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Tomáš Kisela

ON ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF A LINEAR FRACTIONAL DIFFERENTIAL EQUATION WITH A VARIABLE COEFFICIENT

Abstract. The paper deals with qualitative analysis of solutions of a test linear differential equation involving variable coefficient and derivative of non-integer order. We formulate upper and lower estimates for these solutions depending on boundedness of the variable coefficient. In the special case of asymptotically constant coefficient, we present the sufficient (and nearly necessary) conditions for the convergence of solutions to zero.*

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რეზიუმე. ნაშრომში განხილულია ცვლადკოეფიციენტიანი წილად-წარმოებულიანი მოდელური წრფივი დიფერენციალური განტოლების ამონახსნების ხარისხობრივი ანალიზი. ამ ამონახსნებისთვის ჩამოყალიბებულია ცვლადი კოეფიციენტის შემოსაზღვრულობაზე დამოკიდებული ზედა და ქვედა შეფასებები. კერძო შემთხვევაში, როცა კოეფიციენტი ასიმპტოტურად მუდმივია, წარმოდგენილია ამონახსნის ნულისკენ კრებადობის საკმარისი (და თითქმის აუცილებელი) პირობები.

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1 Introduction

During several last decades, derivatives and integrals of non-integer orders, the so-called fractional derivatives and integrals, serve as an effective tool for modelling of many interesting technical and physical problems originating, e.g., in control theory, rheology, anomalous diffusion, chemistry (see, e.g., [4,7]). The extensive applications of this theory bring the need to understand well basic behaviour of the solutions of differential equations containing fractional derivatives.

Starting point for introductory investigation of the qualitative properties of fractional differential equations is the test equation of the form

$$\mathrm{D}_0^{\alpha} y(t) = \lambda y(t), \quad \alpha \in (0,1), \ \lambda \in \mathbb{R}, \tag{1.1}$$

$$\mathrm{D}_0^{\alpha-1} y(0) = y_0, \quad y_0 \in \mathbb{R}. \tag{1.2}$$

The asymptotic behaviour of (1.1), (1.2) was extensively studied by many authors (see, e.g., [6–8]) and their results can be summarized as

Theorem 1.1. *Let $\alpha \in (0,1)$, $\lambda \in \mathbb{R}$. Then the following statements hold:*

(i) *All solutions of* (1.1) *eventually tend to zero if and only if $\lambda \le 0$.*

(ii) *All non-trivial solutions of* (1.1) *are eventually unbounded if and only if $\lambda > 0$.*

Analogous results were obtained for the modifications of (1.1) including vector cases [6,8], delay [1] or discretized operators [2].

Although the statement of Theorem 1.1 seems to be quite similar to the results known from the classical analysis of the equation $y'(t) = \lambda y(t)$, fractional differential equations show several distinguish properties. Most apparent difference occurs for $\lambda = 0$, where in the integer-order case the solutions are known to be bounded but they do not tend to zero. Theorem 1.1 does not discuss the decay rate of solutions. If $\lambda < 0$, unlike for the integer-order differential equations, the solutions of (1.1) do not tend to zero exponentially, but algebraically (this decay depends on the derivative order α).

The goal of this paper is to generalize Theorem 1.1 for the linear fractional differential equation with variable coefficient, i.e.,

$$\mathrm{D}_0^{\alpha} y(t) = f(t) y(t), \quad \alpha \in (0,1), \quad \lambda \in \mathbb{R}, \tag{1.3}$$

where f is a continuous bounded real function and (1.2) is supplied as the initial condition.

Fractional differential equations with variable coefficients are usually studied in the literature from the viewpoint of constructing the solutions with no particular stress put on qualitative properties of such solutions (see, e.g., [10]). In [8,9], the authors considered (1.3) in the vector form and attempted to employ Grönwall's inequality to perform qualitative analysis, however the resulting assertions and proof techniques contain some unfeasible conditions and incorrect assumptions.

