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p-MOMENT EXPONENTIAL STABILITY OF DIFFERENTIAL EQUATIONS WITH RANDOM IMPULSES AND THE ERLANG DISTRIBUTION

Abstract. The investigation of differential equations with random impulses combines ideas in the qualitative theory of differential equations and probability theory. The p-moment exponential stability of the solutions is defined and studied when the waiting time between two consecutive impulses is Erlang distributed. The study is based on the application of Lyapunov functions. Some examples are given to illustrate the results.

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რეზიუმე. შემთხვევით იმპულსებიანი დიფერენციალური განტოლებების გამოკვლევა აერთიანებს დიფერენციალურ განტოლებათა თვისებრივი თეორიისა და ალბათობის თეორიის იდეებს. ამონახსნთა p-მომენტური ექსპონენციალური მდგრადობა განსაზღვრულია და შესწავლილია იმ შემთხვევაში, როცა ორ მომდევნო იმპულსს შორის ლოდინის დრო არის ერლანგის წესით განაწილებული. გამოკვლევა ეფუძნება ლიაპუნოვის ფუნქციების გამოყენებას. შედეგების საილუსტრაციოდ მოყვანილია მაგალითები.

1 Introduction

In some real world phenomena the investigated process changes instantaneously at uncertain moments. In modeling such processes it is necessarily to combine deterministic differential equations with random variables presenting the moments of impulses. The presence of randomness in the jump condition changes the behavior of solutions of differential equations significantly. The study of properties of solutions combines methods of deterministic differential equations and probability theory. Impulsive differential equations with random impulsive moments differ from stochastic differential equations with jumps [5, 9–11]. Investigations concerning deterministic differential equations with random impulses were considered in [2, 3, 7, 8, 12], but there are some inaccuracies there concerning properties of deterministic variables and random variables.

In this paper we study nonlinear differential equations subject to impulses occurring at random moments. Inspired by queuing theory and the distribution for the waiting time, we study the case of Erlang distributed random variables between two consecutive moments of impulses. The p-moment exponential stability of the solution is investigated by employing Lyapunov's functions.

2 Random impulses in differential equations

Let the increasing sequence of nonnegative points $\{T_k\}_{k=0}^{\infty}$ be given with $\lim_{k\to\infty}\{T_k\} = \infty$. Consider the initial value problem for the system of *impulsive differential equations* (IDE) with fixed points of impulses

$$\begin{gathered} x' = f(t, x(t)) \ \text{ for } \ t \in (T_k, T_{k+1}], \ \ k = 0, 1, 2, \dots, \\ x(T_k + 0) = I_k(x(T_k - 0)) \ \text{ for } \ k = 1, 2, \dots, \\ x(T_0) = x_0, \end{gathered} \tag{2.1}$$

where $x, x_0 \in \mathbb{R}^n$, $f \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$, $I_k : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$.

We will assume the following condition is satisfied

H1. *$f(t, 0) = 0$ and $I_k(t, 0) = 0$ for $t \geq 0$, $k = 1, 2, \dots$.*

Let the probability space $(\Omega, \mathcal{F}, P)$ be given. Let $\{\tau_k\}_{k=1}^{\infty}$ be a sequence of independent random variables that are defined on the sample space Ω. We will call the random variables τ_k waiting times. Assume $\sum_{k=1}^{\infty} \tau_k = \infty$ with probability 1.

