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ON THE REDUCTION OF THE DIFFERENTIAL MULTI-FREQUENCY SYSTEM WITH SLOWLY VARYING PARAMETERS TO A SPECIAL KIND

Abstract. For the multi-frequency system of the differential equations the right-hand sides of which are represented by a multiple Fourier series with slowly varying coefficients, the conditions are obtained under which there exists the transformation with the coefficients of similar structure leading this system to a system with slowly varying right-hand sides.

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1 Introduction

In the nonlinear mechanics, the problem of reducing a multi-frequency system of differential equations

$$\frac{d\theta}{dt} = \omega + f(\theta), \tag{1.1}$$

where $\theta \in \mathbb{R}^m$, $\omega \in \mathbb{R}^m$, $f(\theta) \in \mathbb{R}^m$ is a 2π-periodic vector-function, by the transformation of kind

$$\theta = \varphi + w(\varphi), \tag{1.2}$$

where $w(\varphi)$ is also a 2π-periodic vector-function, to the form

$$\frac{d\varphi}{dt} = \nu, \tag{1.3}$$

ν is a constant vector, is well known.

This problem is the subject of numerous studies (see, e.g., [1,3,4]). As is known, the main difficulty here is the problem of small denominators: the scalar product (k,ω) ($k = \operatorname{colon}(k_1,\dots,k_m)$, $k_j \in \mathbb{Z}$) may be arbitrarily small and it turns out to be in the denominators of the expressions representing the solution in terms of some series or iterative processes. Therefore, the vector ω is imposed the condition

$$|(k,\omega)| \geq \frac{C}{\|k\|^{m+1}}\,, \tag{1.4}$$

C is a positive constant, $\|k\| = |k_1| + \dots + |k_m|$. The use of this condition in turn generates "large numerators" that can lead to the divergence of these series and processes. This difficulty is overcome by the method of accelerated convergence [1].

In this paper we consider the system of kind

$$\frac{dx}{dt} = (\Lambda(t) + A(t,\theta))x, \quad \frac{d\theta}{dt} = \omega(t) + b(t,\theta), \tag{1.5}$$

in which t belongs to a finite, but arbitrarily large interval, $\Lambda(t)$ is a diagonal matrix, and the elements of a small matrix $A(t,\theta)$ and a small vector $b(t,\theta)$ are represented by an absolutely and uniformly convergent multiple Fourier-series with respect to θ, with slowly varying coefficients, and the variable vector $\omega(t)$ is not subject to the condition of kind (1.4). For system (1.5), under certain conditions we have proved the existence of the transformation of kind

$$x = (E + W(t,\varphi))y, \quad \theta = \varphi + w(t,\varphi), \tag{1.6}$$

where the elements of the matrix $W(t,\theta)$ and vector $w(t,\theta)$ are of similar structure leading system (1.5) to the form

$$\frac{dy}{dt} = (\Lambda(t) + D(t))y, \quad \frac{d\varphi}{dt} = \omega(t) + \nu(t), \tag{1.7}$$

where the elements of the diagonal matrix $D(t)$ and vector $\nu(t)$ are slowly varying and do not depend on φ. The properties of $W(t,\varphi)$ and $w(t,\varphi)$ are investigated depending on the properties of $A(t,\theta)$ and $b(t,\theta)$. However, the ideas of the method of accelerated convergence are still used, because instead of the small denominators, due to the vector $\omega(t)$, here arise small denominators generated by another circumstances.

2 Basic notation and definitions

Let $\varepsilon \in (0,1]$, $\tau = \varepsilon t \in [0,L]$ ($L \in (0,+\infty)$), $G = [0,L] \times (0,1]$.

Definition 2.1. We say that a scalar function $p(\tau,\varepsilon)$, generally complex-valued, belongs to the class S, if it continuous with respect to $\tau \in [0,L]$ and bounded with respect to $\varepsilon \in (0,1]$.

Thus, $\sup\limits_{G} |p(\tau,\varepsilon)| < +\infty$.

Slowly variability of the function is understood here in the sense of its belonging to the class S.

Definition 2.2. We say that a vector-function $h(\tau,\varepsilon)=\mathrm{colon}(h_1(\tau,\varepsilon),\dots,h_m(\tau,\varepsilon))$ belongs to the class S_1, if $h_j(\tau,\varepsilon)\in S$ $(j=1,\dots,m)$.

Under the norm of a vector $h(\tau,\varepsilon)\in S_1$ is understood

$$\|h(\tau,\varepsilon)\|_0=\max_{1\le j\le N}\sup_{\tau\in[0,L]}|h_j(\tau,\varepsilon)|.$$

This norm may depend on ε.

Definition 2.3. We say that a scalar real function $f(\tau,\varepsilon,\theta)$ belongs to the class $F(M;\alpha;\theta)$, if

$$f(\tau,\varepsilon,\theta)=\sum_{k\in Z_m} f_n(\tau,\varepsilon)\exp(i(k,\theta)),$$

$Z_m=\{k=\mathrm{colon}(k_1,\dots,k_m),\ k_j\in\mathbb{Z}\}$, $\theta=\mathrm{colon}(\theta_1,\dots,\theta_m)$ is the real vector, $(k,\theta)=k_1\theta_1+\dots+k_m\theta_m$, $f_k(\tau,\varepsilon)\in S$, and

$$\sup_{\tau\in[0,L]}|f_k(\tau,\varepsilon)|\le M\exp\Big(-\frac{\alpha}{\varepsilon}\,\|k\|\Big),$$

$\|n\|=|k_1|+\dots+|k_m|$; $M\in(0,+\infty)$, $\alpha\in(0,1)$ is a constant not depending on ε.

Definition 2.4. We say that a real vector-function $h(\tau,\varepsilon,\theta)=\mathrm{colon}(h_1(\tau,\varepsilon,\theta),\dots,h_m(\tau,\varepsilon,\theta))$ belongs to the class $F_1(M;\alpha,\theta)$, if $h_j(\tau,\varepsilon,\theta)\in F(M;\alpha;\theta)$ $(j=1,\dots,m)$.

For the vector-function $h(\tau,\varepsilon,\theta)\in F_1(M;\alpha;\theta)$ and vector $k\in Z_m$ we denote

$$\Gamma_k[h(\tau,\varepsilon,\theta)]=\frac{1}{(2\pi)^m}\int_0^{2\pi}\cdots\int_0^{2\pi} h(\tau,\varepsilon,\theta)e^{-i(k,\theta)}\,d\theta_1\cdots\theta_m,$$

$$\overline{h(\tau,\varepsilon,\theta)}=\Gamma_{\vec 0}[h(\tau,\varepsilon,\theta)],\quad \widetilde{h(\tau,\varepsilon,\theta)}=h(\tau,\varepsilon,\theta)-\overline{h(\tau,\varepsilon,\theta)},$$

where $\vec 0$ is a null-vector of dimension m.

Definition 2.5. We say that a real matrix-function $A(\tau,\varepsilon,\theta)=(a_{jk}(\tau,\varepsilon,\theta))_{j,k=1,\dots,n}$ belongs to the class $F_2(M;\alpha,\theta)$, if $a_{jk}(\tau,\varepsilon,\theta)\in F(M;\alpha;\theta)$ $(j,k=1,\dots,n)$.

For the matrix $A(\tau,\varepsilon,\theta)\in F_2(M;\alpha;\theta)$ and vector $h(\tau,\varepsilon)\in S_1$ we denote

$$\Big(\frac{\partial A}{\partial\theta},h\Big)=\sum_{j=1}^m\frac{\partial A}{\partial\theta_j}\,h_j(\tau,\varepsilon).$$

3 Statement of the problem

Consider the following system of differential equations:

$$\frac{dx}{dt}=(\Lambda(\tau,\varepsilon)+A(\tau,\varepsilon,\theta))x,\quad \frac{d\theta}{dt}=\omega(\tau,\varepsilon)+b(\tau,\varepsilon,\theta),\tag{3.1}$$

where $\tau,\varepsilon\in G$, $x\in\mathbb{R}^n$, $\theta\in\mathbb{R}^m$, $\Lambda(\tau,\varepsilon)=\mathrm{diag}(\lambda_1(\tau,\varepsilon),\dots,\lambda_n(\tau,\varepsilon))$, the real functions $\lambda_j(\tau,\varepsilon)$ belong to the class S, $A(\tau,\varepsilon,\theta)\in F_2(M;\alpha;\theta)$, $\omega(\tau,\varepsilon)\in\ \mathbb{R}^m$, $\omega(\tau,\varepsilon)\in S_1$, $b(\tau,\varepsilon,\theta)\in F_1(M;\alpha;\theta)$ $(M\in(0,1))$.

We study the problem on the existence, construction and properties of the transformation of kind

$$x=(E_n+W(\tau,\varepsilon,\varphi))y,\quad \theta=\varphi+w(\tau,\varepsilon,\varphi),\tag{3.2}$$

where $y\in\mathbb{R}^n$, $\varphi\in\mathbb{R}^m$, E_n is a unit $(n\times n)$-matrix, $w(\tau,\varepsilon,\varphi)\in F_1(M_1^*;\alpha^*;\varphi)$, $W(\tau,\varepsilon,\varphi)\in F_2(M_2^*,\alpha^*,\varphi)$ $(M_1^*,M_2^*,\alpha^*$ are to be defined), whicht leads system (3.1) to the form

$$\frac{dy}{dt}=(\Lambda(\tau,\varepsilon)+D(\tau,\varepsilon))y,\quad \frac{d\varphi}{dt}=\omega(\tau,\varepsilon)+\Delta(\tau,\varepsilon),\tag{3.3}$$

where $D(\tau,\varepsilon)=\mathrm{diag}(d_1(\tau,\varepsilon),\dots,d_n(\tau,\varepsilon))$, $d_j(\tau,\varepsilon)\in S$ $(j=1,\dots,n)$, $\Delta(\tau,\varepsilon)\in S_1$.

4 Auxiliary results

Lemma 4.1. *Let the functions $p(\tau;\varepsilon)$, $q(\tau,\varepsilon)$ belong to the class S, $c = const$. Then the functions $cp(\tau,\varepsilon)$, $p(\tau,\varepsilon) \pm q(\tau,\varepsilon)$, $p(\tau,\varepsilon)q(\tau,\varepsilon)$ belong to the class S, as well.*

Lemma 4.2. *Let $0 < M_1 < M_2$, $0 < \alpha_1 < \alpha_2 < 1$. Then $F(M_1;\alpha;\theta) \subset F(M_2;\alpha;\theta)$, $F(M;\alpha_1,\theta) \supset F(M;\alpha_2;\theta)$.*

Lemma 4.3. *Let $f_j(\tau,\varepsilon;\theta) \in F(M_j;\alpha;\theta)$ $(j = 1,\dots,p)$, $c_1,\dots,c_p$ be the constants. Then*

$$\sum_{j=1}^{p} c_j f_j(\tau,\varepsilon,\theta) \in F\Big(\sum_{j=1}^{p} |c_j| M_j;\alpha;\theta\Big).$$

Lemma 4.4. *Let $p(\tau,\varepsilon) \in S$, $f(\tau,\varepsilon,\theta) \in F(M;\alpha;\theta)$, and $\sup\limits_G |p(\tau,\varepsilon)| \le P$. Then*

$$p(\tau,\varepsilon)f(\tau,\varepsilon,\theta) \in F(PM;\alpha;\theta).$$

The validity of Lemmas 4.1–4.4 is obvious.

