Memoirs on Differential Equations and Mathematical Physics

Volume 79, 2020, 93–105

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MATHEMATICAL STUDY TO A REGULARIZED 3D-BOUSSINESQ SYSTEM

Abstract. We prove existence of weak solution to a regularized Boussinesq system in Sobolev spaces under the minimal regularity to the initial data. Continuous dependence on initial data (and then uniqueness) is proved provided that the initial fluid velocity is mean free. If the temperature is also mean free, we prove that the solution decays exponentially fast, as time goes to infinity. Moreover, we show that the unique solution converges to a Leray–Hopf solution of the three-dimensional Boussinesq system, as the regularizing parameter alpha vanishes. The mean free technical condition appears because the nonlinear part of the fluid equation is subject to regularization. The main tools are the energy methods, the compactness method, the Poincaré inequality and some Grönwall type inequalities. To handle the long time behaviour, a time dependent change of function is used.

2010 Mathematics Subject Classification. Primary 35A05, 35B30, 35B40; Secondary 35B10, 35B45.

Key words and phrases. Three-dimensional periodic Boussinesq system, weak solution, regularization, existence, uniqueness, convergence, asymptotic behavior, long time behavior, mean free.

რეზიუმე. დამტკიცებულია რეგულარიზებული ბუსინესკის სისტემის სუსტი ამონახსნის არსებობა სობოლევის სივრცეებში საწყისი მონაცემების მინიმალური რეგულარობის პირობებში. დამტკიცებულია ამონახსნის უწყვეტი დამოკიდებულება საწყის მონაცემებზე (ხოლო შემდეგ ერთადერთობა), თუ სითხის საწყისი სიჩქარე საშუალოდ თავისუფალია. თუ ტემპერატურაც საშუალოდ თავისუფალია, მაშინ ჩვენ ვამტკიცებთ, რომ ამონახსნი ექსპონენციალურად სწრაფად ქრება, როცა დრო უსასრულობისკენ მიისწრაფის. გარდა ამისა, დამტკიცებულია, რომ ერთადერთი ამონახსნი კრებადია სამგანზოლებიანი ბუსინესკის სისტემის ლერეი-ჰოფის ამონახსნისკენ, როცა მარეგულირებელი ალფა პარამეტრი ნულისკენ მიისწრაფის. საშუალო თავისუფლების ტექნიკური პირობა გამოჩნდება იმიტომ, რომ ხდება სითხის განტოლების არაწრფივი ნაწილის რეგულარიზაცია. კვლევის მთავარი ინსტრუმენტებია ენერგეტიკული მეთოდები, კომპაქტურობის მეთოდი, პუანკარეს უტოლობა და გრონველის ტიპის უტოლობები. იმისათვის, რომ შევისწავლოთ ყოფაქცევა ხანგრძლივი დროის განმავლობაში, გამოყენებულია ფუნქციის ცვალებადობის დროზე დამოკიდებულება.

1 Introduction

We consider the following system denoted by (Bq_α):

$$\begin{gathered}\partial_t\theta - \Delta\theta + (u\cdot\nabla)\theta = 0, \quad (t,x)\in\mathbb{R}_+\times\mathbb{T}^3,\\ \partial_t v - \Delta v + (v\cdot\nabla)u = -\nabla p + \theta e_3, \quad (t,x)\in\mathbb{R}_+\times\mathbb{T}^3,\\ v = u - \alpha^2\Delta u, \quad (t,x)\in\mathbb{R}_+\times\mathbb{T}^3,\\ \operatorname{div} u = \operatorname{div} v = 0, \quad (t,x)\in\mathbb{R}_+\times\mathbb{T}^3,\\ (u,\theta)\big|_{t=0} = (u^0,\theta^0), \quad x\in\mathbb{T}^3,\end{gathered}$$

where the unknown vector field u, the scalars p and θ denote, respectively, the velocity, the pressure and the temperature of the fluid at the point $(t,x)\in\ \mathbb{R}_+\times\mathbb{T}^3$. Here, $\mathbb{T}^3$ is the three-dimensional torus and $\alpha>0$ is a real parameter that has to go to zero. The data θ^0 and u^0 are initial temperature and initial divergence free velocity. In [7], the author explained motivations behind considering regularized systems such as (Bq_α), and he gave a wide review of related literature. Here, we just recall that alpha-regularization consists in replacing the velocity u in some of its occurrences by the most regular field $v=u-\alpha^2\Delta u$. So, contrarily to the non-regularized fluid mechanic equation, we have the existence of a unique three-dimensional solution that depends continuously on initial data. Moreover, as explained in [2], these models can be implemented in a relatively simple way in numerical computation of the three-dimensional fluid equations. Thus, they are to be known as regularization stimulated by numerical motivations. In the framework of computational fluid dynamics, for zero valued temperature, it was proved in [4] that the model we are actually considering, provides a computationally sound analytical subgrid scale model for large eddy simulation of turbulence. More important is that when the regularizing parameter α tends to zero, the solution of (Bq_α) coincides with the solution of Boussinesq system $(Bq_{\alpha=0})$. Furthermore, as time tends to infinity, the system $(Bq_{\alpha>0})$ behaves like $(Bq_{\alpha=0})$.

In this paper, we will investigate the weak solution to the modified Leray-alpha model for the Boussinesq system. More than the linear part, the nonlinear part of the fluid equation is to be regularized as well. This is one of the main differences between systems we considered in [7] and [3], where we regularized only the linear part and studied, respectively, the weak and the strong solutions.

Our first result is the existence of the weak solution to the system (Bq_α) in the context of the minimal regularity to the initial data.

Theorem 1.1. *Let $\theta^0\in L^2(\mathbb{T}^3)$ and let $u^0\in H^1(\mathbb{T}^3)$ be a divergence-free vector field. Then there exists a unique weak solution (u_α,θ_α) of system (Bq_α) such that u_α belongs to $C(\mathbb{R}_+,H^1(\mathbb{T}^3))\cap L^2(\mathbb{R}_+,H^2(\mathbb{T}^3))$ and θ_α belongs to $C(\mathbb{R}_+,L^2(\mathbb{T}^3))\cap L^2(\mathbb{R}_+,H^1(\mathbb{T}^3))$. Moreover, this solution satisfies the energy estimate*

$$\begin{aligned}\|\theta_\alpha\|_{L^2}^2 + \|u_\alpha\|_{L^2}^2 + \alpha^2\|\nabla u_\alpha\|_{L^2}^2 + 2\int_0^t \|\nabla\theta_\alpha\|_{L^2(\mathbb{T}^3)}^2\,d\tau\\ + 2\int_0^t \big(\|\nabla u_\alpha\|_{L^2}^2 + \alpha^2\|\Delta u_\alpha\|_{L^2}^2\big)\,d\tau \le \|\theta^0\|_{L^2}^2 + \|u^0\|_{L^2}^2 + \alpha^2\|\nabla u^0\|_{L^2}^2 + \sigma_\alpha(t),\end{aligned} \quad (1.1)$$

where

$$\sigma_\alpha(t) = (e^{2t}-1)\big(\|\theta^0\|_{L^2}^2 + \|u^0\|_{L^2}^2 + \alpha^2\|\nabla u^0\|_{L^2}^2\big).$$

If the initial velocity is mean free, the solution is continuously dependent on the initial data on any bounded interval $[0,T]$. In particular, it is unique.