We will assume the following condition is satisfied:

H2. *The random variables $\{\tau_k\}_{k=1}^{\infty}$, $\tau_k \in Erlang(\alpha_k, \lambda)$ are independent random variables.*

We will recall some properties of Erlang distribution:

(i) If $X \in Erlang(\alpha_1, \lambda)$ and $Y \in Erlang(\alpha_2, \lambda)$ are independent random variables, then $X + Y \in Erlang(\alpha_1 + \alpha_2, \lambda)$;

(ii) The cumulative distribution function (CDF) of $Erlang(\alpha, \lambda)$ is

$$F(x; \alpha, \lambda) = 1 - \sum_{j=1}^{\alpha-1} \frac{(\lambda x)^j}{j!}\, e^{-\lambda x}, \ \ x \geq 0,$$

and the density function is

$$f(x; \alpha, \lambda) = \lambda\, \frac{(\lambda x)^{\alpha-1}}{(\alpha-1)!}\, e^{-\lambda x}, \ \ x > 0.$$

Define the sequence of random variables $\{\xi_k\}_{k=0}^{\infty}$ such that

$$\xi_k = T_0 + \sum_{i=1}^{k} \tau_i, \ \ k = 1, 2, \dots,$$

where $T_0 \geq 0$ is a fixed point.

We note that $\{\xi_k\}_{k=0}^{\infty}$ is an increasing sequence of random variables defined by the recurrence formula $\xi_0 = T_0$, $\xi_k = \xi_{k-1} + \tau_k$, $k = 1, 2, \dots$. The random variable ξ_n will be called the waiting time and it gives the arrival time of n-th impulses.

Remark 2.1. The random variables $\Xi_n = \xi_n - T_0 = \sum_{i=1}^{n} \tau_i \in Erlang\big(\sum_{i=1}^{n} \alpha_i, \lambda\big)$ are continuous with CDF

$$F\Big(t; \sum_{i=1}^{n} \alpha_i, \lambda\Big) = P(\Xi_n < t) = 1 - \sum_{j=1}^{\sum_{i=1}^{n} \alpha_i - 1} \frac{(\lambda t)^j}{j!} e^{-\lambda t} \ \text{ for } \ t \geq 0.$$

Let the points t_k be arbitrary values of the random variables τ_k, $k = 1, 2, \dots$, correspondingly. Define the increasing sequence of points $T_k = T_0 + \sum_{i=1}^{k} t_i$, $k = 1, 2, 3 \dots$, that are values of the random variables ξ_k.

Consider the initial value problem (IVP) for the system of impulsive fractional differential equations (IFrDE) with fixed points of impulses (2.1). The solution of IVP for IDE (2.1) depends not only on the initial condition (T_0, x_0) but on the moments of impulses T_k, $k = 1, 2, \dots$, i.e. the solution depends on the initially chosen arbitrary values t_k of the random variables τ_k, $k = 1, 2, \dots$. We denote the solution of the initial value problem (2.1) by $x(t; T_0, x_0, \{T_k\})$. We will assume that $x(T_k; T_0, x_0, \{t_k\}) = \lim_{t \to T_k - 0} x(t; T_0, x_0, \{t_k\})$ for any $k = 1, 2, \dots$.

The set of all solutions $x(t; T_0, x_0, \{T_k\})$ of IVP for IDE (2.1) for any values t_k of the random variables τ_k, $k = 1, 2, \dots$, generates a stochastic process with state space $\mathbb{R}^n$. We denote it by $x(t; T_0, x_0, \{\tau_k\})$ and we will say that it is a solution of the following initial value problem for impulsive differential equations with random moments of impulses (RIDE)

$$\begin{aligned} x'(t) &= f(t, x(t)) \ \text{ for } \ t \geq T_0, \ \ \xi_k < t < \xi_{k+1}, \ \ k = 0, 1, \dots, \\ x(\xi_k + 0) &= I_k(x(\xi_k - 0)) \ \text{ for } \ k = 1, 2, \dots, \\ x(T_0) &= x_0. \end{aligned} \tag{2.2}$$

Definition 2.1. Let t_k be a value of the random variable τ_k, $k = 1, 2, 3, \dots$, and $T_k = T_0 + \sum_{i=1}^{k} t_i$, $k = 1, 2, \dots$. Then the solution $x(t; T_0, x_0, \{T_k\})$ of the IVP for the IDE with fixed points of impulses (2.1) is called *a sample path solution* of the IVP for the RIDE (2.2).