Lemma 4.5. *Let $f(\tau,\varepsilon,\theta) \in F(M_1;\alpha;\theta)$, $g(\tau,\varepsilon,\theta) \in F(M_2;\alpha;\theta)$. Then*

$$f(\tau,\varepsilon,\theta)g(\tau,\varepsilon,\theta) \in F\Big(\frac{3^m M_1 M_2}{\delta^m}\,;\alpha-\delta;\theta\Big),$$

where $\delta \in (0,\alpha)$.

Proof. We have

$$f(\tau,\varepsilon,\theta) = \sum_{k\in Z_m} f_k(\tau,\varepsilon)e^{i(k,\theta)}, \quad g(\tau,\varepsilon,\theta) = \sum_{k\in Z_m} g_k(\tau,\varepsilon)e^{i(k,\theta)},$$

and

$$\sup_{\tau\in[0,L]} |f_k(\tau,\varepsilon)| \le M_1 e^{-\frac{\alpha}{\varepsilon}\|k\|}, \quad \sup_{\tau\in[0,L]} |g_k(\tau,\varepsilon)| \le M_2 e^{-\frac{\alpha}{\varepsilon}\|k\|}.$$

Hence

$$f(\tau,\varepsilon,\theta)g(\tau,\varepsilon,\theta) = \sum_{k\in Z_m}\Big(\sum_{l\in Z_m} f_{k-l}(\tau,\varepsilon)g_l(\tau,\varepsilon)\Big)e^{i(k,\theta)},$$

where $l = \mathrm{colon}(l_1,\dots,l_m)$, $k-l = \mathrm{colon}(k_1-l_1,\dots,k_m-l_m)$.

We have

$$\begin{aligned}
\sum_{l\in Z_m} \sup_{\tau\in[0,L]} |f_{k-l}(\tau,\varepsilon)| \sup_{\tau\in[0,L]} |g_l(\tau,\varepsilon)| &\le M_1M_2 \sum_{l\in Z_m} \exp\Big(-\frac{\alpha}{\varepsilon}(\|k-l\|+\|l\|)\Big)\\
&= M_1M_2 \sum_{l_1=-\infty}^{\infty}\cdots\sum_{l_m=-\infty}^{\infty} \exp\Big(-\frac{\alpha}{\varepsilon}(|k_1-l_1|+|l_1|+\dots+|k_m-l_m|+|l_m|)\Big)\\
&= M_1M_2\Big(\sum_{l_1=-\infty}^{\infty}\exp\Big(-\frac{\alpha}{\varepsilon}(|k_1-l_1|+|l_1|)\Big)\Big)\cdots\Big(\sum_{l_m=-\infty}^{\infty}\exp\Big(-\frac{\alpha}{\varepsilon}(|k_m-l_m|+|l_m|)\Big)\Big).
\end{aligned}$$

We denote

$$A(k_j) = \sum_{s=-\infty}^{\infty}\exp\Big(-\frac{\alpha}{\varepsilon}(|k_j-s|+|s|)\Big).$$

1. Let $k_j = 0$. We have

$$\begin{aligned}
A(0) &= \sum_{s=-\infty}^{\infty}\exp\Big(-\frac{2\alpha}{\varepsilon}|s|\Big) = 1+2\sum_{s=1}^{\infty}\exp\Big(-\frac{2\alpha}{\varepsilon}s\Big)\\
&= 1+\frac{2e^{-\frac{2\alpha}{\varepsilon}}}{1-e^{-\frac{2\alpha}{\varepsilon}}} = 1+\frac{2}{e^{\frac{2\alpha}{\varepsilon}}-1} < 1+\frac{1}{\frac{\alpha}{\varepsilon}} = 1+\frac{\varepsilon}{\alpha} < 1+\frac{1}{\alpha} = \frac{\alpha+1}{\alpha} < \frac{2}{\alpha} < \frac{2}{\delta}\,.
\end{aligned}$$

2. Let $k_j > 0$. Then

$$A(k_j) = \sum_{s=-\infty}^{-1} \exp\Big(-\frac{\alpha}{\varepsilon}(|k_j - s| + |s|)\Big) + \sum_{s=0}^{k_j} \exp\Big(-\frac{\alpha}{\varepsilon}(|k_j - s| + |s|)\Big)$$

$$+ \sum_{s=k_j+1}^{\infty} \exp\Big(-\frac{\alpha}{\varepsilon}(|k_j - s| + |s|)\Big) = \sum_{s=-\infty}^{-1} \exp\Big(-\frac{\alpha}{\varepsilon}(k_j - s - s)\Big)$$

$$+ \sum_{s=0}^{k_j} \exp\Big(-\frac{\alpha}{\varepsilon}(k_j - s + s)\Big) + \sum_{s=k_j+1}^{\infty} \exp\Big(-\frac{\alpha}{\varepsilon}(s - k_j + s)\Big)$$

$$= e^{-\frac{\alpha}{\varepsilon} n_j} \sum_{s=1}^{\infty} e^{-\frac{2\alpha}{\varepsilon} s} + (k_j + 1) e^{-\frac{\alpha}{\varepsilon} k_j} + e^{\frac{\alpha}{\varepsilon} k_j} \sum_{s=k_j+1}^{\infty} e^{-\frac{2\alpha}{\varepsilon} s} = \frac{2e^{-\frac{\alpha}{\varepsilon} k_j}}{e^{\frac{2\alpha}{\varepsilon}} - 1} + (k_j + 1) e^{-\frac{\alpha}{\varepsilon} k_j}.$$

3. Let $k_j < 0$. Similarly to the previous arguments, we show that

$$A(k_j) = \frac{2e^{\frac{\alpha}{\varepsilon} k_j}}{e^{\frac{2\alpha}{\varepsilon}} - 1} + (1 - k_j) e^{\frac{\alpha}{\varepsilon} k_j}.$$

Thus, in case $k_j \neq 0$,

$$A(k_j) = \frac{2e^{-\frac{\alpha}{\varepsilon} |k_j|}}{e^{\frac{2\alpha}{\varepsilon}} - 1} + (1 + |k_j|) e^{-\frac{\alpha}{\varepsilon} |k_j|}.$$

Hence

$$A(k_j) < \frac{e^{-\frac{\alpha}{\varepsilon} |k_j|}}{\frac{\alpha}{\varepsilon}} + (1 + |k_j|) e^{-\frac{\alpha}{\varepsilon} |k_j|} = \Big(\frac{1}{\frac{\alpha}{\varepsilon}} + 1 + |k_j|\Big) e^{-\frac{\alpha}{\varepsilon} |k_j|}.$$

We choose a constant M_0 from the condition

$$\Big(\frac{\varepsilon}{\alpha} + 1 + |k_j|\Big) e^{-\frac{\alpha}{\varepsilon} |k_j|} \leq M_0 e^{-\frac{\alpha - \delta}{\varepsilon} |k_j|},$$

where $\delta \in (0, \alpha)$. We estimate

$$\max_{|k_j| \geq 1} \Big(\frac{\varepsilon}{\alpha} + 1 + |k_j|\Big) e^{-\frac{\delta}{\varepsilon} |k_j|}.$$

For the case $x \geq 1$, let us investigate the function

$$u(x) = \Big(\frac{\varepsilon}{\alpha} + 1 + x\Big) e^{-\frac{\delta}{\varepsilon} x}.$$

We have

$$u(1) = \Big(\frac{\varepsilon}{\alpha} + 2\Big) e^{-\frac{\delta}{\varepsilon}}, \quad u'(x) = \Big(1 - \frac{\delta}{\alpha} - \frac{\delta}{\varepsilon} - \frac{\delta}{\varepsilon} x\Big) e^{-\frac{\delta}{\varepsilon} x}.$$

The critical point is $x_0 = -1 + \varepsilon/\delta - \varepsilon/\alpha$. It is easy to establish that this is the maximum point of the function $u(x)$. In case $x_0 \leq 1$, i.e., $\varepsilon\alpha/(2\alpha + \varepsilon) \leq \delta < \alpha$, we get

$$\max_{[1, +\infty)} u(x) = u(1) = \Big(\frac{\varepsilon}{\alpha} + 2\Big) e^{-\frac{\delta}{\varepsilon}}.$$

In case $x_0 > 1$, i.e., $0 < \delta < \varepsilon\alpha/(2\alpha + \varepsilon)$, we obtain

$$\max_{[1, +\infty)} u(x) = u(x_0) = \frac{\varepsilon}{\delta} e^{-(1 - \frac{\delta}{\alpha} - \frac{\delta}{\varepsilon})}.$$

Anyway,

$$\max_{[1, +\infty)} u(x) < \frac{3}{\delta},$$

therefore we can state that $M_0 = 3/\delta$.

Thus, if $k_j \neq 0$, then

$$A(k_j) < \frac{3}{\delta}\, e^{-\frac{\alpha-\delta}{\varepsilon}\,|k_j|}. \tag{4.1}$$

By virtue of the estimation for $A(0)$, we find that estimation (4.1) is true for all $k_j \in \mathbb{Z}$.

We now obtain

$$\sum_{l\in Z_m} \sup_{\tau\in[0,L]} |f_{k-l}(\tau,\varepsilon)| \sup_{\tau\in[0,L]} |g_l(\tau,\varepsilon)| \le M_1 M_2\, \frac{3^m}{\delta^m}\, e^{-\frac{\alpha-\delta}{\varepsilon}\,\|k\|},$$

and thus Lemma 4.5 is proved. □

Lemma 4.6. *Let $f(\tau,\varepsilon,\theta) \in F(M_1;\alpha;\theta)$, $g(\tau,\varepsilon,\theta) \in F(M_2;\alpha-\delta;\theta)$, $\delta \in (0,\alpha)$. Then*

$$f(\tau,\varepsilon,\theta)g(\tau,\varepsilon,\theta) \in F\Big(\frac{4^m}{\delta^m}\, M_1M_2;\alpha-\delta;\theta\Big).$$

The proof is analogous to that of Lemma 4.5.

