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THEOREMS ON DIFFERENTIAL INEQUALITIES AND PERIODIC BOUNDARY VALUE PROBLEM FOR SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS

Abstract. The aim of the present article is to get efficient conditions for the solvability of the periodic boundary value problem

$$u'' = f(t, u); \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega),$$

where the function $f\colon [0,\omega]\times\,]0,+\infty[\, \to \mathbb{R}$ satisfies local Carathéodory conditions, i.e., it may have "singularity" for $u = 0$. For this purpose, first the technique of differential inequalities is developed and the question on existence and uniqueness of a positive solution of the linear problem

$$u'' = p(t)u + q(t); \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega)$$

is studied. A systematic application of the above-mentioned technique enables one to derive sufficient and in certain cases also necessary conditions for the solvability of the nonlinear problem considered.

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რეზიუმე. ნაშრომში შესწავლილია

$$u'' = f(t, u); \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega)$$

პერიოდული სასაზღვრო ამოცანის ამონახსნის არსებობისა და ერთადერთობის საკითხი, სადაც $f\colon [0,\omega]\times\,]0,+\infty[\, \to \mathbb{R}$ არის კარათეოდორის ფუნქცია, რომელსაც შეიძლება ჰქონდეს სინგულა-რობა $u = 0$ წერტილში. ამ მიზნით ჯერ განვითარებულია დიფერენციალურ უტოლობათა ტექნიკა და შესწავლილია

$$u'' = p(t)u + q(t); \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega)$$

წრფივი ამოცანის დადებითი ამონახსნის არსებობის საკითხი. შემდგომ, აღნიშნული ტექნიკის გამოყენებით დადგენილია არაწრფივი ამოცანის ამონახსნის არსებობისა და ერთადერთობის ეფექტური საკმარისი პირობები.

Introduction

The aim of the present work is to study solvability of the periodic boundary value problem

$$u'' = f(t,u); \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega), \tag{0.1}$$

where the function f satisfies either

$$f \in K([0,\omega] \times \mathbb{R}; \mathbb{R}) \tag{0.2}$$

(i.e., $f\colon [0,\omega] \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions), or

$$f \in K_{loc}([0,\omega] \times]0,+\infty[\,; \mathbb{R}) \tag{0.3}$$

(i.e., $f\colon [0,\omega] \times]0,+\infty[\to \mathbb{R}$ satisfies local Carathéodory conditions).

Under a solution of the problem (0.1) in the case when (0.2) holds we understand a function $u \in AC'([0,\omega])$ satisfying given equation for almost all $t \in [0,\omega]$ and boundary conditions in (0.1), while in the case when (0.3) is fulfilled, solutions of (0.1) are supposed to be **positive**.

Among the earlier works playing an important role in the development of the theory of the periodic boundary value problem for differential equations and their systems, we refer to [7, 15, 11, 13]. In particular, all these works contributed significantly to the study of problem (0.1) with f satisfying (0.2). A comprehensive exposition of the topic with relevant historical and bibliographical notes up to 2006 can be found in [4] (see also the survey [20]). The last mentioned book, which is devoted mainly to boundary value problems for second order equations, is the first monographic publication dealing with "phase singular" periodic problems (0.1), i.e., when f satisfies (0.3). The further development of the theory of singular periodic problems is described in [21] (mainly in Section 8). The theory of phase singular problems (0.1) is currently under active development and is far from being complete. A number of recent results are contained, in particular, in [1, 6, 3, 9] and papers cited therein.

The present work is organized as follows. Chapter 1 is of technical character and contains several known results in a suitable for us form for the convenience of references.

Chapter 2 is devoted to the description of the sets $\mathcal{V}^-(\omega)$ and $\mathcal{V}^+(\omega)$ introduced in Definition 0.1. Both these sets play crucial role for the whole article with two reasons. First, each of the inclusions $p \in \mathcal{V}^-(\omega)$ and $p \in \mathcal{V}^+(\omega)$ yield the unique solvability of the linear problem

$$u'' = p(t)u + q(t); \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega), \tag{0.4}$$

and the second, that the condition $p \in \mathcal{V}^-(\omega)$, resp. $p \in \mathcal{V}^+(\omega)$, implies the validity of a certain theorem on differential inequalities which is widely used further for the study of nonlinear problem (0.1) (see Remarks 0.5 and 0.6).

In the case when p is a constant function, say $p(t) \overset{\text{def}}{=} c$, the inclusion $p \in \mathcal{V}^-(\omega)$ holds if and only if $c \in]0,+\infty[$ while the inclusion $p \in \mathcal{V}^+(\omega)$ is equivalent with $c \in [-\frac{\pi^2}{\omega^2}, 0[$ (if $c = 0$ then $p \in \mathcal{V}_0(\omega)$, where the set $\mathcal{V}_0(\omega)$ is introduced in Definition 0.2). However, in general, description of the sets $\mathcal{V}^-(\omega)$ and $\mathcal{V}^+(\omega)$ is not so simple and is far from to be complete. In Section 8, resp. Section 9, we state necessary and sufficient conditions for the validity of the the inclusion $p \in \mathcal{V}^-(\omega)$, resp. $p \in \mathcal{V}^+(\omega)$, while in Sections 11 and 12 several optimal efficient conditions are stated. Section 10 is devoted to the properties of the sets $\mathcal{V}^-(\omega)$ and $\mathcal{V}^+(\omega)$. For example, Proposition 10.7 states that the set $\mathcal{V}^+(\omega)$ is unbounded from above. This property of the set $\mathcal{V}^+(\omega)$ is not possible to realise for the constant function p. In Section 13, difference of the sets $\mathcal{V}^-(\omega)$ and $\mathcal{V}^+(\omega)$ is shown with respect to Lyapunov stability. In particular, it is shown that if $p \in \mathcal{V}^-(\omega)$ then the equation

$$u'' = p(t)u \tag{0.5}$$

is exponentially dichotomic while if $p \in \mathcal{V}^+(\omega)$ then it is Lyapunov stable. As an off product of Chapter 2 there are efficient conditions for stability, resp. unstability of the equation (0.5).

Chapter 3 is devoted to the periodic boundary value problem (0.1). First, in Section 16, the existence of a **positive** solution of the linear problem (0.4) is studied. As it follows from the definitions of the sets $\mathcal{V}^-(\omega)$ and $\mathcal{V}^+(\omega)$, if $p \in \mathcal{V}^-(\omega)$, resp. $p \in \mathcal{V}^+(\omega)$, and q is nontrivial and nonpositive, resp. nonnegative, then then the problem (0.4) has a unique solution and this solution is positive. In Section 16 optimal efficient conditions are stated guaranteeing the existence of a (unique) positive solution of (0.4) even in the case when the function q may change its sign.

Section 17 is rather technical, however Theorem 17.1 stated therein together with the results of Section 11 generalize, resp. make more complete, previously known results on the solvability of the problem (0.1) with f satisfying (0.2) (see Remark 17.5).

The rest of the article is devoted to the solvability of the problem (0.1) in the case when (0.3) is fulfilled. Clearly, the assumption (0.3) include the case when the function f is defined only for positive values of the second variable and may have “singularity” for $u = 0$. In this case the problem (0.1) is referred as “phase singular”. A typical example of phase singular problem is

$$u'' = p(t)u + \frac{h(t)}{u^\lambda} + q(t); \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega) \tag{0.6}$$

with $\lambda > 0$. If $\lambda < 0$ in (0.6) then the “phase singularity” disappears however, as it was mentioned in very beginning of the introduction now we will be interested in the existence of a **positive** solution of the given problem.

In Sections 19 and 22, general theorems on the solvability of (0.1) and their consequences for the problem

$$u'' = p(t)u + h(t,u); \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega) \tag{0.7}$$

are established. Applications of these general results for the problem (0.6) can be found in Sections 20 and 23.

Roughly speaking, results of Sections 19 and 20 concern the problems (0.7) and (0.6) with $p \in \mathcal{V}^-(\omega)$ while the case when $p \in \mathcal{V}^+(\omega)$ is considered in Sections 22 and 23.

The problem (0.7) with $p \in \mathcal{V}_0(\omega)$ we refer as “resonant like case” and it is considered in Sections 21 and 24. Interest on the resonant like problems became from historical development of the theory of singular periodic problems. Although phase singular periodic problems have been studied even in earlier 60’th of 20’th century, actually its systematic treatment began from 1987 after the paper [16] by Lazer and Solimini.

In [16], the authors considered the problems

$$u'' = g(u) + q(t); \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega) \tag{0.8}$$

and

$$u'' = -g(u) + q(t); \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega), \tag{0.9}$$

where $g\colon\ ]0,+\infty[\ \to\ ]0,+\infty[$ is a continuous function and

$$\lim_{x\to 0+} g(x) = +\infty, \qquad \lim_{x\to +\infty} g(x) = 0.$$

In our terminology problems (0.8) and (0.9) belongs to the resonant like case. Results of Section 21 and 24 generalize and make more complete results of [16].

An important step in development of the theory of phase singular periodic problems was the paper [5]. One of the main results of this paper is a Fredholm alternative like result for the problem

$$u'' = -cu + \frac{1}{u^\lambda} + q(t); \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega), \tag{0.10}$$

where $c > 0$, $\lambda \geq 1$, and $q \in C([0,\omega];\mathbb{R})$. Theorem 1.1 of [5] states that the problem (0.10) is solvable provided $c \neq \frac{\pi^2 n^2}{\omega^2}$ for $n \in \mathbb{N}$. Theorem 23.5 below partially make more complete mentioned results of [5] (see Corollaries 23.8 and 23.9).

Notations and Main Definitions

The following notations are used below:

- $\mathbb{N}$ is a set of natural numbers.
- $\mathbb{R} =]-\infty, +\infty[\,$, $\mathbb{R}_+ = [0, +\infty[\,$.
- If $x \in \mathbb{R}$ then $[x]_+ = \frac{1}{2}\big(|x| + x\big)$ and $[x]_- = \frac{1}{2}\big(|x| - x\big)$.
- $C(A; B)$, where $A, B \subseteq \mathbb{R}$, is the set of continuous functions $u\colon A \to B$.
- For $u \in C([a, b]; D)$ we put $\|u\|_C = \max\big\{|u(t)| : t \in [a, b]\big\}$.
- C_ω is a Banach space of all continuous and ω-periodic real functions $u\colon \mathbb{R} \to \mathbb{R}$ equipped with the norm $\|u\|_C = \max\{|u(t)| :\ t \in [0, \omega]\}$.
- $AC(I)$, where $I \subseteq \mathbb{R}$, is a set of absolutely continuous functions $u\colon I \to \mathbb{R}$.
- $AC'(I)$, where $I \subseteq \mathbb{R}$, is a set of functions $u\colon I \to \mathbb{R}$ which are absolutely continuous on I together with their first derivatives.
- $\widetilde{AC}'(I)$, where $I \subseteq \mathbb{R}$, is a set of functions $\gamma \in AC(I)$ such that the function γ' admits the representation $\gamma'(t) = \alpha_0(t) + \alpha_1(t)$ for $t \in I$, where $\alpha_0 \in AC(I)$ and α_1 is a nondecreasing function whose derivative is equal to zero almost everywhere on I.
- $L([a, b])$ is the Banach space of Lebesgue integrable functions $p\colon [a, b] \to \mathbb{R}$ endowed with the norm $\|p\|_L = \int\limits_a^b |p(s)|\,\mathrm{d}s$.
- $L^\nu([a, b])$ is the set of functions $p\colon [a, b] \to \mathbb{R}$ such that $|p|^\nu \in L([a, b])$.
- For $p \in L^\nu([a, b])$ we put $\|p\|_{L^\nu} = (\int\limits_a^b |p(s)|^\nu\,\mathrm{d}s)^{1/\nu}$.
- L_ω is a Banach space of all ω-periodic real functions, which are Lebesgue integrable on $[0, \omega]$, equipped with the norm $\|p\|_L = \int\limits_0^\omega |p(s)|\,\mathrm{d}s$.
- For $A \subseteq L_\omega$ the symbols $\overline{A}$, $\operatorname{Int} A$, and ∂A denote closure, interior part, and boundary of the set A.
- For any $\delta > 0$, we denote $B(p, \delta) \overset{\text{def}}{=} \big\{g \in L_\omega :\ \|p - g\|_L < \delta\big\}$.
- If $p \in L_\omega$ then we put
$$\overline{p} \overset{\text{def}}{=} \frac{1}{\omega}\int\limits_0^\omega p(s)\,\mathrm{d}s. \tag{0.11}$$
and
$$\rho(p) \overset{\text{def}}{=} \exp\left(\frac{\omega}{4}\int\limits_0^\omega [p(s)]_+\,\mathrm{d}s\right). \tag{0.12}$$
- If $q \in L_\omega$ then we put
$$Q_+ \overset{\text{def}}{=} \int\limits_0^\omega [q(s)]_+\,\mathrm{d}s, \quad Q_- \overset{\text{def}}{=} \int\limits_0^\omega [q(s)]_-\,\mathrm{d}s. \tag{0.13}$$

- $\ell\colon L_\omega \to C_\omega$ is an operator defined by
$$\ell(p)(t) \overset{\text{def}}{=} -\frac{1}{\omega}\int\limits_t^{t+\omega}\int\limits_t^s \big(p(\xi)-\overline{p}\big)\,\mathrm{d}\xi\,\mathrm{d}s \quad \text{for } t\in\mathbb{R} \tag{0.14}$$
and
$$\ell \overset{\text{def}}{=} \max\big\{|\ell(p)(t)| :\ t\in[0,\omega]\big\}. \tag{0.15}$$
- If $p\in L_\omega$ then
$$p^* \overset{\text{def}}{=} \begin{cases} \dfrac{1}{M} & \text{if } 0<M<+\infty,\\ 0 & \text{if } M=0 \text{ or } M=+\infty, \end{cases} \tag{0.16}$$
where $M \overset{\text{def}}{=} \operatorname{ess\,sup}\big\{[p(t)]_- :\ t\in[0,\omega]\big\}$.
- $K([a,b]\times A;B)$, where $A,B\subseteq\mathbb{R}$, is the set Carathéodory functions, i.e., the set functions $f\colon [a,b]\times A\to B$ such that:
  (1) for any $x\in A$, the function $f(\,\cdot\,,x)\colon [a,b]\to\mathbb{R}$ is a measurable;
  (2) for almost all $t\in[a,b]$, the function $f(t,\,\cdot\,)\colon A\to B$ is continuous;
  (3) for any $r>0$, there exists $q_r\in L([a,b])$ such that
$$|f(t,x)|\le q_r(t) \quad \text{for } t\in[a,b],\ \ x\in A\cap[-r,r].$$
- $K_{loc}([0,\omega]\times\,]0,+\infty[\,;B)$, where $B\subseteq\mathbb{R}$, is the set of function $f\colon [0,\omega]\times\,]0,+\infty[\,\to B$ such that $f\in K([0,\omega]\times[\varepsilon,+\infty[\,;B)$ for any $\varepsilon>0$.
- $K_{sl}([0,\omega]\times\mathbb{R};\mathbb{R})$ is the set of sublinear functions, i.e., the set of functions $q\in K([0,\omega]\times\mathbb{R};\mathbb{R})$ satisfying
$$\lim_{r\to+\infty}\frac{1}{r}\int\limits_0^\omega |q(s,r)|\,\mathrm{d}s = 0.$$
- Having a function $h\in K_{loc}([0,\omega]\times\,]0,+\infty[\,;\mathbb{R})$, we put
$$H(x) \overset{\text{def}}{=} \int\limits_0^\omega h(s,x)\,\mathrm{d}s \quad \text{for } x>0. \tag{0.17}$$
- Under a solution of the equation
$$u'' = p(t)u + q(t),$$
where $p,q\in L([a,b])$, we understand a function $u\in AC'([a,b])$ satisfying given equation almost everywhere on $[a,b]$.
- Let $u\in AC'([a,b])$ be such that $u(0)=u(\omega)$ and $u'(0)=u'(\omega)$. If it will be needed we will extend the function u periodically and denote it by the same letter.

Definition 0.1. We say that the function $p\in L_\omega$ belongs to the set $\mathcal{V}^-(\omega)$ (resp., $\mathcal{V}^+(\omega)$) if, for any function $u\in AC'([0,\omega])$ satisfying
$$u''(t)\ge p(t)u(t) \quad \text{for } t\in[0,\omega]; \quad u(0)=u(\omega),\ \ u'(0)=u'(\omega),$$
the inequality
$$u(t)\le 0 \quad \text{for } t\in[0,\omega] \quad \big(\text{resp., } u(t)\ge 0 \quad \text{for } t\in[0,\omega]\big)$$
is fulfilled.

Definition 0.2. We say that the function $p\in L_\omega$ belongs to the set $\mathcal{V}_0(\omega)$ if the problem
$$u'' = p(t)u; \quad u(0)=u(\omega),\ \ u'(0)=u'(\omega) \tag{0.18}$$
has a (nontrivial) sign-constant solution.

Definition 0.3. We say that the function $p \in L_\omega$ belongs to the set $\mathcal{D}$ if any nontrivial solution of the equation $u'' = p(t)u$ has at most one zero in $\mathbb{R}$.

Definition 0.4. We say that the function $p \in L_\omega$ belongs to the set $\mathcal{D}(\omega)$ if the problem

$$u'' = p(t)u; \quad u(\alpha) = 0, \ \ u(\beta) = 0$$

has no nontrivial solution for any $\alpha < \beta$ satisfying $\beta - \alpha < \omega$.

Remark 0.5. It is clear that if $p \in \mathcal{V}^-(\omega)$ (resp., $p \in \mathcal{V}^+(\omega)$) then the problem (0.18) has no nontrivial solution. Therefore, by virtue of Fredholm's alternative, for any $q \in L_\omega$, the problem

$$u'' = p(t)u + q(t); \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega)$$

has a unique solution u. Moreover, if $q(t) \geq 0$ (resp., $q(t) \leq 0$) for $t \in \mathbb{R}$ then $u(t) \leq 0$ (resp., $u(t) \geq 0$) for $t \in \mathbb{R}$.

Remark 0.6. One can easily verify that if $p \in \mathcal{V}^-(\omega)$ then a certain assertion on differential inequalities holds. More precisely, let $p \in \mathcal{V}^-(\omega)$, $q \in L_\omega$, and the functions $u, v \in AC'([0,\omega])$ satisfy differential inequalities

$$u''(t) \leq p(t)u(t) + q(t), \quad v''(t) \geq p(t)v(t) + q(t) \quad \text{for } \ t \in [0,\omega]$$

and boundary conditions

$$u(0) = u(\omega), \quad v(0) = v(\omega), \quad u'(0) = u'(\omega), \quad v'(0) = v'(\omega).$$

Then the inequality

$$u(t) \geq v(t) \quad \text{for } \ t \in [0,\omega]$$

is fulfilled. Analogously, if $p \in \mathcal{V}^+(\omega)$, $q \in L_\omega$, and the functions u and v are the same as above then the inequality

$$u(t) \leq v(t) \quad \text{for } \ t \in [0,\omega]$$

holds.

Remark 0.7. The inclusion $p \in \mathcal{V}_0(\omega)$ holds if and only if the function p admits the representation

$$p(t) = g(t) + \big(\ell(g)(t)\big)^2 \quad \text{for } \ t \in \mathbb{R}, \tag{0.19}$$

where $g \in L_\omega$ and $\overline{g} = 0$. Indeed, if p admits the representation (0.19) then one can easily verify that $p \in L_\omega$ and the function

$$u(t) \overset{\text{def}}{=} \exp\left[\int_0^t \ell(g)(s)\,\mathrm{d}s\right] \quad \text{for } \ t \in \mathbb{R}$$

is a solution of the problem (0.18). Let now $p \in \mathcal{V}_0(\omega)$ and u be a sign-constant solution of the problem (0.18). Put

$$\varrho(t) \overset{\text{def}}{=} \frac{u'(t)}{u(t)}, \quad g(t) \overset{\text{def}}{=} \varrho'(t) \quad \text{for } \ t \in \mathbb{R}.$$

It is clear that $\overline{g} = 0$ and $p(t) = \varrho'(t) + \varrho^2(t)$ for $t \in \mathbb{R}$. On the other hand, one can easily verify that $\varrho(t) = \ell(\varrho')(t)$ for $t \in \mathbb{R}$, and thus (0.19) is fulfilled.

From Definitions 0.2–0.4 and Sturm's separation theorem if follows

Proposition 0.8. *$\mathcal{V}_0(\omega) \subseteq \mathcal{D}$ and $\mathcal{D} \subseteq \mathcal{D}(\omega)$.*

Chapter 1

Auxiliary Propositions

1. On the Set $\mathcal{D}$

The next proposition follows from Definition 0.3.

Proposition 1.1. *Let $p \in \mathcal{D}$. Then the problem*

$$u'' = p(t)u; \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega) \tag{1.1}$$

has no more then one, up to a constant multiple, nontrivial solution.

Lemma 1.2. *Let $p \in L_\omega$ and there exist a function $\beta \in \widetilde{AC}'(\mathbb{R}_+)$ such that*

$$\beta''(t) \leq p(t)\beta(t) \quad \textit{for} \;\; t \geq 0,$$
$$\beta(t) > 0 \quad \textit{for} \;\; t \geq 0.$$

Then the equation $v'' = p(t)v$ possesses a solution v satisfying

$$0 < v(t) \leq \beta(t) \quad \textit{for} \;\; t \geq 0, \quad v(0) = \beta(0). \tag{1.2}$$

In particular, $p \in \mathcal{D}$.