This paper is organized as follows. Section 2 presents basic definitions and preliminary results. Main results are contained in Section 3 including the corresponding proofs. Section 4 concludes the paper by some comments and remarks.

2 Preliminaries

Throughout this paper, we employ the Riemann–Liouville derivative of order α. It is introduced as follows: First, let y be a real scalar function defined on $(0,\infty)$. For $\gamma \in (0,\infty)$, the fractional integral of y is defined as

$$\mathrm{D}_0^{-\gamma} y(t) = \int\limits_0^t \frac{(t-\xi)^{\gamma-1}}{\Gamma(\gamma)} y(\xi)\,\mathrm{d}\xi, \quad t \in (0,\infty),$$

and, for $\alpha \in (0,\infty)$, the Riemann–Liouville fractional derivative of y is defined as

$$\mathrm{D}_0^{\alpha} y(t) = \frac{\mathrm{d}^{\lceil \alpha \rceil}}{\mathrm{d}t^{\lceil \alpha \rceil}} \left(\mathrm{D}_0^{-(\lceil \alpha \rceil - \alpha)} y(t) \right), \quad t \in (0,\infty),$$

where $\lceil \cdot \rceil$ denotes the ceiling function (also called upper integer part). We put $\mathrm{D}_0^0 y(t) = y(t)$ (for more on fractional calculus see, e.g., [5,7]).

It is well-known that the solution of (1.1), (1.2) is given by

$$y(t) = y_0 t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^\alpha),$$

where $E_{\alpha,\alpha}$ denotes the two-parameter Mittag–Leffler function introduced generally via the series

$$E_{\eta,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\eta j + \beta)}, \quad z \in \mathbb{C}, \quad \eta, \beta \in (0,\infty). \tag{2.1}$$

The Mittag–Leffler function is known to play a role of generalized exponential function within fractional calculus. Hence, asymptotic behaviour of (2.1) is essential with respect to the qualitative analysis of fractional differential equations. For some of these properties relevant for this paper see, e.g., [3,7,11].

Lemma 2.1. *Let $\eta, \beta \in (0,\infty)$. Then $E_{\eta,\beta}(z)$ is positive and increasing for $z \in \mathbb{R}$.*

Lemma 2.2. *Let $\eta, \beta \in (0,\infty)$, $\lambda \in \mathbb{R}$.*

(i) *If $\lambda > 0$, then*

$$t^{\beta-1} E_{\eta,\beta}(\lambda t^\eta) = \frac{\lambda^{(1-\beta)/\eta}}{\eta} \exp(\lambda^{1/\eta} t) + \mathcal{O}(t^{\beta-2\eta-1}) \quad \text{as } t \to \infty.$$

(ii) *If $\lambda = 0$, then*

$$t^{\beta-1} E_{\eta,\beta}(\lambda t^\eta) = \frac{t^{\eta-1}}{\Gamma(\eta)}.$$

(iii) *If $\lambda < 0$, then*

$$t^{\beta-1} E_{\eta,\beta}(\lambda t^\eta) = \begin{cases} \dfrac{-t^{\beta-\eta-1}}{\lambda \Gamma(\beta-\eta)} + \mathcal{O}(t^{\beta-3\eta-1}), & \beta \neq \eta, \\ \dfrac{-t^{-\eta-1}}{\lambda^2 \Gamma(-\eta)} + \mathcal{O}(t^{-2\eta-1}), & \beta = \eta \end{cases} \quad \text{as } t \to \infty.$$

We note that the $\mathcal{O}$-symbol for any functions g, h is introduced as $g(t) = \mathcal{O}(h(t))$ as $t \to \infty$ if and only if there exist reals t_0, M such that $|g(t)| \le M|h(t)|$ for all $t \ge t_0$.

3 Main results

In this section we study asymptotic properties of solutions of (1.3) based on the boundedness of the variable coefficient f. We supply (1.3) with the initial condition (1.2) where, without loss of generality, we assume $y_0 \in (0,\infty)$ throughout this section.