Any sample path solution $x(t; T_0, x_0, \{T_k\}) \in C^1((T_k, T_{k+1}], \mathbb{R}^n)$, $k = 0, 1, 2, \dots$.

Definition 2.2. A stochastic process $x(t; T_0, x_0, \{\tau_k\})$ with an uncountable state space $\mathbb{R}^n$ is said to be a solution of the IVP for the system of RIDE (2.2) if for any values t_k of the random variable τ_k, $k = 1, 2, 3, \dots$, and $T_k = T_0 + \sum_{i=1}^{k} t_i$, $k = 1, 2, \dots$, the corresponding function $x(t; T_0, x_0, \{T_k\})$ is a sample path solution of the IVP for RIDE (2.2).

Example.

Case 1 (*differential equation*). Consider the following scalar ordinary differential equation (ODE) $x' = 0$, $x(0) = x_0 \neq 0$. Its solution $x(t) = x_0$, $t \geq 0$, is stable but does not approach 0.

Case 2 (*impulsive differential equations with fixed points of impulses*). Consider the following IVP for the scalar IDE (2.1)

$$\begin{aligned} x' &= 0 \ \text{ for } \ t \geq 0, \ \ t \neq T_k, \\ x(T_k + 0) &= a x(T_k - 0) \ \text{ for } \ k = 1, 2, \dots, \\ x(0) &= x_0 \neq 0, \end{aligned} \tag{2.3}$$

where a is a constant.

The solution of IVP (2.3) is the piecewise continuous function $x(t; x_0) = a^k x_0$ for $t \in (T_k, T_{k+1}]$. The behavior of $x(t; x_0)$ depends significantly on the amplitude a of the impulses.

If $|a| < 1$ then $|x(t;x_0)|$ approaches 0.

Case 3 (*differential equation with random points of impulses*). Consider the following partial case of the IVP for RIDE (2.2)

$$\begin{aligned} x' &= 0 \text{ for } t \geq 0, \ \xi_k < t < \xi_{k+1}, \\ x(\xi_k + 0) &= ax(\xi_k - 0) \text{ for } k = 1, 2, \dots, \\ x(0) &= x_0 \neq 0, \end{aligned} \tag{2.4}$$

where $x \in \mathbb{R}$, a is a constant and the random variables ξ_k are defined above.

Let for any $k = 1, 2, \dots$ the point t_k be an arbitrary value of the random variable τ_k and $T_k = \sum\limits_{i=0}^{k} t_i$, $k = 1, 2, 3 \dots$, i.e. T_k is a value of the random variable ξ_k. Consider the IVP for the corresponding IDE

$$\begin{aligned} x' &= 0 \text{ for } t \geq 0, \ t \neq T_k, \\ x(T_k + 0) &= ax(T_k - 0) \text{ for } k = 1, 2, \dots, \\ x(0) &= x_0. \end{aligned} \tag{2.5}$$

The solution of (2.5) is $x(t; 0, x_0, \{T_k\}) = a^k x_0$ for $T_k < t \leq T_{k+1}$. It depends on both initial value x_0 and the moments of impulses T_k, i.e. on the initially chosen arbitrary values t_k of the random variables τ_k, $k = 1, 2, \dots$.

The set of all solutions of the IVP (2.5) for any values t_k of the random variables τ_k generates a stochastic process $x(t; x_0, \{\tau_k\}) = a^k x_0$ for $\xi_k < t \leq \xi_{k+1}$ which has an expected value

$$E\big|x(t; x_0, \{\tau_k\})\big| = |x_0| P(T_0 < t < \xi_1) + \sum_{k=1}^{\infty} |a^k x_0| P(\xi_k < t < \xi_{k+1}),$$

i.e. it depends significantly on the distribution of the random variables τ_k.