The proof is done in the frequency space and uses the compactness method. To close the energy estimates, the buoyancy force presents some difficulties that we have overcome by Grönwall' s lemma, without useless sharpness. More than the uniqueness, we have continuous dependence of the weak solution on the initial data. This is the main advantage provided by alpha regularization, since such dependence plays an important role in numerical schemes.

To prove continuous dependence with respect to the initial data, we consider the system satisfied by the difference of two solutions and apply energy methods. The Young product inequalities and suitable Sobolev products allow to estimate the nonlinear terms. Grönwall's type differential inequality finishes the proof. In particular, we infer the uniqueness of solution. Compared to [7] and [3], the mean free condition is compulsory, since we are regularizing the nonlinear term and thus the Poincaré inequality turns to be a necessary tool to run the argument of the continuous dependence to initial data.

Our next result asserts that for long time, the regularized temperature and the regularized velocity fields vanish exponentially fast as time tends to infinity. This convergence is uniform with respect to α. One recovers, for $\alpha > 0$, a similar property of the long time behavior to the Leray–Hopf solution of the non-regularized system.

Theorem 1.2. *Let $a \in (0,1)$. Let θ_α and u_α be the family of solutions from Theorem 1.1. If θ^0 and u^0 are both mean free and satisfy the inequality*

$$\|\theta^0\|_{L^2}^2 + \|u^0\|_{L^2}^2 + \alpha^2\|\nabla u^0\|_{L^2}^2 \leq 1 - a,$$

then θ_α and u_α decay exponentially fast to zero as time tends to infinity as soon as the initial data (hence the solution) are mean free:

$$\|\theta_\alpha(t)\|_{L^2} + \|u_\alpha(t)\|_{H^1} \leq (1-a)e^{-at} \quad \forall\, t \geq 0.$$

To prove this result, we use a change of the function that depends explicitly on time. This leads to an energy estimate that is sharper than the one of the existence result. For zero-mean valued temperature and velocity, this estimation allows to derive the vanishing limit and the rate of convergence, as time tends to infinity.

Our last result describes the weak and strong convergence, as $\alpha \to 0$, of the unique weak solution of the regularized system (Bq_α) to the Leray–Hopf solution of the system (Bq_0). This convergence asserts that as smaller is alpha, as better we describe reality.

Theorem 1.3. *Let $T > 0$, $(u_\alpha, \theta_\alpha)$ be the unique solution of system (Bq_α). Then there exist the subsequences u_{α_k}, v_{α_k} and θ_{α_k}, a scalar function θ, and a divergence-free vector field u, both belonging to $L^\infty([0,T], L^2(\mathbb{T}^3)) \cap L^2([0,T], H^1(\mathbb{T}^3))$, such that as $\alpha_k \to 0^+$, we have:*

1. *The sequence u_{α_k} converges to u and θ_{α_k} converges to θ weakly in $L^2([0,T], H^1(\mathbb{T}^3))$ and strongly in $L^2([0,T], L^2(\mathbb{T}^3))$.*
2. *The sequence v_{α_k} converges to u weakly in $L^2([0,T], L^2(\mathbb{T}^3))$ and strongly in $L^2([0,T], H^{-1}(\mathbb{T}^3))$.*
3. *The sequence u_{α_k} converges to u and θ_{α_k} converges to θ weakly in $L^2(\mathbb{T}^3)$ and uniformly over $[0,T]$. Furthermore, (u,θ) is the weak solution of the Boussinesq system (Bq_0) on $[0,T]$ associated with the initial data (u^0, θ^0) satisfying for all $t \in [0,T]$ the energy inequality*

$$\|\theta\|_{L^2}^2 + \|u\|_{L^2}^2 + \int_0^t \|\nabla\theta\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\, d\tau \leq \|\theta^0\|_{L^2}^2 + \|u^0\|_{L^2}^2 + \sigma_0(t). \tag{1.2}$$

Here, (Bq_0) and σ_0 denote, respectively, (Bq_α) and σ_α for $\alpha = 0$.

The purpose of the proof is to extract subsequences that converge to the solution of (Bq) as $\alpha \to 0^+$. First, we derive a uniform bound independent of the parameter α. This gives the weak convergence. Then, following the lines of the existence proof, we establish strong convergence of such subsequences in suitable spaces. This strong convergence allows to take the limit in the quadratic terms, and hence a weak convergence of the unique weak solution of (Bq) to a weak solution of (Bq) is proved and the associated energy estimate is derived.

The remainder of the paper is organized as follows. We start with recalling some useful background. Section 3 is devoted to the proof of the existence result and the continuous dependence of the weak solution on the initial data, in particular, uniqueness. In Section 4, we investigate the long time behaviour of the regularized temperature and the regularized velocity. Section 5 is devoted to proving several convergence results, as the regularizing parameter α vanishes.

2 Preliminary results

For $n \in N$, let P_n denote the projection into the Fourier modes of order up to n, that is,

$$P_n\Big(\sum_{k\in Z^3} \widehat{u}_k e^{ik\cdot x}\Big) = \sum_{|k|\le n} \widehat{u}_k e^{ik\cdot x}.$$

We define for $s \geq 0$ the operator Λ^s acting on $H^s(\mathbb{T}^3)$ by

$$\Lambda^s u(x) = \sum_{k\in Z^3} |k|^s \widehat{u}_k e^{ik\cdot x} \in L^2(\mathbb{T}^3).$$

Moreover, we denote by $\|\cdot\|_{\dot H^s}$ the seminorm $\|\cdot\|_{L^2}$. This is, of course, compatible with the definition of the Sobolev norm that $\|\cdot\|_{H^s}$ is equivalent to $\|\cdot\|_{L^2} + \|\cdot\|_{\dot H^s}$. We will also make use of the fact that $\|u\|_{\dot H^s} \le \|u\|_{\dot H^t}$ if $0 < s \le t$ and $\Lambda^2 = -\Delta$. Moreover, if $\operatorname{div} u = 0$, we have $(v\cdot\nabla u, u)_{L^2(\mathbb{T}^3)} = 0$ and $(u\cdot\nabla\theta, \theta)_{L^2(\mathbb{T}^3)} = 0$. Finally, we recall the version of the Aubin–Lions Theorem that will be used.