Proof. For any $k \in \mathbb{N}$, consider on $[0, k]$ the Dirichlet problem

$$v'' = p(t)v; \quad v(0) = \beta(0), \;\; v(k) = 0. \tag{1.3$_k$}$$

Since the functions $\alpha_1 \equiv 0$ and β are, respectively, lower and upper functions of the problem (1.3_1), there is a solution $v_1 \in AC'([0, 1])$ of this problem satisfying

$$0 < v_1(t) \leq \beta(t) \quad \text{for} \;\; t \in [0, 1[$$

(see, e. g., [4] or [14, Lemma 3.7]). Moreover,

$$v_1'(0) \leq \beta'(0), \quad v_1'(1) < 0.$$

Let now v_k be a solution of the problem (1.3_k) and

$$0 < v_k(t) \leq \beta(t) \quad \text{for} \;\; t \in [0, k[\,.$$

Clearly, $v_k'(k) < 0$ and the function

$$\alpha_{k+1}(t) \overset{\text{def}}{=} \begin{cases} v_k(t) & \text{for} \;\; t \in [0, k], \\ 0 & \text{for} \;\; t \in]k, k+1] \end{cases}$$

is a lower function of the problem (1.3_{k+1}), while the function β is its upper function. Then there is a solution $v_{k+1} \in AC'([0, k+1])$ of the problem (1.3_{k+1}) such that

$$\alpha_{k+1}(t) \leq v_{k+1}(t) \leq \beta(t) \quad \text{for} \;\; t \in [0, k+1].$$

Clearly, $v_k'(0) \leq v_{k+1}'(0) \leq \beta'(0)$. Therefore, for any $k \in \mathbb{N}$, we have

$$0 < v_k(t) \leq v_{k+1}(t) \leq \beta(t) \quad \text{for} \;\; t \in [0, k[$$

and

$$v_k(0) = \beta(0), \quad v_1'(0) \leq v_k'(0) \leq v_{k+1}'(0) \leq \beta'(0).$$

Hence, we easily get that the sequences $\{v_k^{(i)}\}_{k=1}^{+\infty}$, $i=0,1$, are uniformly bounded and equicontinuous in $[0,+\infty[$ (i.e., on every closed subinterval of $[0,+\infty[$). Then, by virtue of the Arzelá–Ascolli lemma, there is a $v \in AC'(\mathbb{R}_+)$ and a subsequence $\{v_{k_n}\}_{n=1}^{+\infty}$ such that

$$\lim_{n\to+\infty} v_{k_n}^{(i)}(t) = v^{(i)}(t) \quad \text{uniformly in } [0,+\infty[\,, \;\; i=0,1.$$

It is clear that the function v is a solution of the equation $v'' = p(t)v$ and satisfies (1.2). As for the inclusion $p \in \mathcal{D}$, it follows immediately from Sturm's (separation) theorem. □

Lemma 1.3. *Let $p \in \mathcal{D}$, $q \in L_\omega$, $q(t) \geq 0$ for $t \in \mathbb{R}$, and u be a solution of the problem*

$$u'' = p(t)u + q(t); \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega)$$

satisfying $u(t) > 0$ for $t \in \mathbb{R}$. Let, moreover, $a \geq 0$, v be a solution of the problem

$$v'' = p(t)v; \quad v(a) = u(a), \;\; v'(a) \leq u'(a),$$

and

$$u'(a) - v'(a) + \operatorname{mes}\big\{t \in [0,\omega] : \; q(t) > 0\big\} > 0. \tag{1.4}$$

Then the function v does not preserve its sign in $[a,+\infty[$.

Proof. Suppose the contrary that

$$v(t) > 0 \quad \text{for } t \geq a. \tag{1.5}$$

Evidently,

$$u'(t)v(t) - u(t)v'(t) = u(a)\big(u'(a) - v'(a)\big) + \int_a^t q(s)v(s)\,\mathrm{d}s \quad \text{for } t \geq a. \tag{1.6}$$

Hence, in view of (1.4) and (1.5), we get

$$u'(t)v(t) - u(t)v'(t) \geq \delta > 0 \quad \text{for } t \geq a+\omega, \tag{1.7}$$

where $\delta = u(a)(u'(a) - v'(a)) + \int_a^{a+\omega} q(s)v(s)\,\mathrm{d}s$. Therefore,

$$\left(\frac{u(t)}{v(t)}\right)' > 0 \quad \text{for } t \geq a+\omega \tag{1.8}$$

and, consequently,

$$0 < v(t) < c_0 u(t) \quad \text{for } t > a+\omega, \tag{1.9}$$

where $c_0 = \frac{v(a+\omega)}{u(a+\omega)}$.

Introduce the notation

$$v_k(t) \overset{\text{def}}{=} v(t+k\omega) \quad \text{for } t \in [a, a+\omega], \;\; k \in \mathbb{N}. \tag{1.10}$$

It follows from (1.8) (since $u(t+k\omega) = u(t)$ for $t \in \mathbb{R}$) that

$$\frac{u(t)}{v_k(t)} = \frac{u(t+k\omega)}{v(t+k\omega)} < \frac{u(t+(k+1)\omega)}{v(t+(k+1)\omega)} = \frac{u(t)}{v_{k+1}(t)} \quad \text{for } t \in [a, a+\omega], \;\; k \in \mathbb{N}.$$

Thus

$$0 < v_{k+1}(t) < v_k(t) \quad \text{for } t \in [a, a+\omega], \;\; k \in \mathbb{N}. \tag{1.11}$$

On the other hand, (1.9) and (1.10) yield

$$0 < v_k(t) < c_0 u(t) \quad \text{for } t \in [a, a+\omega], \;\; k \in \mathbb{N}. \tag{1.12}$$

It is clear that, for any $k \in \mathbb{N}$, there is a $\xi_k \in [a, a+\omega]$ such that

$$v_k(a+\omega) - v_k(a) = v_k'(\xi_k)\omega.$$

Hence, in view of (1.12),

$$|v_k'(\xi_k)| \leq \frac{2c_0}{\omega}\, u(a) \quad \text{for } k \in \mathbb{N}. \tag{1.13}$$

By virtue of (1.12) and (1.13), we get

$$
\begin{aligned}
|v_k'(t)| &\leq |v_k'(\xi_k)| + \int\limits_a^{a+\omega} |p(t)| v_k(t)\,\mathrm{d}t \\
&\leq \frac{2c_0}{\omega}\, u(a) + c_0\|u\|_C\|p\|_L \quad \text{for } t \in [a, a+\omega], \quad k \in \mathbb{N}
\end{aligned} \tag{1.14}
$$

and

$$
|v_k''(t)| \leq c_0|p(t)|\|u\|_C \quad \text{for } t \in [a, a+\omega], \quad k \in \mathbb{N}. \tag{1.15}
$$

Now it follows from (1.12), (1.14), and (1.15) that the sequences $\{v_k^{(i)}\}_{k=1}^{+\infty}$, $i = 0, 1$, are uniformly bounded and equicontinuous on $[a, a+\omega]$. Therefore, by virtue of the Arzelá–Ascolli lemma, there is a function $v_0 \in AC'([a, a+\omega])$ and a subsequence $\{v_{k_n}\}_{n=1}^{+\infty}$ such that

$$
\lim_{n\to+\infty} v_{k_n}^{(i)}(t) = v_0^{(i)}(t) \quad \text{uniformly on } [a, a+\omega], \quad i = 0, 1. \tag{1.16}
$$

Taking, moreover, into account (1.11), we get

$$
\lim_{k\to+\infty} v_k(t) = v_0(t) \quad \text{uniformly on } [a, a+\omega], \tag{1.17}
$$

as well. Moreover, (1.11) and (1.17) imply

$$
v_k(t) \geq v_0(t) \quad \text{for } t \in [a, a+\omega], \quad k \in \mathbb{N}, \tag{1.18}
$$

$$
v_0(t) \geq 0 \quad \text{for } t \in [a, a+\omega]. \tag{1.19}
$$

On the other hand, in view of (1.7) and (1.10), we have

$$
u'(t)v_{k_n}(t) - v_{k_n}'(t)u(t) \geq \delta > 0 \quad \text{for } t \in [a, a+\omega],
$$

which, together with (1.16), results in

$$
u'(t)v_0(t) - v_0'(t)u(t) \geq \delta > 0 \quad \text{for } t \in [a, a+\omega], \tag{1.20}
$$

Taking, moreover, into account (1.19) we get

$$
v_0(t) > 0 \quad \text{for } t \in [a, a+\omega]. \tag{1.21}
$$

On account of (1.10) and (1.18), it is clear that

$$
\begin{aligned}
\int\limits_{a+\omega}^{a+(k+1)\omega} q(s)v(s)\,\mathrm{d}s &= \sum_{i=1}^{k} \int\limits_{a+i\omega}^{a+(i+1)\omega} q(s)v(s)\,\mathrm{d}s \\
&= \sum_{i=1}^{k} \int\limits_a^{a+\omega} q(s)v_i(s)\,\mathrm{d}s \geq k \int\limits_a^{a+\omega} q(s)v_0(s)\,\mathrm{d}s \quad \text{for } k \in \mathbb{N}.
\end{aligned}
$$

Consequently, (1.6) implies that

$$
u'(a)v_{k+1}(a) - v_{k+1}'(a)u(a) \geq k \int\limits_a^{a+\omega} q(s)v_0(s)\,\mathrm{d}s \quad \text{for } k \in \mathbb{N}.
$$

The latter inequality (with $k = k_n - 1$), together with (1.16) and (1.21), yields that

$$
q \equiv 0. \tag{1.22}
$$

Hence, in view of (1.4), (1.6), and (1.10) we get

$$
u'(t)v_k(t) - v_k'(t)u(t) = c \quad \text{for } t \in [a, a+\omega], \quad k \in \mathbb{N}, \tag{1.23}
$$

where $c = u(a)(u'(a) - v'(a)) > 0$. Since $u(t) > 0$ for $t \in \mathbb{R}$, it easily follows from (1.23), on account of (1.17), that the sequence $\{v_k'\}_{k=1}^{+\infty}$ is uniformly convergent. However, its subsequence $\{v_{k_n}'\}_{n=1}^{+\infty}$ uniformly converges to v_0' and thus

$$
\lim_{k\to+\infty} v_k'(t) = v_0'(t) \quad \text{uniformly on } [a, a+\omega]. \tag{1.24}
$$

By virtue of (1.10), (1.17) and (1.24), it is clear that the function v_0 is a solution of the equation

$$v'' = p(t)v.$$

On the other hand, in view of (1.10),

$$v_{k+1}^{(i)}(a) = v_k^{(i)}(a+\omega) \quad \text{for } k \in \mathbb{N}, \ \ i = 0, 1,$$

which, together with (1.17) and (1.24), results in

$$v_0(a) = v_0(a+\omega), \quad v_0'(a) = v_0'(a+\omega).$$

Moreover, in view of (1.17) and (1.24), we get from (1.23) that

$$u'(t)v_0(t) - v_0'(t)u(t) = c > 0 \quad \text{for } t \in [a, a+\omega]. \tag{1.25}$$

Therefore, we have proved that the equation $v'' = p(t)v$ has a (nontrivial) periodic solution v_0 satisfying (1.25). On the other hand, in view of (1.22), the function u is a nontrivial periodic solution of the same equation. Hence, by virtue of Proposition 1.1, there is a $\lambda > 0$ such that $u = \lambda v_0$. However, this contradicts (1.25). □

Lemma 1.4. *Let $p \in L_\omega$, v be a solution of the equation*

$$u'' = p(t)u \tag{1.26}$$

and

$$0 < v(t) \le M \quad \textit{for } t \ge 0, \tag{1.27}$$

where $M > 0$. Then

$$v(t+\omega) = \frac{v(\omega)}{v(0)}\, v(t) \quad \textit{for } t \ge 0. \tag{1.28}$$

Proof. It is clear that for any $k \in \mathbb{N}$ there is a $\xi_k \in [(k-1)\omega, k\omega]$ such that

$$v(k\omega) - v((k-1)\omega) = v'(\xi_k)\omega.$$

Hence, in view of (1.27), we get

$$|v'(\xi_k)| \le \frac{2M}{\omega} \quad \text{for } k \in \mathbb{N}. \tag{1.29}$$

On the other hand,

$$|v'(t)| \le |v'(\xi_k)| + \int\limits_{(k-1)\omega}^{k\omega} |p(s)| v(s)\, \mathrm{d}s \quad \text{for } t \in [(k-1)\omega, k\omega], \ \ k \in \mathbb{N}.$$

This inequality together with (1.27) and (1.29) implies

$$|v'(t)| \le M_1 \quad \text{for } t \ge 0, \tag{1.30}$$

where $M_1 = \frac{2M}{\omega} + M\|p\|_L$.

Introduce the notation

$$v_1(t) \overset{\text{def}}{=} v(t+\omega) \quad \text{for } t \in \mathbb{R}.$$

Since the function v_1 is a solution of the equation (1.26), there is a $c \in \mathbb{R}$ such that

$$v_1'(t)v(t) - v_1(t)v'(t) = c \quad \text{for } t \in \mathbb{R}. \tag{1.31}$$

Hence, $\left(\frac{v_1(t)}{v(t)}\right)' = \frac{c}{v^2(t)}$ for $t \ge 0$ and thus

$$\frac{v_1(t)}{v(t)} = \frac{v_1(0)}{v(0)} + c\int\limits_0^t \frac{1}{v^2(s)}\, \mathrm{d}s \quad \text{for } t \ge 0. \tag{1.32}$$

Suppose that $c < 0$. Then it follows from (1.32), in view of (1.27), that

$$\frac{v_1(t)}{v(t)} \le \frac{v_1(0)}{v(0)} - \frac{|c|}{M^2}\, t \quad \text{for } t \ge 0,$$

which contradicts first inequality in (1.27).

Let now $c > 0$. Then it follows from (1.32), in view of (1.27), that

$$M \geq v_1(t) \geq \Big(\frac{v_1(0)}{v(0)} + \frac{c\,t}{M^2}\Big)v(t) \quad \text{for } \ t \geq 0.$$

Hence

$$\lim_{t\to+\infty} v(t) = 0. \tag{1.33}$$

Now we get from (1.31), in view of (1.30) and (1.33) that $c = 0$, which contradicts our assumption.

Thus we have proved that $c = 0$, which together with (1.32) implies (1.28). □

2. On the Set $\mathcal{D}(\omega)$

Proposition 2.1. $\overline{\mathcal{D}(\omega)} = \mathcal{D}(\omega)$.

Proof. Suppose the contrary, let $\overline{\mathcal{D}(\omega)} \neq \mathcal{D}(\omega)$. Then there exist $p \in \overline{\mathcal{D}(\omega)}$, $\alpha \in [0,\omega[$, and $\beta \in\,]\alpha, \alpha+\omega[$ such that the problem

$$u'' = p(t)u; \quad u(\alpha) = 0, \ \ u(\beta) = 0$$

possesses a nontrivial solution u such that

$$u(t) > 0 \quad \text{for } \ t \in\,]\alpha,\beta[\,.$$

Clearly,

$$u'(\alpha) > 0 \tag{2.1}$$

and there exists $\beta_0 \in]\beta, \alpha+\omega[$ such that $u(\beta_0) < 0$.

On the other hand, there is a sequence $\{p_n\}_{n=1}^{+\infty} \subset \mathcal{D}(\omega)$ such that

$$\lim_{n\to+\infty} \|p_n - p\|_L = 0. \tag{2.2}$$

For any $n \in \mathbb{N}$ consider on $[\alpha, \alpha+\omega]$ the Cauchy problem

$$v'' = p_n(t)v; \quad v(\alpha) = 0, \ \ v'(\alpha) = u'(\alpha)$$

and denote its solution by v_n. In view of (2.1) it is clear that

$$v_n(t) > 0 \quad \text{for } \ t \in\,]\alpha, \alpha+\omega[\ \ n \in \mathbb{N}. \tag{2.3}$$

Let now $\varepsilon \in]0, -u(\beta_0)[\,$. Then, by virtue of (2.2) and well-posedness of the Cauchy problem there exists $n_0 \in \mathbb{N}$ such that

$$|v_n(t) - u(t)| < \varepsilon \quad \text{for } \ t \in [\alpha, \alpha+\omega], \ \ n > n_0.$$

Consequently, $v_n(\beta_0) < u(\beta_0) + \varepsilon < 0$ for $n > n_0$, which contradicts (2.3). □

Proposition 2.2. *Let $p \in L_\omega$. Then the inclusion $p \in \operatorname{Int}\mathcal{D}(\omega)$ holds if and only if the problem*

$$u'' = p(t)u; \quad u(\alpha) = 0, \ \ u(\beta) = 0 \tag{2.4}$$

has no nontrivial solution for any $\alpha < \beta$ satisfying $\beta - \alpha \leq \omega$.

Proof. Denote by A the set of $p \in L_\omega$ such that the problem (2.4) has no nontrivial solution for any $\alpha < \beta$ satisfying $\beta - \alpha \leq \omega$.

Let $p \in \operatorname{Int}\mathcal{D}(\omega)$ and $p \notin A$. Then there is a $\alpha \in [0,\omega[$ such that the problem

$$u'' = p(t)u; \quad u(\alpha) = 0, \ \ u(\alpha+\omega) = 0$$

possesses a solution u and

$$u(t) > 0 \quad \text{for } \ t \in]\alpha, \alpha+\omega[\,.$$

Clearly,

$$u'(\alpha+\omega) < 0. \tag{2.5}$$

Since $p \in \operatorname{Int}\mathcal{D}(\omega)$, there is a $\delta > 0$ such that $p - \delta \in \mathcal{D}(\omega)$. Let v be a solution of the problem

$$v'' = \big(p(t) - \delta\big)v; \quad v(\alpha) = 0, \ \ v'(\alpha) = 1.$$

Clearly,

$$v(t) > 0 \quad \text{for } \ t \in\,]\alpha, \alpha+\omega[$$

and, consequently,

$$v(\alpha+\omega) \geq 0. \tag{2.6}$$

On the other hand,

$$\big(u'(t)v(t) - u(t)v'(t)\big)' = \delta u(t)v(t) > 0 \quad \text{for } \ t \in \,]\alpha, \alpha+\omega[\,.$$

Integrating the latter inequality on $[\alpha, \alpha+\omega]$ we get

$$u'(\alpha+\omega)v(\alpha+\omega) > 0,$$

which contradicts (2.5) and (2.6). Therefore, we have proved that $\operatorname{Int}\mathcal{D}(\omega) \subseteq A$.

Let now $p \in A$ and $p \notin \operatorname{Int}\mathcal{D}(\omega)$. Then there is a sequence $\{p_n\}_{n=1}^{+\infty} \subset L_\omega$ such that $p_n \notin \mathcal{D}(\omega)$ for $n \in \mathbb{N}$ and

$$\lim_{n\to+\infty} \|p_n - p\|_L = 0. \tag{2.7}$$

Hence, for some $\alpha_n \in [0, \omega[$ and $\beta_n \in \,]\alpha_n, \alpha_n+\omega[$ the problem

$$u'' = p_n(t)u; \quad u(\alpha_n) = 0, \ \ u(\beta_n) = 0$$

possesses a nontrivial solution u_n. Suppose without loss of generality that

$$\lim_{n\to+\infty} \alpha_n = \alpha, \qquad \lim_{n\to+\infty} \beta_n = \beta. \tag{2.8}$$

Clearly, $\alpha \in [0, \omega]$ and $\beta \in [\alpha, \omega]$.

Let v_n is a solution of the Cauchy problem

$$v'' = p_n(t)v; \quad v(\alpha_n) = 0, \ \ v'(\alpha_n) = 1.$$

Clearly, $v_n(t) = \frac{1}{u_n'(a)}\, u_n(t)$ and, consequently,

$$v_n(\beta_n) = 0 \quad \text{for } \ n \in \mathbb{N}. \tag{2.9}$$

In view of (2.7), (2.8) and well-posedness of the Cauchy problem we get

$$\lim_{n\to+\infty} v_n^{(i)}(t) = v^{(i)}(t) \quad \text{uniformly on } \ [0, 2\omega], \ \ i = 0, 1, \tag{2.10}$$

where v is a solution of the Cauchy problem

$$v'' = p(t)v; \quad v(\alpha) = 0, \ \ v'(\alpha) = 1.$$

Since $v_n(\alpha_n) = 0$ and $v_n(\beta_n) = 0$, there is a $\xi_n \in \,]\alpha_n, \beta_n[$ such that $v_n'(\xi_n) = 0$. Hence, $\beta > \alpha$, since otherwise, on account of (2.10), we get the contradiction $v'(\alpha) = 0$. On the other hand, (2.9) and (2.10) imply $v(\beta) = 0$. Therefore, v is a nontrivial solution of the equation $v'' = p(t)v$ satisfying $v(\alpha) = 0$ and $v(\beta) = 0$, where $\alpha < \beta$ and $\beta - \alpha \leq \omega$. However, this contradicts our assumption that $p \in A$. Therefore, we have proved that $A \subseteq \operatorname{Int}\mathcal{D}(\omega)$ as well. □

Proposition 2.3. *Let $p \in L_\omega$. Then the inclusion $p \in \partial\mathcal{D}(\omega)$ holds if and only if the problem* (2.4) *has no nontrivial solution for any $\alpha < \beta$ satisfying $\beta - \alpha < \omega$ and there is a $\alpha_0 \in [0, \omega[$ such that the problem*

$$u'' = p(t)u; \quad u(\alpha_0) = 0, \ \ u(\alpha_0+\omega) = 0$$

has a nontrivial solution.