3 Preliminary probability results

Lemma 3.1. *Let the condition* (H2) *be satisfied. Then the probability that there will be exactly k impulses until time t, $t \geq T_0$, is given by*

$$P(S_k(t)) = e^{-\lambda(t-T_0)} \sum_{j=\sum_{i=1}^{k-1} \alpha_i}^{\sum_{i=1}^{k} \alpha_i - 1} \frac{(\lambda(t-T_0))^j}{j!}, \quad t \geq T_0,$$

where the events $S_k(t) = \{\omega \in \Omega : \ \xi_k(\omega) < t < \xi_{k+1}(\omega)\}$, $k = 1, 2, \dots$.

Proof. According to Remark 2.1 we get

$$\begin{aligned} P(S_k(t)) &= P(\xi_k < t < \xi_{k+1}) = P(\xi_k - T_0 < t - T_0 < \xi_{k+1} - T_0) \\ &= F_{\Xi_k}\Big(t - T_0; \sum_{i=1}^{k} \alpha_i, \lambda\Big) - F_{\Xi_{k+1}}\Big(t - T_0; \sum_{i=1}^{k+1} \alpha_i, \lambda\Big). \end{aligned}$$

□

Corollary. *Let the condition* (H2) *be satisfied with $\alpha_i = \alpha$, $i = 1, 2, \dots$. Then the probability that there will be exactly k impulses until time t, $t \geq T_0$, is given by*

$$P(S_k(t)) = e^{-\lambda(t-T_0)} \sum_{j=(k-1)\alpha}^{k\alpha - 1} \frac{(\lambda(t-T_0))^j}{j!}.$$

We now obtain a formula for the solution of the initial value problem for a scalar linear differential equation with random moments of impulses:

$$\begin{aligned} u' &= -m_k u \quad \text{for} \quad t \geq T_0, \quad \xi_k < t < \xi_{k+1}, \\ u(\xi_k + 0) &= b_k u(\xi_k - 0) \quad \text{for} \quad k = 1, 2, \dots, \\ u(T_0) &= u_0, \end{aligned} \tag{3.1}$$

where $u_0 \in \mathbb{R}$, $m_k > 0$, $k = 0, 1, 2, \dots$, and $b_k \neq 1$, $k = 1, 2, \dots$, are real constants.

Lemma 3.2. *Let the condition* (H2) *be satisfied and the nonincreasing sequence of real positive numbers* $\{m_i\}_{i=0}^{\infty}$ *be such that*

$$\sum_{k=0}^{\infty} e^{-m_k(t-T_0)} \prod_{i=1}^{k} |b_i| < \infty.$$

Then the solution of the IVP for the linear RIDE (3.1) *is given by the formula*

$$u(t; T_0, u_0, \{\tau_k\}) = \begin{cases} u_0 e^{-m_0(t-T_0)} & \text{for } T_0 < t < \tau_1 \\ u_0 \Big(\prod_{i=1}^{k} b_i \Big) e^{-\sum_{i=1}^{k} m_{i-1}\tau_i} e^{-m_k(t-\xi_k)} & \text{for } \xi_k < t < \xi_{k+1}, \quad k = 1, 2, \dots, \end{cases} \tag{3.2}$$

and the expected value of the solution satisfies the inequality

$$E\big(|u(t; T_0, u_0, \{\tau_k\})|\big) \leq |u_0| \sum_{k=0}^{\infty} e^{-m_k(t-T_0)} \prod_{i=1}^{k} |b_i| \quad \text{for } t \geq T_0.$$

Proof. The formula for the solution follows from the formula for the solution of the corresponding IVP for the linear IDE with fixed points of impulses and Definition 2.2.