Corollary of Lemmas 4.5 and 4.6. *Let $f_j(\tau,\varepsilon,\theta) \in F(M_j;\alpha;\theta)$ $(j=1,\dots,p,\ p\ge 2)$. Then $f_1(\tau,\varepsilon,\theta)\cdots f_p(\tau,\varepsilon,\theta) \in F(V_p;\alpha-\delta;\theta)$, where*

$$V_p = \frac{4^{m(p-1)}}{\delta^{m(p-1)}}\, M_1\cdots M_p.$$

Lemma 4.7. *Let*

$$f(\tau,\varepsilon,\theta) = \sum_{k\in Z_m} f_k(\tau,\varepsilon)e^{i(k,\theta)} \in F(M;\alpha;\theta).$$

Then

$$\frac{\partial^s f(\tau,\varepsilon,\theta)}{\partial\theta^s} = \frac{\partial^s f(\tau,\varepsilon,\theta)}{\partial\theta_1^{s_1}\cdots\partial\theta_m^{s_m}} \in F\Big(\frac{s^s}{\delta^s e^s}\, M;\alpha-\delta;\theta\Big),$$

where $s = s_1+\cdots+s_m \ge 1$, $\delta\in(0,\alpha)$.

Proof. We have

$$\frac{\partial^s f(\tau,\varepsilon,\theta)}{\partial\theta_1^{s_1}\cdots\partial\theta_m^{s_m}} = \sum_{\substack{k\in Z_m\\ (\|k\|\ge 1)}} (ik_1)^{s_1}\cdots(ik_m)^{s_m} f_k(\tau,\varepsilon)e^{i(k,\theta)},$$

$$\sum_{\substack{k\in Z_m\\ (\|k\|\ge 1)}} |k_1|^{s_1}\cdots|k_m|^{s_m} \sup_{\tau\in[0,L]} |f_k(\tau,\varepsilon)| \le M \sum_{\substack{k\in Z_m\\ (\|k\|\ge 1)}} \|k\|^s e^{-\frac{\alpha}{\varepsilon}\,\|k\|}.$$

It is easy to show that if $x\ge 1$, $s\ge 1$, then

$$x^s e^{-\frac{\delta}{\varepsilon}\,x} < \frac{s^s}{\delta^s e^s}\,.$$

Hence

$$M\|k\|e^{-\frac{\alpha}{\varepsilon}\,\|k\|} < \frac{s^s}{\delta^s e^s}\, M e^{-\frac{\alpha-\delta}{\varepsilon}\,\|k\|},$$

and Lemma 4.7 is proved. □

Lemma 4.8. *Let the vector-function $w(\tau,\varepsilon,\theta) = \operatorname{colon}(w_1(\tau,\varepsilon,\theta),\dots,w_m(\tau,\varepsilon,\theta)) \in F_1(M_1;\alpha;\theta)$, and the vector-function $v(\tau,\varepsilon,\theta) = \operatorname{colon}(v_1(\tau,\varepsilon,\theta),\dots,v_m(\tau,\varepsilon,\theta)) \in F_1(M_2;\alpha-\delta;\theta)$, where $0<\delta<\alpha$. If $\delta\in(0,\alpha/2)$ and*

$$\mu = \frac{m\cdot 4^m}{\delta^{m+1}}\, M_2 < \frac{1}{2}\,, \tag{4.2}$$

then the vector-function $w(\tau,\varepsilon,\varphi+v(\tau,\varepsilon,\varphi)) - w(\tau,\varepsilon,\varphi)$, where $\varphi = \operatorname{colon}(\varphi_1,\dots,\varphi_m)$, belongs to the class

$$F_1\Big(\frac{m^2\cdot 4^m}{\delta^{2m+2}}\, M_1(M_2+M_2^2);\alpha-2\delta;\varphi\Big).$$

Proof. We expand the scalar functions

$$\begin{aligned} & w_j(\tau,\varepsilon,\varphi+v(\tau,\varepsilon,\varphi)) \\ & \quad = w_j\big(\tau,\varepsilon,\varphi_1+v_1(\tau,\varepsilon,\varphi_1,\dots,\varphi_m),\dots,\varphi_m+v_m(\tau,\varepsilon,\varphi_1,\dots,\varphi_m)\big) \quad (j=1,\dots,N) \end{aligned}$$

with respect to $v_1,\dots,v_m$ in the Taylor series

$$w_j(\tau,\varepsilon,\varphi+v(\tau,\varepsilon,\varphi))-w_j(\tau,\varepsilon,\varphi)=dw_j(\tau,\varepsilon,\varphi)+\sum_{s=2}^{\infty}\frac{1}{s!}\,d^s w_j(\tau,\varepsilon,\varphi), \tag{4.3}$$

where

$$dw_j(\tau,\varepsilon,\varphi)=\sum_{l=1}^{m}\frac{\partial w_j(\tau,\varepsilon,\varphi)}{\partial\varphi_l}\,v_l(\tau,\varepsilon,\varphi),$$
$$d^s w_j(\tau,\varepsilon,\varphi)=\sum_{\substack{s_1+\dots+s_m=s\\(0\le s_\nu\le s)}}\frac{\partial^s w_j(\tau,\varepsilon,\varphi)}{\partial\varphi^s}\,(v(\tau,\varepsilon,\varphi))^s,$$

and

$$\frac{\partial^s w_j(\tau,\varepsilon,\varphi)}{\partial\varphi^s}=\frac{\partial^s w_j(\tau,\varepsilon,\varphi)}{\partial\varphi_1^{s_1}\cdots\partial\varphi_m^{s_m}},$$
$$(v(\tau,\varepsilon,\varphi))^s=(v_1(\tau,\varepsilon,\varphi))^{s_1}\cdots(v_m(\tau,\varepsilon,\varphi))^{s_m}, \quad s=2,3,\dots; \quad j=1,\dots,m.$$

By virtue of Lemma 4.7,

$$\frac{\partial w_j(\tau,\varepsilon,\varphi)}{\partial\varphi_\nu}\in F\Big(\frac{M_1}{\delta e}\,;\alpha-\delta;\varphi\Big), \quad \nu=1,\dots,m.$$

Due to Lemma 4.5,

$$\frac{\partial w_j(\tau,\varepsilon,\varphi)}{\partial\varphi_\nu}\,v_\nu(\tau,\varepsilon,\varphi)\in F\Big(\frac{3^N M_1 M_2}{\delta^{N+1}e}\,;\alpha-2\delta;\varphi\Big), \quad \nu=1,\dots,m,$$

if $\delta\in(0,\alpha/2)$. Therefore

$$dw_j(\tau,\varepsilon,\varphi)\in F\Big(\frac{m3^m M_1 M_2}{\delta^{m+1}e}\,;\alpha-2\delta;\varphi\Big).$$

By virtue of Lemma 4.7,

$$\frac{\partial^s w_j(\tau,\varepsilon,\varphi)}{\partial\varphi^s}\in F\Big(\frac{s^s M_1}{\delta^s e^s}\,;\alpha-\delta;\varphi\Big),$$

if $s\ge 2$, $\delta\in(0,\alpha)$.

By virtue of Corollary of Lemmas 4.5 and 4.6, if $s\ge 2$, $\delta\in(0,\alpha/2)$, we have

$$(v(\tau,\varepsilon,\varphi))^s\in F\Big(\frac{4^{m(s-1)}}{\delta^{m(s-1)}}\,M_2^s;\alpha-2\delta;\varphi\Big).$$

Then by Lemma 4.6,

$$\frac{\partial^s w_j(\tau,\varepsilon,\varphi)}{\partial\varphi^s}\,(v(\tau,\varepsilon,\varphi))^s\in F\Big(\frac{4^{ms}s^s}{\delta^{(m+1)s}e^s}\,M_1M_2^s;\alpha-2\delta;\varphi\Big).$$

Hence

$$d^s w_j(\tau,\varepsilon,\varphi)\in F(W_s;\alpha-2\delta;\varphi),$$

where

$$W_s=\frac{m^s 4^{ms}s^s}{\delta^{(m+1)s}e^s}\,M_1M_2^s.$$

We consider the expression

$$\sum_{s=2}^{\infty} \frac{W_s}{s!} = M_1 \sum_{s=2}^{\infty} \frac{\mu^s s^s}{s!\, e^s},$$

where μ is defined by formula (4.2). By virtue of the Stirling's formula [2, p. 371], we have

$$\frac{s^s}{s!\, e^s} < \frac{1}{\sqrt{2\pi s}},$$

hence

$$\sum_{s=2}^{\infty} \frac{W_s}{s!} < M_1 \sum_{s=2}^{\infty} \frac{\mu^s}{\sqrt{2\pi}\cdot\sqrt{s}} < \frac{M_1}{2\sqrt{\pi}} \sum_{s=2}^{\infty} \mu^s.$$

Due to inequality (4.2), this series is convergent, and we obtain

$$\sum_{s=2}^{\infty} \frac{W_s}{s!} < \frac{M_1}{2\sqrt{\pi}}\, \frac{\mu^2}{1-\mu} < \frac{m^2 4^{2m}}{\delta^{2m+2}}\, M_1 M_2^2.$$

Hence

$$\sum_{s=2}^{\infty} \frac{1}{s!}\, d^s w_j(\tau,\varepsilon,\varphi) \in F\Big(\frac{m^2 4^{2m}}{\delta^{2m+2}}\, M_1 M_2^2; \alpha - 2\delta; \varphi\Big).$$

Now, by virtue of (4.3), we obtain

$$w_j(\tau,\varepsilon,\varphi + v(\tau,\varepsilon,\varphi)) - w_j(\tau,\varepsilon,\varphi) \in F\bigg(M_1\Big(\frac{m 3^m}{\delta^{m+1} e}\, M_2 + \frac{m^2 4^{2m}}{\delta^{2m+2}}\, M_2^2\Big); \alpha - 2\delta; \varphi\bigg),$$

and Lemma 4.8 is proved. □

Corollary. *If, in addition to the conditions of Lemma* 4.8*, the condition* $M_2 < 1$ *is satisfied, then*

$$w_j(\tau,\varepsilon,\varphi + v(\tau,\varepsilon,\varphi)) - w_j(\tau,\varepsilon,\varphi) \in F\Big(\frac{2m^2 4^{2m}}{\delta^{2m+2}}\, M_1 M_2; \alpha - 2\delta; \varphi\Big).$$

Lemma 4.9. *Let the matrix-function* $A(\tau,\varepsilon,\theta) \equiv (a_{jk}(\tau,\varepsilon,\theta))_{j,k=1,\dots,m} \in F_2(M;\alpha;\theta)$*. Suppose that the conditions*

$$0 < \delta < \alpha, \quad \frac{m\cdot 4^m}{\delta^m}\, M < \frac{1}{2} \tag{4.4}$$

hold. Then

$$(E_m + A(\tau,\varepsilon,\theta))^{-1} \in F_2(2;\alpha-\delta;\theta).$$

Proof. Let $A^p = (a_{jk}^{(p)})_{j,k=1,\dots,m}$, $p = 2,3,\dots$. Then

$$a_{jk}^{(2)} = \sum_{s=1}^{m} a_{js} a_{sk}, \quad j,k = 1,\dots,m.$$

By virtue of Lemmas 4.3 and 4.5,

$$a_{jk}^{(2)} \in F\Big(\frac{m 4^m}{\delta^m}\, M^2; \alpha-\delta; \theta\Big), \quad 0 < \delta < \alpha.$$