Proof. The assertion follows immediately from Proposition 2.1 and 2.2, and the formula $\partial\mathcal{D}(\omega) = \overline{\mathcal{D}(\omega)} \setminus \operatorname{Int}\mathcal{D}(\omega)$. □

Proposition 2.4. *Let $p \in \partial\mathcal{D}(\omega)$. Then the problem* (1.1) *has no nontrivial solution.*

Proof. The assertion of the proposition follows immediately from Proposition 2.3 and Sturm's separation theorem. □

Proposition 2.5. *Let $p \in \mathcal{D}(\omega)$ $(p \in \operatorname{Int}\mathcal{D}(\omega))$. Then for any $\alpha < \beta$ and $u \in AC'([\alpha,\beta])$ satisfying $\beta - \alpha < \omega$ $(\beta - \alpha \leq \omega)$ and*

$$u''(t) \geq p(t)u(t) \quad \text{for } t \in [\alpha,\beta], \tag{2.11}$$

$$u(\alpha) \leq 0, \quad u(\beta) \leq 0, \tag{2.12}$$

the inequality

$$u(t) \leq 0 \quad \text{for } t \in [\alpha,\beta]$$

holds.

Proof. Suppose the contrary, let the assertion of the proposition is violated. Then there are $p \in \mathcal{D}(\omega)$ $(p \in \operatorname{Int}\mathcal{D}(\omega))$, $\alpha < \beta$, and $u \in AC'([\alpha,\beta])$ such that $\beta - \alpha < \omega$ $(\beta - \alpha \leq \omega)$, (2.11) is fulfilled and

$$u(t) > 0 \quad \text{for } t \in]\alpha,\beta[\,, \quad u(\alpha) = 0, \quad u(\beta) = 0.$$

It is clear that

$$u'(\beta) \leq 0. \tag{2.13}$$

Moreover, either $u'(\beta) < 0$ or

$$u'(\beta) = 0 \text{ and } \operatorname{mes}\big\{t \in [\alpha,\beta] :\ u''(t) > p(t)u(t)\big\} > 0. \tag{2.14}$$

Let v is a solution of the problem

$$v'' = p(t)v; \quad v(\alpha) = 0, \ \ v'(\alpha) = 1.$$

The inclusion $p \in \mathcal{D}(\omega)$ $(p \in \operatorname{Int}\mathcal{D}(\omega))$ implies that

$$v(t) > 0 \quad \text{for } t \in]\alpha,\alpha+\omega[\quad \big(v(t) > 0 \quad \text{for } t \in]\alpha,\alpha+\omega]\big). \tag{2.15}$$

On the other hand, it is clear that

$$\big(u'(t)v(t) - u(t)v'(t)\big)' = v(t)\big(u''(t) - p(t)u(t)\big) \quad \text{for } t \in [\alpha,\beta].$$

Integration of the latter equality on $[\alpha,\beta]$ results in

$$u'(\beta)v(\beta) = \int_\alpha^\beta v(s)\big(u''(s) - p(s)u(s)\big)\,\mathrm{d}s. \tag{2.16}$$

Hence, in view of (2.11) and (2.15) we get that $u'(\beta) \geq 0$ which, together with (2.13), implies that (2.14) is fulfilled. However, (2.14) and (2.15) contradicts (2.16). $\square$

Proposition 2.6. *Let $p \in L_\omega$. Then the inclusion $p \in \operatorname{Int}\mathcal{D}(\omega)$ holds if and only if for any $\alpha \in [0,\omega[$ there exists $\gamma_\alpha \in AC'([\alpha,\alpha+\omega])$ satisfying*

$$\gamma_\alpha''(t) \leq p(t)\gamma_\alpha(t) \quad \text{for } t \in [\alpha,\alpha+\omega], \tag{2.17}$$

$$\gamma_\alpha(t) > 0 \quad \text{for } t \in]\alpha,\alpha+\omega[\,, \tag{2.18}$$

and

$$\gamma_\alpha(\alpha) + \gamma_\alpha(\alpha+\omega) + \operatorname{mes}\big\{t \in [\alpha,\alpha+\omega] : \gamma_\alpha''(t) < p(t)\gamma_\alpha(t)\big\} > 0. \tag{2.19}$$

Proof. Let $p \in \operatorname{Int}\mathcal{D}(\omega)$ and $\alpha \in [0,\omega[$. By virtue of Proposition 2.2, the problem

$$u'' = p(t)u; \quad u(\alpha) = 0, \ \ u(\alpha+\omega) = 0$$

has no nontrivial solution. Therefore, by virtue of Fredholm's alternative, the problem

$$\gamma'' = p(t)\gamma; \quad \gamma(\alpha) = 1, \ \ \gamma(\alpha+\omega) = 1$$

possesses a (unique) solution γ_α. Clearly, (2.17) and (2.19) hold. It is also evident that $\min\{\gamma_\alpha(t) : t \in [\alpha,\alpha+\omega]\} \neq 0$ because otherwise there is a $t_0 \in]\alpha,\alpha+\omega[$ such that $\gamma_\alpha(t_0) = 0$, $\gamma_\alpha'(t_0) = 0$ and therefore $\gamma_\alpha \equiv 0$.

Suppose that $\min\{\gamma_\alpha(t) : t \in [\alpha,\alpha+\omega]\} < 0$. Then there are $\alpha_0 \in]\alpha,\alpha+\omega[$ and $\beta_0 \in]\alpha_0,\alpha_0+\omega[$ such that $\gamma_\alpha(\alpha_0) = 0$ and $\gamma_\alpha(\beta_0) = 0$, which contradicts the assumption $p \in \operatorname{Int}\mathcal{D}(\omega)$. Thus the function γ_α satisfies (2.18) as well.

Let now for any $\alpha \in [0,\omega[$ there is a $\gamma_\alpha \in AC'([\alpha,\alpha+\omega])$ satisfying (2.17)–(2.19) and $p \notin \operatorname{Int}\mathcal{D}(\omega)$. Then, by virtue of Proposition 2.2, there are $\alpha \in [0,\omega[$, $\beta \in]\alpha,\alpha+\omega]$ and $u \in AC'([\alpha,\beta])$ such that

$$u''(t) = p(t)u(t) \quad \text{for } t \in [\alpha,\beta], \quad u(\alpha) = 0, \ u(\beta) = 0,$$
$$u(t) > 0 \quad \text{for } t \in]\alpha,\beta[\,. \tag{2.20}$$

It is clear that,

$$u'(\alpha) > 0, \quad u'(\beta) < 0. \tag{2.21}$$

On the other hand,

$$\big(u'(t)\gamma_\alpha(t) - u(t)\gamma_\alpha'(t)\big)' = u(t)\big(p(t)\gamma_\alpha(t) - \gamma_\alpha''(t)\big) \quad \text{for } t \in [\alpha,\beta].$$

Integration of this inequality on $[\alpha,\beta]$ results in

$$u'(\beta)\gamma_\alpha(\beta) - u'(\alpha)\gamma_\alpha(\alpha) = \int_\alpha^\beta u(s)\big(p(s)\gamma_\alpha(s) - \gamma_\alpha''(s)\big)\,\mathrm{d}s. \tag{2.22}$$

Hence, in view of (2.17), (2.20), and (2.21), we get $\gamma_\alpha(\alpha) = 0$, $\gamma_\alpha(\beta) = 0$. Taking, moreover, into account (2.18) we get that $\beta = \alpha + \omega$. Thus

$$\gamma_\alpha(\alpha) = 0, \quad \gamma_\alpha(\alpha+\omega) = 0, \quad \int_\alpha^{\alpha+\omega} u(s)\big(p(s)\gamma_\alpha(s) - \gamma_\alpha''(s)\big)\,\mathrm{d}s = 0$$

which, together with (2.19) and (2.20), yields the contradiction $0 > 0$. □

Lemma 2.7. *Let $p \in \mathcal{D}(\omega)$ and u be a solution of the problem*

$$u'' = p(t)u + q(t); \quad u(0) = u(\omega), \ u'(0) = u'(\omega), \tag{2.23}$$

where $q \in L_\omega$ and

$$q(t) \geq 0 \quad \textit{for } t \in \mathbb{R}. \tag{2.24}$$

Then either $u(t) \geq 0$ for $t \in \mathbb{R}$ or $u(t) \leq 0$ for $t \in \mathbb{R}$. If, moreover, $q \not\equiv 0$ then either $u(t) > 0$ for $t \in \mathbb{R}$ or $u(t) < 0$ for $t \in \mathbb{R}$.

Proof. Let u be a solution of (2.23) and

$$\max\big\{u(t) : \ t \in [0,\omega]\big\} > 0, \quad \min\big\{u(t) : \ t \in [0,\omega]\big\} < 0.$$

Then there are $\alpha \in [0,\omega[$ and $\beta \in]\alpha,\alpha+\omega[$ such that

$$u(\alpha) = 0, \quad u(\beta) = 0, \tag{2.25}$$
$$u(t) > 0 \quad \text{for } t \in]\alpha,\beta[\,. \tag{2.26}$$

In view of (2.24) and (2.25), it follows from Proposition 2.5 that $u(t) \leq 0$ for $t \in [\alpha,\beta]$, which contradicts (2.26). Therefore, either $u(t) \geq 0$ for $t \in \mathbb{R}$ or $u(t) \leq 0$ for $t \in \mathbb{R}$.

Suppose, moreover, that $q \not\equiv 0$ and there is a $t_0 \in [0,\omega[$ such that $u(t_0) = 0$. Then, in view of the above-proved we get that $u'(t_0) = 0$ as well. However, u is an ω-periodic function and therefore

$$u(t_0) = 0 = u(t_0+\omega), \quad u'(t_0) = 0 = u'(t_0+\omega). \tag{2.27}$$

Let v be a solution of the problem

$$v'' = p(t)v; \quad v(t_0) = 0, \ v'(t_0) = 1.$$

Since $p \in \mathcal{D}(\omega)$, we get $v(t) > 0$ for $t \in]t_0,t_0+\omega[$. Consequently,

$$\int_{t_0}^{t_0+\omega} q(s)v(s)\,\mathrm{d}s > 0. \tag{2.28}$$

On the other hand, clearly

$$\big(u'(t)v(t) - u(t)v'(t)\big)' = q(t)v(t) \quad \text{for } t \in [t_0,t_0+\omega].$$

Integrating the latter equality on $[t_0, t_0+\omega]$ and taking (2.27) into account, we get $\int\limits_{t_0}^{t_0+\omega} q(s)v(s)\,\mathrm{d}s = 0$ which contradicts (2.28). Therefore, either $u(t) > 0$ for $t \in \mathbb{R}$ or $u(t) < 0$ for $t \in \mathbb{R}$. $\square$

Lemma 2.8. *Let $p \in \partial\mathcal{D}(\omega)$, $q \in L_\omega$ satisfy* (2.24)*, and $q \not\equiv 0$. Let, moreover, u be a solution of the problem* (2.23)*. Then $u(t) > 0$ for $t \in \mathbb{R}$.*

Proof. By virtue of Proposition 2.1 and Lemma 2.7 either $u(t) > 0$ for $t \in \mathbb{R}$ or

$$u(t) < 0 \quad \text{for} \ \ t \in \mathbb{R}. \tag{2.29}$$

Suppose that (2.29) is fulfilled. Then it is clear that the function $\gamma_\alpha(t) \overset{\text{def}}{=} -u(t)$ for $t \in [\alpha, \alpha+\omega]$ and $\alpha \in [0, \omega[$ satisfies (2.17)–(2.19). Therefore, by virtue of Proposition 2.6, we get $p \in \operatorname{Int}\mathcal{D}(\omega)$, which contradicts the assumption $p \in \partial\mathcal{D}(\omega)$. $\square$

3. On a Sequence of Periodic Problems

First of all we recall that a linear periodic problem is well-posed. More precisely, consider the problem

$$u'' = p(t)u + q(t); \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega), \tag{3.1}$$

and a sequence of the problems

$$u'' = p_n(t)u + q_n(t); \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega), \tag{3.1$_n$}$$

where $p, q \in L_\omega$ and $p_n, q_n \in L_\omega$ for $n \in \mathbb{N}$.

From the general theory of boundary value problems for the systems of linear equations it follows that (see [13, Theorem 1.2])

Lemma 3.1. *Let the problem* (3.1) *have a unique solution u. Let, moreover,*

$$\sup\big\{\|p_n\|_L : n \in \mathbb{N}\big\} < +\infty,$$

$$\lim_{n\to+\infty} \int\limits_0^t p_n(s)\,\mathrm{d}s = \int\limits_0^t p(s)\,\mathrm{d}s \quad \textit{uniformly on} \ \ [0,\omega],$$

$$\lim_{n\to+\infty} \int\limits_0^t q_n(s)\,\mathrm{d}s = \int\limits_0^t q(s)\,\mathrm{d}s \quad \textit{uniformly on} \ \ [0,\omega].$$

Then there is a $n_0 \in \mathbb{N}$ such that, for any $n > n_0$, the problem (3.1$_n$) *has a unique solution u_n and*

$$\lim_{n\to+\infty} \|u_n - u\|_C = 0.$$

Next proposition immediately follows from Lemma 3.1.

Proposition 3.2. *Let the problem* (3.1) *have a unique solution u. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that, for any $g, \widetilde{q} \in L_\omega$ satisfying*

$$\|g - p\|_L < \delta,$$

$$\bigg|\int\limits_0^t \big(g(s) - \widetilde{q}(s)\big)\,\mathrm{d}s\bigg| < \delta \quad \textit{for} \ \ t \in [0,\omega],$$

the problem

$$v'' = g(t)v + \widetilde{q}(t); \quad v(0) = v(\omega), \ \ v'(0) = v'(\omega)$$

has a unique solution v and

$$\|u - v\|_C < \varepsilon.$$

By the standard arguments using in the proof of well-posedness of a periodic boundary value problem one can show that

Proposition 3.3. *Let*

$$\lim_{n\to+\infty}\|p_n-p\|_L=0,\qquad \lim_{n\to+\infty}\|q_n-q\|_L=0.$$

Let, moreover, for any $n\in\mathbb{N}$, u_n be a solution of the problem (3.1_n) *and the sequence $\{\|u_n\|_C\}_{n=1}^{+\infty}$ is bounded. Then there is a subsequence $\{u_{n_k}\}_{k=1}^{+\infty}$ such that*

$$\lim_{k\to+\infty}u_{n_k}^{(i)}(t)=u^{(i)}(t)\quad \textit{uniformly on}\ \ [0,\omega],\ \ i=0,1,$$

where u is a solution of the problem (3.1).

4. Some Technical Estimates

Let $p\in L_\omega$, $a\in[0,\omega[$ and p^* be the number defined by (0.16). Let, moreover,

$$\begin{aligned} F_1(t)&\overset{\text{def}}{=}\int_a^t\int_a^s[p(\xi)]_-\,\mathrm{d}\xi\,\mathrm{d}s\quad \text{for }\ t\in[a,a+\omega],\\ F_2(t)&\overset{\text{def}}{=}\int_t^{a+\omega}\int_s^{a+\omega}[p(\xi)]_-\,\mathrm{d}\xi\,\mathrm{d}s\quad \text{for }\ t\in[a,a+\omega]. \end{aligned}\tag{4.1}$$

Proposition 4.1. *The inequality*

$$F_1(t)+F_2(t)\geq\frac{p^*}{4}\,\big\|[p]_-\big\|_L^2\quad \textit{for}\ \ t\in[a,a+\omega].\tag{4.2}$$

is satisfied.

Proof. Set

$$\begin{aligned} h_1(t)&\overset{\text{def}}{=}F_1(t)-\frac{p^*}{2}\bigg(\int_a^t[p(s)]_-\,\mathrm{d}s\bigg)^2\quad \text{for }\ t\in[a,a+\omega],\\ h_2(t)&\overset{\text{def}}{=}F_2(t)-\frac{p^*}{2}\bigg(\int_t^{a+\omega}[p(s)]_-\,\mathrm{d}s\bigg)^2\quad \text{for }\ t\in[a,a+\omega]. \end{aligned}$$

It is clear that, $h_1(a)=0$, $h_2(a+\omega)=0$ and

$$\begin{aligned} h_1'(t)&=\big(1-p^*[p(t)]_-\big)\int_a^t[p(s)]_-\,\mathrm{d}s\geq0\quad \text{for }\ t\in[a,a+\omega],\\ h_2'(t)&=\big(p^*[p(t)]_--1\big)\int_t^{a+\omega}[p(s)]_-\,\mathrm{d}s\leq0\quad \text{for }\ t\in[a,a+\omega]. \end{aligned}$$

Hence,

$$h_1(t)\geq0,\quad h_2(t)\geq0\quad \text{for }\ t\in[a,a+\omega].\tag{4.3}$$

On the other hand, in view of the inequality $x^2+(c-x)^2\geq\frac{c^2}{2}$ for $x\in[0,c]$, we get

$$\bigg(\int_a^t[p(s)]_-\,\mathrm{d}s\bigg)^2+\bigg(\int_t^{a+\omega}[p(s)]_-\,\mathrm{d}s\bigg)^2\geq\frac12\,\big\|[p]_-\big\|_L^2\quad \text{for }\ t\in[a,a+\omega].\tag{4.4}$$

Inequality (4.2) now follows from (4.3) and (4.4). □

Proposition 4.2. *Let $p \in L_\omega$ and*

$$I(t) \overset{\text{def}}{=} \int_0^\omega \exp\left(2\int_s^t \ell(p)(\xi)\,\mathrm{d}\xi\right)\mathrm{d}s \quad \textit{for } t \in \mathbb{R},$$

where $\ell(p)$ is defined by (0.14). *Then the estimate*

$$I(t) \le \frac{\mathrm{e}^{\omega\ell} - 1}{\ell} \quad \textit{for } t \in \mathbb{R} \tag{4.5}$$

holds.

Proof. Introduce the notation

$$h(t,s) = 2\int_s^t \ell(p)(\xi)\,\mathrm{d}\xi \quad \text{for } t,s \in \mathbb{R}.$$

Since

$$\int_0^\omega \ell(p)(\xi)\,\mathrm{d}\xi = 0 \tag{4.6}$$

it is sufficient to prove the validity of (4.5) only for $t \in [0,\omega]$.

Assume that $t \in [0,\omega/2]$. Then it is clear

$$\begin{aligned} &h(t,s) \le 2\ell(t-s) \quad \text{for } s \in [0,t],\\ &h(t,s) \le 2\ell(s-t) \quad \text{for } s \in [t,\omega/2+t]. \end{aligned}$$

Moreover, in view of (4.6), clearly

$$\begin{aligned} h(t,s) = 2\int_0^t \ell(p)(\xi)\,\mathrm{d}\xi - 2\int_0^s \ell(p)(\xi)\,\mathrm{d}\xi &= 2\int_0^t \ell(p)(\xi)\,\mathrm{d}\xi + 2\int_s^\omega \ell(p)(\xi)\,\mathrm{d}\xi\\ &\le 2\ell(t+\omega-s) \quad \text{for } s \in [t+\omega/2,\omega]. \end{aligned}$$

Hence

$$\int_0^t \mathrm{e}^{h(t,s)}\,\mathrm{d}s \le \frac{\mathrm{e}^{2\ell t} - 1}{2\ell}, \quad \int_t^{t+\frac{\omega}{2}} \mathrm{e}^{h(t,s)}\,\mathrm{d}s \le \frac{\mathrm{e}^{\omega\ell} - 1}{2\ell}, \quad \int_{t+\frac{\omega}{2}}^\omega \mathrm{e}^{h(t,s)}\,\mathrm{d}s \le \frac{\mathrm{e}^{\omega\ell} - \mathrm{e}^{2\ell t}}{2\ell}$$

and therefore (4.5) is fulfilled for $t \in [0,\omega/2]$.