Lemma 3.1. *Let $\alpha \in (0,1)$, $U, L \in \mathbb{R}$ and let f be a continuous real function such that*

$$L \le f(t) \le U \quad \text{for all } t \in (0,\infty).$$

Then every solution y of (1.3), (1.2) satisfies

$$y_0 t^{\alpha-1} E_{\alpha,\alpha}(L t^\alpha) \le y(t) \le y_0 t^{\alpha-1} E_{\alpha,\alpha}(U t^\alpha) \quad \text{for all } t \in (0,\infty).$$

Proof. First we show that $y(t) \geq y_0 t^{\alpha-1} E_{\alpha,\alpha}(Lt^\alpha)$ for all $t > 0$. We introduce an auxiliary function ε^- via the relation $\varepsilon^-(t) = f(t) - L$, i.e. $L = f(t) - \varepsilon^-(t)$. Clearly, ε^- is non-negative and bounded by $U - L$. This enables us to rewrite (1.3) as

$$\mathrm{D}_0^\alpha y(t) = Ly(t) + \varepsilon^-(t)y(t).$$

We denote by y_h^L the solution of $\mathrm{D}_0^\alpha y(t) = Ly(t)$, $\mathrm{D}_0^\alpha y(0) = 1$. Hence, based on the variation of constants formula, the solution y of (1.3), (1.2) satisfies

$$y(t) = y_0 y_h^L(t) + \int_0^t y_h^L(t-\xi)\varepsilon^+(\xi)y(\xi)\,\mathrm{d}\xi. \tag{3.1}$$

Due to Lemma 2.1 we have $0 < y_h^L(t)$ for all $t \in (0,\infty)$. Assume that there exists $\widehat{t}$ such that $y(\widehat{t}) < y_0 y_h^L(\widehat{t})$. Relation (3.1) implies that there exists $t_0 \in (0, \widehat{t})$ such that

$$y(t) > y_0 y_h^L(t) \ \text{ for all } \ t \in (0, t_0).$$

Since y is a continuous function, t_0 can be chosen so that $y(t_0) = y_0 y_h^L(t_0)$. Therefore, by (3.1), we get

$$\int_0^{t_0} y_h^L(t_0-\xi)\varepsilon^-(\xi)y(\xi)\,\mathrm{d}\xi = 0. \tag{3.2}$$

Since y_h^L and ε^- are non-negative functions, (3.2) implies that there exists a subset of non-zero measure of $(0, t_0)$ where y is negative, which leads to a contradiction. Hence, $y(t) > y_0 y_h^L(t) = y_0 t^{\alpha-1} E_{\alpha,\alpha}(Lt^\alpha)$ for all $t > 0$.

The second part of the inequality, i.e., $y(t) \leq y_0 t^{\alpha-1} E_{\alpha,\alpha}(Ut^\alpha)$ for all $t > 0$, is proved analogously by using the auxiliary non-negative function ε^+ defined via the relation $\varepsilon^+(t) = U - f(t)$. That concludes the proof. □

This enables us to formulate

Theorem 3.2. *Let $\alpha \in (0,1)$, $U, L \in \mathbb{R}$, $t_0 \in (0,\infty)$ and let f be a bounded continuous function. Further, let $L < f(t) < U$ for all $t \in (t_0, \infty)$.*

(i) *If $U < 0$, then all solutions of (1.3) tend to zero. Moreover, every non-trivial solution y of (1.3), (1.2) satisfies $\widehat{K}^L t^{-\alpha-1} \leq y(t) \leq \widehat{K}^U t^{-\alpha-1}$ as $t \to \infty$ for suitable positive real constants $\widehat{K}^L$, $\widehat{K}^U$.*

(ii) *If $U = 0$, then all solutions of (1.3) tend to zero. Moreover, every non-trivial solution y of (1.3), (1.2) satisfies $\widehat{K}^L t^{-\alpha-1} \leq y(t) \leq \widehat{K}^U t^{\alpha-1}$ as $t \to \infty$ for suitable positive real constants $\widehat{K}^L$, $\widehat{K}^U$.*

(iii) *If $L > 0$, then all non-trivial solutions of (1.3) are unbounded.*

Proof. (i) Since f is bounded, Lemma 3.1 implies that the solution y of (1.3) is positive.