Further,

$$a_{jk}^{(3)} = \sum_{s=1}^{m} a_{js}^{(2)} a_{sk}, \quad j,k = 1,\dots,m.$$

By virtue of Lemmas 4.3 and 4.6,

$$a_{jk}^{(3)} \in F\Big(\frac{m^2 4^{2m}}{\delta^{2m}}\, M^3; \alpha-\delta; \theta\Big), \quad 0 < \delta < \alpha.$$

By the method of mathematical induction, we obtain

$$a_{jk}^{(p)} \in F\Big(\frac{m^{p-1}4^{m(p-1)}}{\delta^{m(p-1)}}\,M^p;\alpha-\delta;\theta\Big),\ \ 0<\delta<\alpha,$$

hence

$$A^p \in F_2\Big(\frac{m^{p-1}4^{m(p-1)}}{\delta^{m(p-1)}}\,M^p;\alpha-\delta;\theta\Big),\ \ 0<\delta<\alpha.$$

Consider the numerical series

$$1+\sum_{p=1}^{\infty}\frac{m^{p-1}4^{m(p-1)}}{\delta^{m(p-1)}}\,M^p = 1+M\sum_{p=1}^{\infty}\Big(\frac{m\cdot 4^m}{\delta^m}\,M\Big)^{p-1}.$$

By virtue of (4.4), this series is convergent, and its sum is less than $1+2M$. Since $2M<1$ (this also follows from (4.4)), thus we obtain what was required. □

5 The basic result

Theorem. *Let system* (3.1) *satisfy the following conditions:*

1)
$$|\lambda_j(\tau,\varepsilon)-\lambda_s(\tau,\varepsilon)| \ge \frac{\alpha}{L}>0,\ \ j,s=1,\dots,n,\ \ j\ne s;$$

2)
$$r=\frac{H_1 M}{q^2}<1,$$

where
$$H_1 = 3^{5m+5}2n^2m^4 4^{4m+1}(L^2+L+1),\ \ q=\Big(\frac{\alpha}{\alpha+2}\Big)^{5m+5}.$$

Then there exists the transformation of kind (3.2) *in which*

$$W(\tau,\varepsilon,\varphi)\in F_2\Big(M_2^*;\frac{\alpha}{2};\varphi\Big),\quad w(\tau,\varepsilon,\varphi)\in F_1\Big(M_1^*;\frac{\alpha}{2};\varphi\Big),$$

where
$$M_1^* = Q(r,1)\exp\Big(\frac{H_1}{q}\,Q(r,q)\Big),\quad M_2^*=Q\Big(r,\frac{1}{4}\Big),$$

$Q(r,q)$ *is the sum of the numerical series*
$$\sum_{j=0}^{\infty}\frac{r^{2^j}}{q^j}\,,$$

convergent if $r,q\in(0,1)$*, which leads system* (3.1) *to kind* (3.3)*, in which*

$$\sup_G |d_j(\tau,\varepsilon)|\le Q(r,1),\qquad \sup_{\varepsilon\in(0,1]}\|\Delta(\tau,\varepsilon)\|_0\le Q(r,1).$$

Proof. We denote
$$\beta_k=\Big(\frac{\alpha}{\alpha+2}\Big)^k,\quad \delta_k=\frac{\beta_k}{3}\,,\ \ k=1,2,\dots.$$

Obviously,
$$\delta_k^{5m+5}=\frac{1}{3^{5m+5}}\,q^k,\quad \sum_{k=1}^{\infty}\beta_k=\frac{\alpha}{2}\,,$$

and
$$\forall\, s\in\mathbb{N}:\ \ 0<\beta_1+\dots+\beta_s<\frac{\alpha}{2}\,,\ \ \frac{\alpha}{2}<\alpha-\beta_1-\dots-\beta_s<\alpha.$$

Following the method described in [1], we represent the process of reducing system (3.1) to form (3.3) as a sequence of steps. At the first step we make in system (3.1) the following substitution:

$$x = \big(E_n + U^{(1)}(\tau,\varepsilon,\varphi^{(1)})\big)y^{(1)}, \quad \theta = \varphi^{(1)} + v^{(1)}(\tau,\varepsilon,\varphi^{(1)}), \tag{5.1}$$

where $y^{(1)} \in \mathbb{R}^n$, $\varphi^{(1)} \in \mathbb{R}^m$, vector $v^{(1)}(\tau,\varepsilon,\varphi^{(1)})$ is defined from the vector partial differential equation

$$\frac{\partial v^{(1)}}{\partial \varphi^{(1)}}\big(\omega(\tau,\varepsilon) + \Delta^{(1)}(\tau,\varepsilon)\big) + \varepsilon\frac{\partial v^{(1)}}{\partial \tau} = b(\widetilde{\tau,\varepsilon,\varphi^{(1)}}), \tag{5.2}$$

where

$$\Delta^{(1)}(\tau,\varepsilon) = \overline{b(\tau,\varepsilon,\varphi^{(1)})}.$$

It is obvious that $\Delta^{(1)}(\tau,\varepsilon) \in \mathbb{R}^m$ and belongs to the class S_1.

The matrix $U^{(1)}(\tau,\varepsilon,\varphi^{(1)})$ is defined from the matrix partial differential equation

$$\begin{aligned}\Big(\frac{\partial U^{(1)}}{\partial \varphi^{(1)}}, \omega(\tau,\varepsilon) + \Delta^{(1)}(\tau,\varepsilon)\Big) + \varepsilon\,\frac{\partial U^{(1)}}{\partial \tau}&\\ = \big(\Lambda(\tau,\varepsilon) + D^{(1)}(\tau,\varepsilon)\big)U^{(1)} - U^{(1)}\big(\Lambda(\tau,\varepsilon) + D^{(1)}(\tau,\varepsilon)\big) &+ C^{(0)}(\tau,\varepsilon,\varphi^{(1)}),\end{aligned} \tag{5.3}$$

where

$$D^{(1)}(\tau,\varepsilon) = \operatorname{diag}\Big(\overline{a_{11}(\tau,\varepsilon,\varphi^{(1)})}, \dots, \overline{a_{nn}(\tau,\varepsilon,\varphi^{(1)})}\Big),$$
$$C^{(0)}(\tau,\varepsilon,\varphi^{(1)}) = A(\tau,\varepsilon,\varphi^{(1)}) - D^{(1)}(\tau,\varepsilon).$$

As a result of substitution (5.1), system (3.1) is reduced to the form

$$\begin{aligned}\frac{dy^{(1)}}{dt} &= \Big(\Lambda(\tau,\varepsilon) + D^{(1)}(\tau,\varepsilon) + A^{(1)}(\tau,\varepsilon,\varphi^{(1)})\Big)y^{(1)},\\ \frac{d\varphi^{(1)}}{dt} &= \omega(\tau,\varepsilon) + \Delta^{(1)}(\tau,\varepsilon) + b^{(1)}(\tau,\varepsilon,\varphi^{(1)}),\end{aligned} \tag{5.4}$$

where the vector $b^{(1)}(\tau,\varepsilon,\varphi^{(1)})$ is defined from the equation

$$\Big(E_m + \frac{\partial v^{(1)}}{\partial \varphi^{(1)}}\Big)b^{(1)} = b\big(\tau,\varepsilon,\varphi^{(1)} + v^{(1)}(\tau,\varepsilon,\varphi^{(1)})\big) - b(\tau,\varepsilon,\varphi^{(1)}), \tag{5.5}$$

and the matrix $A^{(1)}(\tau,\varepsilon,\varphi^{(1)})$ is defined from the equation

$$\begin{aligned}\big(E_n + U^{(1)}(\tau,\varepsilon,\varphi^{(1)})\big)A^{(1)} &= -\Big(\frac{\partial U^{(1)}}{\partial \varphi^{(1)}}, b^{(1)}(\tau,\varepsilon,\varphi^{(1)})\Big) + C^{(0)}(\tau,\varepsilon,\varphi^{(1)})U^{(1)}(\tau,\varepsilon,\varphi^{(1)})\\ &+ \Big[A\big(\tau,\varepsilon,\varphi^{(1)} + v^{(1)}(\tau,\varepsilon,\varphi^{(1)})\big) - A(\tau,\varepsilon,\varphi^{(1)})\Big]\big(E_n + U^{(1)}(\tau,\varepsilon,\varphi^{(1)})\big).\end{aligned} \tag{5.6}$$

Taking into account (5.2), we set

$$v^{(1)}(\tau,\varepsilon,\varphi^{(1)}) = \sum_{\substack{k\in Z_m\\(\|k\|>0)}} v_k^{(1)}(\tau,\varepsilon)\exp(i(k,\varphi^{(1)})), \tag{5.7}$$

where

$$\begin{aligned}v_k^{(1)}(\tau,\varepsilon) &= \frac{1}{\varepsilon}\exp\Big(-\frac{i}{\varepsilon}\int_0^\tau \big(k,\omega(\xi,\varepsilon) + \Delta^{(1)}(\xi,\varepsilon)\big)\,d\xi\Big)\\ &\times \int_0^\tau b_k(z,\varepsilon)\exp\Big(\frac{i}{\varepsilon}\int_0^z \big(k,\omega(\xi,\varepsilon) + \Delta^{(1)}(\xi,\varepsilon)\big)\,d\xi\Big)\,dz, \quad b_k(z,\varepsilon) = \Gamma_k[b(z,\varepsilon,\varphi^{(1)})].\end{aligned} \tag{5.8}$$

Thus $v_k^{(1)}(\tau,\varepsilon)\in S_1$, and

$$\|v_k^{(1)}(\tau,\varepsilon)\|_0 \le \frac{LM}{\varepsilon}\exp\Big(-\frac{\alpha}{\varepsilon}\,\|k\|\Big),\ \ \|k\|>0.$$

We define the constant M_0 by the condition

$$\frac{1}{\varepsilon}\,e^{-\frac{\alpha}{\varepsilon}\,\|k\|}\le M_0 e^{-\frac{\alpha-\delta_1}{\varepsilon}\,\|k\|},\ \ \|k\|>0$$

$\forall\,\varepsilon\in(0,1]$, where $\delta_1\in(0,\alpha)$ and does not depend on ε. Obviously, if $x\ge 1$, then

$$\frac{1}{\varepsilon}\,e^{-\frac{\delta_1}{\varepsilon}\,x}\le\frac{1}{\varepsilon}\,e^{-\frac{\delta_1}{\varepsilon}}.$$

Since

$$\lim_{\varepsilon\to+0}\frac{1}{\varepsilon}\,e^{-\frac{\delta_1}{\varepsilon}}=0,$$

and $\forall\,\varepsilon\in(0,1]$

$$\frac{1}{\varepsilon}\,e^{-\frac{\delta_1}{\varepsilon}}\le\frac{1}{\delta_1 e}<\frac{1}{\delta_1}$$

is valid, we can state that $M_0=1/\delta_1$. Thus, if $\varepsilon\in(0,1]$ and $\|k\|\ge 1$, we obtain

$$\|v_k^{(1)}(\tau,\varepsilon)\|_0\le\frac{LM}{\delta_1}\,e^{-\frac{\alpha-\delta_1}{\varepsilon}\,\|k\|}.$$

It follows that

$$v^{(1)}(\tau,\varepsilon,\varphi^{(1)})\in F_1\Big(\frac{L}{\delta_1}\,M;\alpha-\delta_1;\varphi^{(1)}\Big).$$

By virtue of Lemma 4.7,

$$\frac{\partial v^{(1)}}{\partial\varphi^{(1)}}\in F_2\Big(\frac{L}{\delta_1^2}\,M;\alpha-2\delta_1;\varphi^{(1)}\Big),$$

if $\delta_1\in(0,\alpha/2)$. In view of Lemma 4.9, we can conclude that if $\delta\in(0,\alpha/3)$ and

$$\frac{m4^mL}{\delta_1^{m+2}}\,M<\frac{1}{2}\,, \tag{5.9}$$

then the matrix $(E_m+\partial v^{(1)}/\partial\varphi^{(1)})^{-1}$ exists and belongs to the class $F_2(2;\alpha-3\delta_1;\varphi^{(1)})$.