According to Lemma 3.1, formula (3.2), the independence of the random variables τ_k and inequality $E(\eta) \leq E(\xi)$ for the random variables $\eta, \xi: \ 0 \leq \eta \leq \xi$ we have

$$\begin{aligned} E\big(|u(t; T_0, u_0, \{\tau_k\})|\big) &= |u_0| e^{-m_0(t-T_0)} P(\tau_1 > t - T_0) \\ &+ \sum_{k=1}^{\infty} |u_0| \Big(\prod_{i=1}^{k} |b_i| \Big) e^{-m_k(t-T_0)} \prod_{i=1}^{k} E\big(e^{-(m_{i-1}-m_k)\tau_i}\big) P(S_k(t)) \quad \text{for } t \geq T_0. \end{aligned} \tag{3.3}$$

Using the definition of the density function of the Erlang distribution and substituting $(m_i - m_k + \lambda)x = s$ we get

$$\begin{aligned} E e^{(m_k - m_i)\tau_i} &= \int_0^{\infty} e^{(m_k-m_i)x} \lambda \frac{(\lambda x)^{\alpha_i - 1}}{(\alpha_i - 1)!} e^{-\lambda x} \, dx = \frac{(\lambda)^{\alpha_i}}{(\alpha_i - 1)!} \int_0^{\infty} e^{-(m_i - m_k + \lambda)x} x^{\alpha_i - 1} \, dx \\ &= \frac{1}{(m_i - m_k + \lambda)^{\alpha_i}} \frac{(\lambda)^{\alpha_i}}{(\alpha_i - 1)!} \int_0^{\infty} e^{-s} s^{\alpha_i - 1} \, ds = \Big(\frac{\lambda}{m_i - m_k + \lambda} \Big)^{\alpha_i}. \end{aligned} \tag{3.4}$$

Substitute (3.4) in (3.3), use Lemma 3.1 and obtain

$$\begin{aligned} E\big(|u(t; T_0, u_0, \{\tau_k\})|\big) &= |u_0| e^{-(m_0+\lambda)(t-T_0)} \sum_{j=1}^{\alpha_1 - 1} \frac{(\lambda(t-T_0))^j}{j!} \\ &+ \sum_{k=1}^{\infty} |u_0| e^{-(m_k+\lambda)(t-T_0)} \Big(\prod_{i=1}^{k} |b_i| \Big) \prod_{i=1}^{k} \Big(\frac{\lambda}{m_i - m_k + \lambda} \Big)^{\alpha_i} \sum_{j=\sum_{i=1}^{k-1} \alpha_i}^{\sum_{i=1}^{k} \alpha_i - 1} \frac{(\lambda(t-T_0))^j}{j!}. \end{aligned} \tag{3.5}$$

Inequalities (3.5) and $\frac{\lambda}{m_i - m_k + \lambda} \leq 1$ prove the lemma. □

Remark 3.1. Note that the conditions of Lemma 3.2 are satisfied for $m_k = m$ and $|b_i| = \frac{1}{2}$.

4 p-exponential stability

In this paper we will use Lyapunov functions $V(t,x) : J \times \Delta \to \mathbb{R}_+$, which are continuous on $J \times \Delta$ and locally Lipschitzian with respect to its second argument, where $J \subset \mathbb{R}_+$ and $\Delta \subset \mathbb{R}^n$, $0 \in \Delta$, and their Dini derivatives.

Definition 4.1. Let $p > 0$. Then the trivial solution ($x_0 = 0$) of the RIDE (2.2) is said to be p-moment exponentially stable if for any $x_0 \in \mathbb{R}^n$ there exist constants α, $\mu > 0$ such that $E[\|x(t;T_0,x_0,\{\tau_k\})\|^p] < \alpha\|x_0\|^p e^{-\mu(t-T_0)}$ for all $t > T_0$, where $x(t;T_0,x_0,\{\tau_k\}$ is the solution of the IVP for the RIDE (2.2).