Analogously, if $t \in [\omega/2,\omega]$ then

$$\begin{aligned} h(t,s) &= 2\int_0^t \ell(p)(\xi)\,\mathrm{d}\xi - 2\int_0^s \ell(p)(\xi)\,\mathrm{d}\xi\\ &= -2\int_t^\omega \ell(p)(\xi)\,\mathrm{d}\xi - 2\int_0^s \ell(p)(\xi)\,\mathrm{d}\xi \le 2\ell(\omega-t+s) \quad \text{for } s \in [0,t-\omega/2], \end{aligned}$$

$$h(t,s) \le 2\ell(t-s) \quad \text{for } s \in [t-\omega/2,t], \quad h(t,s) \le 2\ell(s-t) \quad \text{for } s \in [t,\omega]$$

and by direct calculation one can verify that (4.5) is fulfilled for $t \in [\omega/2,\omega]$. □

Proposition 4.3. *Let $p \in L_\omega$, $\nu \in \,]0,1/2[\,$, and*

$$\omega\big\|[p]_+\big\|_L < \frac{(1-\nu)(1-2\nu)}{\nu^2}\,.$$

Let, moreover, $a \in [0,\omega]$ and v be a solution of the equation

$$v'' = p(t)v \tag{4.7}$$

satisfying

$$v(t) > 0 \quad \textit{for } t \in \,]a,a+\omega[\,.$$

Then

$$\int\limits_a^{a+\omega} \frac{1}{[v(s)]^{\frac{\nu}{1-\nu}}}\,\mathrm{d}s \le \frac{\omega}{\|v\|_{C([a,a+\omega])}^{\frac{\nu}{1-\nu}}}\,r, \tag{4.8}$$

where

$$r \stackrel{\text{def}}{=} (1-\nu)\Big((1-\nu)^2 - \nu\sqrt{(1-\nu)^2 + (1-\nu)\omega\big\|[p]_+\big\|_L}\Big)^{-1}.$$

Proof. First of all mention that

$$\lim_{t\to a+} \frac{t-a}{[v(t)]^{\frac{\nu}{1-\nu}}} = 0, \qquad \lim_{t\to(a+\omega)-} \frac{a+\omega-t}{[v(t)]^{\frac{\nu}{1-\nu}}} = 0. \tag{4.9}$$

Choose $c \in [a, a+\omega]$ such that

$$v(c) = \|v\|_{C([a,a+\omega])}.$$

It is clear that, either

$$c \ne a \ \text{ and } \ v'(c) \ge 0, \tag{4.10}$$

or

$$c \ne a+\omega \ \text{ and } \ v'(c) \le 0.$$

Suppose that $c \ne a$. Introduce the notations

$$I \stackrel{\text{def}}{=} \int\limits_a^c \frac{1}{[v(s)]^{\frac{\nu}{1-\nu}}}\,\mathrm{d}s, \quad A \stackrel{\text{def}}{=} \int\limits_a^c \frac{(s-a)|v'(s)|}{[v(s)]^{\frac{1}{1-\nu}}}\,\mathrm{d}s,$$

$$B \stackrel{\text{def}}{=} \int\limits_a^c \frac{(s-a)^2[v'(s)]^2}{[v(s)]^{\frac{2-\nu}{1-\nu}}}\,\mathrm{d}s.$$

By virtue of Hölder's inequality, we have

$$A^2 \le IB. \tag{4.11}$$

In view of (4.9), clearly

$$I \le \frac{c-a}{\|v\|_{C([a,a+\omega])}^{\frac{\nu}{1-\nu}}} + \frac{\nu}{1-\nu}\,A. \tag{4.12}$$

Multiplying both sides of (4.7) by $\frac{(t-a)^2}{[v(t)]^{\frac{1}{1-\nu}}}$ and integrating it on $[a,c]$, we get

$$\frac{(c-a)^2 v'(c)}{\|v\|_{C([a,a+\omega])}^{\frac{1}{1-\nu}}} - \int\limits_a^c v'(s)\left(\frac{2(s-a)}{[v(s)]^{\frac{1}{1-\nu}}} - \frac{(s-a)^2 v'(s)}{(1-\nu)[v(s)]^{\frac{2-\nu}{1-\nu}}}\right)\mathrm{d}s = \int\limits_a^c \frac{(s-a)^2}{[v(s)]^{\frac{\nu}{1-\nu}}}\,p(s)\,\mathrm{d}s.$$

Hence,on account of (4.10), we have

$$\frac{1}{1-\nu}\,B \le 2A + \int\limits_a^c \frac{(s-a)^2}{[v(s)]^{\frac{\nu}{1-\nu}}}\,[p(s)]_+\,\mathrm{d}s. \tag{4.13}$$

It is clear that,

$$\begin{aligned} &\int\limits_a^c \frac{(s-a)^2}{[v(s)]^{\frac{\nu}{1-\nu}}}\,[p(s)]_+\,\mathrm{d}s \\ &= \frac{1}{\|v\|_{C([a,a+\omega])}^{\frac{\nu}{1-\nu}}}\int\limits_a^c (s-a)^2[p(s)]_+\,\mathrm{d}s + \frac{\nu}{1-\nu}\int\limits_a^c \frac{v'(s)}{[v(s)]^{\frac{1}{1-\nu}}}\Big(\int\limits_a^s (\xi-a)^2[p(\xi)]_+\,\mathrm{d}\xi\Big)\mathrm{d}s \\ &\le \Big(\frac{c-a}{\|v\|_{C([a,a+\omega])}^{\frac{\nu}{1-\nu}}} + \frac{\nu}{1-\nu}\,A\Big)\omega\big\|[p]_+\big\|_L. \end{aligned} \tag{4.14}$$

It follows from (4.11), by virtue of (4.12)–(4.14), that

$$A^2 \leq \left(\frac{\nu}{1-\nu}\,A + \frac{c-a}{\|v\|^{\frac{\nu}{1-\nu}}_{C([a,a+\omega])}}\right)\left[\left(2-2\nu+\nu\omega\big\|[p]_+\big\|_L\right)A + \frac{(1-\nu)(c-a)}{\|v\|^{\frac{\nu}{1-\nu}}_{C([a,a+\omega])}}\,\omega\big\|[p]_+\big\|_L\right]$$

and, consequently,

$$A \leq \frac{1-\nu}{(1-\nu)(1-2\nu)-\nu^2\omega\big\|[p]_+\big\|_L} \\ \times \left(1-\nu+\nu\omega\big\|[p]_+\big\|_L + \sqrt{(1-\nu)^2+(1-\nu)\omega\big\|[p]_+\big\|_L}\right)\frac{c-a}{\|v\|^{\frac{\nu}{1-\nu}}_{C([a,a+\omega])}}\,.$$

The latter inequality, together with (4.12), results in

$$I \leq \frac{c-a}{\|v\|^{\frac{\nu}{1-\nu}}_{C([a,a+\omega])}}\;\frac{1-\nu}{(1-\nu)^2-\nu\sqrt{(1-\nu)^2+(1-\nu)\omega\big\|[p]_+\big\|_L}}\,.$$

Thus we have proved that

$$\int_a^c \frac{1}{[v(s)]^{\frac{\nu}{1-\nu}}}\,\mathrm{d}s \leq \frac{c-a}{\|v\|^{\frac{\nu}{1-\nu}}_{C([a,a+\omega])}}\,r \quad \text{if } \ c \neq a.$$

Analogously, one can show that

$$\int_c^{a+\omega} \frac{1}{[v(s)]^{\frac{\nu}{1-\nu}}}\,\mathrm{d}s \leq \frac{a+\omega-c}{\|v\|^{\frac{\nu}{1-\nu}}_{C([a,a+\omega])}}\,r \quad \text{if } \ c \neq a+\omega.$$

Clearly, the latter two inequalities imply (4.8). □

The next proposition is an analog of the well-known Gronwall–Bellman lemma.

Proposition 4.4. *Let* $v \in C([0,\omega];\mathbb{R})$, $a \in [0,\omega]$, $\lambda \in]0,1[$, *and* $\mu > 0$ *be such that*

$$0 \leq v(t) \leq \mu\left|\int_a^t v^{\lambda}(s)\,\mathrm{d}s\right| \quad \textit{for } \ t \in [0,\omega]. \tag{4.15}$$

Then

$$v(t) \leq \big[(1-\lambda)\mu\big]^{\frac{1}{1-\lambda}}\,|t-a|^{\frac{1}{1-\lambda}} \quad \textit{for } \ t \in [0,\omega]. \tag{4.16}$$

Proof. Let $\varepsilon > 0$ is arbitrary and $w(t) \overset{\text{def}}{=} \varepsilon + \mu|\int_a^t v^{\lambda}(s)\,\mathrm{d}s|$ for $t \in [0,\omega]$. It is clear that, $w \in AC([0,\omega])$ and

$$w'(t)\,\mathrm{sgn}(t-a) = \mu v^{\lambda}(t) \quad \text{for } \ t \in [0,\omega].$$

Hence, in view of (4.15), we get that

$$w'(t)\,\mathrm{sgn}(t-a) \leq \mu w^{\lambda}(t) \quad \text{for } \ t \in [0,\omega].$$

Therefore,

$$\frac{1}{1-\lambda}\left(w^{1-\lambda}(t)-\varepsilon^{1-\lambda}\right) \leq \mu|t-a| \quad \text{for } \ t \in [0,\omega]$$

and, consequently,

$$v(t) \leq \big[(1-\lambda)\mu|t-a| + \varepsilon^{1-\lambda}\big]^{\frac{1}{1-\lambda}} \quad \text{for } \ t \in [0,\omega].$$

Since $\varepsilon > 0$ was arbitrary, the latter inequality implies (4.16). □

5. Efficient Conditions for Inclusion $p \in \operatorname{Int}\mathcal{D}(\omega)$

Remark 5.1. It is well known that, for any $\omega > 0$, there is a (best) constant $k^*(\omega)$ such that, for any $a \in \mathbb{R}$ and $v \in AC'([a, a+\omega])$ satisfying $v(a) = 0$ and $v(a+\omega) = 0$, the inequality

$$\int_a^{a+\omega} v^4(t)\,\mathrm{d}t \le k^*(\omega)\bigg(\int_a^{a+\omega} \big(v'(t)\big)^2\,\mathrm{d}t\bigg)^2 \tag{5.1}$$

holds (see, e. g., [2, 22]). It is also known that

$$\frac{1}{k^*(\omega)} = \frac{\pi^2}{12\omega^3}\left(\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}\right)^4 \quad \text{or alternatively} \quad \frac{1}{k^*(\omega)} = \frac{64}{3\omega^3}\,\chi^4,$$

where Γ is a Gamma function of Euler and $\chi \overset{\text{def}}{=} \int_0^1 \frac{\mathrm{d}s}{\sqrt{1-s^4}}$.

Proposition 5.2. *Let $p \in L_\omega$, $[p]_-^2 \in L_\omega$, and*

$$k^*(\omega)\big\|[p]_-^2\big\|_L < 1. \tag{5.2}$$

Then $p \in \operatorname{Int}\mathcal{D}(\omega)$.

One can prove Proposition 5.2 using inequality (5.1). Proof based on elementary arguments can be found in [12]. Mention also that the inequality (5.2) is optimal in Proposition 5.2 and cannot be weaken to $k^*(\omega)\big\|[p]_-^2\big\|_L \le 1$ (see Lemmas 2.1 and 2.2 in [12]).

Proposition 5.3. *Let $p \in L_\omega$ be such that*

$$\big\|[p]_-\big\|_L \le \frac{4}{\omega} + \frac{p^*}{4\omega}\,\big\|[p]_-\big\|_L^2, \tag{5.3}$$

where the number p^* is defined by (0.16)*. Then $p \in \operatorname{Int}\mathcal{D}(\omega)$.*

Proof. Let $a \in [0, \omega[$ and

$$I_a(t) \overset{\text{def}}{=} (a+\omega-t)\int_a^t (s-a)[p(s)]_-\,\mathrm{d}s + (t-a)\int_t^{a+\omega}(a+\omega-s)[p(s)]_-\,\mathrm{d}s$$

for $t \in [a, a+\omega]$. It is proved in [18, Theorem 1.1] that if

$$\sup\big\{I_a(t):\ t \in [a, a+\omega]\big\} \le \omega \tag{5.4}$$

then any nontrivial solution of the equation $u'' = p(t)u$ has at most one zero in $[a, a+\omega]$. Therefore, if (5.4) is fulfilled for any $a \in [0, \omega[$ then $p \in \operatorname{Int}\mathcal{D}(\omega)$ (see Proposition 2.2). Now we show that (5.3) implies that (5.4) is fulfilled for any $a \in [0, \omega[$. Clearly,

$$\begin{aligned} I_a(t) &= (a+\omega-t)(t-a)\big\|[p]_-\big\|_L - (a+\omega-t)F_1(t) - (t-a)F_2(t) \\ &\le (t-a)(a+\omega-t)\Big(\big\|[p]_-\big\|_L - \frac{1}{\omega}\big(F_1(t)+F_2(t)\big)\Big) \quad \text{for } t \in [a, a+\omega], \end{aligned}$$

where the functions F_1 and F_2 are defined by (4.1). Hence, in view of Proposition 4.1, we get

$$I_a(t) \le \frac{\omega^2}{4}\left(\big\|[p]_-\big\|_L - \frac{p^*}{4\omega}\,\big\|[p]_-\big\|_L^2\right) \quad \text{for } t \in [a, a+\omega]$$

which, together with (5.3), implies (5.4). ☐

6. On the Set $\operatorname{Int}\mathcal{D}(\omega)$

Let $p \in \operatorname{Int}\mathcal{D}(\omega)$. Then, in view of Proposition 2.2 and Fredholm's first theorem, for any $a \in [0, \omega[$ and $q \in L_\omega$ the problem

$$u'' = p(t)u + q(t); \quad u(a) = 0, \;\; u(a+\omega) = 0 \tag{6.1}$$

possesses a unique solution u. Let $\Omega_a \colon L([a, a+\omega]) \to C([a, a+\omega]; \mathbb{R})$ is a Green's operator of the problem (6.1), i.e.,

$$\Omega_a(q)(t) \stackrel{\text{def}}{=} u(t) \quad \text{for } \; t \in [a, a+\omega].$$

It is known that Ω_a is a bounded linear operator (this actually follows from well-posedness of Dirichlet problem). Denote by $\|\Omega_a\|$ the norm of the operator Ω_a and set

$$A \stackrel{\text{def}}{=} \big\{\|\Omega_a\| : \; a \in [0, \omega]\big\}.$$

Proposition 6.1. *Let $p \in \operatorname{Int}\mathcal{D}(\omega)$. Then the set A is bounded.*

Proof. Suppose the contrary, let the set A be unbounded from above. Then, for any $n \in \mathbb{N}$, there are $a_n \in [0, \omega[$ and $q_n \in L_\omega$ such that

$$\|q_n\|_L \leq 1 \tag{6.2}$$

and

$$\|v_n\|_C \geq n, \tag{6.3}$$

where $v_n \in AC'([a_n, a_n+\omega])$ is a solution of the problem

$$v_n'' = p(t)v_n + q_n(t); \quad v_n(a_n) = 0, \;\; v_n(a_n+\omega) = 0.$$

Assume without loss of generality that

$$\lim_{n \to +\infty} a_n = a_0, \tag{6.4}$$

where $a_0 \in [0, \omega]$. Put

$$\widetilde{v}_n(t) = \frac{1}{\|v_n\|_C}\, v_n(t), \quad \widetilde{q}_n(t) = \frac{1}{\|v_n\|_C}\, q_n(t) \quad \text{for } \; t \in [a_n, a_n+\omega], \;\; n \in \mathbb{N}.$$

Clearly,

$$\|\widetilde{v}_n\|_C = 1, \quad \lim_{n \to +\infty} \|\widetilde{q}_n\|_L = 0, \tag{6.5}$$

and for any $n \in \mathbb{N}$

$$\begin{gathered} \widetilde{v}_n''(t) = p(t)\widetilde{v}_n(t) + \widetilde{q}_n(t) \quad \text{for } \; t \in [a_n, a_n+\omega], \\ \widetilde{v}_n(a_n) = 0, \quad \widetilde{v}_n(a_n+\omega) = 0. \end{gathered} \tag{6.6}$$

Moreover, for any $n \in \mathbb{N}$, there is a $t_n \in [a_n, a_n+\omega]$ such that $\widetilde{v}_n'(t_n) = 0$. Hence, in view of (6.2), (6.3), and (6.6), we get

$$|\widetilde{v}_n'(t)| = \left| \int_{t_n}^{t} \widetilde{v}_n''(s)\,\mathrm{d}s \right| \leq \int_{a_n}^{a_n+\omega} |\widetilde{v}_n''(s)|\,\mathrm{d}s \leq \|p\|_L + \frac{1}{n} \quad \text{for } \; t \in [a_n, a_n+\omega], \;\; n \in \mathbb{N}. \tag{6.7}$$

Introduce the notation

$$\widetilde{\widetilde{v}}_n(t) = \begin{cases} \widetilde{v}_n(t) & \text{for } \; t \in [a_n, a_n+\omega], \\ 0 & \text{for } \; t \in [0, 2\omega] \setminus [a_n, a_n+\omega]. \end{cases}$$

In view of (6.5) and (6.7), the sequence $\{\widetilde{\widetilde{v}}_n\}_{n=1}^{+\infty}$ is uniformly bounded and equicontinuous on $[0, 2\omega]$. Hence, by virtue of the Arzelá–Ascoli lemma we can assume without loss of generality that

$$\lim_{n \to +\infty} \widetilde{\widetilde{v}}_n(t) = v_0(t) \quad \text{uniformly on } \; [0, 2\omega], \tag{6.8}$$

where $v_0 \in C([0, 2\omega]; \mathbb{R})$. It is clear that,

$$v_0(a_0) = 0, \quad v_0(a_0+\omega) = 0, \;\; \text{and } \; \|v_0\|_C = 1. \tag{6.9}$$

On the other hand, in view of (6.6), we have

$$\widetilde{v}_n(t) = -\frac{1}{\omega}\bigg[(a_n+\omega-t)\int\limits_{a_n}^{t}(s-a_n)\big(p(s)\widetilde{v}_n(s)+\widetilde{q}_n(s)\big)\,\mathrm{d}s$$

$$+(t-a_n)\int\limits_{t}^{a_n+\omega}(a_n+\omega-s)\big(p(s)\widetilde{v}_n(s)+\widetilde{q}_n(s)\big)\,\mathrm{d}s\bigg] \quad \text{for } t\in[a_n,a_n+\omega].$$

Hence, on account of (6.4), (6.5), and (6.8), we get

$$v_0(t) = -\frac{1}{\omega}\bigg[(a_0+\omega-t)\int\limits_{a_0}^{t}(s-a_0)p(s)v_0(s)\,\mathrm{d}s$$

$$+(t-a_0)\int\limits_{t}^{a_0+\omega}(a_0+\omega-s)p(s)v_0(s)\,\mathrm{d}s\bigg] \quad \text{for } t\in[a_0,a_0+\omega].$$

Thus $v_0 \in AC'([a_0,a_0+\omega])$ and

$$v_0''(t) = p(t)v_0(t) \quad \text{for } t\in[a_0,a_0+\omega].$$

However the latter equality, together with (6.9), contradicts the assumption $p \in \operatorname{Int}\mathcal{D}(\omega)$ (see Proposition 2.2). □

Introduce the definition

Definition 6.2. Let $p \in \operatorname{Int}\mathcal{D}(\omega)$. Set

$$\rho_0(p) \stackrel{\mathrm{def}}{=} \sup\big\{\|\Omega_a\| : \ a\in[a,a+\omega]\big\},$$

where Ω_a is defined as above.

Remark 6.3. Let $p \equiv$ *Const.* and $p \in \ ]-\frac{\pi^2}{\omega^2},0[$. Then, by direct calculation one can easily show that

$$\rho_0(p) \le \frac{\omega^2\sqrt{|p|}}{4\sin(\omega\sqrt{|p|})}\,.$$

Remark 6.4. In view of Proposition 6.1, the number $\rho_0(p)$ is finite and, for any $a\in[0,\omega]$ and $q\in L([a,a+\omega])$, the (unique) solution v of the problem (6.1) satisfies the estimate

$$|v(t)| \le \rho_0(p)\|q\|_L \quad \text{for } t\in[a,a+\omega].$$

Bellow we will establish some estimates of the number $\rho_0(p)$. First of all mention that, by virtue of Proposition 2.5, if $p\in\operatorname{Int}\mathcal{D}(\omega)$ then the operator Ω_a is nonpositive, i.e., transforms the set of nonnegative functions to the set of nonpositive functions. Therefore,

$$\|\Omega_a\| = \sup\Big\{\|\Omega_a(q)\|_C : \ q(t)\le 0 \text{ for } t\in[a,a+\omega],\ \|q\|_L\le 1\Big\}. \tag{6.10}$$

Proposition 6.5. *Let $p\in L_\omega$ and*

$$\big\|[p]_-\big\|_L < \frac{4}{\omega}+\frac{p^*}{4\omega}\big\|[p]_-\big\|_L^2, \tag{6.11}$$

where p^* is defined by (0.16)*. Then*

$$\rho_0(p) \le \Big[\frac{4}{\omega}+\frac{p^*}{4\omega}\big\|[p]_-\big\|_L^2-\big\|[p]_-\big\|_L\Big]^{-1}. \tag{6.12}$$

Proof. In view of (6.11) and Proposition 5.3, we have $-[p]_-\in\operatorname{Int}\mathcal{D}(\omega)$. On account of Proposition 2.6, we easily get that $p\in\operatorname{Int}\mathcal{D}(\omega)$ as well. Let $a\in[0,\omega[$ be arbitrary and let $u\in AC'([a,a+\omega])$ be a solution of the problem (6.1), where $q\in L([a,a+\omega])$,

$$q(t)\le 0 \quad \text{for } t\in[a,a+\omega], \quad \|q\|_L\le 1. \tag{6.13}$$

On account of Proposition 2.5 we have

$$u(t) \geq 0 \quad \text{for } \ t \in [a, a+\omega]. \tag{6.14}$$

By direct calculation one can verify that

$$\begin{aligned} u(t) = -\frac{1}{\omega}\bigg[(a+\omega-t)\int\limits_a^t (s-a)\big(p(s)u(s)+q(s)\big)\,\mathrm{d}s \\ + (t-a)\int\limits_t^{a+\omega}(a+\omega-s)\big(p(s)u(s)+q(s)\big)\,\mathrm{d}s\bigg] \quad \text{for } \ t \in [a, a+\omega]. \end{aligned}$$

Hence, in view of (6.13) and (6.14) we get

$$0 \leq u(t) \leq \frac{\omega}{4} + \frac{\|u\|_C}{\omega}\, I(t) \quad \text{for } \ t \in [a, a+\omega], \tag{6.15}$$

where

$$I(t) \stackrel{\text{def}}{=} (a+\omega-t)\int\limits_a^t (s-a)[p(s)]_-\,\mathrm{d}s + (t-a)\int\limits_t^{a+\omega}(a+\omega-s)[p(s)]_-\,\mathrm{d}s \quad \text{for } \ t \in [a, a+\omega].$$

It is clear that,

$$\begin{aligned} I(t) &= (t-a)(a+\omega-t)\big\|[p]_-\big\|_L - (a+\omega-t)F_1(t) - (t-a)F_2(t) \\ &\leq (t-a)(a+\omega-t)\Big(\big\|[p]_-\big\|_L - \frac{1}{\omega}\left(F_1(t)+F_2(t)\right)\Big) \quad \text{for } \ t \in [a, a+\omega], \end{aligned}$$

where the functions F_1 and F_2 are defined by (4.1). Hence, by virtue of Proposition 4.1 we get

$$I(t) \leq \frac{\omega^2}{4}\left(\big\|[p]_-\big\|_L - \frac{p^*}{4\omega}\big\|[p]_-\big\|_L^2\right) \quad \text{for } \ t \in [a, a+\omega].$$

The latter inequality together with (6.11) and (6.15) imply that

$$\|u\|_C \leq \left[\frac{4}{\omega} + \frac{p^*}{4\omega}\big\|[p]_-\big\|_L^2 - \big\|[p]_-\big\|_L\right]^{-1},$$

which, in view of (6.10), leads to the desired estimate (6.12). □

Proposition 6.6. *Let* $p \in L_\omega$*,* $[p]_-^2 \in L_\omega$*, and*

$$k^*(\omega)\big\|[p]_-^2\big\|_L < 1, \tag{6.16}$$

where $k^*(\omega)$ *is a number appearing in Remark* 5.1*. Then*

$$\rho_0(p) \leq \frac{\omega}{4}\left(1 - \sqrt{k^*(\omega)\big\|[p]_-^2\big\|_L}\right)^{-1}. \tag{6.17}$$

Proof. In view of (6.16) we have $-[p]_- \in \operatorname{Int}\mathcal{D}(\omega)$ (see Proposition 5.2). Hence, on account of Proposition 2.6, we get $p \in \operatorname{Int}\mathcal{D}(\omega)$ as well. Let $a \in [0, \omega[$ be arbitrary and let $u \in AC'([a, a+\omega])$ be a solution of the problem (6.1), where $q \in L([a, a+\omega])$ satisfies (6.13). On account of Proposition 2.5, we have that (6.14) holds.