From inequality (5.9), it follows that $Lm/\delta_1<1$, therefore, by virtue of Corollary from Lemma 4.8, we can conclude that

$$b\Big(\tau,\varepsilon,\varphi^{(1)}+v^{(1)}(\tau,\varepsilon,\varphi^{(1)})\Big)\in F_1\Big(\frac{2m^24^{2m}L}{\delta_1^{2m+3}}\,M^2;\alpha-2\delta_1;\varphi^{(1)}\Big).$$

Now, by virtue of Lemma 4.6 and equation (5.5),

$$b_1(\tau,\varepsilon,\varphi^{(1)})\in F_1\Big(\frac{m^34^{3m+1}L}{\delta_1^{3m+3}}\,M^2;\alpha-\beta_1;\varphi^{(1)}\Big).$$

We now construct the matrix $U^{(1)}(\tau,\varepsilon,\varphi^{(1)})=(u_{js}^{(1)}(\tau,\varepsilon,\varphi^{(1)}))_{j,s=1,\dots,n}$. We write equation (5.3) componentwise,

$$\begin{aligned}&\sum_{l=1}^{n}\frac{\partial u_{js}^{(1)}}{\partial\varphi_l^{(1)}}\,(\omega(\tau,\varepsilon)+\Delta_l^{(1)}(\tau,\varepsilon))+\varepsilon\,\frac{\partial u_{js}^{(1)}}{\partial\tau}\\&\qquad=\big(\lambda_j(\tau,\varepsilon)-\lambda_s(\tau,\varepsilon)+d_j^{(1)}(\tau,\varepsilon)-d_s^{(1)}(\tau,\varepsilon)\big)u_{js}^{(1)}+c_{js}^{(0)}(\tau,\varepsilon,\varphi^{(1)}),\ \ j,s=1,\dots,n.\end{aligned}\tag{5.10}$$

Consider first the case $j = s$. We set

$$u_{jj}^{(1)}(\tau,\varepsilon,\varphi^{(1)}) = \sum_{\substack{k\in Z_m\\(\|k\|>0)}} u_{jj,k}^{(1)}(\tau,\varepsilon)\exp(i(k,\varphi^{(1)})), \tag{5.11}$$

where

$$\begin{aligned} u_{jj,k}^{(1)}(\tau,\varepsilon) &= \frac{1}{\varepsilon}\exp\bigg(-\frac{i}{\varepsilon}\int_0^\tau \big(k,\omega(\xi,\varepsilon)+\Delta^{(1)}(\xi,\varepsilon)\big)\,d\xi\bigg) \\ &\quad\times\int_0^\tau c_{jj,k}^{(0)}(z,\varepsilon)\exp\bigg(\frac{i}{\varepsilon}\int_0^z \big(k,\omega(\xi,\varepsilon)+\Delta^{(1)}(\xi,\varepsilon)\big)\,d\xi\bigg)\,dz, \\ c_{jj,k}^{(0)}(z,\varepsilon) &= \frac{1}{(2\pi)^m}\int_0^{2\pi}\cdots\int_0^{2\pi} c_{jj}^{(0)}(z,\varepsilon,\varphi)e^{-i(k,\varphi)}\,d\varphi_1\cdots d\varphi_m, \quad j=1,\dots,n, \quad k\in Z_m. \end{aligned} \tag{5.12}$$

Hence

$$u_{jj}^{(1)}(\tau,\varepsilon,\varphi^{(1)}) \in F\Big(\frac{L}{\delta_1}\,M;\alpha-\delta_1;\varphi^{(1)}\Big), \quad j=1,\dots,n,$$

where $\delta_1 \in (0,\alpha)$ and does not depend on ε.

Let now $j \neq s$. We choose M insomuch small that

$$2M < \frac{\alpha-\delta_1}{L}\,. \tag{5.13}$$

Then, by virtue of condition 1) of the theorem, we have

$$\big|\lambda_j(\tau,\varepsilon)-\lambda_s(\tau,\varepsilon)+d_j^{(1)}(\tau,\varepsilon)-d_s^{(1)}(\tau,\varepsilon)\big| \geq \frac{\alpha}{L}-\frac{\alpha-\delta_1}{L} = \frac{\delta_1}{L} > 0. \tag{5.14}$$

Here in turn, we consider two cases.

Case 1. Let $\lambda_j(\tau,\varepsilon)-\lambda_s(\tau,\varepsilon) \leq -\alpha/L < 0$. Then

$$\lambda_j(\tau,\varepsilon)-\lambda_s(\tau,\varepsilon)+d_j^{(1)}(\tau,\varepsilon)-d_s^{(1)}(\tau,\varepsilon) \leq -\frac{\delta_1}{L} < 0.$$

We define the elements $u_{js}^{(1)}$ of matrix $U^{(1)}$ by the formulas

$$u_{js}^{(1)}(\tau,\varepsilon,\varphi^{(1)}) = \sum_{k\in Z_m} u_{js,k}^{(1)}(\tau,\varepsilon)\exp(i(k,\varphi^{(1)})),$$

where

$$\begin{aligned} u_{js,k}^{(1)}(\tau,\varepsilon) &= \frac{1}{\varepsilon}\exp\bigg(\frac{1}{\varepsilon}\int_0^\tau (\lambda_j(\xi,\varepsilon)-\lambda_s(\xi,\varepsilon)+d_j^{(1)}(\xi,\varepsilon)-d_s^{(1)}(\xi,\varepsilon)-i\big(k,\omega(\xi,\varepsilon)+\Delta^{(1)}(\xi,\varepsilon)\big)\,d\xi\bigg) \\ &\quad\times\int_0^\tau c_{js,k}^{(0)}(z,\varepsilon)\exp\bigg(-\frac{1}{\varepsilon}\int_0^z \Big(\lambda_j(\xi,\varepsilon)-\lambda_s(\xi,\varepsilon) \\ &\quad+d_j^{(1)}(\xi,\varepsilon)-d_s^{(1)}(\xi,\varepsilon)-i\big(k,\omega(\xi,\varepsilon)+\Delta^{(1)}(\xi,\varepsilon)\big)\Big)\,d\xi\bigg)\,dz, \quad j,s=1,\dots,n, \quad k\in Z_m. \end{aligned} \tag{5.15}$$

We estimate

$$|u_{js,k}^{(1)}(\tau,\varepsilon)| \le \frac{1}{\varepsilon}\int\limits_0^\tau |c_{js,k}^{(0)}(z,\varepsilon)| \exp\left(\frac{1}{\varepsilon}\int\limits_z^\tau (\lambda_j(\xi,\varepsilon)-\lambda_s(\xi,\varepsilon)+d_j^{(1)}(\xi,\varepsilon)-d_s^{(1)}(\xi,\varepsilon))\,d\xi\right)dz$$
$$\le \frac{1}{\varepsilon} M \exp\Big(-\frac{\alpha}{\varepsilon}\|k\|\Big)\int\limits_0^\tau \exp\Big(-\frac{\delta_1}{L\varepsilon}(\tau-z)\Big)\,dz$$
$$= \frac{1}{\varepsilon} M \exp\Big(-\frac{\alpha}{\varepsilon}\|k\|\Big)\frac{1}{\frac{\delta_1}{L\varepsilon}}\Big(1-\exp\Big(-\frac{\delta_1}{L\varepsilon}\tau\Big)\Big) \le \frac{LM}{\delta_1}\exp\Big(-\frac{\alpha-\delta_1}{\varepsilon}\|k\|\Big).$$

Hence

$$u_{js}^{(1)}(\tau,\varepsilon,\varphi^{(1)}) \in F\Big(\frac{LM}{\delta_1}\,;\alpha-\delta_1;\varphi^{(1)}\Big).$$

Case 2. Let $\lambda_j(\tau,\varepsilon)-\lambda_s(\tau,\varepsilon) \ge \alpha/L > 0$. Then $\lambda_j(\tau,\varepsilon)-\lambda_s(\tau,\varepsilon)+d_j^{(1)}(\tau,\varepsilon)-d_s^{(1)}(\tau,\varepsilon) \ge \delta_1/L > 0$. We define the elements $u_{js}^{(1)}$ of matrix $U^{(1)}$ by the formulas

$$u_{js}^{(1)}(\tau,\varepsilon,\varphi^{(1)}) = \sum_{k\in Z_m} u_{js,k}^{(1)}(\tau,\varepsilon)\exp(i(k,\varphi^{(1)})),$$

where

$$u_{js,k}^{(1)}(\tau,\varepsilon)$$
$$= -\frac{1}{\varepsilon}\exp\left(\frac{1}{\varepsilon}\int\limits_0^\tau \Big(\lambda_j(\xi,\varepsilon)-\lambda_s(\xi,\varepsilon)+d_j^{(1)}(\xi,\varepsilon)-d_s^{(1)}(\xi,\varepsilon)-i\big(k,\omega(\xi,\varepsilon)+\Delta^{(1)}(\xi,\varepsilon)\big)\Big)\,d\xi\right)$$
$$\times \int\limits_\tau^L c_{js,k}^{(0)}(z,\varepsilon)\exp\Bigg(-\frac{1}{\varepsilon}\int\limits_0^z \Big(\lambda_j(\xi,\varepsilon)-\lambda_s(\xi,\varepsilon)$$
$$+ d_j^{(1)}(\xi,\varepsilon)-d_s^{(1)}(\xi,\varepsilon)-i\big(k,\omega(\xi,\varepsilon)+\Delta^{(1)}(\xi,\varepsilon)\big)\Big)\,d\xi\Bigg)\,dz, \quad j,s=1,\dots,n, \quad k\in Z_m. \quad (5.16)$$

As in the first case, we show that

$$u_{js}^{(1)}(\tau,\varepsilon,\varphi^{(1)}) \in F\Big(\frac{LM}{\delta_1}\,;\alpha-\delta_1;\varphi^{(1)}\Big).$$