Theorem 4.1. *Let the following conditions be satisfied:*

1. *The conditions* (H1)*,* (H2) *hold.*
2. *The function $V \in \Lambda(\mathbb{R}_+,\mathbb{R}^n)$ and there exist positive constants a, b such that*
   (i) *$a\|x\|^p \le V(t,x) \le b\|x\|^p$ for $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$;*
   (ii) *there exists a constant $m \ge 0$ such that*
   $$D^+_{(2.1)}V(t,x) \le -mV(t,x) \ \text{ for } \ t \in \mathbb{R}_+, \ \ x \in \mathbb{R}^n;$$
   (iii) *for any $k = 1,2,\dots$ there exist functions $w_k \in C(\mathbb{R}_+,\mathbb{R}_+)$ and constants $C_k > 0$, $w_k(t) \le C_k$ for $t \ge 0$ such that $\sum_{k=0}^{\infty}\prod_{i=1}^{k} C_i = C < \infty$ and*
   $$V(t,I_k(x)) \le w_k(t)V(t,x) \ \text{ for } \ t \in \mathbb{R}_+, \ \ x \in \mathbb{R}^n. \tag{4.1}$$

Then the trivial solution of the RIDE (2.2) *is p-moment exponentially stable.*

Proof. Let $(T_0,x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ be an arbitrary initial data and the stochastic process $x_\tau(t) = x(t;T_0,x_0,\{\tau_k\})$ be a solution of the IVP for the RIDE (2.2).

Now consider the IVP for the scalar linear RIDE (3.1) with $m_k = m$, $b_k = C_k$ and $u_0 = V(T_0,x_0)$ with a solution $u_\tau(t) = u(t;T_0,x_0,\{\tau_k\})$. According to Lemma 3.2 the inequality

$$E\big(|u_\tau(t)|\big) \le |u_0|e^{-m(t-T_0)}\sum_{k=0}^{\infty}\prod_{i=1}^{k}|C_i| = C|u_0|e^{-m(t-T_0)}$$

holds.

Let t_k be arbitrary values of the random variables τ_k, $k = 1,2,\dots$, and $T_k = T_0 + \sum_{i=1}^{k} t_i$, $k = 1,2,\dots$. Consider the sample path solutions $x(t) = x(t;T_0,x_0,\{T_k\})$ and $u(t) = u(t;T_0,x_0,\{T_k\})$.

Let $v(t) = V(t,x(t))$ for $t \ge T_0$. The function $v(t)$ satisfies the linear impulsive differential inequalities with fixed points of impulses

$$\begin{aligned} &D_+v(t) \le -m\,v_\tau(t) \ \text{ for } \ T_k < t < T_{k+1},\\ &v(T_k+) \le C_k v(T_k), \ \ k = 1,2,\dots,\\ &\ v(T_0) = V(T_0,x_0). \end{aligned} \tag{4.2}$$

The function $m(t) = v(t) - u(t)$, $t \ge T_0$, is a piecewise continuous function and satisfies IVP (4.2) with a zero initial condition. Therefore $m(t) \le 0$ on $[T_0,\infty)$ (for details see the books [4,6]). Therefore $v_\tau(t) \le u_\tau(t)$ where the set of $v(t;T_0,x_0,\{T_k\})$ for any values t_k of the random variables τ_k, $k = 1,2,\dots$, generates a stochastic process $v_\tau(t)$ with state space $\mathbb{R}^n$.

From the condition 2 (i) of Theorem 4.1 we obtain

$$\begin{aligned} E\big(\|x_\tau(t)\|^p\big) &= \frac{1}{a}\,E\big(a\|x_\tau(t)\|^p\big) \le \frac{1}{a}\,E(V(t,x_\tau(t))) = \frac{1}{a}\,E(v_\tau(t)) \le \frac{1}{a}\,E(u_\tau(t))\\ &\le \frac{C}{a}\,V(T_0,x_0)e^{-m(t-T_0)} \le \frac{Cb}{a}\,\|x_0\|^p e^{-m(t-T_0)}, \ \ t \ge T_0. \end{aligned}$$

□

Remark 4.1. If $\alpha_k = 1$ for all k, i.e. the random variables $\tau_k \in Exp(\lambda)$, the p-moment exponential stability is studied in [1].

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