;\mathbb{R}^{n\times n})$ $(m=1,2,\dots)$ *and* $W_m\in B(T;\mathbb{R}^{n\times n})$ $(m=1,2,\dots)$ *such that the conditions* (6.2.19) *and*

$$\limsup_{m\to+\infty}\bigg(\int_a^b\|P_m(t)-Q_m(t)\|\,dt+\sum_{l=1}^{\infty}\|G_m(\tau_l)-W_m(\tau_l)\|\bigg)<+\infty \tag{6.3.10}$$

hold, and the conditions (6.1.13),

$$\lim_{m\to+\infty}\mathcal{B}_\iota(Z_m^{-1};P_m,G_m)(t)=\int\limits_a^t Z_0^{-1}(\tau)P_0(\tau)\,d\tau, \tag{6.3.11}$$

$$\lim_{m\to+\infty}\mathcal{B}_\iota(Z_m^{-1};q_m,u_m)(t)=\int\limits_a^t Z_0^{-1}(\tau)q_0(\tau)\,d\tau \tag{6.3.12}$$

hold uniformly on I, where Z_m ($Z_m(a)=I_n$) is a fundamental matrix of the homogeneous system (6.2.22), (6.2.23) *for any $m\in\widetilde{\mathbb{N}}$.*

Corollary 6.3.6. *Let conditions* (6.3.4) *and* (6.3.5) *hold and let there exist sequences of matrix-functions $Q_m\in L(I;\mathbb{R}^{n\times n})$ $(m=1,2,\dots)$ and $W_m\in B(T;\mathbb{R}^{n\times n})$ $(m=1,2,\dots)$ such that the pairs (Q_m,W_m) $(m=1,2,\dots)$ satisfy the Lappo–Danilevskiĭ condition at the point a, condition* (6.3.10) *holds and the conditions*

$$\lim_{m\to+\infty}\int\limits_a^t Q_m(\tau)\,d\tau=\int\limits_a^t Q_0(\tau)\,d\tau,$$

$$\lim_{m\to+\infty}\sum_{\tau_l\in[a,t[}W_m(\tau_l)=O_{n\times n}, \tag{6.3.13}$$

$$\lim_{m\to+\infty}\bigg(\int\limits_a^t Z_m^{-1}(\tau)P_m(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}Z_m^{-1}(\tau_l)(I_n+W_m(\tau_l))^{-1}G_m(\tau_l)\bigg)=\int\limits_a^t Z_0^{-1}(\tau)P_0(\tau)\,d\tau, \tag{6.3.14}$$

$$\lim_{m\to+\infty}\bigg(\int\limits_a^t Z_m^{-1}(\tau)q_m(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}Z_m^{-1}(\tau_l)(I_n+W_m(\tau_l))^{-1}u_m(\tau_l)\bigg)=\int\limits_a^t Z_0^{-1}(\tau)q_0(\tau)\,d\tau \tag{6.3.15}$$

hold uniformly on I, where Z_m ($Z_m(a)=I_n$) is a fundamental matrix of the homogeneous system (6.2.22), (6.2.23) *for any sufficiently large m. Then inclusion* (6.3.6) *holds.*

Remark 6.3.3. In Corollary 6.3.6, due to (6.3.13), condition (6.2.19) holds for every sufficiently large m and, therefore, conditions (6.3.14) and (6.3.15) of the corollary are correct.

Remark 6.3.4. In Corollaries 6.3.5 and 6.3.6, if we assume that $W_m(\tau_l)=O_{n\times n}$ $(m=1,2,\dots;$ $l=1,2,\dots)$, then condition (6.2.19) holds. Moreover, due to the definition of the operator $\mathcal{B}_\iota$, each of conditions (6.3.11) and (6.3.14) has the form

$$\lim_{m\to+\infty}\bigg(\int\limits_a^t Z_m^{-1}(\tau)P_m(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}Z_m^{-1}(\tau_l)G_m(\tau_l)\bigg)=\int\limits_a^t Z_0^{-1}(\tau)P_0(\tau)\,d\tau$$

and each of conditions (6.3.12) and (6.3.15) has the form

$$\lim_{m\to+\infty}\bigg(\int\limits_a^t Z_m^{-1}(\tau)q_m(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}Z_m^{-1}(\tau_l)u_m(\tau_l)\bigg)=\int\limits_a^t Z_0^{-1}(\tau)q_0(\tau)\,d\tau.$$