Thus

$$U^{(1)}(\tau,\varepsilon,\varphi^{(1)}) \in F_2\Big(\frac{LM}{\delta_1}\,;\alpha-\delta_1;\varphi^{(1)}\Big).$$

By virtue of Corollary from Lemma 4.8, we can conclude that under condition (5.9),

$$A(\tau,\varepsilon,\varphi^{(1)}+v^{(1)}(\tau,\varepsilon,\varphi^{(1)}))-A(\tau,\varepsilon,\varphi^{(1)}) \in F_2\Big(\frac{2m^2 4^{2m}L}{\delta_1^{2m+3}}\,M^2;\alpha-2\delta_1;\varphi^{(1)}\Big).$$

From (5.9) we have $LM/\delta_1 < 1$, hence

$$E_n + U^{(1)}(\tau,\varepsilon,\varphi^{(1)}) \in F_2(2;\alpha-\delta_1;\varphi^{(1)}),$$

and, by virtue of Lemma 4.6,

$$\big(A(\tau,\varepsilon,\varphi^{(1)}+v^{(1)}(\tau,\varepsilon,\varphi^{(1)}))-A(\tau,\varepsilon,\varphi^{(1)})\big)(E_n+U^{(1)}(\tau,\varepsilon,\varphi^{(1)}))$$
$$\in F_2\Big(\frac{nm^2 4^{3m+1}}{\delta_1^{3m+3}}\,M^2;\alpha-2\delta_1;\varphi^{(1)}\Big), \quad (5.17)$$

$$C(\tau,\varepsilon,\varphi^{(1)})U^{(1)}(\tau,\varepsilon,\varphi^{(1)}) \in F_2\Big(\frac{n4^m L}{\delta_1^{m+1}}\,M^2;\alpha-\delta_1;\varphi^{(1)}\Big). \quad (5.18)$$

Due to Lemma 4.7,

$$\frac{\partial U^{(1)}}{\partial \varphi^{(1)}} \in F_2\Big(\frac{L}{\delta_1^2}\, M; \alpha - 2\delta_1; \varphi^{(1)}\Big),$$
$$\sum_{k=1}^{m} \frac{\partial U^{(1)}}{\partial \varphi_k^{(1)}}\, b_k^{(1)} \in F_2\Big(\frac{m^4 4^{4m+1} L^2}{\delta_1^{4m+5}}\, M^3; \alpha - 2\delta_1; \varphi^{(1)}\Big).$$

By virtue of Lemma 4.9 and condition (5.9), we can conclude that

$$\big(E_n + U^{(1)}(\tau, \varepsilon, \varphi^{(1)})\big)^{-1} \in F_2(2; \alpha - 2\delta_1; \varphi^{(1)}).$$

Hence, (5.6), (5.17) and (5.18) yield

$$A^{(1)}(\tau, \varepsilon, \varphi^{(1)}) \in F_2\Big(\frac{2n^2 m^4 4^{5m+1}}{\delta_1^{5m+5}}\,(L^2 + L + 1) M^2; \alpha - \beta_1; \varphi^{(1)}\Big).$$

Thus, under conditions (5.9) and (5.13), we have

$$v^{(1)}(\tau, \varepsilon, \varphi^{(1)}) \in F_1\Big(\frac{L}{\delta_1}\,; \alpha - \delta_1; \varphi^{(1)}\Big),$$
$$U^{(1)}(\tau, \varepsilon, \varphi^{(1)}) \in F_2\Big(\frac{L}{\delta_1}\,; \alpha - \delta_1; \varphi^{(1)}\Big),$$
$$b^{(1)}(\tau, \varepsilon, \varphi^{(1)}) \in F_1\Big(\frac{2n^2 m^4 4^{5m+1}}{\delta_1^{5m+5}}\,(L^2 + L + 1) M^2; \alpha - \beta_1; \varphi^{(1)}\Big),$$
$$A^{(1)}(\tau, \varepsilon, \varphi^{(1)}) \in F_2\Big(\frac{2n^2 m^4 4^{5m+1}}{\delta_1^{5m+5}}\,(L^2 + L + 1) M^2; \alpha - \beta_1; \varphi^{(1)}\Big).$$

This completes the first step of the process.

At the step with number $l-1$ of the process, we obtain the system

$$\begin{aligned} \frac{dy^{(l-1)}}{dt} &= \Big(\Lambda(\tau, \varepsilon) + D^{(1)}(\tau, \varepsilon) + \cdots + D^{(l-1)}(\tau, \varepsilon) + A^{(l-1)}(\tau, \varepsilon, \varphi^{(l-1)})\Big) y^{(l-1)}, \\ \frac{d\varphi^{(l-1)}}{dt} &= \omega(\tau, \varepsilon) + \Delta^{(1)}(\tau, \varepsilon) + \cdots + \Delta^{(l-1)}(\tau, \varepsilon) + b^{(l-1)}(\tau, \varepsilon, \varphi^{(l-1)}), \end{aligned} \tag{5.19}$$

where $D^{(1)}, \dots, D^{(l-1)}$ are the diagonal $(n \times n)$-matrices with elements from the class S, the vectors $\Delta^{(1)}, \dots, \Delta^{(l-1)}$ belong to the class S_1, $b^{(l-1)} \in F_1(K_{l-1}; \alpha - \beta_1 - \cdots - \beta_{l-1}; \varphi^{(l-1)})$, $A^{(l-1)} \in F_2(K_{l-1}; \alpha - \beta_1 - \cdots - \beta_{l-1}; \varphi^{(l-1)})$,

$$K_l = \frac{H^{2^l - 1}}{\delta_l^{5m+5} (\delta_{l-1}^{5m+5})^2 \cdots (\delta_1^{5m+5})^{2^l - 1}}\, M^{2^l}, \quad H = 2n^2 m^4 4^{5m+1} (L^2 + L + 1).$$

At the step with number l, we make in system (5.19) the following substitution:

$$y^{(l-1)} = \big(E_n + U^{(l)}(\tau, \varepsilon, \varphi^{(l)})\big) y^{(l)}, \quad \varphi^{(l-1)} = \varphi^{(l)} + v^{(l)}(\tau, \varepsilon, \varphi^{(l)}), \tag{5.20}$$

where $y^{(l)} \in \mathbb{R}^n$, $\varphi^{(l)} \in \mathbb{R}^m$. The vector $v^{(1)}(\tau, \varepsilon, \varphi^{(1)})$ is defined from the vector partial differential equation

$$\frac{\partial v^{(l)}}{\partial \varphi^{(l)}} \Big(\omega(\tau, \varepsilon) + \Delta^{(1)}(\tau, \varepsilon) + \cdots + \Delta^{(l)}(\tau, \varepsilon)\Big) + \varepsilon \frac{\partial v^{(l)}}{\partial \tau} = \widetilde{b^{(l-1)}(\tau, \varepsilon, \varphi^{(1)})}, \tag{5.21}$$

where

$$\Delta^{(l)}(\tau, \varepsilon) = \overline{b^{(l-1)}(\tau, \varepsilon, \varphi^{(l)})}.$$

Obviously, $\Delta^{(l)}(\tau, \varepsilon) \in \mathbb{R}^m$ and belongs to the class S_1.

The matrix $U^{(l)}(\tau,\varepsilon,\varphi^{(l)})$ is defined from the matrix partial differential equation

$$\begin{aligned}\Big(\frac{\partial U^{(l)}}{\partial\varphi^{(l)}}\,,\omega(\tau,\varepsilon)+\Delta^{(1)}(\tau,\varepsilon)+\cdots+\Delta^{(l)}(\tau,\varepsilon)\Big)+\varepsilon\frac{\partial U^{(l)}}{\partial\tau}\\ =\Big(\Lambda(\tau,\varepsilon)+D^{(1)}(\tau,\varepsilon)+\cdots+D^{(l)}(\tau,\varepsilon)\Big)U^{(l)}\\ -U^{(l)}\Big(\Lambda(\tau,\varepsilon)+D^{(1)}(\tau,\varepsilon)+\cdots+D^{(l)}(\tau,\varepsilon)\Big)+C^{(l-1)}(\tau,\varepsilon,\varphi^{(l)}),\end{aligned}\tag{5.22}$$

where

$$D^{(l)}(\tau,\varepsilon)=\mathrm{diag}\Big(\overline{a_{11}^{(l-1)}(\tau,\varepsilon,\varphi^{(l)})},\dots,\overline{a_{nn}^{(l-1)}(\tau,\varepsilon,\varphi^{(l)})}\Big),$$
$$C^{(l-1)}(\tau,\varepsilon,\varphi^{(1)})=A^{(l-1)}(\tau,\varepsilon,\varphi^{(1)})-D^{(l)}(\tau,\varepsilon).$$

Taking into account (5.21), we set

$$v^{(l)}(\tau,\varepsilon,\varphi^{(1)})=\sum_{\substack{k\in Z_m\\(\|k\|>0)}}v_k^{(l)}(\tau,\varepsilon)\exp(i(k,\varphi^{(l)})),\tag{5.23}$$

where

$$\begin{aligned}v_k^{(l)}(\tau,\varepsilon)=\frac{1}{\varepsilon}\exp\Big(-\frac{i}{\varepsilon}\int_0^\tau\big(k,\omega(\xi,\varepsilon)+\Delta^{(1)}(\xi,\varepsilon)+\cdots+\Delta^{(l)}(\xi,\varepsilon)\big)\,d\xi\Big)\\ \times\int_0^\tau b_k^{(l-1)}(z,\varepsilon)\exp\Big(\frac{i}{\varepsilon}\int_0^z\big(k,\omega(\xi,\varepsilon)+\Delta^{(1)}(\xi,\varepsilon)+\cdots+\Delta^{(l)}(\xi,\varepsilon)\big)\,d\xi\Big)\,dz,\end{aligned}\tag{5.24}$$

$$b_k^{(l-1)}(z,\varepsilon)=\Gamma_k[b^{(l-1)}(z,\varepsilon,\varphi^{(l)})].$$

Taking into account (5.22), we set

$$U^{(l)}(\tau,\varepsilon,\varphi^{(1)})=(u_{js}^{(l)}(\tau,\varepsilon,\varphi^{(l)}))_{j,s=1,\dots,n},$$
$$u_{jj}^{(l)}(\tau,\varepsilon,\varphi^{(l)})=\sum_{\substack{k\in Z_m\\(\|k\|>0)}}u_{jj,k}^{(l)}(\tau,\varepsilon)\exp(i(k,\varphi^{(1)})),$$

where

$$\begin{aligned}u_{jj,k}^{(l)}(\tau,\varepsilon)=\frac{1}{\varepsilon}\exp\Big(-\frac{i}{\varepsilon}\int_0^\tau\big(k,\omega(\xi,\varepsilon)+\Delta^{(1)}(\xi,\varepsilon)+\cdots+\Delta^{(l)}(\xi,\varepsilon)\big)\,d\xi\Big)\\ \times\int_0^\tau c_{jj,k}^{(l-1)}(z,\varepsilon)\exp\Big(\frac{i}{\varepsilon}\int_0^z\big(k,\omega(\xi,\varepsilon)+\Delta^{(1)}(\xi,\varepsilon)+\cdots+\Delta^{(l)}(\xi,\varepsilon)\big)\,d\xi\Big)\,dz,\end{aligned}$$

$$c_{jj,k}^{(l-1)}(z,\varepsilon)=\frac{1}{(2\pi)^m}\int_0^{2\pi}\cdots\int_0^{2\pi}c_{jj}^{(l-1)}(z,\varepsilon,\varphi)e^{-i(k,\varphi)}d\varphi_1\cdots d\varphi_m,\quad j=1,\dots,n,\quad k\in Z_m.$$