First let us prove that $y(t) \leq \widehat{K}^U t^{-\alpha-1}$ as $t \to \infty$ for suitable real $\widehat{K}^U$. We denote $\varepsilon^+(t) = U - f(t)$ and, using similar approach as in the proof of Lemma 3.1, rewrite the solution of (1.3) as

$$y(t) = y_0 y_h^U(t) - \int_0^{t_0} y_h^U(t-\xi)\varepsilon^+(\xi)y(\xi)\,\mathrm{d}\xi - \int_{t_0}^{t} y_h^U(t-\xi)\varepsilon^+(\xi)y(\xi)\,\mathrm{d}\xi. \tag{3.3}$$

Now, we investigate each term of (3.3) separately. The asymptotic behaviour of the first term is known, indeed, due to $U < 0$ and Lemma 2.2, we have

$$y_0 y_h^U(t) = y_0 t^{\alpha-1} E_{\alpha,\alpha}(Ut^\alpha) = \frac{-y_0 t^{-\alpha-1}}{U^2\Gamma(-\alpha)} + \mathcal{O}(t^{-2\alpha-1}) \ \text{ as } \ t \to \infty. \tag{3.4}$$

The middle term of (3.3) contains positive functions y_h^U, y and the function ε^+ which is allowed to change its sign on $(0, t_0)$, but is bounded, i.e., there exists m such that $|\varepsilon^+(t)| < m$ for all $t \in (0, t_0)$. Thus, we get

$$\left| -\int_0^{t_0} y_h^U(t-\xi)\varepsilon^+(\xi)y(\xi)\,\mathrm{d}\xi \right| \le \int_0^{t_0} y_h^U(t-\xi)|\varepsilon^+(\xi)|y(\xi)\,\mathrm{d}\xi$$

$$\le m\Big(\frac{-y_0 t^{-\alpha-1}}{U^2\Gamma(-\alpha)} + \mathcal{O}(t^{-2\alpha-1})\Big)\int_0^{t_0} y(\xi)\,\mathrm{d}\xi \le K t^{-\alpha-1} \text{ as } t\to\infty, \quad (3.5)$$

where we have used the fact that the solution y of (1.3) is integrable (see, e.g., [5,7]).

The third term of (3.3) contains only positive functions y_h^U, y and ε^+ (more precisely, ε^+ is non-negative for $t \in (0,\infty)$). Considering this along with (3.4), (3.5), we can estimate (3.3) as

$$y(t) \le \widehat{K}^U t^{-\alpha-1} \text{ as } t\to\infty,$$

where $\widehat{K}^U$ is a suitable positive real constant.

The second part of the inequality, i.e., $y(t) \ge \widehat{K}^L t^{-\alpha-1}$, can be proved analogously.

The assertions (ii), (iii) can be proved by using similar steps as for (i). □

Theorem 3.2 directly implies the following results for f being asymptotically constant.

Corollary 3.3. *Let $\alpha \in (0,1)$, $P \in \mathbb{R}$ and let f be a bounded continuous function such that*

$$\lim_{t\to\infty} f(t) = P.$$

Then the following statements hold:

(i) *All solutions of* (1.3) *eventually tend to zero if $P < 0$.*

(ii) *All non-trivial solutions of* (1.3) *are eventually unbounded if $P > 0$.*

We can see that Corollary 3.3 is nearly in the effective form. The only case holding us from formulating not only sufficient but also necessary conditions, is $P = 0$. Theorem 1.1 indicates that $\lambda = 0$ plays a role of stability boundary for (1.1), (1.2). Corollary 3.3 therefore further highlights the special importance of the zero right-hand side of fractional differential equations.