Lemma 2.1. *Let X_0, X and X_1 be three Banach spaces with $X_0 \subset X \subset X_1$. Suppose that X_0 is compactly embedded in X and X is continuously embedded in X_1. For $1 \le p,\ q \le \infty$, let*

$$\mathcal{W} = \Big\{u \in L^p([0,T], X_0) :\ \frac{du}{dt} \in L^q([0,T], X_1)\Big\}.$$

- *If $p < +\infty$, then the embedding of $\mathcal{W}$ into $L^p([0,T]; X)$ is compact.*
- *If $p = +\infty$ and $q > 1$, then the embedding of $\mathcal{W}$ into $C([0,T]; X)$ is compact.*

Also, we need the following inequalities:

$$\|\vartheta\|_{L^3} \le \|\vartheta\|_{L^2}^{1/2} \|\nabla\vartheta\|_{L^2}^{1/2}, \tag{2.1}$$

$$\|\vartheta\|_{L^\infty} \le \|\vartheta\|_{\dot H^1}^{1/2} \|\vartheta\|_{\dot H^2}^{1/2}, \tag{2.2}$$

$$\|\vartheta\|_{L^6} \le \|\nabla\vartheta\|_{L^2}. \tag{2.3}$$

3 Existence and uniqueness results

Let $u_n = P_n u$. One approximates the continuous problem (Bq_α) by the following problem denoted by $(Bq_\alpha)_n$:

$$\partial_t\theta_n - \Delta\theta_n + P_n \operatorname{div}(\theta_n u_n) = 0, \tag{3.1}$$

$$\partial_t v_n - \Delta v_n + P_n \operatorname{div}(v_n u_n) - \theta_n e_3 = P_n \nabla\Delta^{-1}\Big(\sum_{i,j=1}^{3} \partial_i\partial_j(v_n^i u_n^j) - \partial_3\theta_n\Big), \tag{3.2}$$

$$v_n = u_n - \alpha^2 \Delta u_n, \tag{3.3}$$

$$\operatorname{div} u_n = \operatorname{div} v_n = 0, \tag{3.4}$$

$$(u_n, \theta_n)_{t=0} = (u_n^0, \theta_n^0) = (P_n u^0, P_n \theta^0). \tag{3.5}$$

The ordinary differential equation theory implies that there exists some maximal $T_n^* > 0$ and a unique local solution $u_n \in C^\infty([0, T_n^*) \times \mathbb{T}^3)$ to $(Bq_\alpha)_n$. Taking the inner product of (3.1) by θ_n and (3.2) by u_n, applying the Cauchy–Schwarz inequality to the forcing term $< \theta_n e_3, u_n >_{L^2}$ and dropping the viscous term, we obtain

$$\frac{d}{dt}\big(\|\theta_n\|_{L^2}^2 + \|u_n\|_{L^2}^2 + \alpha^2\|\nabla u_n\|_{L^2}^2\big) \le 2\big(\|\theta_n\|_{L^2}^2 + \|u_n\|_{L^2}^2 + \alpha^2\|\nabla u_n\|_{L^2}^2\big).$$

Let

$$\phi(t) = \|\theta_n\|_{L^2}^2 + \|u_n\|_{L^2}^2 + \alpha^2\|\nabla u_n\|_{L^2}^2,$$

then the above equation reads $\phi'(t) \leq 2\phi(t)$. Applying Grönwall' s inequality and integrating over $[0,t]$, we obtain $\phi(t) \leq \phi(0)e^{2t}$. Thus,

$$\|\theta_n(t)\|_{L^2}^2 + \|u_n(t)\|_{L^2}^2 + \alpha^2\|\nabla u_n(t)\|_{L^2}^2 \leq (\|\theta_n^0\|_{L^2}^2 + \|u_n^0\|_{L^2}^2 + \alpha^2\|\nabla u_n^0\|_{L^2}^2)e^{2t}.$$

This implies that

$$\|\theta_n(t)\|_{L^2}^2 + \|u_n(t)\|_{L^2}^2 + \alpha^2\|\nabla u_n(t)\|_{L^2}^2 + 2\int_0^t \|\nabla\theta_n(\tau)\|_{L^2(\mathbb{T}^3)}^2\, d\tau$$
$$+ 2\int_0^t (\|\nabla u_n(\tau)\|_{L^2}^2 + \alpha^2\|\Delta u_n(\tau)\|_{L^2}^2)\, d\tau \leq \|\theta_n^0\|_{L^2}^2 + \|u_n^0\|_{L^2}^2 + \alpha^2\|\nabla u_n^0\|_{L^2}^2 + \sigma_\alpha(t),$$

where

$$\sigma_\alpha(t) = (e^{2t}-1)\big(\|\theta_n^0\|_{L^2}^2 + \|u_n^0\|_{L^2}^2 + \alpha^2\|\nabla u_n^0\|_{L^2}^2\big).$$

So, the maximal solution to problem (3.1)–(3.5) is global and $T_n^* = +\infty$.

Using the product laws and interpolation inequality, we obtain

$$\|\operatorname{div}(v_n \otimes u_n)\|_{\dot{H}^{-2}} \leq \|v_n\|_{L^2}\|u_n\|_{L^2}^{1/2}\|u_n\|_{\dot{H}^1}^{1/2}.$$

Hence, $\frac{d}{dt}v_n \in L^2([0,T],\dot{H}^{-2})$. We denote by $\mathcal{W}$ the set of functions defined by

$$\mathcal{W} = \Big\{u_n:\ u_n \in L^2([0,T],\ \dot{H}^2(\mathbb{T}^3)),\ \frac{du_n}{dt} \in L^2([0,T],L^2(\mathbb{T}^3))\Big\}.$$