Multiplying both sides of (6.1) by u and integrating it on $[a, a+\omega]$ we get

$$\int\limits_a^{a+\omega}\big[u'(s)\big]^2\,\mathrm{d}s = -\int\limits_a^{a+\omega} p(s)u^2(s)\,\mathrm{d}s - \int\limits_a^{a+\omega} q(s)u(s)\,\mathrm{d}s$$

which, together with (6.13) and (6.14), results in

$$\int\limits_a^{a+\omega}\big[u'(s)\big]^2\,\mathrm{d}s \leq \int\limits_a^{a+\omega}[p(s)]_-u^2(s)\,\mathrm{d}s + \|u\|_C. \tag{6.18}$$

By virtue of Hölder's inequality and Remark 5.1, we have

$$\Big(\int_a^{a+\omega}[p(s)]_-u^2(s)\,\mathrm{d}s\Big)^2 \le \int_a^{a+\omega}[p(s)]_-^2\,\mathrm{d}s\int_a^{a+\omega}u^4(s)\,\mathrm{d}s \le k^*(\omega)\big\|[p]_-^2\big\|_L\Big(\int_a^{a+\omega}\big[u'(s)\big]^2\,\mathrm{d}s\Big)^2$$

which, together with (6.16) and (6.18), implies

$$\int_a^{a+\omega}\big[u'(s)\big]^2\,\mathrm{d}s \le \Big(1-\sqrt{k^*(\omega)\big\|[p]_-^2\big\|_L}\Big)^{-1}\|u\|_C. \tag{6.19}$$

Let now $t_0 \in\,]a, a+\omega[$ be such that $u(t_0)=\|u\|_C$. Then, by virtue of Hölder's inequality we obtain

$$\|u\|_C^2=\Big(\int_a^{t_0}u'(s)\,\mathrm{d}s\Big)^2 \le (t_0-a)\int_a^{t_0}\big[u'(s)\big]^2\,\mathrm{d}s,$$

$$\|u\|_C^2=\Big(\int_{t_0}^{a+\omega}u'(s)\,\mathrm{d}s\Big)^2 \le (a+\omega-t_0)\int_{t_0}^{a+\omega}\big[u'(s)\big]^2\,\mathrm{d}s.$$

Hence, using the inequality $4xy\le(x+y)^2$ we get

$$\|u\|_C^2 \le \frac{\omega}{4}\int_a^{a+\omega}\big[u'(s)\big]^2\,\mathrm{d}s,$$

which, together with (6.19), implies that

$$\|u\|_C \le \frac{\omega}{4}\Big(1-\sqrt{k^*(\omega)\big\|[p]_-^2\big\|_L}\Big)^{-1}.$$

Taking now (6.10) into account we get that the desired estimate (6.17) is fulfilled. □

Let now again $p\in \operatorname{Int}\mathcal{D}(\omega)$ and $a\in[0,\omega]$. Then, in view of Proposition 2.2 and Fredholm's first theorem the problem

$$u''=p(t)u;\quad u(a)=1,\ \ u(a+\omega)=1 \tag{6.20}$$

possesses a unique solution u_a and

$$u_a(t)>0\quad\text{for }\ t\in[a,a+\omega]. \tag{6.21}$$

Introduce the notation

$$\nu^*(p)\overset{\text{def}}{=}\sup\big\{\|u_a\|_C:\ a\in[0,\omega]\big\}. \tag{6.22}$$

Remark 6.7. By direct calculation one can easily verify that if $p\equiv$ *Const.* and $p\in\big]-\frac{\pi^2}{\omega^2},0\big]$ then $\nu^*(p)=\frac{1}{\cos\frac{\omega\sqrt{|p|}}{2}}$.

Proposition 6.8. *Let $p\in\operatorname{Int}\mathcal{D}(\omega)$. Then*

$$\nu^*(p)\le 1+\rho_0(p)\big\|[p]_-\big\|_L. \tag{6.23}$$

Proof. Let $a\in[0,\omega[$ be arbitrary, u_a be a solution of the problem (6.20), and let v_a be a solution of the problem

$$v''=p(t)v-[p(t)]_-;\quad v(a)=0,\ \ v(a+\omega)=0.$$

Put $w(t)\overset{\text{def}}{=}u_a(t)-v_a(t)-1$ for $t\in[a,a+\omega]$. It is clear that,

$$w''(t)=p(t)w(t)+[p(t)]_+\quad\text{for }\ t\in[a,a+\omega],$$
$$w(a)=0,\quad w(a+\omega)=0.$$

Hence, by virtue of Proposition 2.5, we get that $w(t)\le 0$ for $t\in[a,a+\omega]$. Consequently,

$$u_a(t)\le 1+v_a(t)\quad\text{for }\ t\in[a,a+\omega].$$

Taking now (6.21) and Remark 6.4 into account, we get

$$\|u_a\|_C\le 1+\rho_0(p)\big\|[p]_-\big\|_L$$

which, together with (6.22), implies (6.23). $\square$

Recall that the numbers $\overline{p}$ and ℓ are defined by (0.11) and (0.15), respectively.

Proposition 6.9. *Let $p \in L_\omega$ and*

$$\ell^2\Big(1 - \frac{\pi^2}{(\mathrm{e}^{\omega\ell} - 1)^2}\Big) < \overline{p} < \ell^2. \tag{6.24}$$

Then $p \in \operatorname{Int} \mathcal{D}(\omega)$ and

$$\nu^*(p) \le \frac{\mathrm{e}^{\omega\ell}}{1 - c_0}\,, \tag{6.25}$$

where $c_0 \stackrel{\text{def}}{=} \frac{\mathrm{e}^{\omega\ell} - 1}{\pi\ell} \sqrt{\ell^2 - \overline{p}}\,.$

Proof. First of all mention that, by virtue of the first inequality in (6.24), $c_0 < 1$. Put

$$p_0(t) \stackrel{\text{def}}{=} p(t) - \overline{p} + \ell^2(p)(t), \quad u_0(t) \stackrel{\text{def}}{=} \mathrm{e}^{\int_0^t \ell(p)(s)\,\mathrm{d}s} \qquad \text{for } t \in [0,\omega].$$

It is clear that, $p_0 \in L_\omega$, $u_0(t+\omega) = u_0(t)$ for $t \in \mathbb{R}$,

$$u_0(t) < \mathrm{e}^{\omega\ell} \quad \text{for } t \in \mathbb{R}, \tag{6.26}$$

and

$$u_0''(t) = p_0(t)u_0(t) \quad \text{for } t \in \mathbb{R}. \tag{6.27}$$

Moreover, it follows from Proposition 4.2 that

$$u_0^2(t)\int_0^\omega \frac{1}{u_0^2(s)}\,\mathrm{d}s \le \frac{\mathrm{e}^{\omega\ell} - 1}{\ell} \quad \text{for } t \in \mathbb{R}. \tag{6.28}$$

Let $a \in [0,\omega[$ and

$$\lambda_0 \stackrel{\text{def}}{=} c_0\pi\bigg(\int_0^\omega \frac{1}{u_0^2(s)}\,\mathrm{d}s\bigg)^{-1},$$

$$\varphi_a(t) \stackrel{\text{def}}{=} \frac{1}{\sin c_0\pi}\bigg[\sin\bigg(\lambda_0\int_a^t \frac{1}{u_0^2(s)}\,\mathrm{d}s\bigg) + \sin\bigg(\lambda_0\int_t^{a+\omega} \frac{1}{u_0^2(s)}\,\mathrm{d}s\bigg)\bigg]$$

for $t \in [a, a+\omega]$. One can easily verify that

$$0 < \varphi_a(t) \le \frac{1}{\cos\frac{c_0\pi}{2}} < \frac{1}{1-c_0} \quad \text{for } t \in [a, a+\omega] \tag{6.29}$$

and

$$\begin{gathered}\varphi_a''(t) = -\frac{\lambda_0^2}{u_0^4(t)}\,\varphi_a(t) - 2\ell(p)(t)\varphi_a'(t) \quad \text{for } t \in [a, a+\omega],\\ \varphi_a(a) = 1, \quad \varphi_a(a+\omega) = 1.\end{gathered} \tag{6.30}$$

Let now

$$v_a(t) \stackrel{\text{def}}{=} u_0(t)\varphi_a(t) \quad \text{for } t \in [a, a+\omega]. \tag{6.31}$$

In view of (6.27) and (6.30), we easily get that

$$v_a''(t) = \Big(p_0(t) - \frac{\lambda_0^2}{u_0^4(t)}\Big)v_a(t) \quad \text{for } t \in [a, a+\omega], \tag{6.32}$$

$$v_a(a) = 1, \quad v_a(a+\omega) = 1. \tag{6.33}$$

On the other hand, by virtue of (6.24) and (6.28), the inequality

$$\overline{p} - \ell^2(p)(t) \ge \overline{p} - \ell^2 = -\frac{c_0^2\pi^2}{\left(\mathrm{e}^{\omega\ell} - 1\right)^2}\,\ell^2 \ge -\frac{\lambda_0^2}{u_0^4(t)} \quad \text{for } t \in \mathbb{R}$$

holds. Hence, it follows from (6.32), by virtue of (6.29) and (6.31), that

$$v_a''(t) \le p(t)v_a(t) \quad \text{for } t \in [a, a+\omega], \tag{6.34}$$

$$v_a(t) > 0 \quad \text{for } t \in [a, a+\omega]. \tag{6.35}$$

Consequently, by virtue of Proposition 2.6, we get that $p \in \operatorname{Int}\mathcal{D}(\omega)$.

Let now u_a be a solution of the problem

$$u'' = p(t)u; \quad u(a) = 1, \;\; u(a+\omega) = 1.$$

Then, in view of (6.33), (6.34), and Proposition 2.5, we get that

$$u_a(t) \le v_a(t) \quad \text{for } t \in [a, a+\omega].$$

Hence, on account of (6.26), (6.29), and (6.31), the inequality

$$u_a(t) \le \frac{\mathrm{e}^{\omega\ell}}{1-c_0} \quad \text{for } t \in [a, a+\omega]$$

holds and, consequently, (6.25) is fulfilled. □

Proposition 6.10. *Let $p \in \operatorname{Int}\mathcal{D}(\omega)$, $q \in L_\omega$, and*

$$q(t) \le 0 \quad \text{for } t \in [0, \omega], \quad q \not\equiv 0.$$

Then, for any $a \in [0, \omega]$, the unique solution u of the problem

$$u'' = p(t)u + q(t); \quad u(a) = 0, \;\; u(a+\omega) = 0$$

satisfies

$$u(t) > 0 \quad \text{for } t \in]a, a+\omega[\,. \tag{6.36}$$

Proof. In view of Proposition 2.5 we have that

$$u(t) \ge 0 \quad \text{for } t \in [0, \omega]. \tag{6.37}$$

Let there is a $t_0 \in]a, a+\omega[$ such that

$$u(t_0) = 0. \tag{6.38}$$

Then, in view of (6.37), we have

$$u'(t_0) = 0. \tag{6.39}$$

Denote by u_1, resp. u_2 solutions of the problems

$$u_1'' = p(t)u_1; \quad u_1(a) = 0, \;\; u_1'(a) = 1, \tag{6.40}$$

$$u_2'' = p(t)u_2; \quad u_2(a+\omega) = 0, \;\; u_2'(a+\omega) = -1. \tag{6.41}$$

Since $p \in \operatorname{Int}\mathcal{D}(\omega)$, we have

$$u_1(t) > 0 \quad \text{for } t \in]a, a+\omega], \qquad u_2(t) > 0 \quad \text{for } t \in [a, a+\omega[\,. \tag{6.42}$$

On the other hand, it is clear that

$$\big(u'(t)u_i(t) - u(t)u_i'(t)\big)' = q(t)u_i(t) \quad \text{for } t \in [a, a+\omega], \;\; i = 1, 2.$$

Integrating the latter equalities on $[a, t_0]$ and $[t_0, a+\omega]$, and taking into account (6.38) and (6.39) we get

$$\int_a^{t_0} q(s)u_1(s)\,\mathrm{d}s = 0, \qquad \int_{t_0}^{a+\omega} q(s)u_2(s)\,\mathrm{d}s = 0.$$

Hence, on account of (6.42) we get $q \equiv 0$, which contradicts an assumption of the proposition. □

Proposition 6.11. *Let $p \in L_\omega$, $p(t) \ge 0$ for $t \in [0, \omega]$, and $p \not\equiv 0$. Then, for any $a \in [0, \omega]$, the unique solution u of the problem* (6.20) *satisfies the estimates*

$$\frac{\omega}{u_2(a)} < u(t) < \frac{\omega}{u_2(a)}\,\rho(p) \quad \text{for } t \in]a, a+\omega[\,, \tag{6.43}$$

$$u(t) > \frac{1}{\rho(p)} \quad \text{for } t \in]a, a+\omega[\,, \tag{6.44}$$

where u_2 is a solution of the problem (6.41) *and $\rho(p)$ is defined by* (0.12).

Proof. In view of Proposition 2.6 (with $\gamma_a \equiv 1$), clearly $p \in \operatorname{Int}\mathcal{D}(\omega)$. Hence, for any $a \in [0,\omega]$, the problem (6.20) possesses a unique solution u. Let u_1 be a solution of the problem (6.40). Since $p \in \operatorname{Int}\mathcal{D}(\omega)$, it is clear that (6.42) holds. On the other hand, $u_1'(t)u_2(t) - u_1(t)u_2'(t) = Const.$ and thus $u_1(a+\omega) = u_2(a)$. Now it is clear that

$$u(t) = \frac{1}{u_2(a)}\big(u_1(t) + u_2(t)\big) \quad \text{for } t \in [a, a+\omega].$$

Let us estimate the functions u_1 and u_2. It follows from (6.40) and (6.41) that

$$\begin{aligned} u_1(t) &= t-a + \int_a^t (t-s)p(s)u_1(s)\,\mathrm{d}s \quad \text{for } t \in [a, a+\omega], \\ u_2(t) &= a+\omega-t + \int_t^{a+\omega} (s-t)p(s)u_2(s)\,\mathrm{d}s \quad \text{for } t \in [a, a+\omega]. \end{aligned} \tag{6.45}$$

Hence, on account of (6.42) and the conditions $p(t) \geq 0$, $p \not\equiv 0$, we get

$$u_1(t) + u_2(t) > \omega \quad \text{for } t \in [a, a+\omega]$$

and, consequently, the first inequality in (6.43) holds.

On the other hand, by virtue of the inequalities

$$\begin{aligned} \frac{t-s}{t-a} &< \frac{a+\omega-s}{\omega} \quad \text{for } s \in \,]a,t[\,, \\ \frac{s-t}{a+\omega-t} &< \frac{s-a}{\omega} \quad \text{for } s \in \,]t, a+\omega[\,, \end{aligned}$$

it follows from (6.45) that

$$\begin{aligned} \frac{u_1(t)}{t-a} &= 1 + \frac{1}{\omega}\int_a^t (a+\omega-s)(s-a)p(s)\frac{u_1(s)}{s-a}\,\mathrm{d}s \quad \text{for } t \in \,]a, a+\omega[\,, \\ \frac{u_2(t)}{a+\omega-t} &= 1 + \frac{1}{\omega}\int_t^{a+\omega} (a+\omega-s)(s-a)p(s)\frac{u_2(s)}{a+\omega-s}\,\mathrm{d}s \quad \text{for } t \in \,]a, a+\omega[\,. \end{aligned}$$

Hence, by virtue of Gronwall–Bellman's lemma we get

$$u_1(t) \leq (t-a)\exp\left[\frac{\omega}{4}\int_a^t p(s)\,\mathrm{d}s\right], \quad u_2(t) \leq (a+\omega-t)\exp\left[\frac{\omega}{4}\int_t^{a+\omega} p(s)\,\mathrm{d}s\right]$$

for $t \in [a, a+\omega]$. The latter inequalities, together with the condition $p \not\equiv 0$, imply that

$$u_1(t) \leq (t-a)\rho(p), \quad u_2(t) \leq (a+\omega-t)\rho(p) \quad \text{for } t \in [a, a+\omega] \tag{6.46}$$

and, for any $t \in \,]a, a+\omega[\,$, at least one of the previous inequalities is strict. Consequently,

$$u_1(t) + u_2(t) < \omega\rho(p) \quad \text{for } t \in \,]a, a+\omega[$$

and therefore, the second inequality in (6.43) is fulfilled.

As for the inequality (6.44), it immediately follows from the first inequality in (6.43) and the second inequality in (6.46). □

Proposition 6.12. *Let $p \in \operatorname{Int}\mathcal{D}(\omega)$ and $[p]_- \not\equiv 0$. Then, for any $a \in [0,\omega]$, the unique solution u of the problem* (6.20) *satisfies the estimate*

$$u(t) > \frac{1}{\rho(p)} \quad \textit{for } t \in \,]a, a+\omega[\,, \tag{6.47}$$

where $\rho(p)$ is a number defined by (0.12).

Proof. Let $a \in [0, \omega]$ be arbitrary and let u be a solution of the problem (6.20). It is clear that

$$u(t) > 0 \quad \text{for } t \in [a, a + \omega]. \tag{6.48}$$

Denote by v the solution of the problem

$$v'' = [p(t)]_+ v; \quad v(a) = 1, \; v(a + \omega) = 1$$

and put $w(t) \overset{\text{def}}{=} u(t) - v(t)$ for $t \in [a, a + \omega]$. It is clear that

$$w''(t) = [p(t)]_+ w(t) + q(t) \quad \text{for } t \in [a, a + \omega],$$
$$w(a) = 0, \quad w(a + \omega) = 0,$$

where $q(t) \overset{\text{def}}{=} -[p(t)]_- u(t)$ for $t \in [a, a+\omega]$. By virtue of (6.48) and the assumption $[p]_- \not\equiv 0$, we have that

$$q(t) \le 0 \quad \text{for } t \in [a, a + \omega], \quad q \not\equiv 0.$$

Taking, moreover, into account that $[p]_+ \in \operatorname{Int} \mathcal{D}(\omega)$ we get, by virtue of Proposition 6.10 that

$$u(t) > v(t) \quad \text{for } t \in \,]a, a + \omega[\,. \tag{6.49}$$

If $[p]_+ \equiv 0$ then clearly $v \equiv 1$ and, consequently, in view of (6.49), the desired estimate (6.47) holds. If $[p]_+ \not\equiv 0$ then, by virtue of proposition 6.11, we get $v(t) > \frac{1}{\rho(p)}$ pro $t \in \,]a, a + \omega[$ which, together with (6.49), yields the desired estimate (6.47). □

In the next proposition we will establish estimates of the numbers $\rho_0(p)$ and $\nu^*(p)$ (see Definition 6.2 and (6.22)) in the case when $p \in \mathcal{V}_0(\omega)$. It is clear that, $\mathcal{V}_0(\omega) \subset \operatorname{Int} \mathcal{D}(\omega)$ and therefore $\rho_0(p)$ and $\nu^*(p)$ are defined correctly.