Lemma 3.1 implies that if f is allowed to change its sign, the solutions of (1.3) can tend to zero and be unbounded. In particular, we can see that if f is non-positive and tends to zero, the solutions of (1.3) tend to zero (see Theorem 3.2(ii)). None of Theorems 1.1, 3.2 discusses situations when f tends to zero and is positive or oscillates. To illustrate the range of possible behaviours of solutions (1.3) in such cases, we consider the following examples:

(A) Let f be a bounded continuous function satisfying

$$\lim_{t\to\infty} f(t) = 0 \text{ and } f(t) > K t^{-\gamma},$$

where $\gamma \in (0,\infty)$, $\alpha \in (0,1)$ and K is a positive real. Then the solution y of (1.3), (1.2) can be estimated as

$$y(t) = y_0 \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \int_0^t \frac{(t-\xi)^{\alpha-1}}{\Gamma(\alpha)} f(\xi)y(\xi)\,\mathrm{d}\xi$$

$$\ge y_0 \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \int_0^t \frac{(t-\xi)^{\alpha-1}}{\Gamma(\alpha)} \frac{K\xi^{\alpha-\gamma-1}}{\Gamma(\alpha)}\,\mathrm{d}\xi = y_0 \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{K\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}\,\frac{t^{2\alpha-\gamma-1}}{\Gamma(2\alpha-\gamma)}\,.$$

Obviously, if $\gamma \in (0, 2\alpha-1)$ and $\alpha \in (1/2, 1)$, then y is eventually unbounded.

(B) Let f be a bounded continuous function such that

$$f(t) \geq 0 \text{ for } t \in (0, \infty) \text{ and } f(t) = 0 \text{ for } t \in (t_0, \infty),$$

where $t_0 \in (0, \infty)$, $\alpha \in (0, 1)$. As in the proof of Theorem 3.2, the solution y of (1.3) can be estimated as

$$y(t) = y_0 \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \int_0^{t_0} \frac{(t-\xi)^{\alpha-1}}{\Gamma(\alpha)} f(\xi) y(\xi) \, \mathrm{d}\xi$$

$$\leq y_0 \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{K_1 (t-t_0)^{\alpha-1}}{\Gamma(\alpha)} \int_0^{t_0} y(\xi) \, \mathrm{d}\xi \leq y_0 \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{K_2 (t-t_0)^{\alpha-1}}{\Gamma(\alpha)},$$

where K_1, K_2 are suitable positive reals. Obviously, y tends to zero.

Remark. The assumption of $y_0 \in (0, \infty)$ made throughout this section is not essential. Clearly, if $y_0 \in (-\infty, 0)$, then the resulting inequalities only change their orientation.

4 Conclusions

We have studied asymptotic properties of solutions of the linear fractional differential equation with variable coefficient (1.3)).

Lemma 3.1 implies that if f is bounded, then the corresponding solution of (1.3) is bounded by the solutions of (1.1) for particular choices of λ depending on the bounds of f. Consequently, Theorem 1.1 shows that the solutions of (1.3) pose algebraic decay or exponential growth if f is bounded and non-positive or positive, respectively.

The assumptions on the sign of f needed in Theorem 1.1 were weakened in Theorem 3.2 where the fixed sign of f is required only for sufficiently large t. Finally, Corollary 3.3 outlines the specific case of asymptotically constant coefficient f. In particular, if f tends to a non-zero constant, the full discussion of asymptotic behaviour is presented. If f tends to zero, solutions can be eventually unbounded or tending to zero depending on decay rate of f as illustrated by the examples.

Possible future research directions are a deeper analysis of the case when f asymptotically approaches zero, and various generalizations of (1.3) to multi-term equations or vector forms.

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Author's address:

Institute of Mathematics, Brno University of Technology, Technická 2, 616 69 Brno, Czech Republic.

E-mail: kisela@fme.vutbr.cz