By the Aubin–Lions Theorem, we conclude that there is a subsequence $u_{n'}$ such that $u_{n'} \rightharpoonup u_\alpha$ weakly in $L^2([0,T],\dot{H}^2(\mathbb{T}^3))$, and $u_{n'} \to u_\alpha$ strongly in $L^2([0,T],\dot{H}^1(\mathbb{T}^3))$, moreover, $u_{n'} \to u_\alpha$ in $C([0,T],L^2(\mathbb{T}^3))$. Likewise, if we denote

$$\mathcal{W}' = \Big\{\theta_n:\ \theta_n \in L^2([0,T],\ \dot{H}^1(\mathbb{T}^3)),\ \frac{d\theta_n}{dt} \in L^2([0,T],\dot{H}^{-1}(\mathbb{T}^3))\Big\},$$

then there exists θ_α such that $\theta_{n'} \rightharpoonup \theta_\alpha$ weakly in $L^2([0,T],\dot{H}^1(\mathbb{T}^3))$, and $\theta_{n'} \to \theta_\alpha$ strongly in $L^2([0,T],L^2(\mathbb{T}^3))$, moreover, $\theta_{n'} \to \theta_\alpha$ in $C([0,T],\dot{H}^{-1}(\mathbb{T}^3))$. Further, we relabel $u_{n'}$, $v_{n'}$ and $\theta_{n'}$ by u_n, v_n and θ_n and note that the strong convergence is compulsory when taking the limit in the nonlinear term. Let us begin with proving that

$$\lim_{n\to+\infty} P_n[(u_n\nabla)\theta_n] = [(u_\alpha\nabla)\theta_\alpha]$$

in $\mathcal{D}'(\mathbb{R}_+^* \times \mathbb{T}^3)$. Let $\Psi \in \dot{H}^2$ be a vector divergence-free test function, $\Phi \in \dot{H}^1$ be a scalar test function, and $\forall t \in \mathbb{R}^+$,

$$I_n^1 = \int_0^t \big\langle P_n\big[(u_n - u_\alpha)\nabla\theta_n\big],\Phi\big\rangle_{L^2}\, d\tau,$$

$$I_n^2 = \int_0^t \big\langle P_n\big[(u_\alpha)\nabla(\theta_n - \theta_\alpha)\big],\Phi\big\rangle_{L^2}\, d\tau,$$

$$I_n^3 = \int_0^t \big\langle (P_n - I)(u_\alpha\nabla)\theta_\alpha,\Phi\big\rangle_{L^2}\, d\tau.$$

Using, respectively, the Cauchy–Schwarz inequality and Sobolev product laws, we obtain

$$|I_n^1| \leq \|u_n - u_\alpha\|_{L^2([0,T],\dot{H}^1)}\|\theta_n\|_{L^2([0,T],\dot{H}^1)}\|\Phi\|_{\dot{H}^1},$$
$$|I_n^2| \leq \|u_\alpha\|_{L^2([0,T],\dot{H}^2)}\|\theta_n - \theta_\alpha\|_{L^2([0,T],L^2)}\|\Phi\|_{\dot{H}^1}.$$

As for I_n^3, first, we estimate the term

$$\big\langle (P_n - I)(u_\alpha \nabla)\theta_\alpha, \Phi\big\rangle_{L^2} = \int_{\mathbb{T}^3} \sum_{|k|>n} \widehat{(u_{\alpha,k}\nabla)\theta}_{\alpha,k} e^{ik\cdot x}\Phi\,dx$$
$$\leq \int_{\mathbb{T}^3} \sum_{|k|>n} \frac{|k|}{n} \widehat{(u_{\alpha,k}\nabla)\theta}_{\alpha,k} e^{ik\cdot x}\Phi\,dx \leq \frac{1}{n}\int_{\mathbb{T}^3} \Lambda(\operatorname{div}(u_\alpha\theta_\alpha))\Phi\,dx.$$

Then, by inequality (2.2) and Hölder's inequality, we obtain

$$|I_n^3| \leq \frac{1}{n}\int_0^t \big\|\Lambda(\operatorname{div}(u_\alpha\theta_\alpha))\big\|_{\dot H^{-1}}\|\Phi\|_{\dot H^1}\,d\tau \leq \frac{1}{n}\|u_\alpha\|_{L^2([0,T],\dot H^2)}\|\theta_\alpha\|_{L^2([0,T],\dot H^1)}\|\Phi\|_{\dot H^1}.$$

Now, let us prove that

$$\lim_{n\to+\infty} P_n(v_n\cdot\nabla)u_n = (v_\alpha\cdot\nabla)u_\alpha$$

in $\mathcal{D}'(\mathbb{R}_+^*\times\mathbb{T}^3)$. Let

$$J_n^1 = \int_0^t \big\langle P_n(v_n - v_\alpha)\cdot\nabla u_n, \Psi\big\rangle_{L^2}\,d\tau,$$
$$J_n^2 = \int_0^t \big\langle P_n v_\alpha\cdot\nabla(u_n - u_\alpha), \Psi\big\rangle_{L^2}\,d\tau,$$
$$J_n^3 = \int_0^t \big\langle (P_n - I)(v_\alpha\cdot\nabla)u_\alpha, \Psi\big\rangle_{L^2}\,d\tau.$$

As for J_n^1, we have

$$|J_n^1| \leq \int_0^t \big\|(v_n - v_\alpha)\cdot\nabla u_n\big\|_{\dot H^{-2}}\|\Psi\|_{\dot H^2}\,d\tau$$
$$\leq c\int_0^t \|v_n - v_\alpha\|_{\dot H^{-1}}\|\nabla u_n\|_{\dot H^{1/2}}\|\Psi\|_{\dot H^2}\,d\tau \leq c\|v_n - v_\alpha\|_{L^2([0,T],\dot H^{-1})}\|u_n\|_{L^2([0,T],\dot H^2)}\|\Psi\|_{\dot H^2}.$$

Since u_n is bounded in $L^2([0,T],\dot H^2)$ and $v_n\to v_\alpha$ in $L^2([0,T],\dot H^{-1})$, we get $\lim\limits_{n\to+\infty} J_n^1 = 0$. Applying the Cauchy–Schwarz inequality and Sobolev product laws, we have

$$|J_n^2| \leq \int_0^t \big\|v_\alpha\cdot\nabla(u_n - u_\alpha)\big\|_{\dot H^{-2}}\|\Psi\|_{\dot H^2}\,d\tau$$
$$\leq \int_0^t \|v_\alpha\|_{\dot H^{-1/2}}\big\|\nabla(u_n - u_\alpha)\big\|_{L^2}\|\Psi\|_{\dot H^2}\,d\tau \leq \|v_\alpha\|_{L^2([0,T],L^2)}\|u_n - u_\alpha\|_{L^2([0,T],\dot H^1)}\|\Psi\|_{\dot H^2}.$$

Since v_α is bounded in $L^2([0,T],L^2)$ and $u_n\to u_\alpha$ strongly in $L^2([0,T],\dot H^1)$, we get $\lim\limits_{n\to+\infty} J_n^2 = 0$. As for J_n^3, at a first step, we estimate the term

$$\big\langle (P_n - I)(v_\alpha\cdot\nabla)u_\alpha, \Psi\big\rangle_{L^2} = \int_{\mathbb{T}^3}(P_n - I)(v_\alpha\cdot\nabla)u_\alpha\Psi\,dx \leq \frac{1}{n}\int_{\mathbb{T}^3}\Lambda(\operatorname{div}(v_\alpha\otimes u_\alpha))\Psi\,dx,$$

where we have used the divergence-free condition and a standard calculation. Then, by the Cauchy–Schwarz inequality and Sobolev product laws, we get

$$
\begin{aligned}
|J_n^3| &\le \frac{1}{n}\int_0^t \big\langle \Lambda(\mathrm{div}(v_\alpha\otimes u_\alpha)),\Psi\big\rangle_{L^2}\,d\tau\\
&\le \frac{1}{n}\int_0^t \big\|\Lambda(\mathrm{div}(v_\alpha\otimes u_\alpha))\big\|_{\dot H^{-2}}\|\Psi\|_{\dot H^2}\,d\tau \le \frac{1}{n}\,\|v_\alpha\|_{L^2([0,T],L^2)}\|u_\alpha\|_{L^2([0,T],\dot H^2)}\|\Psi\|_{\dot H^2}.
\end{aligned}
$$