Proposition 6.13. *Let $p \in \mathcal{V}_0(\omega)$. Then the estimates*

$$\nu^*(p) \le \mathrm{e}^{\frac{\omega}{2}\sqrt{\overline{p}}}, \quad \rho_0(p) \le \frac{\omega}{4}\,\mathrm{e}^{\omega\sqrt{\overline{p}}} \tag{6.50}$$

are fulfilled, where $\overline{p}$ is a number defined by (0.11).

Proof. Let $a \in [0, \omega]$ be fixed and u_n be a solution of the problem

$$u_a'' = p(t) u_a, \tag{6.51}$$
$$u_a(a) = 1, \quad u_a(a + \omega) = 1. \tag{6.52}$$

Since $p \in \mathcal{V}_0(\omega)$, it is clear that $u_a(t) > 0$ for $t \in [a, a + \omega]$ and

$$u_a'(a) = u_a'(a + \omega) \tag{6.53}$$

as well. Extend the function u_a periodically and denote it by the same letter. Put

$$\rho_a(t) = \frac{u_a'(t)}{u_a(t)} \quad \text{for } t \in \mathbb{R},$$
$$M_a = \max\big\{u_a(t) : t \in [0, \omega]\big\}, \quad m_a = \min\big\{u_a(t) : t \in [0, \omega]\big\},$$

and choose $\alpha \in [a, a + \omega[$ and $\beta \in \,]\alpha, \alpha + \omega[$ such that

$$u_a(\alpha) = M_a, \quad u_a(\beta) = m_a.$$

It is clear that,

$$\rho_a'(t) = p(t) - \rho_a^2(t) \quad \text{for } t \in \mathbb{R}$$

and

$$\int\limits_{\alpha}^{\alpha+\omega} \rho_a^2(s)\,\mathrm{d}s = \omega\overline{p}. \tag{6.54}$$

On the other hand, by virtue of Hölder's inequality, we have that

$$\ln^2 \frac{M_a}{m_a} = \bigg(\int\limits_{\alpha}^{\beta} \rho_a(s)\,\mathrm{d}s\bigg)^2 \le (\beta - \alpha)\int\limits_{\alpha}^{\beta} \rho_a^2(s)\,\mathrm{d}s$$

and

$$\ln^2 \frac{M_a}{m_a} = \Bigg(\int\limits_{\beta}^{\alpha+\omega} \rho_a(s)\,\mathrm{d}s \Bigg)^2 \le (\alpha+\omega-\beta) \int\limits_{\beta}^{\alpha+\omega} \rho_a^2(s)\,\mathrm{d}s.$$

Hence, in view of the inequality $4xy \le (x+y)^2$ for $x, y \in \mathbb{R}$, we get that

$$\ln^4 \frac{M_a}{m_a} \le \frac{\omega^2}{16} \Bigg(\int\limits_{\alpha}^{\alpha+\omega} \rho_a^2(s)\,\mathrm{d}s \Bigg)^2$$

which, together with (6.54), implies that

$$\frac{M_a}{m_a} \le \mathrm{e}^{\frac{\omega}{2}\sqrt{p}}\,. \tag{6.55}$$

In view of (6.52), we have that $m_a \le 1$. Consequently, (6.55) implies that $M_a \le \mathrm{e}^{\frac{\omega}{2}\sqrt{p}}$ and therefore, the firs inequality in (6.50) is fulfilled.

Now we will show that the second estimate in (6.50) is fulfilled. Let $q \in L_\omega$, $q(t) \le 0$ for $t \in \mathbb{R}$ and $\|q\|_L \le 1$. Denote by u a solution of the problem

$$u'' = p(t)u + q(t); \quad u(a) = 0, \ \ u(a+\omega) = 0.$$

Let, moreover, u_1 and u_2 be solutions of the problems (6.40) and (6.41), respectively. It is clear that $u(t) > 0$ for $t \in \,]a, a+\omega[\,$. By direct calculations one can easily verify that

$$u_1(t) = u_a(t) \int\limits_a^t \frac{1}{u_a^2(s)}\,\mathrm{d}s, \quad u_2(t) = u_a(t) \int\limits_t^{a+\omega} \frac{1}{u_a^2(s)}\,\mathrm{d}s \quad \text{for } \ t \in [a, a+\omega]$$

and

$$u(t) = \frac{1}{u_1(a+\omega)} \Bigg(u_2(t) \int\limits_a^t u_1(s)|q(s)|\,\mathrm{d}s + u_1(t) \int\limits_t^{a+\omega} u_2(s)|q(s)|\,\mathrm{d}s \Bigg)$$

for $t \in [a, a+\omega]$. Hence we get that

$$\begin{aligned} 0 \le u(t) &\le \frac{u_a(t)}{u_1(a+\omega)} \int\limits_a^t \frac{1}{u_a^2(s)}\,\mathrm{d}s \int\limits_t^{a+\omega} \frac{1}{u_a^2(s)}\,\mathrm{d}s \int\limits_a^{a+\omega} u_a(s)|q(s)|\,\mathrm{d}s \\ &\le \frac{u_a(t)}{4u_1(a+\omega)} \Bigg(\int\limits_a^{a+\omega} \frac{1}{u_a^2(s)}\,\mathrm{d}s \Bigg)^2 \int\limits_a^{a+\omega} u_a(s)|q(s)|\,\mathrm{d}s \\ &\le \frac{M_a^2}{4} \int\limits_a^{a+\omega} \frac{1}{u_a^2(s)}\,\mathrm{d}s \|q\|_L \le \frac{\omega}{4} \Big(\frac{M_a}{m_a} \Big)^2 \|q\|_L \quad \text{for } \ t \in [a, a+\omega] \end{aligned}$$

which, together with (6.55), implies that

$$0 \le u(t) \le \frac{\omega}{4}\, \mathrm{e}^{\omega\sqrt{p}}\, \|q\|_L \quad \text{for } \ t \in [a, a+\omega].$$

Now, in view of Definition 6.2 and (6.10), it is clear that the second estimate in (6.50) is fulfilled. □

Proposition 6.14. *Let* $p_n, p \in \operatorname{Int} \mathcal{D}(\omega)$ *and*

$$\lim_{n\to+\infty} \|p_n - p\|_L = 0. \tag{6.56}$$

Then

$$\lim_{n\to+\infty} \rho_0(p_n) = \rho_0(p), \quad \lim_{n\to+\infty} \nu^*(p_n) = \nu^*(p). \tag{6.57}$$

Proof. Let $a \in [0,\omega[$ and u_{na} and v_a be solutions of the problems

$$u'' = p_n(t)u + q(t); \quad u(a) = 0, \ \ u(a+\omega) = 0,$$
$$v'' = p(t)v + q(t); \quad v(a) = 0, \ \ v(a+\omega) = 0,$$

where $q \in L_\omega$, $q(t) \geq 0$ for $t \in \mathbb{R}$, $\|q\|_L \leq 1$. It is clear, the function u_{na} is a solution of the problem

$$u'' = p(t)u + q(t) + \big(p_n(t) - p(t)\big)u_{na}(t); \quad u(a) = 0, \ \ u(a+\omega) = 0$$

as well. Hence,

$$u_{na}(t) = v_a(t) + \Omega_a\big((p_n - p)u_{na}\big)(t) \quad \text{for } \ t \in [a, a+\omega], \ \ n \in \mathbb{N}, \tag{6.58}$$

where Ω_a is a Green's operator of the problem (6.1). It follows from (6.58) that

$$\begin{gathered} \|u_{na}\|_C \leq \|v_a\|_C + \|\Omega_a\| \, \|p_n - p\|_L \, \|u_{na}\|_C \quad \text{for } \ n \in \mathbb{N}, \\ \|v_a\|_C \leq \|u_{na}\|_C + \|\Omega_a\| \, \|p_n - p\|_L \, \|u_{na}\|_C \quad \text{for } \ n \in \mathbb{N}. \end{gathered} \tag{6.59}$$

Hence, in view of the inequality $\|\Omega_a\| \leq \rho_0(p)$ (see Definition 6.2), we get that

$$\begin{gathered} \|u_{na}\|_C\Big(1 - \rho_0(p)\|p_n - p\|_L\Big) \leq \|v_a\|_C \quad \text{for } \ n \in \mathbb{N}, \\ \|v_a\|_C \leq \|u_{na}\|_C\Big(1 + \rho_0(p)\|p_n - p\|_L\Big) \quad \text{for } \ n \in \mathbb{N}. \end{gathered} \tag{6.60}$$

Taking now into account that $\|v_a\|_C \leq \rho_0(p)$ and $\|u_{na}\|_C \leq \rho_0(p_n)$ (see Remark 6.4), we get

$$\|u_{na}\|_C\Big(1 - \rho_0(p)\|p_n - p\|_L\Big) \leq \rho_0(p) \quad \text{for } \ n \in \mathbb{N},$$
$$\|v_a\|_C \leq \rho_0(p_n)\Big(1 + \rho_0(p)\|p_n - p\|_L\Big) \quad \text{for } \ n \in \mathbb{N}.$$

Consequently, in view of (6.56) and Definition 6.2, the inequalities

$$\frac{\rho_0(p)}{1 + \rho_0(p)\|p_n - p\|_L} \leq \rho_0(p_n) \leq \frac{\rho_0(p)}{1 - \rho_0(p)\|p_n - p\|_L}$$

hold for $n \in \mathbb{N}$ large enough which, in view of (6.56), implies that $\lim\limits_{n\to+\infty} \rho_0(p_n) = \rho_0(p)$.

Let now u_{na} and v_a be solutions of the problems

$$u'' = p_n(t)u; \quad u(a) = 1, \ \ u(a+\omega) = 1,$$
$$v'' = p(t)v; \quad v(a) = 1, \ \ v(a+\omega) = 1.$$

Clearly, u_{na} is a solution of the problem

$$u'' = p(t)u + \big(p_n(t) - p(t)\big)u_{na}(t); \quad u(a) = 1, \ \ u(a+\omega) = 1$$

as well. In view of Green's formula, we have that (6.58) holds and, consequently, (6.59) and (6.60) are fulfilled. Taking now into account that $\|v_a\|_C \leq \nu^*(p)$ and $\|u_{na}\|_C \leq \nu^*(p_n)$ (see (6.22)), we get from (6.60) that

$$\|u_{na}\|_C\Big(1 - \rho_0(p)\|p_n - p\|_L\Big) \leq \nu^*(p) \quad \text{for } \ n \in \mathbb{N},$$
$$\|v_a\|_C \leq \nu^*(p_n)\Big(1 + \rho_0(p)\|p_n - p\|_L\Big) \quad \text{for } \ n \in \mathbb{N}.$$

Consequently, the inequalities

$$\|u_{na}\|_C \leq \frac{\nu^*(p)}{1 - \rho_0(p)\|p_n - p\|_L}$$

and

$$\|v_a\|_C \leq \frac{\nu^*(p_n)}{1 + \rho_0(p)\|p_n - p\|_L}$$

hold for $n \in \mathbb{N}$ large enough. The latter inequalities, in view of (6.22), implies that for $n \in \mathbb{N}$ large enough,

$$\nu^*(p)\Big(1 + \rho_0(p)\|p_n - p\|_L\Big) \leq \nu^*(p_n) \leq \frac{\nu^*(p)}{1 - \rho_0(p)\|p_n - p\|_L}$$

which, together with (6.56), implies that $\lim\limits_{n\to+\infty} \nu^*(p_n) = \nu^*(p)$. □

7. On the Floquet Theory

In this chapter, for convenience of references, we recall Floquet theorems for the equation

$$u'' = p(t)u, \tag{7.1}$$

where $p \in L_\omega$.

Denote by v_1 and v_2 solutions of the problems

$$v_1'' = p(t)v_1; \quad v_1(0) = 1, \ \ v_1'(0) = 0,$$
$$v_2'' = p(t)v_2; \quad v_2(0) = 0, \ \ v_2'(0) = 1.$$

The number $A \overset{\text{def}}{=} v_1(\omega) + v_2'(\omega)$ is called Lyapunov constant and the equation

$$x^2 - Ax + 1 = 0$$

is called a characteristic equation for (7.1). Roots of the characteristic equation are called Floquet's multipliers of equation (7.1).

Floquet's first theorem states that

Theorem 7.1. *Equation* (7.1) *is stable if and only if either*

(1) *Floquet multipliers of equation* (7.1) *are complex valued,*

or

(2) *Floquet multipliers μ_1 and μ_2 of equation* (7.1) *are real valued, $\mu_1 = \mu_2$, $|\mu_1| = 1$, and any solution u of the equation* (7.1) *satisfies*

$$u(t+\omega) = \mu_1 u(t) \quad \textit{for } \ t \in \mathbb{R}.$$

Floquet's second theorem states that

Theorem 7.2. *The number $\mu \in \mathbb{R}$ is a Floquet multiplier of the equation* (7.1) *if and only if there is a nontrivial solution u of the equation* (7.1) *satisfying*

$$u(t+\omega) = \mu u(t) \quad \textit{for } \ t \in \mathbb{R}.$$

Theorem 7.3. *The complex number $\mu \notin \mathbb{R}$ with real and imaginary parts α and β, respectively, is a Floquet multiplier of the equation* (7.1) *if and only if*

$$\alpha^2 + \beta^2 = 1, \tag{7.2}$$

and there are linearly independent solutions u and v of the equation (7.1) *satisfying*

$$u(t+\omega) = \alpha u(t) - \beta v(t), \quad v(t+\omega) = \beta u(t) + \alpha v(t) \quad \textit{for } \ t \in \mathbb{R}. \tag{7.3}$$

It is well known that the stability of the equation (7.1) is connected with the solvability of a certain periodic boundary value problem. More precisely, consider the problem

$$w'' = p(t)w + \frac{1}{w^3}\,; \quad w(0) = w(\omega), \ \ w'(0) = w'(\omega), \tag{7.4}$$

where $p \in L_\omega$. Under a solution of the problem (7.4) we understand a **positive** function $w \in AC'([0,\omega])$ satisfying given equation almost everywhere on $[0,\omega]$ and boundary conditions in (7.4).

Proposition 7.4. *Equation* (7.1) *is stable if and only if the problem* (7.4) *is solvable.*

Proof. Let equation (7.1) is stable. Then, by virtue of Theorems 7.1–7.3 there are linearly independent solutions u and v of the equation (7.1) satisfying either (7.2) and (7.3), or

$$u(t+\omega) = \mu u(t), \quad v(t+\omega) = \mu v(t) \quad \text{for } \ t \in \mathbb{R},$$

where $\mu \in \mathbb{R}$ and $|\mu| = 1$. Assume without loss of generality that

$$u'(t)v(t) - u(t)v'(t) = 1 \quad \text{for } \ t \in \mathbb{R} \tag{7.5}$$

and put

$$w(t) = \sqrt{u^2(t) + v^2(t)} \quad \text{for } \ t \in \mathbb{R}.$$

It is clear that $w \in AC'([0,\omega])$,

$$w(t) > 0, \quad w(t+\omega) = w(t) \quad \text{for } \ t \in \mathbb{R}. \tag{7.6}$$

On the other hand, one can easily verify that

$$w''(t) = p(t)w(t) + \frac{1}{w^3(t)} \quad \text{for } \ t \in \mathbb{R}. \tag{7.7}$$

Hence, w is a solution of the problem (7.4).

Let now the problem (7.4) possess a solution w. Extend the function w periodically and denote it by the same letter. Clearly, (7.6) and (7.7) are fulfilled. It is also evident that there is a $t_0 \in [0,\omega[$ such that

$$w'(t_0) = 0.$$

Denote by u and v solutions of the problems

$$\begin{aligned} u'' &= p(t)u; \quad u(t_0) = 0, \ \ u'(t_0) = \frac{1}{w(t_0)}\,, \\ v'' &= p(t)v; \quad v(t_0) = w(t_0), \ \ v'(t_0) = 0 \end{aligned}$$

and put

$$w_0(t) \stackrel{\text{def}}{=} \sqrt{u^2(t) + v^2(t)} \quad \text{for } \ t \in \mathbb{R}. \tag{7.8}$$

It is clear that (7.5) holds. One can easily verify that

$$\begin{gathered} w_0''(t) = p(t)w_0(t) + \frac{1}{w_0^3(t)} \quad \text{for } \ t \in [0,\omega], \\ w_0(t_0) = w(t_0), \quad w_0'(t_0) = 0. \end{gathered} \tag{7.9}$$

Let now

$$\alpha(t) \stackrel{\text{def}}{=} w(t) - w_0(t), \quad \widetilde{p}(t) \stackrel{\text{def}}{=} p(t) - \frac{w^2(t) + w_0(t)w(t) + w_0^2(t)}{\big(w(t)w_0(t)\big)^3} \quad \text{for } \ t \in \mathbb{R}.$$

It follows from (7.7) and (7.9) that the function α is a solution of the initial value problem

$$\alpha'' = \widetilde{p}(t)\alpha; \quad \alpha(t_0) = 0, \ \ \alpha'(t_0) = 0.$$

Hence, $\alpha \equiv 0$ and, consequently, $w_0(t) = w(t)$ for $t \in \mathbb{R}$. Therefore, in view of (7.6), the function w_0 is bounded. Taking now into account (7.8) we get that any solution of the equation (7.1) is bounded and thus the equation (7.1) is stable. $\square$

Chapter 2

Theorems on Differential Inequalities

8. On the Set $\mathcal{V}^-(\omega)$

Theorem 8.1. $\mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega) = \mathcal{D}$.

Proof. Show that $\mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega) \subseteq \mathcal{D}$. In view of Proposition 0.8, we have $\mathcal{V}_0(\omega) \subseteq \mathcal{D}$. Thus, it is sufficient to show that $\mathcal{V}^-(\omega) \subseteq \mathcal{D}$. Let $p \in \mathcal{V}^-(\omega)$. By virtue of Remark 0.5, the problem

$$u'' = p(t)u - |p(t)| - 1; \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega) \tag{8.1}$$

has a unique solution u and

$$u(t) \geq 0 \quad \text{for } \; t \in \mathbb{R}. \tag{8.2}$$

By direct calculation one can easily verify that the function β defined by

$$\beta(t) = 1 + u(t - k\omega) \quad \text{for } \; t \in [(k-1)\omega, k\omega], \;\; k \in \mathbb{N}$$

satisfies assumptions of Lemma 1.2 and, therefore, $p \in \mathcal{D}$.

Now we will show that $\mathcal{D} \subseteq \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$. Let $p \in \mathcal{D}$. Suppose first that the problem

$$u'' = p(t)u; \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega) \tag{8.3}$$

has a nontrivial solution u. Since $p \in \mathcal{D}$, the function u is of a constant sign and thus $p \in \mathcal{V}_0(\omega)$.

Now suppose that the problem (8.3) has no nontrivial solution. We will show that $p \in \mathcal{V}^-(\omega)$ in this case. Assume the contrary, let $p \notin \mathcal{V}^-(\omega)$. Then there is a $q \in L_\omega$ such that

$$q(t) \geq 0 \quad \text{for } \; t \in \mathbb{R}, \;\; q \not\equiv 0, \tag{8.4}$$

and the solution u of the problem

$$u'' = p(t)u + q(t); \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega)$$

does not satisfy the inequality $u(t) \leq 0$ for some $t \in \mathbb{R}$. Then, by virtue of Proposition 0.8 and Lemma 2.7 we get

$$u(t) > 0 \quad \text{for } \; t \in \mathbb{R}. \tag{8.5}$$

Let v be a solution of the initial value problem

$$v'' = p(t)v; \quad v(0) = 0, \;\; v'(0) = 1.$$

Since $p \in \mathcal{D}$, we have

$$v(t) > 0 \quad \text{for } \; t > 0. \tag{8.6}$$

Therefore,

$$\big(u'(t)v(t) - u(t)v'(t)\big)' = q(t)v(t) \geq 0 \quad \text{for } \; t \geq 0. \tag{8.7}$$

However $u'(0)v(0) - u(0)v'(0) = -u(0) < 0$. Hence, we get from (8.7) that either

$$u'(t)v(t) - u(t)v'(t) < 0 \quad \text{for } \; t > 0 \tag{8.8}$$

or there is a $a > 0$ such that

$$u'(a)v(a) - u(a)v'(a) = 0. \tag{8.9}$$

First assume that (8.8) is fulfilled. Then, in view of (8.6), we get

$$\left(\frac{u(t)}{v(t)}\right)' < 0 \quad \text{for } \; t > 0.$$

Therefore, there are $c > 0$ and $t_0 > 0$ such that

$$v(t) > cu(t) \quad \text{for } \ t \geq t_0.$$

Hence, it follows from (8.7) that

$$u'(t)v(t) - u(t)v'(t) = \delta + \int_{t_0}^{t} q(s)v(s)\,\mathrm{d}s \geq \delta + c\int_{t_0}^{t} q(s)u(s)\,\mathrm{d}s \quad \text{for } \ t \geq t_0, \tag{8.10}$$

where

$$\delta = -u(0) + \int_{0}^{t_0} q(s)v(s)\,\mathrm{d}s.$$

On the other hand, on account of (8.4) and (8.5), we have

$$\int_{t_0}^{+\infty} q(s)u(s)\,\mathrm{d}s = +\infty$$

which, together with (8.10), contradicts (8.8).

Now assume that (8.9) holds for a certain $a > 0$. Then, in view of (8.5) and (8.6), there is a $\lambda > 0$ such that

$$v(a) = \lambda u(a), \quad v'(a) = \lambda u'(a).$$

However, conditions (8.4), (8.5) and Lemma 1.3 imply that the function $\frac{1}{\lambda}\,v$ does not preserve its sign in $[a, +\infty[$, which contradicts (8.6). □

Remark 8.2. It follows from Theorem 8.1, Proposition 0.8, Lemma 2.7, and Remark 0.5 that if $p \in \mathcal{V}^-(\omega)$, $q \in L_\omega$, $q(t) \geq 0$ for $t \in \mathbb{R}$, and $q \not\equiv 0$, then the (unique) solution u of the problem

$$u'' = p(t)u + q(t); \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega)$$

satisfies $u(t) < 0$ for $t \in \mathbb{R}$.