If $j\neq s$, then we set

$$u_{js}^{(l)}(\tau,\varepsilon,\varphi^{(l)})=\sum_{k\in Z_m}u_{js,k}^{(l)}(\tau,\varepsilon)\exp(i(k,\varphi^{(l)})),$$

where in case $\lambda_j(\tau,\varepsilon)-\lambda_s(\tau,\varepsilon)\le -\alpha/L<0$,

$$u^{(l)}_{js,k}(\tau,\varepsilon)=\frac{1}{\varepsilon}\exp\bigg(\frac{1}{\varepsilon}\int_0^\tau\Big(\lambda_j(\xi,\varepsilon)-\lambda_s(\xi,\varepsilon)+d_j^{(1)}(\xi,\varepsilon)-d_s^{(1)}(\xi,\varepsilon)+\cdots+d_j^{(l)}(\xi,\varepsilon)-d_s^{(l)}(\xi,\varepsilon)$$
$$-i\big(k,\omega(\xi,\varepsilon)+\Delta^{(1)}(\xi,\varepsilon)+\cdots+\Delta^{(l)}(\xi,\varepsilon)\big)\Big)\,d\xi\bigg)$$
$$\times\int_0^\tau c^{(l-1)}_{js,k}(z,\varepsilon)\exp\bigg(-\frac{1}{\varepsilon}\int_0^z\Big(\lambda_j(\xi,\varepsilon)-\lambda_s(\xi,\varepsilon)+d_j^{(1)}(\xi,\varepsilon)-d_s^{(1)}(\xi,\varepsilon)+\cdots+d_j^{(l)}(\xi,\varepsilon)-d_s^{(l)}(\xi,\varepsilon)$$
$$-i\big(k,\omega(\xi,\varepsilon)+\Delta^{(1)}(\xi,\varepsilon)+\cdots+\Delta^{(l)}(\xi,\varepsilon)\big)\Big)\,d\xi\bigg)\,dz,\quad j,s=1,\dots,n,\quad k\in Z_m,$$

and in case $\lambda_j(\tau,\varepsilon)-\lambda_s(\tau,\varepsilon)\ge \alpha/L>0$,

$$u^{(l)}_{js,k}(\tau,\varepsilon)=-\frac{1}{\varepsilon}\exp\bigg(\frac{1}{\varepsilon}\int_0^\tau\Big(\lambda_j(\xi,\varepsilon)-\lambda_s(\xi,\varepsilon)+d_j^{(1)}(\xi,\varepsilon)-d_s^{(1)}(\xi,\varepsilon)+\cdots+d_j^{(l)}(\xi,\varepsilon)-d_s^{(l)}(\xi,\varepsilon)$$
$$-i\big(k,\omega(\xi,\varepsilon)+\Delta^{(1)}(\xi,\varepsilon)+\cdots+\Delta^{(l)}(\xi,\varepsilon)\big)\Big)\,d\xi\bigg)$$
$$\times\int_\tau^L c^{(l-1)}_{js,k}(z,\varepsilon)\exp\bigg(-\frac{1}{\varepsilon}\int_0^z\Big(\lambda_j(\xi,\varepsilon)-\lambda_s(\xi,\varepsilon)+d_j^{(1)}(\xi,\varepsilon)-d_s^{(1)}(\xi,\varepsilon)+\cdots+d_j^{(l)}(\xi,\varepsilon)-d_s^{(l)}(\xi,\varepsilon)$$
$$-i\big(k,\omega(\xi,\varepsilon)+\Delta^{(1)}(\xi,\varepsilon)+\cdots+\Delta^{(l)}(\xi,\varepsilon)\big)\Big)\,d\xi\bigg)\,dz,\quad j,s=1,\dots,n,\quad k\in Z_m.$$

Here we suppose M insomuch small that

$$2K_{l-1}<\frac{\delta_{l-1}-\delta_l}{L}\,, \tag{5.25}$$
$$\frac{n4^mL}{\delta_l^{m+2}}\,K_{l-1}<\frac{1}{2}\,. \tag{5.26}$$

Then

$$\big|\lambda_j(\tau,\varepsilon)-\lambda_s(\tau,\varepsilon)+d_j^{(1)}(\tau,\varepsilon)-d_s^{(1)}(\tau,\varepsilon)+\cdots+d_j^{(l)}(\tau,\varepsilon)-d_s^{(l)}(\tau,\varepsilon)\big|\ge\frac{\delta_l}{L}\,.$$

We have

$$\delta_l^{5m+5}(\delta_{l-1}^{5m+5})^2\cdots(\delta_1^{5m+5})^{2^l-1}=\frac{1}{3^{5m+5}}\,q^l\,\frac{1}{(3^{5m+5})^2}\,(q^{l-1})^2\cdots\frac{1}{(3^{5m+5})^{2^l-2}}\,(q^2)^{2^{l-2}}$$
$$=\frac{1}{(3^{5m+5})^{1+2+\cdots+2^{l-1}}}\,q^{l+2(l-1)+\cdots+2\cdot2^{l-2}+2^{l-1}}=\frac{1}{(3^{5m+5})^{2^l-1}}\,q^{2^{l+1}-l-2},$$

where q is defined in the statement of the theorem. Therefore

$$K_l=\frac{H_1^{2^l-1}}{q^{2^{l+1}-l-2}}\,M^{2^l},$$

where $H_1=2n^2m^43^{5m+5}4^{5m+1}(L^2+L+1)$. Hence $K_l<r^{2^l}$, where $r=\frac{H_1}{q^2}\,M$.

The condition $r<1$ guarantees the convergence of the series $\sum\limits_{l=1}^{\infty}K_l$. It is easy to verify that this condition ensures that inequalities (5.25), (5.26) hold.

As a result of substitution (5.20), system (5.19) is reduced to the form

$$\begin{aligned}\frac{dy^{(l)}}{dt} &= \Big(\Lambda(\tau,\varepsilon) + D^{(1)}(\tau,\varepsilon) + \dots + D^{(l)}(\tau,\varepsilon) + A^{(l)}(\tau,\varepsilon,\varphi^{(l-1)})\Big)y^{(l)},\\ \frac{d\varphi^{(l)}}{dt} &= \omega(\tau,\varepsilon) + \Delta^{(1)}(\tau,\varepsilon) + \dots + \Delta^{(l)}(\tau,\varepsilon) + b^{(l)}(\tau,\varepsilon,\varphi^{(l)}).\end{aligned} \tag{5.27}$$

Carrying out the arguments analogous to those of the first step, we show that

$$\begin{aligned}v^{(l)}(\tau,\varepsilon,\varphi^{(l)}) &\in F_1\Big(\frac{K_{l-1}}{\delta_l}\,;\alpha-\beta_1-\dots-\beta_{l-1}-\delta_l;\varphi^{(l)}\Big),\\ U^{(l)}(\tau,\varepsilon,\varphi^{(l)}) &\in F_2\Big(\frac{K_{l-1}}{\delta_l}\,;\alpha-\beta_1-\dots-\beta_{l-1}-\delta_l;\varphi^{(l)}\Big),\\ b^{(l)}(\tau,\varepsilon,\varphi^{(l)}) &\in F_1\Big(K_l;\alpha-\beta_1-\dots-\beta_l;\varphi^{(l)}\Big),\\ A^{(l)}(\tau,\varepsilon,\varphi^{(l)}) &\in F_2\Big(K_l;\alpha-\beta_1-\dots-\beta_l;\varphi^{(l)}\Big).\end{aligned}$$

Hence, the iterative process

$$\begin{gathered}x = \big(E_n + U^{(1)}(\tau,\varepsilon,\varphi^{(1)})\big)y^{(1)}, \quad \theta = \varphi^{(1)} + v^{(1)}(\tau,\varepsilon,\varphi^{(1)}),\\ y^{(l-1)} = \big(E_n + U^{(l)}(\tau,\varepsilon,\varphi^{(l)})\big)y^{(l)}, \quad \varphi^{(l-1)} = \varphi^{(l)} + v^{(l)}(\tau,\varepsilon,\varphi^{(l)}),\ \ l=2,3,\dots\,,\end{gathered} \tag{5.28}$$

in case it is convergent, leads system (3.1) to kind (3.3) in which

$$D(\tau,\varepsilon) = \sum_{l=1}^{\infty} D^{(l)}(\tau,\varepsilon), \quad \Delta(\tau,\varepsilon) = \sum_{l=1}^{\infty} \Delta^{(l)}(\tau,\varepsilon),$$

where

$$\begin{gathered}\Delta^{(l)}(\tau,\varepsilon) \in S_1, \quad \|\Delta^{(l)}(\tau,\varepsilon)\|_0 \le K_{l-1}, \quad D^{(l)}(\tau,\varepsilon) = \operatorname{diag}\big(d_1^{(l)}(\tau,\varepsilon),\dots,d_n^{(l)}(\tau,\varepsilon)\big),\\ d_j^{(l)}(\tau,\varepsilon) \in S, \quad \sup_G |d_j^{(l)}(\tau,\varepsilon)| \le K_{l-1}\ \ (j=1,\dots,n).\end{gathered}$$

We prove the convergence of process (5.28). Towards this end, we represent process (5.28) in the form

$$x = \big(E_n + W^{(l)}(\tau,\varepsilon,\varphi^{(l)})\big)y^{(l)}, \quad \theta = \varphi^{(l)} + w^{(l)}(\tau,\varepsilon,\varphi^{(l)}),\ \ l=1,2,\dots, \tag{5.29}$$

where

$$\begin{aligned}W^{(1)}(\tau,\varepsilon,\varphi^{(1)}) &= U^{(1)}(\tau,\varepsilon,\varphi^{(1)}), \quad w^{(1)}(\tau,\varepsilon,\varphi^{(1)}) = v^{(1)}(\tau,\varepsilon,\varphi^{(1)}),\\ W^{(l)}(\tau,\varepsilon,\varphi^{(l)}) &= \big(E_n + W^{(l-1)}(\tau,\varepsilon,\varphi^{(l)} + v^{(l)}(\tau,\varepsilon,\varphi^{(l)}))\big)U^{(l)}(\tau,\varepsilon,\varphi^{(l)})\\ &\quad + W^{(l-1)}(\tau,\varepsilon,\varphi^{(l)} + v^{(l)}(\tau,\varepsilon,\varphi^{(l)})),\end{aligned} \tag{5.30}$$

$$w^{(l)}(\tau,\varepsilon,\varphi^{(l)}) = v^{(l)}(\tau,\varepsilon,\varphi^{(l)}) + w^{(l-1)}(\tau,\varepsilon,\varphi^{(l)} + v^{(l)}(\tau,\varepsilon,\varphi^{(l)})),\ \ l=2,3,\dots\,. \tag{5.31}$$