To prove the continuity of the solution, it suffices to prove at a first step that for all $t_0\in\mathbb{R}_+$,

$$\|\theta_\alpha(t)-\theta_\alpha(t_0)\|_{L^2(\mathbb{T}^3)}\to 0 \ \text{ as } \ t\to t_0.$$

Towards this end, we have to prove that the function $t\longmapsto\|\theta_\alpha(t)\|_{L^2}$ is continuous and the function $t\longmapsto\theta_\alpha(t)$ is weakly continuous with value in $L^2(\mathbb{T}^3)$. We have $\theta_\alpha\in L^\infty(\mathbb{R}_+,L^2(\mathbb{T}^3))\cap L^2(\mathbb{R}_+,\dot H^1(\mathbb{T}^3))$, so, $\frac{d}{dt}\|\theta_\alpha(t)\|_{L^2}^2$ belongs to $L^1([0,T])$. Hence, $\|\theta_\alpha(t)\|_{L^2}^2$ belongs to $C([0,T])$. Since $\theta_\alpha\in L^2(\mathbb{R}_+,\dot H^1(\mathbb{T}^3))$ and $\Phi\in\dot H^1$, we find that as t tends to t_0, the inequality

$$\Big|\int_{t_0}^t \big\langle\nabla\theta_\alpha,\nabla\Phi\big\rangle_{L^2}\,d\tau\Big|\le\Big(\int_{t_0}^t\|\nabla\theta_\alpha(\tau)\|_{L^2}^2\,d\tau\Big)^{1/2}\Big(\int_{t_0}^t\|\nabla\Phi(\tau)\|_{L^2}^2\,d\tau\Big)^{1/2}$$

tends to zero. Using inequality (2.2) and the Cauchy–Schwarz and Hölder inequalities, we find that

$$\Big|\int_{t_0}^t \big\langle\,\mathrm{div}(\theta_\alpha u_\alpha),\Phi >_{L^2}\ d\tau\Big|\le\Big(\int_{t_0}^t\|\theta_\alpha\|_{L^2}^2\,d\tau\Big)^{1/2}\Big(\int_{t_0}^t\|u_\alpha\|_{\dot H^2}^2\,d\tau\Big)^{1/2}\|\Phi\|_{\dot H^1}$$

tends to zero as t tends to t_0. Therefore $langle\theta_\alpha(t),\Phi\rangle_{L^2}\to\langle\theta(t_0),\Phi\rangle_{L^2}$ as $t\to t_0$ for every $\Phi\in\dot H^1$. In particular, $\theta_\alpha(t)\in L^2$ and $\Phi\in\dot H^1\subset L^2$. Since the Sobolev space $\dot H^1$ is dense in L^2, we have for $t\in[0,T]$, $\langle\theta_\alpha(t),\Phi\rangle_{L^2}\to\langle\theta(t_0),\Phi\rangle_{L^2}$ as $t\to t_0$ for every $\Phi\in L^2$. Hence, $\theta_\alpha\in C([0,T),L^2)$. Similarly, we obtain $\|\nabla u_\alpha(t)-\nabla u_\alpha(t_0)\|_{L^2}^2\to 0$ as $t\to t_0$.

To prove continuous dependence of solutions on initial data, we assumer that (u,θ) and $(\overline{u},\overline{\theta})$ are any two solutions of the system (Bq_α) on the interval $[0,T]$, with initial values (u^0,θ^0) and $(\overline{u}^0,\overline{\theta}^0)$, respectively. Let us denote $v=u-\alpha^2\Delta u$, $\overline{v}=\overline{u}-\alpha^2\overline{\Delta u}$, $\delta u=u-\overline{u}$, $\delta v=v-\overline{v}$, $\delta\theta=\theta-\overline{\theta}$, and by $\delta p=p-\overline{p}$. Then

$$
\begin{aligned}
\partial_t\delta\theta-\Delta\delta\theta+(\delta u\cdot\nabla)\theta+(\overline{u}\cdot\nabla)\delta\theta&=0,\\
\partial_t\delta v-\Delta\delta v+(\delta v\cdot\nabla)u+(\overline{v}\cdot\nabla)\delta u&=-\nabla\delta p+\delta\theta e_3,\\
\delta v&=\delta u-\alpha^2\Delta\delta u,\\
\mathrm{div}\,\delta u=\mathrm{div}\,\delta v&=0,\\
(\delta u,\delta\theta)_{t=0}&=(u^0-\overline{u}^0,\theta^0-\overline{\theta}^0).
\end{aligned}
$$

We have $\frac{d}{dt}\,\delta\theta\in L^2([0,T],\dot H^{-1})$ and $\delta\theta\in L^2([0,T],\dot H^1)$. Moreover, $\frac{d}{dt}\,\delta v$ belongs to $L^2([0,T],\dot H^{-2})$ and $\delta u\in L^2([0,T],\dot H^2)$. By appropriate duality action, for almost every time t in $[0,T]$ we have

$$
\begin{aligned}
\Big\langle\frac{d}{dt}\,\delta\theta,\delta\theta\Big\rangle_{\dot H^{-1}}+\|\nabla\delta\theta\|_{L^2}^2+\big\langle\delta u\cdot\nabla\theta,\delta\theta\big\rangle_{\dot H^{-1}}&=0,\\
\Big\langle\frac{d}{dt}\,\delta v,\delta u\Big\rangle_{\dot H^{-2}}+\big(\|\nabla\delta u\|_{L^2}^2+\alpha^2\|\Delta\delta u\|_{L^2}^2\big)+\langle\delta v\cdot\nabla u,\delta u\rangle_{\dot H^{-2}}&=\langle\delta\theta,\delta u\rangle_{\dot H^{-1}}.
\end{aligned}
$$