Theorem 8.3. *Let $p \in L_\omega$. Then the inclusion $p \in \mathcal{V}^-(\omega)$ holds if and only if there exists a function $\gamma \in AC'([0,\omega])$ satisfying*

$$\gamma''(t) \leq p(t)\gamma(t) \quad \textit{for } \ t \in [0,\omega], \tag{8.11}$$

$$\gamma(t) > 0 \quad \textit{for } \ t \in [0,\omega], \tag{8.12}$$

$$\gamma(0) \geq \gamma(\omega), \quad \frac{\gamma'(\omega)}{\gamma(\omega)} \geq \frac{\gamma'(0)}{\gamma(0)}\,, \tag{8.13}$$

and

$$\gamma(0) - \gamma(\omega) + \frac{\gamma'(\omega)}{\gamma(\omega)} - \frac{\gamma'(0)}{\gamma(0)} + \mathrm{mes}\big\{t \in [0,\omega] : \; \gamma''(t) < p(t)\gamma(t)\big\} > 0. \tag{8.14}$$

Proof. Let $p \in \mathcal{V}^-(\omega)$. Then, on account of Remark 0.5, the problem (8.1) has a unique solution u and (8.2) is fulfilled. Evidently, the function $\gamma(t) \overset{\text{def}}{=} 1 + u(t)$ for $t \in [0,\omega]$ satisfies (8.11)–(8.14).

Suppose now that there is a $\gamma \in AC'([0,\omega])$ satisfying (8.11)–(8.14). Introduce the function β by

$$\beta(t) \overset{\text{def}}{=} \Big(\frac{\gamma(\omega)}{\gamma(0)}\Big)^{k-1}\gamma(t - (k-1)\omega) \quad \text{for } \ t \in [(k-1)\omega, k\omega[\,, \;\; k \in \mathbb{N}.$$

In view (8.13), it is clear that $\beta \in \widetilde{AC}'(\mathbb{R}_+)$. Moreover, by virtue of (8.11) and (8.12), the function β satisfies assumptions of Lemma 1.2 and therefore

$$p \in \mathcal{D}. \tag{8.15}$$

Consequently, by virtue of Theorem 8.1, we have $p \in \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$. Let us show that $p \notin \mathcal{V}_0(\omega)$. Assume the contrary, let $p \in \mathcal{V}_0(\omega)$. Then, there is a $u \in AC'([0,\omega])$ satisfying

$$u''(t) = p(t)u(t) \quad \text{for } t \in [0,\omega], \tag{8.16}$$
$$u(0) = u(\omega), \quad u'(0) = u'(\omega), \tag{8.17}$$
$$u(0) = \beta(0), \tag{8.18}$$
$$u(t) > 0 \quad \text{for } t \in [0,\omega]. \tag{8.19}$$

By virtue of Lemma 1.2, there is a function $v \in AC'(\mathbb{R}_+)$ satisfying

$$v''(t) = p(t)v(t) \quad \text{for } t \in [0,\omega] \tag{8.20}$$

and (1.2). In view of (1.2), (8.15), (8.18), and (8.19), it follows from Lemma 1.3 (with $a = 0$ and $q \equiv 0$) that

$$v'(0) \geq u'(0). \tag{8.21}$$

Relations (1.2), together with the first inequality in (8.13), imply that $v(\omega) \leq v(0)$. Therefore, either

$$v(\omega) < v(0), \tag{8.22}$$

or

$$v(\omega) = v(0). \tag{8.23}$$

Assume first that (8.22) holds. Put $w(t) = v(t) - u(t)$ for $t \geq 0$. Clearly, the function w is a solution of the equation $w'' = p(t)w$. On the other hand, by virtue of (1.2), (8.18), and (8.21), we have $w(0) = 0$ and $w'(0) \geq 0$. Taking , moreover, into account (8.15) we get

$$w(t) \geq 0 \quad \text{for } t \geq 0. \tag{8.24}$$

However, in view of (1.2), (8.17), (8.18) and (8.22) we have $w(\omega) < 0$, which contradicts (8.24).

Now assume that (8.23) is fulfilled. Then, it follows from (1.2) and the first inequality in (8.13) that

$$\gamma(0) = \gamma(\omega), \tag{8.25}$$
$$v'(\omega) \geq \gamma'(\omega), \quad v'(\omega) \leq \gamma'(0).$$

Taking, moreover, into account the second inequality in (8.13) we get

$$\gamma'(0) = \gamma'(\omega). \tag{8.26}$$

By virtue of (8.11), (8.14), (8.25), and (8.26), we have

$$\gamma''(t) = p(t)\gamma(t) - q(t) \quad \text{for } t \in [0,\omega], \tag{8.27}$$

where

$$q(t) \overset{\text{def}}{=} p(t)\gamma(t) - \gamma''(t) \quad \text{for } t \in [0,\omega],$$
$$q(t) \geq 0 \quad \text{for } t \in [0,\omega], \ \ q \not\equiv 0. \tag{8.28}$$

In view of (8.16) and (8.27), we have

$$\big(u'(t)\gamma(t) - u(t)\gamma'(t)\big)' = q(t)u(t) \geq 0 \quad \text{for } t \geq 0.$$

Hence, on account of (8.17), (8.25), and (8.26), we get

$$\int_0^\omega q(s)u(s)\,\mathrm{d}s = 0.$$

However, the latter equality contradicts (8.19) and (8.28). □

Remark 8.4. Theorem 8.3 (with $\gamma \equiv 1$) implies, in particular, that if $p(t) \geq 0$ for $t \in [0,\omega]$ and $p \not\equiv 0$ then $p \in \mathcal{V}^-(\omega)$. More general, if $p_0 \in \mathcal{V}_0(\omega)$, $p(t) \geq p_0(t)$ for $t \in [0,\omega]$, and $p \not\equiv p_0$ then $p \in \mathcal{V}^-(\omega)$.

Remark 8.5. It follows from Theorem 8.3 that if $p_0 \in \mathcal{V}^-(\omega)$ and $p(t) \geq p_0(t)$ for $t \in [0,\omega]$ then $p \in \mathcal{V}^-(\omega)$ as well.

Remark 8.6. Let $p \in L_\omega$, $a \in]0, \omega[$, and $p_a(t) \stackrel{\text{def}}{=} p(t+a)$ for $t \in \mathbb{R}$. Then the inclusion $p_a \in \mathcal{V}^-(\omega)$ implies the inclusion $p \in \mathcal{V}^-(\omega)$. Indeed, let $p_a \in \mathcal{V}^-(\omega)$. Then, in view of Remark 8.2, the problem

$$u'' = p_a(t)u - 1; \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega)$$

has a unique solution u and $u(t) > 0$ for $t \in \mathbb{R}$. Clearly, the function

$$\gamma(t) \stackrel{\text{def}}{=} u(t-a) \quad \text{for } \; t \in [0, \omega]$$

satisfies the assumptions of Theorem 8.3 and thus $p \in \mathcal{V}^-(\omega)$ as well.

9. On the et $\mathcal{V}^+(\omega)$

Theorem 9.1. *Let $p \in L_\omega$. Then the inclusion $p \in \mathcal{V}^+(\omega)$ holds if and only if $p \in \mathcal{D}(\omega)$ and there exists a function $\gamma \in AC'([0, \omega])$ satisfying*

$$\gamma''(t) \geq p(t)\gamma(t) \quad \textit{for } \; t \in [0, \omega], \tag{9.1}$$

$$\gamma(t) > 0 \quad \textit{for } \; t \in [0, \omega], \tag{9.2}$$

$$\gamma(0) = \gamma(\omega), \quad \gamma'(0) \geq \gamma'(\omega), \tag{9.3}$$

and

$$\gamma'(0) - \gamma'(\omega) + \operatorname{mes}\big\{t \in [0, \omega] : \; \gamma''(t) > p(t)\gamma(t)\big\} > 0. \tag{9.4}$$

Proof. Let $p \in \mathcal{V}^+(\omega)$. Then, in view of Remark 0.5, the problem

$$u'' = p(t)u + |p(t)| + 1; \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega)$$

has a unique solution u_0 and $u_0(t) \geq 0$ for $t \in \mathbb{R}$. By direct calculations one can easily verify that the function $\gamma(t) \stackrel{\text{def}}{=} 1 + u_0(t)$ for $t \in [0, \omega]$ satisfies (9.1)–(9.4).

Now we will show that $p \in \mathcal{D}(\omega)$. Suppose the contrary, let $p \notin \mathcal{D}(\omega)$. Then there are $\alpha < \beta$, $\beta - \alpha < \omega$, and a solution v of the equation

$$v'' = p(t)v$$

such that

$$v(t) > 0 \quad \text{for } \; t \in]\alpha, \beta[, \quad v(\alpha) = 0, \;\; v(\beta) = 0.$$

Clearly,

$$v'(\alpha) > 0, \quad v'(\beta) < 0, \tag{9.5}$$

and there is a $\beta_0 \in]\beta, \alpha + \omega[$ such that

$$v(t) < 0 \quad \text{for } \; t \in]\beta, \beta_0]. \tag{9.6}$$

Put

$$q(t) = \begin{cases} 0 & \text{for } \; t \in [\alpha, \beta] \cup]\beta_0, \alpha + \omega], \\ 1 & \text{for } \; t \in]\beta, \beta_0] \end{cases}$$

and extend it ω-periodically. Since $p \in \mathcal{V}^+(\omega)$, the problem

$$u'' = p(t)u + q(t); \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega)$$

has a unique solution u and

$$u(t) \geq 0 \quad \text{for } \; t \in \mathbb{R}. \tag{9.7}$$

Let

$$w(t) \stackrel{\text{def}}{=} u'(t)v(t) - u(t)v'(t) \quad \text{for } \; t \in \mathbb{R}. \tag{9.8}$$

It is clear that,

$$w'(t) = q(t)v(t) \leq 0 \quad \text{for } \; t \in [\alpha, \alpha + \omega], \tag{9.9}$$

$$w' \not\equiv 0 \quad \text{on } [\alpha, \alpha + \omega]. \tag{9.10}$$

However,

$$w'(t) = 0 \quad \text{for } \; t \in [\alpha, \beta]$$

and
$$w(\alpha) = -v'(\alpha)u(\alpha), \quad w(\beta) = -v'(\beta)u(\beta).$$
Taking, together with this, into account (9.5) and (9.7), we get $u(\alpha) = 0$ and $u'(\alpha) = 0$. Since u is an ω-periodic function, we have $u(\alpha+\omega) = 0$ and $u'(\alpha+\omega) = 0$, as well. Consequently, in view of (9.8), we get $w(\alpha) = 0$ and $w(\alpha+\omega) = 0$, which contradicts (9.9) and (9.10). Therefore, $p \in \mathcal{D}(\omega)$.

Now let $p \in \mathcal{D}(\omega)$ and there is a function $\gamma \in \widetilde{AC}'([0,\omega])$ satisfying (9.1)–(9.4). We will show that $p \in \mathcal{V}^+(\omega)$. Suppose the contrary, let $p \notin \mathcal{V}^+(\omega)$. Then there are $u \in \widetilde{AC}'(\mathbb{R})$ and $q \in L_\omega$ such that
$$u''(t) = p(t)u(t) + q(t) \quad \text{for } t \in \mathbb{R}, \tag{9.11}$$
$$u(0) = u(\omega), \quad u'(0) = u'(\omega), \tag{9.12}$$
$$q(t) \geq 0 \quad \text{for } t \in \mathbb{R}, \tag{9.13}$$
and the inequality $u(t) \geq 0$ does not hold for some $t \in \mathbb{R}$ (consequently, $u \not\equiv 0$). Then, in view of Lemma 2.7, we get
$$u(t) \leq 0 \quad \text{for } t \in \mathbb{R}, \ u \not\equiv 0. \tag{9.14}$$
By virtue of (9.11) and (9.14) it is clear that
$$\big(u'(t)\gamma(t) - u(t)\gamma'(t)\big)' = q(t)\gamma(t) + |u(t)|\big(\gamma''(t) - p(t)\gamma(t)\big) \quad \text{for } t \in [0,\omega].$$
The integration of the latter equality on $[0,\omega]$, together with (9.3), (9.12), and (9.14), implies
$$|u(0)|\big(\gamma'(\omega) - \gamma'(0)\big) = \int_0^\omega \Big[q(t)\gamma(t) + |u(t)|\big(\gamma''(t) - p(t)\gamma(t)\big)\Big]\,dt.$$
Hence, in view of (9.1), (9.2), (9.13), and the second inequality in (9.3), we get
$$\int_0^\omega \Big[q(t)\gamma(t) + |u(t)|\big(\gamma''(t) - p(t)\gamma(t)\big)\Big]\,dt = 0.$$
Consequently,
$$q(t) = 0 \quad \text{for } t \in [0,\omega], \tag{9.15}$$
$$|u(t)|\big(\gamma''(t) - p(t)\gamma(t)\big) = 0 \quad \text{for } t \in [0,\omega], \tag{9.16}$$
$$|u(0)|\big(\gamma'(\omega) - \gamma'(0)\big) = 0. \tag{9.17}$$
However, (9.11), (9.14), and (9.15) yield that $u(t) < 0$ for $t \in [0,\omega]$. Hence, it follows from (9.16) and (9.17) that
$$\gamma''(t) = p(t)\gamma(t) \quad \text{for } t \in [0,\omega], \quad \gamma'(\omega) = \gamma'(0),$$
which contradicts (9.4). □

Remark 9.2. It follows from Theorem 9.1, Lemma 2.7, and Remark 0.5 that if $p \in \mathcal{V}^+(\omega)$, $q \in L_\omega$, $q(t) \geq 0$ for $t \in \mathbb{R}$, and $q \not\equiv 0$, then the (unique) solution u of the problem
$$u'' = p(t)u + q(t); \quad u(0) = u(\omega), \ u'(0) = u'(\omega)$$
satisfies $u(t) > 0$ for $t \in \mathbb{R}$.

Theorem 9.1′**.** *Let $p \in L_\omega$. Then the inclusion $p \in \operatorname{Int}\mathcal{V}^+(\omega)$ holds if and only if $p \in \operatorname{Int}\mathcal{D}(\omega)$ and there exists $\gamma \in \widetilde{AC}'([0,\omega])$ satisfying* (9.1)–(9.4).

Proof. Let $p \in \operatorname{Int}\mathcal{V}^+(\omega)$. Then, in view of Theorem 9.1, there is a function $\gamma \in \widetilde{AC}'([0,\omega])$ satisfying (9.1)–(9.4). On the other hand, there is an $\varepsilon_0 > 0$ such that $B(p,\varepsilon_0) \subset \mathcal{V}^+(\omega)$. Hence, by virtue of Theorem 9.1, we get $B(p,\varepsilon_0) \subset \mathcal{D}(\omega)$ and therefore $p \in \operatorname{Int}\mathcal{D}(\omega)$.

Let now $p \in \operatorname{Int}\mathcal{D}(\omega)$ and there is a function $\gamma \in \widetilde{AC}'([0,\omega])$ satisfying (9.1)–(9.4). Then, by virtue of Theorem 9.1, $p \in \mathcal{V}^+(\omega)$. Consequently, there is a (unique) solution u of the problem
$$u'' = p(t)u + |p(t)| + 1; \quad u(0) = u(\omega), \ u'(0) = u'(\omega),$$
and moreover (see Remark 0.6)
$$u(t) \geq 1 \quad \text{for } t \in \mathbb{R}. \tag{9.18}$$

On account of Proposition 3.2, there is an $\delta_1 > 0$ such that, for any $g \in B(p, \delta_1)$, the problem

$$v'' = g(t)v + |g(t)| + 1; \quad v(0) = v(\omega), \ \ v'(0) = v'(\omega) \tag{9.19}$$

has a unique solution v and $|u(t) - v(t)| < \frac{1}{2}$ for $t \in \mathbb{R}$. Hence, in view of (9.18), we get

$$v(t) \geq \frac{1}{2} \quad \text{for} \ \ t \in \mathbb{R}. \tag{9.20}$$

On the other hand, since $p \in \operatorname{Int} \mathcal{D}(\omega)$, there is an $\delta_2 > 0$ such that

$$B(p, \delta_2) \subset \mathcal{D}(\omega). \tag{9.21}$$

Let now $\delta = \min\{\delta_1, \delta_2\}$. Then, in view of (9.19)–(9.21), it follows from Theorem 9.1 that $B(p, \delta) \subset \mathcal{V}^+(\omega)$. Consequently, $p \in \operatorname{Int} \mathcal{V}^+(\omega)$. □

Theorem 9.3. *Let $p \in L_\omega$ be such that*

$$p \not\equiv 0, \quad \int_0^\omega p(s)\,\mathrm{d}s \leq 0. \tag{9.22}$$

Then $p \in \mathcal{V}^+(\omega)$ $(p \in \operatorname{Int} \mathcal{V}^+(\omega))$ if and only if $p \in \mathcal{D}(\omega)$ $(p \in \operatorname{Int} \mathcal{D}(\omega))$.

Proof. By virtue of Theorem 9.1 (Theorem 9.1′), we have $\mathcal{V}^+(\omega) \subset \mathcal{D}(\omega)$ $(\operatorname{Int} \mathcal{V}^+(\omega) \subset \operatorname{Int} \mathcal{D}(\omega))$. Thus it is sufficient to prove that the conditions (9.22) and the inclusion $p \in \mathcal{D}(\omega)$ $(p \in \operatorname{Int} \mathcal{D}(\omega))$ imply $p \in \mathcal{V}^+(\omega)$ $(p \in \operatorname{Int} \mathcal{V}^+(\omega))$.

Let

$$\rho(t) \stackrel{\text{def}}{=} -\frac{1}{\omega} \int_t^{t+\omega} \int_t^s \big(p(\xi) - \overline{p}\big)\,\mathrm{d}\xi\,\mathrm{d}s \quad \text{for} \ \ t \in \mathbb{R},$$

where $\overline{p}$ is defined by (0.11). It is clear that

$$\rho'(t) = p(t) - \overline{p} \quad \text{for} \ \ t \in \mathbb{R}, \tag{9.23}$$

$$\rho(t) = \rho(0) + \int_0^t \big(p(s) - \overline{p}\big)\,\mathrm{d}s \quad \text{for} \ \ t \in \mathbb{R}. \tag{9.24}$$

In, particular,

$$\rho(0) = \rho(\omega). \tag{9.25}$$

The integration of (9.24) over $[0, \omega]$ yields

$$\int_0^\omega \rho(s)\,\mathrm{d}s = \omega\rho(0) + \int_0^\omega \int_0^s \big(p(\xi) - \overline{p}\big)\,\mathrm{d}\xi\,\mathrm{d}s = 0. \tag{9.26}$$

Mention also that either $\overline{p} < 0$ or $\overline{p} = 0$. If $\overline{p} = 0$ holds then, in view of the condition $p \not\equiv 0$, we get from (9.23) that $\rho \not\equiv 0$. Thus in both cases

$$\rho^2(t) \geq \overline{p} \quad \text{for} \ \ t \in \mathbb{R}, \quad \operatorname{mes}\big\{t \in [0, \omega] : \ \rho^2(t) > \overline{p}\big\} > 0. \tag{9.27}$$

Now, let,

$$\gamma(t) \stackrel{\text{def}}{=} \exp\bigg(\int_0^t \rho(s)\,\mathrm{d}s\bigg) \quad \text{for} \ \ t \in [0, \omega].$$

On account of (9.23) and (9.25)–(9.27), one can easily verify that (9.1)–(9.4) are fulfilled. Taking, moreover, into account assumption $p \in \mathcal{D}(\omega)$ $(p \in \operatorname{Int} \mathcal{D}(\omega))$ we get from Theorem 9.1 (Theorem 9.1′) that $p \in \mathcal{V}^+(\omega)$ $(p \in \operatorname{Int} \mathcal{V}^+(\omega))$. □

10. Properties of the Sets $\mathcal{V}^-(\omega)$ and $\mathcal{V}^+(\omega)$

Proposition 10.1. *The set $\mathcal{V}^-(\omega)$ is unbounded, open, and convex.*

Proof. Unboundedness of $\mathcal{V}^-(\omega)$ follows from Remark 8.4. Show that the set $\mathcal{V}^-(\omega)$ is open. Let $p \in \mathcal{V}^-(\omega)$. Then, in view of Remark 0.6, there is a unique solution u of the problem

$$u'' = p(t)u - |p(t)| - 1; \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega),$$

and moreover

$$u(t) \geq 1 \quad \text{for } \ t \in \mathbb{R}. \tag{10.1}$$

On account of Proposition 3.2, there is a $\delta > 0$ such that, for any $g \in B(p, \delta)$, the problem

$$v'' = g(t)v - |g(t)| - 1; \quad v(0) = v(\omega), \ \ v'(0) = v'(\omega)$$

has a unique solution v and $|u(t) - v(t)| < \frac{1}{2}$ for $t \in \mathbb{R}$. Hence, in view of (10.1), we get $v(t) \geq \frac{1}{2}$ for $t \in \mathbb{R}$. and therefore, by virtue of Theorem 8.3, we have $B(p, \delta) \subset \mathcal{V}^-(\omega)$.