Corollary 6.3.7. *Let conditions* (6.3.4), (6.3.5) *and*

$$\limsup_{m\to+\infty}\sum_{l=1}^{\infty}\|G_m(\tau_l)\|<+\infty$$

hold. Let, moreover, the matrix-functions P_m $(m=0,1,\dots)$ satisfy the Lappo–Danilevskiĭ condition at the point a and the conditions

$$\lim_{m\to+\infty}\int\limits_a^t P_m(\tau)\,d\tau=\int\limits_a^t P_0(\tau)\,d\tau,$$

$$\lim_{m\to+\infty}\sum_{\tau_l\in[a,t[}G_m(\tau_l)=O_{n\times n},$$

$$\lim_{m\to+\infty}\int_a^t\exp\bigg(-\int_a^\tau P_m(s)\,ds\bigg)P_m(\tau)\,d\tau=\int_a^t\exp\bigg(-\int_a^\tau P_0(s)\,ds\bigg)P_0(\tau)\,d\tau,$$

$$\lim_{m\to+\infty}\int_a^t\exp\bigg(-\int_a^\tau P_m(s)\,ds\bigg)q_m(\tau)\,d\tau=\int_a^t\exp\bigg(-\int_a^\tau P_0(s)\,ds\bigg)q_0(\tau)\,d\tau,$$

$$\lim_{m\to+\infty}\sum_{\tau_l\in[a,t[}\exp\bigg(-\int_a^{\tau_l} P_m(s)\,ds\bigg)u_m(\tau_l)=0_n$$

hold uniformly on I. Then inclusion (6.3.6) *holds.*

Corollary 6.3.8. *Let $P_m=(p_{mij})_{i,j=1}^n\in L(I;\mathbb{R}^{n\times n})$, $q_m=(q_{mi})_{i=1}^n\in L(I;\mathbb{R}^n)$, $G_m=(g_{mij})_{i,j=1}^n\in B(T;\mathbb{R}^{n\times n})$ and $u_m=(g_{mi})_{i=1}^n\in B(T;\mathbb{R}^n)$ $(m=1,2,\dots)$ and let conditions* (6.3.4), (6.3.5) *and*

$$\limsup_{m\to+\infty}\sum_{i,j=1;\,i\neq j}^n\bigg(\int_a^b|p_{mij}(t)|\,dt+\sum_{l=1}^\infty|g_{mij}(\tau_l)|\bigg)<+\infty$$

hold. Let, moreover, the conditions

$$\lim_{m\to+\infty}\bigg(\int_a^t p_{mii}(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}g_{mii}(\tau_l)\bigg)=\int_a^t p_{0ii}(\tau)\,d\tau\ \ (i=1,\dots,n),$$

$$\lim_{m\to+\infty}\bigg(\int_a^t z_{mii}^{-1}(\tau)p_{mij}(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}z_{mii}^{-1}(\tau_l)(1+g_{mii}(\tau_l))^{-1}g_{mij}(\tau_l)\bigg)$$
$$=\int_a^t z_{0ii}^{-1}(\tau)p_{0ij}(\tau)\,d\tau\ \ (i\neq j;\ i,j=1,\dots,n),$$

$$\lim_{m\to+\infty}\bigg(\int_a^t z_{mii}^{-1}(\tau)q_{mi}(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}z_{mii}^{-1}(\tau_l)(1+g_{mii}(\tau_l))^{-1}u_{mi}(\tau_l)\bigg)=\int_a^t z_{0ii}^{-1}(\tau)q_{0i}(\tau)\,d\tau$$
$$(i=1,\dots,n)$$

hold uniformly on I, where

$$z_{mii}(t)\equiv\exp\bigg(\int_a^t p_{mii}(\tau)\,d\tau\bigg)\prod_{s\le\tau_l<t}(1+g_{mii}(\tau_l)),\ \ i\in\{1,\dots,n\}$$

for any sufficiently large m. Then inclusion (6.3.6) *holds.*

Remark 6.3.5. For Corollary 6.3.8, the remark analogous to Remark 1.2.3 is true, i.e.,

$$1+g_{mii}(\tau_l)\neq0\ \ (i=1,\dots,n;\ l=1,2,\dots)$$

for every sufficiently large m and, therefore, all conditions of the corollary are correct.

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