Using the fact that (see, e.g., [8, Chapter 3, p. 169])

$$
\begin{aligned}
\Big\langle\frac{d}{dt}\,\delta\theta,\delta\theta\Big\rangle_{\dot H^{-1}(\mathbb{T}^3)}&=\frac{1}{2}\,\frac{d}{dt}\,\|\delta\theta\|_{L^2(\mathbb{T}^3)}^2,\\
\Big\langle\frac{d}{dt}\,\delta v,\delta u\Big\rangle_{\dot H^{-2}(\mathbb{T}^3)}&=\frac{1}{2}\,\frac{d}{dt}\,\big(\|\delta u\|_{L^2(\mathbb{T}^3)}^2+\alpha^2\|\nabla\delta u\|_{L^2(\mathbb{T}^3)}^2\big),
\end{aligned}
$$

and summing up, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\big(\|\delta u\|^2_{L^2(\mathbb{T}^3)}+\alpha^2\|\nabla\delta u\|^2_{L^2(\mathbb{T}^3)}+\|\delta\theta\|^2_{L^2(\mathbb{T}^3)}\big)\\ +\big(\|\nabla\delta u\|^2_{L^2(\mathbb{T}^3)}+\alpha^2\|\Delta\delta u\|^2_{L^2(\mathbb{T}^3)}\big)+\|\nabla\delta\theta\|^2_{L^2(\mathbb{T}^3)}\\ =\langle\delta\theta,\delta u\rangle_{\dot H^{-1}(\mathbb{T}^3)}\underbrace{-\langle\delta v\cdot\nabla u,\delta u\rangle_{\dot H^{-2}(\mathbb{T}^3)}}_{I_2}\underbrace{-\langle\delta u\cdot\nabla\theta,\delta\theta\rangle_{\dot H^{-1}(\mathbb{T}^3)}}_{I_3}.\end{aligned}$$

Using, respectively, the Cauchy–Schwarz and Young's inequalities, we obtain

$$\big|\langle\delta\theta,\delta u\rangle_{\dot H^{-1}(\mathbb{T}^3)}\big|\le\frac{1}{2}\big(\|\delta u\|^2_{L^2}+\|\delta\theta\|^2_{L^2}\big). \tag{3.6}$$

For I_2, we note that

$$\big|\langle\delta v\cdot\nabla u,\delta u\rangle_{\dot H^{-2}(\mathbb{T}^3)}\big|=\big|\langle\delta v\cdot\nabla u,\delta u\rangle_{L^2(\mathbb{T}^3)}\big|\le\|\delta u\|_{L^\infty(T^3)}\|\nabla u\|_{L^2(T^3)}\|\delta v\|_{L^2(T^3)}.$$

Using inequality (2.2), we obtain

$$|I_2|\le C\|\delta v\|_{L^2(T^3)}\|\nabla u\|_{L^2(T^3)}\|\delta u\|^{1/2}_{\dot H^1(T^3)}\|\delta u\|^{1/2}_{\dot H^2(T^3)}.$$

The velocity has zero average for positive times, thus we have

$$\|\delta v\|_{L^2(T^3)}\le(c+\alpha^2)\|\Delta\delta u\|_{L^2(T^3)}, \tag{3.7}$$

using (3.7) and Young's inequality, we obtain

$$\begin{aligned}|I_2|&\le C(c+\alpha^2)\|\nabla u\|_{L^2(T^3)}\|\delta u\|^{1/2}_{\dot H^1(T^3)}\|\delta u\|^{3/2}_{\dot H^2(T^3)}\\ &\le\frac{C}{\alpha^6}(c+\alpha^2)^4\|\nabla u\|^4_{L^2(T^3)}\|\nabla\delta u\|^2_{L^2(T^3)}+\frac{\alpha^2}{2}\|\Delta\delta u\|^2_{L^2(T^3)}.\end{aligned} \tag{3.8}$$

To estimate I_3, we use the Cauchy–Schwarz inequality twice to obtain

$$\big|\langle\delta u\cdot\nabla\theta,\delta\theta\rangle_{\dot H^{-1}(\mathbb{T}^3)}\big|\le\|\delta u\|_{L^3}\|\nabla\theta\|_{L^2}\|\delta\theta\|_{L^6}.$$

Next, inequalities (2.1), (2.3) and Sobolev's norm definition imply that

$$\big|\langle\delta u\cdot\nabla\theta,\delta\theta\rangle_{\dot H^{-1}(\mathbb{T}^3)}\big|\le\|\delta u\|^{1/2}_{L^2}\|\delta u\|^{1/2}_{\dot H^1}\|\nabla\theta\|_{L^2}\|\delta\theta\|_{\dot H^1}\le\|\delta u\|^{1/2}_{L^2}\|\nabla\delta u\|^{1/2}_{L^2}\|\nabla\theta\|_{L^2}\|\nabla\delta\theta\|_{L^2}.$$

Using twice the Young product inequality, we obtain

$$|I_3|\le\frac{1}{4\alpha}\big(\|\delta u\|^2_{L^2}+\alpha^2\|\nabla\delta u\|^2_{L^2}\big)\|\nabla\theta\|^2_{L^2}+\frac{1}{2}\|\nabla\delta\theta\|^2_{L^2}. \tag{3.9}$$

Summing up estimates (3.6), (3.8) and (3.9), we infer that

$$\begin{aligned}\frac{d}{dt}\big(\|\delta u\|^2_{L^2}+\alpha^2\|\nabla\delta u\|^2_{L^2}+\|\delta\theta\|^2_{L^2}\big)+\big(\|\nabla\delta u\|^2_{L^2}+\alpha^2\|\Delta\delta u\|^2_{L^2}\big)+\|\nabla\delta\theta\|^2_{L^2}\\ \le g(t)\big(\|\delta u\|^2_{L^2(\mathbb{T}^3)}+\alpha^2\|\nabla\delta u\|^2_{L^2(\mathbb{T}^3)}+\|\delta\theta\|^2_{L^2(\mathbb{T}^3)}\big),\end{aligned}$$

where

$$g(t)=\Big(1+C\Big(\frac{1}{\alpha^8}+1\Big)\|\nabla u\|^4_{L^2}+\frac{1}{2\alpha}\|\nabla\theta\|^2_{L^2}\Big).$$

Dropping the dissipative positive term from the left-hand side, we obtain

$$\frac{d}{dt}\big(\|\delta u\|^2_{L^2(\mathbb{T}^3)}+\alpha^2\|\nabla\delta u\|^2_{L^2(\mathbb{T}^3)}+\|\delta\theta\|^2_{L^2(\mathbb{T}^3)}\big)\le g(t)\big(\|\delta u\|^2_{L^2(\mathbb{T}^3)}+\alpha^2\|\nabla\delta u\|^2_{L^2(\mathbb{T}^3)}+\|\delta\theta\|^2_{L^2(\mathbb{T}^3)}\big).$$

Since $\theta\in L^2([0,T],\dot H^1)$ and $u\in L^\infty([0,T],\dot H^1)$, Grönwall's lemma (cf. [5, Appendix A, p. 377]) leads to

$$\big(\|\delta u\|^2_{L^2(\mathbb{T}^3)}+\alpha^2\|\nabla\delta u\|^2_{L^2(\mathbb{T}^3)}+\|\delta\theta\|^2_{L^2(\mathbb{T}^3)}\big)\le\big(\|\delta u^0\|^2_{L^2(\mathbb{T}^3)}+\alpha^2\|\nabla\delta u^0\|^2_{L^2(\mathbb{T}^3)}+\|\delta\theta^0\|^2_{L^2(\mathbb{T}^3)}\big)e^{\int_0^t g(s)\,ds}.$$

This implies the continuous dependence of the weak solution on the initial data in any bounded interval of time $[0,T]$. In particular, the solution is unique.