Then

$$w^{(1)}(\tau,\varepsilon,\varphi^{(1)}) \in F_1(r;\alpha-\beta_1,\varphi^{(1)}), \quad W^{(1)}(\tau,\varepsilon,\varphi^{(1)}) \in F_2(r;\alpha-\beta_1,\varphi^{(1)}).$$

By virtue of Corollary from Lemma 4.8, we successively obtain

$$\begin{gathered}w^{(2)}(\tau,\varepsilon,\varphi^{2)}) \in F_1\Big(r^2 + r\Big(1+\frac{H_1}{q^2}\,r^2\Big);\alpha-\beta_1-\beta_2;\varphi^{(2)}\Big),\\ w^{(l)}(\tau,\varepsilon,\varphi^{l)}) \in F_1\big(w_l^*;\ \alpha-\beta_1-\dots-\beta_l;\varphi^{(l)}\big),\ \ l=3,4,\dots\,,\end{gathered}$$

where

$$\begin{aligned}w_l^* = r^{2^{l-1}} + r^{2^{l-2}}\Big(1+\frac{H_1}{q^l}\,r^{2^{l-1}}\Big) + r^{2^{l-3}}\Big(1+\frac{H_1}{q^{l-1}}\,r^{2^{l-2}}\Big)\Big(1+\frac{H_1}{q^l}\,r^{2^{l-1}}\Big) + \cdots\\ + r\Big(1+\frac{H_1}{q^2}\,r^2\Big)\Big(1+\frac{H_1}{q^3}\,r^4\Big)\cdots\Big(1+\frac{H_1}{q^l}\,r^{2^{l-1}}\Big).\end{aligned}$$

Consider

$$w^{(l+1)}(\tau,\varepsilon,\varphi)-w^{(l)}(\tau,\varepsilon,\varphi)=v^{(l+1)}(\tau,\varepsilon,\varphi)+w^{(l)}(\tau,\varepsilon,\varphi+v^{(l+1)}(\tau,\varepsilon,\varphi))-w^{(l)}(\tau,\varepsilon,\varphi).$$

By virtue of Corollary from Lemma 4.8, we have

$$w^{(l)}(\tau,\varepsilon,\varphi+v^{(l+1)}(\tau,\varepsilon,\varphi))-w^{(l)}(\tau,\varepsilon,\varphi)\in F_1\Big(\frac{H_1}{q^{l+1}}\,r^{2^l}\,w_l^*;\alpha-\beta_1-\cdots-\beta_l-2\delta_{l+1};\varphi\Big).$$

Hence,

$$w^{(l+1)}(\tau,\varepsilon,\varphi)-w^{(l)}(\tau,\varepsilon,\varphi)\in F_1\Big(r^{2^l}\Big(1+\frac{H_1}{q^{l+1}}\,w_l^*\Big);\alpha-\beta_1-\cdots-\beta_{l+1};\varphi\Big). \tag{5.32}$$

We estimate

$$\begin{aligned}
w_l^* &\le \Big(\sum_{j=0}^{l-1} r^{2^j}\Big)\prod_{j=1}^{l-1}\Big(1+\frac{H_1}{q^{j+1}}\,r^{2^j}\Big)\\
&=\Big(\sum_{j=0}^{l-1} r^{2^j}\Big)\exp\Big[\ln\prod_{j=1}^{l-1}\Big(1+\frac{H_1}{q^{j+1}}\,r^{2^j}\Big)\Big]=\Big(\sum_{j=0}^{l-1} r^{2^j}\Big)\exp\Big[\sum_{j=1}^{l-1}\ln\Big(1+\frac{H_1}{q^{j+1}}\,r^{2^j}\Big)\Big]\\
&<\Big(\sum_{j=0}^{l-1} r^{2^j}\Big)\exp\Big(\sum_{j=1}^{l-1}\frac{H_1}{q^{j+1}}\,r^{2^j}\Big)<\Big(\sum_{j=0}^{l-1} r^{2^j}\Big)\exp\Big(\frac{H_1}{q}\sum_{j=0}^{l-1}\frac{r^{2^j}}{q^j}\Big).
\end{aligned} \tag{5.33}$$

The numerical series

$$\sum_{j=0}^{\infty}\frac{r^{2^j}}{q^j}$$

under the condition $r,q\in(0,1)$ is convergent, we denote its sum by $Q(r,q)$. Then, by virtue of (5.33), we obtain

$$w_l^*<Q(r,1)\exp\Big(\frac{H_1}{q}\,Q(r,q)\Big). \tag{5.34}$$

Hence,

$$r^{2^l}\Big(1+\frac{H_1}{q^{l+1}}\,w_l^*\Big)<r^{2^l}\Big(1+\frac{H_1}{q^{l+1}}\,Q(r,1)\exp\Big(\frac{H_1}{q}\,Q(r,q)\Big)\Big),$$

from the latter inequality and (5.32) it follows that

$$w^{(l+1)}(\tau,\varepsilon,\varphi)-w^{(l)}(\tau,\varepsilon,\varphi)\in F_1(c_l^{(1)};\alpha-\beta_1-\cdots-\beta_{l+1};\varphi), \tag{5.35}$$

where $c_l^{(1)}$ is the element of a convergent positive sign numerical series.

Next, we consider the process defined by (5.30). Suppose that

$$W^{(l)}(\tau,\varepsilon,\varphi^{(l)})\in F_2(W_l^*;\alpha-\beta_1-\cdots-\beta_l;\varphi^{(l)}).$$

Then

$$\begin{aligned}
&\big(E_n+W^{(l-1)}(\tau,\varepsilon,\varphi^{(l)}+v^{(l)}(\tau,\varepsilon,\varphi^{(l)}))\big)U^{(l)}(\tau,\varepsilon,\varphi^{(l)})\\
&\qquad\in F_2\Big(r^{2^{l-1}}(1+W_{l-1}^*(1+r^{2^{l-1}}));\alpha-\beta_1-\cdots-\beta_l;\varphi^{(l)}\Big).
\end{aligned}$$

Hence,

$$\begin{aligned}
&W^{(l)}(\tau,\varepsilon,\varphi^{(l)})\\
&\in F_2\Big(r^{2^{l-1}}(1+W_{l-1}^*(1+r^{2^{l-1}}))+W_{l-1}^*(1+r^{2^{l-1}});\alpha-\beta_1-\cdots-\beta_l;\varphi^{(l)}\Big),\quad l=2,3,\ldots.
\end{aligned} \tag{5.36}$$

This implies

$$W_l^* \le r^{2^{l-1}}(1 + W_{l-1}^*(1 + r^{2^{l-1}})) + W_{l-1}^*(1 + r^{2^{l-1}}) \\ < r^{2^{l-1}}(1 + 2W_{l-1}^*) + r^{2^{l-1}} W_{l-1}^* + W_{l-1}^* = r^{2^{l-1}}(1 + 3W_{l-1}^*) + W_{l-1}^*,$$

whence, taking into account that $r < 1$, we succesively obtain

$$W_1^* = r,$$
$$W_2^* < r^2(1 + 3r) + r < r + r^2 + 3r^3 < r + 4r^2,$$
$$W_3^* < r^4(1 + 3(r + 4r^2)) + r + 4r^2 < r + 4r^2 + 16r^4.$$

Further, by the method of mathematical induction, we obtain

$$W_l^* < r + 4r^2 + \cdots + 4^{l-1} r^{2^{l-1}},$$

from which we get

$$W_l^* < Q\Big(r, \frac{1}{4}\Big). \tag{5.37}$$

Consider

$$W^{(l+1)}(\tau, \varepsilon, \varphi) - W^{(l)}(\tau, \varepsilon, \varphi) = \big(E_n + W^{(l)}(\tau, \varepsilon, \varphi + v^{(l+1)}(\tau, \varepsilon, \varphi))\big) U^{(l+1)}(\tau, \varepsilon, \varphi) \\ + W^{(l)}(\tau, \varepsilon, \varphi + v^{(l+1)}(\tau, \varepsilon, \varphi)) - W^{(l)}(\tau, \varepsilon, \varphi). \tag{5.38}$$

By virtue of Corollary from Lemma 4.8, we have

$$W^{(l)}(\tau, \varepsilon, \varphi + v^{(l+1)}(\tau, \varepsilon, \varphi)) - W^{(l)}(\tau, \varepsilon, \varphi) \\ \in F_2\bigg(\frac{2m^2 4^{2m}}{\delta_{l+1}^{2m+2}}\, Q\Big(r, \frac{1}{4}\Big)\, \frac{K_l}{\delta_{l+1}}\,; \alpha - \beta_1 - \cdots - \beta_l - 2\delta_{l+1}; \varphi\bigg),$$

hence,

$$W^{(l)}(\tau, \varepsilon, \varphi + v^{(l+1)}(\tau, \varepsilon, \varphi)) - W^{(l)}(\tau, \varepsilon, \varphi) \in F_2\bigg(Q\Big(r, \frac{1}{4}\Big) f^{2^l}; \alpha - \beta_1 - \cdots - \beta_l - 2\delta_{l+1}; \varphi\bigg).$$

Next, taking into account (5.37),

$$(E_n + W^{(l)}(\tau, \varepsilon, \varphi + v^{(l+1)}(\tau, \varepsilon, \varphi)))U^{(l+1)}(\tau, \varepsilon, \varphi) \\ \in F_2\bigg(r^{2^l}\Big(1 + 2Q\Big(r, \frac{1}{4}\Big)\Big); \alpha - \beta_1 - \cdots - \beta_{l+1}; \varphi\bigg).$$

Hence, by virtue of (5.38),

$$W^{(l+1)}(\tau, \varepsilon, \varphi) - W^{(l)}(\tau, \varepsilon, \varphi) \in F_2(c_l^{(2)}; \alpha - \beta_1 - \cdots - \beta_{l+1}; \varphi), \tag{5.39}$$

where $c_l^{(2)} = r^{2^l}(1 + 3Q(r, 1/4))$ is the element of the convergent positive sign numerical series.

From formulas (5.35), (5.39) follows the convergence of process (5.29). From formulas (5.34) and (5.37) it follows that $w(\tau, \varepsilon, \varphi) \in F_1(M_1^*; \alpha/2; \varphi)$, $W(\tau, \varepsilon, \varphi) \in F_2(M_2^*; \alpha/2; \varphi)$, where

$$M_1^* = Q(r, 1) \exp\Big(\frac{H_1}{q}\, Q(r, q)\Big), \quad M_2^* = Q\Big(r, \frac{1}{4}\Big).$$

□

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