4 Decay results

Following [1], we introduce the change of functions $\varphi_n := \mathcal{F}^{-1}(e^{at|k|}\widehat{\theta}_n)$ and $w_n := \mathcal{F}^{-1}(e^{at|k|}\widehat{u}_n)$. Applying Fourier transform to (3.1) and to (3.2), we obtain

$$\partial_t\widehat{\varphi}_n + |k|(|k|-a)\widehat{\varphi}_n + e^{at|k|}\mathcal{F}(P_n(u_n\cdot\nabla\theta_n)) = 0, \tag{4.1}$$

$$(1+\alpha^2|k|^2)\big(\partial_t\widehat{w}_n + |k|(|k|-a)\widehat{w}_n\big) - \widehat{\varphi}_n e_3 + e^{at|k|}\mathcal{F}(P_n(v_n\cdot\nabla\theta_n)) = 0. \tag{4.2}$$

We note that under the divergence free condition, the pressure term vanishes. The Plancherel identity implies that the trilinear expressions vanish as $(v\cdot\nabla u, u)_{L^2} = 0$ and $(u\cdot\nabla\theta,\theta)_{L^2} = 0$. Taking the combinations $\overline{(4.1)}\widehat{\varphi}_n + (4.1)\overline{\widehat{\varphi}_n}$ and $\overline{(4.2)}\widehat{w}_n + (4.2)\overline{\widehat{w}_n}$, using the Cauchy–Schwarz inequality and the fact that

$$(1-a)|k|^2 \le |k|(|k|-a) \quad \forall\, k\in Z^3,$$

one obtains

$$\partial_t|\widehat{\varphi}_n|^2 + 2(1-a)|k|^2|\widehat{\varphi}_n|^2 = 0 \tag{4.3}$$

and

$$(1+\alpha^2|k|^2)\partial_t|\widehat{w}_n|^2 + 2(1-a)|k|^2(1+\alpha^2|k|^2)|\widehat{w}_n|^2 \le |\widehat{\varphi}_n|\,|\widehat{w}_n|. \tag{4.4}$$

Integrating (4.3) with respect to time and summing up over $k\in Z^3$, we obtain

$$\|\varphi(t,\cdot)\|_{L^2}^2 + (1-a)\int_0^t \|\nabla\varphi(\tau)\|_{L^2}^2\,d\tau \le \|\theta^0\|_{L^2}^2. \tag{4.5}$$

Integrating (4.4) with respect to time and summing up over $k\in Z^3$, we obtain

$$\|w(t)\|_{L^2}^2 + \alpha^2\|\nabla w(t)\|_{L^2}^2 + (1-a)\int_0^t \|\nabla w(s)\|_{L^2}^2 + \alpha^2\|\Delta w(s)\|_{L^2}^2\,ds$$
$$\le \|u^0\|_{L^2}^2 + \alpha^2\|\nabla u^0\|_{L^2}^2 + \|\theta^0\|_{L^2}\int_0^t \|w(\tau)\|_{L^2}\,d\tau.$$

Since $\partial_t|\widehat{w}_n|^2 \le |\widehat{\varphi}_n||\widehat{w}_n|$, we can deduce that

$$\|w(t)\|_{L^2}^2 + \alpha^2\|\nabla w(t)\|_{L^2}^2 + (1-a)\int_0^t \|\nabla w(s)\|_{L^2}^2 + \alpha^2\|\Delta w(s)\|_{L^2}^2\,ds$$
$$\le \big(\|u^0\|_{L^2}^2 + \alpha^2\|\nabla u^0\|_{L^2}^2 + t\|\theta^0\|_{L^2}\big)^2. \tag{4.6}$$

Summing up estimates (4.5) and (4.6), one obtains

$$\|\varphi(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2 + \alpha^2\|\nabla w(t)\|_{L^2}^2 + (1-a)\int_0^t \|\nabla\varphi(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2 + \alpha^2\|\Delta w(t)\|_{L^2}^2$$
$$\le \big(\|\theta^0\|_{L^2}^2 + \|u^0\|_{L^2}^2 + \alpha^2\|\nabla u^0\|_{L^2}^2 + t\|\theta^0\|_{L^2}\big)^2.$$

As for the existence result, this energy estimate allows to run a standard compactness argument and to obtain the existence of (φ, w) such that $\varphi\in C(\mathbb{R}^+, L^2)\cap L^2(\mathbb{R}^+, H^1)$ and $w\in C(\mathbb{R}^+, H^1)\cap L^2(\mathbb{R}^+, H^2)$. In particular,

$$\sum_{k\in Z^3} e^{2at|k|}\big(|\theta(t,k)|^2 + (1+\alpha^2|k|^2)|u(t,k)|^2\big) \le \big(\|\theta^0\|_{L^2}^2 + \|u^0\|_{L^2}^2 + \alpha^2\|\nabla u^0\|_{L^2}^2 + t\|\theta^0\|_{L^2}\big)^2. \tag{4.7}$$

For zero-mean valued (θ, u), multiplying by $\exp(-2at)$, we deduce that θ and u vanish, respectively, in the L^2 and H^1 norm as time tends to infinity. Note that estimation (4.7) does not allow to deduce the decay result, so a sharper estimation is needed.

5 Convergence results

As α is destined to vanish, we can suppose that there exists a fixed α_0 such that $0 < \alpha \le \alpha_0$. It follows that

$$\|\theta_\alpha\|_{L^2}^2 + \|u_\alpha\|_{L^2}^2 + \alpha^2\|\nabla u_\alpha\|_{L^2}^2 + 2\int_0^t \|\nabla\theta_\alpha\|_{L^2(\mathbb{T}^3)}^2\,d\tau + 2\int_0^t \left(\|\nabla u_\alpha\|_{L^2}^2 + \alpha^2\|\Delta u_\alpha\|_{L^2}^2\right)d\tau \le \|\theta^0\|_{L^2}^2 + \|u^0\|_{L^2}^2 + \alpha_0^2\|\nabla u^0\|_{L^2}^2 + \sigma_{\alpha_0}(t). \quad (5.1)$$

This implies that θ_α and u_α are uniformly bounded in $L^2([0,T],\dot{H}^1(\mathbb{T}^3))$ and v_α is uniformly bounded in $L^2([0,T],L^2(\mathbb{T}^3))$, then the Banach–Alaoglu theorem [6] allows to extract subsequences (u_α), (v_α), and (θ_α) such that $(\theta_\alpha,u_\alpha) \rightharpoonup (\theta,u)$ weakly in $L^2([0,T],\dot{H}^1(\mathbb{T}^3))$ and $v_\alpha \rightharpoonup u$ weakly in $L^2([0,T],L^2(\mathbb{T}^3))$ as $\alpha\to 0$. Using the energy estimate, we infer that (u_α,θ_α) converges to (u,θ) weakly in $L^2(\mathbb{T}^3)$ and uniformly over $[0,T]$. At this step, we have proved the two first results of statements 1 and 2 and the third statement of Theorem 1.3.

About time derivatives, since θ_α is uniformly bounded independently on α in the space $L^2([0,T],\dot{H}^1(\mathbb{T}^3))$, we find that $\Delta\theta_\alpha$ belongs to $L^2([0,T],\dot{H}^{-1}(\mathbb{T}^3))$. Furthermore, the energy estimate (5.1) implies that

$$\int_0^T \|\operatorname{div}\theta_\alpha u_\alpha\|_{\dot{H}^{-3/2}}^2 \le \|\theta_\alpha\|_{L^\infty([0,T],L^2)}^2\|u_\alpha\|_{L^2([0,T],\dot{H}^1)}^2 \le \frac{1}{2}\left(\|\theta^0\|_{L^2}^2 + \|u^0\|_{L^2}^2 + \alpha_0^2\|\nabla u^0\|_{L^2}^2 + \sigma_{\alpha_0}(t)\right)^2.$$

Then we obtain

$$\left\|\frac{d}{dt}\,\theta_\alpha\right\|_{L^2([0,T],\dot{H}^{-3/2})} \le K_1,$$

where K_1 is a real positive constant. To handle the velocity derivatives, we apply the operator $(I-\alpha^2\Delta)^{-1}$ to the equation (3.2) and obtain

$$\frac{d}{dt}\,u_\alpha = \Delta u_\alpha - (I-\alpha^2\Delta)^{-1}(v_\alpha\cdot\nabla)u_\alpha + (I-\alpha^2\Delta)^{-1}\nabla p_\alpha + (I-\alpha^2\Delta)^{-1}\theta_\alpha e_3. \quad (5.2)$$

We have that u_α is uniformly bounded independently of α in $L^2([0,T],\dot{H}^1(\mathbb{T}^3))$, and it follows that Δu_α belongs to $L^2([0,T],\dot{H}^{-1}(\mathbb{T}^3))$. First, we note that

$$\big\|\,|(I-\alpha^2\Delta)^{-1}|\,\big\| \le 1.$$

Then we use the Sobolev norms definition and product laws to get

$$\int_0^T \left\|(I-\alpha^2\Delta)^{-1}\operatorname{div}(v_\alpha\otimes u_\alpha)\right\|_{\dot{H}^{-5/2}}^2 \le \int_0^T \left\|\operatorname{div}(v_\alpha\otimes u_\alpha)\right\|_{\dot{H}^{-5/2}}^2 \le \int_0^T \|v_\alpha\|_{L^2}^2\|u_\alpha\|_{L^2}^2 \le \|u_\alpha\|_{L^\infty([0,T],L^2)}^2\|v_\alpha\|_{L^2([0,T],L^2)}^2.$$

Thus, estimate (5.1) allows to bound the convective term. The linear terms are not problematic. Equation (5.2) implies that $\|\frac{d}{dt}\,u_{\alpha_k}\|_{L^2([0,T],\dot{H}^{-5/2}(\mathbb{T}^3))} \le K$, where K is a real positive constant, and so on for $\frac{d}{dt}v_{\alpha_k}$ in the space $L^2([0,T],\dot{H}^{-9/2}(\mathbb{T}^3))$.

At this step, using Aubin's compactness theorem, we can extract subsequences of θ_α, u_α that converge strongly in $L^2([0,T],L^2(\mathbb{T}^3))$ and subsequence of v_α converging strongly in $L^2([0,T],\dot{H}^{-1}(\mathbb{T}^3))$.

Thus, as in the existence section, using Aubin's compactness theorem, we can take the weak limit in the variational formulation associated to the system (Bq_α). For $t \in [0;T]$ one obtains

$$(\theta(t), \Phi) - (\theta(0), \Phi) - \int_0^t (\theta, \Delta\Phi)\, d\tau + \int_0^t ((u\nabla)\theta, \Phi)\, d\tau = 0,$$

$$(u(t), \Psi) - (u(0), \Psi) - \int_0^t (u, \Delta\Psi)\, d\tau + \int_0^t ((u\nabla)u, \Psi)\, d\tau - \int_0^t (\theta e_3, \Psi)\, d\tau = 0$$

for all Φ and Ψ belonging to the space of infinitely differentiable functions with a compact support $\mathcal{D}(\mathbb{T}^3 \times [0,T))$.

On the other hand, θ_α converges weakly to θ and u_α converges weakly to u in $L^2([0,T], L^2(\mathbb{T}^3)) \cap L^2([0,T], \dot{H}^1(\mathbb{T}^3))$, which are Hilbert spaces. So, for all non-negative time t, we have

$$\|\theta\|_{L^2}^2 + \|u\|_{L^2}^2 \leq \liminf_{\alpha\to 0} \left(\|\theta_\alpha\|_{L^2}^2 + \|u_\alpha\|_{L^2}^2 + \alpha^2\|\nabla u_\alpha\|_{L^2}^2\right),$$

and

$$2\int_0^t \|\nabla\theta\|_{L^2(\mathbb{T}^3)}^2\, d\tau + 2\int_0^t \|\nabla u\|_{L^2}^2\, d\tau \leq \liminf_{\alpha\to 0} 2\int_0^t \|\nabla\theta_\alpha\|_{L^2(\mathbb{T}^3)}^2\, d\tau + 2\int_0^t \left(\|\nabla u_\alpha\|_{L^2}^2 + \alpha^2\|\Delta u_\alpha\|_{L^2}^2\right) d\tau.$$

Taking the lower limit as α tends to zero in the energy inequality (1.1), we obtain (1.2).

Acknowledgment

The first author gratefully acknowledge the approval and the support of this research study by the grant # SAT-2017-1-8-F-7433 from the Deanship of Scientific Research at Northern Border University, Arar, K.S.A.

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(Received 02.10.2019)

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