Now we will show that the set $\mathcal{V}^-(\omega)$ is convex. Let $p_0, p_1 \in \mathcal{V}^-(\omega)$. In view of Remark 0.6, the problems

$$u_i'' = p_i(t)u_i - |p_i(t)| - 1; \quad u_i(0) = u_i(\omega), \ \ u_i'(0) = u_i'(\omega), \ \ i = 0, 1$$

possess unique solutions u_0 and u_1 respectively and, moreover, $u_i(t) \geq 1$ for $t \in \mathbb{R}$, $i = 0, 1$. Introduce the notations

$$\rho_i(t) \overset{\text{def}}{=} \frac{u_i'(t)}{u_i(t)}\,, \quad h_i(t) \overset{\text{def}}{=} \frac{1}{u_i(t)}\big(1 + |p_i(t)|\big) \quad \text{for } \ t \in \mathbb{R}, \ \ i = 0, 1.$$

It is clear that,

$$\rho_i'(t) = p_i(t) - h_i(t) - \rho_i^2(t) \quad \text{for } \ t \in \mathbb{R}, \ \ i = 0, 1, \tag{10.2}$$

$$\rho_i(0) = \rho_i(\omega), \qquad \int\limits_0^\omega \rho_i(s)\,\mathrm{d}s = 0, \ \ i = 0, 1. \tag{10.3}$$

Let now $\lambda \in [0, 1]$ and $\rho(t) \overset{\text{def}}{=} (1 - \lambda)\rho_0(t) + \lambda\rho_1(t)$ for $t \in \mathbb{R}$. Then, in view of (10.2) we get

$$\rho'(t) = (1 - \lambda)p_0(t) + \lambda p_1(t) - \big[(1{-}\lambda)h_0(t){+}\lambda h_1(t)\big] - \big[(1{-}\lambda)\rho_0^2(t){+}\lambda\rho_1^2(t)\big] \quad \text{for } \ t{\in}\mathbb{R}. \tag{10.4}$$

However, $(1 - \lambda)x^2 + \lambda y^2 \geq ((1 - \lambda)x + \lambda y)^2$ for $x, y \in \mathbb{R}$. Hence, it follows from (10.4) that

$$\rho'(t) \leq (1 - \lambda)p_0(t) + \lambda p_1(t) - \rho^2(t) \quad \text{for } \ t \in \mathbb{R} \tag{10.5}$$

and

$$\operatorname{mes}\Big\{t \in [0, \omega] : \ \rho'(t) < (1 - \lambda)p_0(t) + \lambda p_1(t) - \rho^2(t)\Big\} > 0.$$

Set $\gamma(t) \overset{\text{def}}{=} \exp(\int\limits_0^t \rho(s)\,\mathrm{d}s)$ for $t \in [0, \omega]$. In view of (10.3) and (10.5), one can easily verify that the function γ satisfies (8.11)–(8.14) with $p(t) \overset{\text{def}}{=} (1 - \lambda)p_0(t) + \lambda p_1(t)$ and therefore, by virtue of Theorem 8.3, we get $(1 - \lambda)p_0 + \lambda p_1 \in \mathcal{V}^-(\omega)$. □

Proposition 10.2. $\overline{\mathcal{V}^-(\omega)} = \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$ *and* $\overline{\mathcal{V}^+(\omega)} = \mathcal{V}^+(\omega) \cup \mathcal{V}_0(\omega)$.

Proof. Let $p \in \overline{\mathcal{V}^-(\omega)}$ ($p \in \overline{\mathcal{V}^+(\omega)}$). Then there is a sequence $\{p_n\}_{n=1}^{+\infty} \subset \mathcal{V}^-(\omega)$ ($\{p_n\}_{n=1}^{+\infty} \subset \mathcal{V}^+(\omega)$) such that

$$\lim_{n \to +\infty} \|p_n - p\|_L = 0. \tag{10.6}$$

By virtue of Remark 0.6, for any $n \in \mathbb{N}$, there is a unique solution v_n of the problem

$$v'' = p_n(t)v - |p_n(t)| - 1; \quad v(0) = v(\omega), \ \ v'(0) = v'(\omega)$$

$$\big(v'' = p_n(t)v + |p_n(t)| + 1; \quad v(0) = v(\omega), \ \ v'(0) = v'(\omega)\big)$$

and, moreover,

$$v_n(t) \geq 1 \quad \text{for } \ t \in \mathbb{R}. \tag{10.7}$$

Without loss of generality we can assume that there exists a finite or infinite limit

$$\lim_{n \to +\infty} \|v_n\|_C = \lambda.$$

In view of (10.7) clearly either $1 \le \lambda < +\infty$ or $\lambda = +\infty$. Introduce the notations

$$u_n(t) \stackrel{\text{def}}{=} \frac{v_n(t)}{\|v_n\|_C}, \quad q_n(t) \stackrel{\text{def}}{=} \frac{1}{\|v_n\|_C}\big(1 + |p_n(t)|\big),$$

and

$$q(t) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } \lambda = +\infty, \\ \dfrac{1}{\lambda}\big(1 + |p(t)|\big) & \text{if } \lambda < +\infty. \end{cases}$$

On account of (10.6), clearly

$$\lim_{n\to+\infty} \|q_n - q\|_L = 0.$$

By virtue of Proposition 3.3, we can assume without loss of generality that

$$\lim_{n\to+\infty} u_n^{(i)}(t) = u^{(i)}(t) \quad \text{uniformly on } [0,\omega], \ \ i = 0, 1,$$

where u is a solution of the problem

$$\begin{gathered} u'' = p(t)u - q(t); \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega) \\ \big(u'' = p(t)u + q(t); \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega)\big). \end{gathered} \tag{10.8}$$

Moreover, it is clear that

$$u(t) \ge 0 \quad \text{for } t \in [0,\omega], \ \ \|u\|_C = 1. \tag{10.9}$$

If $\lambda = +\infty$ then (by definition) $q \equiv 0$ and it follows from (10.8) and (10.9) that $u(t) > 0$ for $t \in [0,\omega]$. Hence, in this case $p \in \mathcal{V}_0(\omega)$.

Let now $\lambda < +\infty$. By direct calculation one can easily verify that the function $\gamma(t) \stackrel{\text{def}}{=} u(t) + \frac{1}{\lambda}$ for $t \in [0,\omega]$ satisfies (8.11)–(8.14) ((9.1)–(9.4)). Then, in view of Theorem 8.3, we get $p \in \mathcal{V}^-(\omega)$ (by virtue of Theorem 9.1 we have $p_n \in \mathcal{D}(\omega)$. Taking, moreover, into account (10.6) and Proposition 2.1, we get $p \in \mathcal{D}(\omega)$. Hence, on account of Theorem 9.1, we get $p \in \mathcal{V}^+(\omega)$). Thus we have proved that $\overline{\mathcal{V}^-(\omega)} \subseteq \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$ ($\overline{\mathcal{V}^+(\omega)} \subseteq \mathcal{V}^+(\omega) \cup \mathcal{V}_0(\omega)$).

To complete the proof it is sufficient to show that $\mathcal{V}_0(\omega) \subseteq \overline{\mathcal{V}^-(\omega)}$ ($\mathcal{V}_0(\omega) \subseteq \overline{\mathcal{V}^+(\omega)}$). Let $p \in \mathcal{V}_0(\omega)$. Then there is a function $\gamma \in \widetilde{AC}'([0,\omega])$ satisfying

$$\begin{gathered} \gamma(t) > 0, \quad \gamma''(t) = p(t)\gamma(t) \quad \text{for } t \in [0,\omega]; \\ \gamma(0) = \gamma(\omega), \ \ \gamma'(0) = \gamma'(\omega). \end{gathered} \tag{10.10}$$

Introduce the notation $p_n(t) \stackrel{\text{def}}{=} p(t) + \frac{1}{n}$ ($p_n(t) \stackrel{\text{def}}{=} p(t) - \frac{1}{n}$). In view of Theorem 8.3 and (10.10), we get $p_n \in \mathcal{V}^-(\omega)$ (by virtue of (10.10) and Sturm's comparison theorem we get $p_n \in \mathcal{D}$. However, $\mathcal{D} \subset \mathcal{D}(\omega)$ and thus $p_n \in \mathcal{D}(\omega)$. Taking moreover into account (10.10), we get, by virtue of Theorem 9.1, that $p_n \in \mathcal{V}^+(\omega)$).On the other hand, clearly (10.6) holds and therefore $p \in \overline{\mathcal{V}^-(\omega)}$ ($p \in \overline{\mathcal{V}^+(\omega)}$). ☐

Remark 10.3. It follows from Propositions 10.1 and 10.2 that $\partial\mathcal{V}^-(\omega) = \mathcal{V}_0(\omega)$.

Proposition 10.4. *$\partial\mathcal{V}^+(\omega) = \partial\mathcal{D}(\omega) \cup \mathcal{V}_0(\omega)$ and $\partial\mathcal{D}(\omega) \subset \mathcal{V}^+(\omega)$.*

Proof. It is clear that $\partial\mathcal{V}^+(\omega) = \overline{\mathcal{V}^+(\omega)} \setminus \operatorname{Int}\mathcal{V}^+(\omega)$. Taking into account Proposition 10.2 and the fact that $\mathcal{V}^+(\omega) \cap \mathcal{V}_0(\omega) = \varnothing$, we get

$$\partial\mathcal{V}^+(\omega) = \big(\mathcal{V}^+(\omega) \setminus \operatorname{Int}\mathcal{V}^+(\omega)\big) \cup \mathcal{V}_0(\omega).$$

By virtue of Theorems 9.1 and 9.1′, the inclusion $p \in \mathcal{V}^+(\omega) \setminus \operatorname{Int}\mathcal{V}^+(\omega)$ holds if and only if there is a function $\gamma \in \widetilde{AC}'([0,\omega])$ satisfying (9.1)–(9.4) and the inclusion

$$p \in \mathcal{D}(\omega) \setminus \operatorname{Int}\mathcal{D}(\omega) \tag{10.11}$$

is fulfilled. On account of Proposition 2.1, the inclusion (10.11) is equivalent with

$$p \in \partial\mathcal{D}(\omega). \tag{10.12}$$

Therefore, to complete the proof it is sufficient to show that if (10.12) is fulfilled then there exists $\gamma \in \widetilde{AC}'([0,\omega])$ satisfying (9.1)–(9.4). Let (10.12) holds. Then, by virtue of Proposition 2.4 and Fredholm's alternative, the problem

$$\gamma'' = p(t)\gamma + |p(t)| + 1; \quad \gamma(0) = \gamma(\omega), \ \ \gamma'(0) = \gamma'(\omega)$$

has a unique solution γ. In view of (10.12) and Proposition 2.1, we have $p \in \mathcal{D}(\omega)$. Hence, by virtue of Lemma 2.8, we get $\gamma(t) > 0$ for $t \in [0,\omega]$. Now, it is clear that, the function γ satisfies (9.1)–(9.4). $\square$

The next proposition immediately follows from the previous one.

Proposition 10.5. $\mathcal{V}^+(\omega) = \partial\mathcal{D}(\omega) \cup \operatorname{Int}\mathcal{V}^+(\omega)$.

Proposition 10.6. $\mathcal{V}^+(\omega) \cup \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega) = \mathcal{D}(\omega)$.

Proof. In view of Theorem 9.1, we have $\mathcal{V}^+(\omega) \subset \mathcal{D}(\omega)$ while, by virtue of Theorem 8.1 (and Proposition 0.8), $\mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega) = \mathcal{D} \subset \mathcal{D}(\omega)$. Hence, $\mathcal{V}^+(\omega) \cup \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega) \subseteq \mathcal{D}(\omega)$.

Let now $p \in \mathcal{D}(\omega)$. Suppose first that the problem

$$u'' = p(t)u; \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega) \tag{10.13}$$

has a nontrivial solution u. Then, in view of Lemma 2.7, we can assume without loss of generality that $u(t) \geq 0$ for $t \in \mathbb{R}$. However, $u \not\equiv 0$ and thus $u(t) > 0$ for $t \in \mathbb{R}$. Therefore, in this case $p \in \mathcal{V}_0(\omega)$.

Suppose now that the problem (10.13) has no nontrivial solution. Then, by virtue of Fredholm's alternative, the problem

$$u'' = p(t)u + |p(t)| + 1; \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega)$$

has a unique solution u. On account of Lemma 2.7, either

$$u(t) > 0 \quad \text{for} \ \ t \in \mathbb{R} \tag{10.14}$$

or

$$u(t) < 0 \quad \text{for} \ \ t \in \mathbb{R}. \tag{10.15}$$

If (10.14) holds then, by virtue of Theorem 9.1 (with $\gamma = u$) we get $p \in \mathcal{V}^+(\omega)$, while if (10.15) is fulfilled then, in view of Theorem 8.3 (with $\gamma = -u$) we get $p \in \mathcal{V}^-(\omega)$. Therefore, $\mathcal{D}(\omega) \subseteq \mathcal{V}^+(\omega) \cup \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$. $\square$

Now we will show that the set $\mathcal{V}^+(\omega)$ is unbounded. More precisely, the following proposition holds.

Proposition 10.7. *For any $c > 0$ there is a $p \in \mathcal{V}^+(\omega)$ such that $\int_0^\omega p(s)\,\mathrm{d}s > c$.*

Proof. Let $c > 0$. Choose $g \in L_\omega$ such that

$$\int_0^\omega g(s)\,\mathrm{d}s = 0, \quad \int_0^\omega (\ell(g)(t))^2\,\mathrm{d}t > c + \omega \tag{10.16}$$

and introduce the notation $p_0(t) \overset{\text{def}}{=} g(t) + (\ell(g)(t))^2$, where ℓ is the operator defined by (0.14). In view of Remark 0.7, we have $p_0 \in \mathcal{V}_0(\omega)$. By virtue of Proposition 0.8, $p_0 \in \mathcal{D}$ as well. In view of Proposition 2.2, we have $\mathcal{D} \subset \operatorname{Int}\mathcal{D}(\omega)$. Hence, there is a $\varepsilon \in]0,1[$ such that $p_0 - \varepsilon \in \mathcal{D}(\omega)$. On the other hand, since $p_0 \in \mathcal{V}_0(\omega)$ the problem

$$\gamma'' = p_0(t)\gamma; \quad \gamma(0) = \gamma(\omega), \ \ \gamma'(0) = \gamma'(\omega)$$

has a positive solution γ. Let now $p(t) \overset{\text{def}}{=} p_0(t) - \varepsilon$. Then, by virtue of Theorem 9.1, we get $p \in \mathcal{V}^+(\omega)$. On the other hand, on account of (10.16), it is clear that $\int_0^\omega p(s)\,\mathrm{d}s = \int_0^\omega p_0(s)\,\mathrm{d}s - \varepsilon\omega > c$. $\square$

Proposition 10.8. *If $p \in \mathcal{V}^-(\omega)$, then $\int_0^\omega p(s)\,\mathrm{d}s > 0$ while if $p \in \mathcal{V}^+(\omega)$, then $\int_0^\omega p(s)\,\mathrm{d}s > -\frac{\pi^2}{\omega}$.*

Proof. Let $p \in \mathcal{V}^-(\omega)$ and u be a solution of the problem

$$u'' = p(t)u - 1; \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega).$$

In view of Remark 8.2, we have $u(t) > 0$ for $t \in [0,\omega]$. Put $\rho(t) = \frac{u'(t)}{u(t)}$ for $t \in [0,\omega]$. It is clear that $\rho(0) = \rho(\omega)$ and

$$\rho'(t) = p(t) - \frac{1}{u(t)} - \rho^2(t) \quad \text{for} \ \ t \in [0,\omega].$$

The integration of the latter equality yields

$$\int_0^\omega p(s)\,\mathrm{d}s = \int_0^\omega \Big(\frac{1}{u(s)} + \rho^2(s)\Big)\,\mathrm{d}s > 0.$$

Let now $p \in \mathcal{V}^+(\omega)$. By virtue of Theorem 9.1, the inclusion $p \in \mathcal{D}(\omega)$ holds. It follows from Corollary 2 of [17] that, for any $a \in [0, \omega[$, the inequality

$$\int_a^{a+\omega} \sin^2 \frac{\pi(s-a)}{\omega}\, p(s)\,\mathrm{d}s > -\frac{\pi^2}{2\omega} \tag{10.17}$$

holds. The latter inequality with $a = 0$ and $a = \frac{\omega}{2}$ implies that

$$\int_0^\omega \sin^2 \frac{\pi s}{\omega}\, p(s)\,\mathrm{d}s > -\frac{\pi^2}{2\omega} \quad \text{and} \quad \int_0^\omega \cos^2 \frac{\pi s}{\omega}\, p(s)\,\mathrm{d}s > -\frac{\pi^2}{2\omega}\,,$$

respectively, and, consequently, we have $\int\limits_0^\omega p(s)\,\mathrm{d}s > -\frac{\pi^2}{\omega}$. □

Remark 10.9. Let $\omega = 2\pi$, $c > 0$, and $p(t) \stackrel{\mathrm{def}}{=} -c(1-\cos t)$. As it was mentioned in the proof of Proposition 10.8, if $p \in \mathcal{V}^+(\omega)$ then (10.17) holds for any $a \in [0, \omega[$. Taking $a = 0$ in (10.17), we get $c < \frac{1}{6}$. Thus the condition $c \in \left]0, \frac{1}{6}\right[$ is necessary for the inclusion $p \in \mathcal{V}^+(\omega)$.

Proposition 10.10. *Let $p \in L_\omega$ and $p \notin \mathcal{V}^-(\omega)$. Then there is a $\widetilde{p} \in \mathcal{V}_0(\omega)$ such that $\widetilde{p}(t) \geq p(t)$ for $t \in \mathbb{R}$.*

Proof. If $p \in \mathcal{V}_0(\omega)$ then the assertion of the proposition holds with $\widetilde{p} \equiv p$.

Suppose that $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$. Introduce the notation

$$p_\lambda(t) \stackrel{\mathrm{def}}{=} p(t) + \lambda\big(|p(t)| + 1\big) \quad \text{for } t \in \mathbb{R}, \quad \lambda > 0,$$
$$A \stackrel{\mathrm{def}}{=} \big\{\lambda > 0 : \ p_\lambda \in \mathcal{V}^-(\omega)\big\}.$$

Since the inequality

$$p_\lambda(t) \geq 1 \quad \text{for } t \in \mathbb{R}$$

holds for $\lambda \geq 1$, it follows from Remark 8.4 that $p_\lambda \in \mathcal{V}^-(\omega)$ for $\lambda \geq 1$. Hence, $[1, +\infty[\subseteq A$ and, consequently, $A \neq \varnothing$. On the other hand, it is clear that $0 \notin A$.

Let now

$$\lambda_* \stackrel{\mathrm{def}}{=} \inf A.$$

First we will show that

$$\lambda_* \notin A. \tag{10.18}$$

Indeed, if $\lambda_* \in A$ then $p_{\lambda_*} \in \mathcal{V}^-(\omega)$. However, $\lambda_* \neq 0$ and, consequently, $\lambda_* > 0$. By virtue of Proposition 10.1, the set $\mathcal{V}^-(\omega)$ is open. Hence, there is an $\varepsilon \in]0, \lambda_*[$ such that $p_\lambda \in \mathcal{V}^-(\omega)$ for $\lambda \in]\lambda_* - \varepsilon, \lambda_*[$ which contradicts the definition of the number λ_*. Thus we have proved that (10.18) holds.

Let now $\{\lambda_n\}_{n=1}^{+\infty} \subset A$ is such that

$$\lambda_{k+1} < \lambda_k \quad \text{for } k \in \mathbb{N}, \qquad \lim_{k\to+\infty} \lambda_k = \lambda_*.$$

Denote by u_k the solution of the problem

$$u_k'' = p_{\lambda_k}(t)u_k - 1; \quad u_k(0) = u_k(\omega), \ \ u_k'(0) = u_k'(\omega). \tag{10.19}$$

By virtue of Remark 8.2, we have

$$u_k(t) > 0 \quad \text{for } t \in [0, \omega], \ \ k \in \mathbb{N}. \tag{10.20}$$

It is clear that,

$$p_{\lambda_{k+1}}(t) \leq p_{\lambda_k}(t) \quad \text{for } t \in [0, \omega], \ \ k \in \mathbb{N}.$$

Taking, moreover, in to account (10.20), it follows from (10.19) that, for any $k \in \mathbb{N}$, we have

$$u_k''(t) \ge p_{\lambda_{k+1}}(t)u_k(t) - 1 \quad \text{for } t \in [0,\omega],$$
$$u_k(0) = u_k(\omega), \quad u_k'(0) = u_k'(\omega).$$

Hence, in view of (10.19), it follows from Remark 0.6 that

$$u_{k+1}(t) \ge u_k(t) \quad \text{for } t \in [0,\omega], \ \ k \in \mathbb{N}. \tag{10.21}$$

Now we will show that

$$\lim_{k\to+\infty} \|u_k\|_C = +\infty. \tag{10.22}$$

Suppose the contrary, let $\lim\limits_{k\to+\infty} \|u_k\|_C = c$. Then, in view of Proposition 3.3, we can assume without loss of generality that

$$\lim_{k\to+\infty} u_k^{(i)}(t) = u_0^{(i)} \quad \text{uniformly on } [0,\omega], \ \ i = 0,1, \tag{10.23}$$

where u_0 is a solution of the problem

$$u_0'' = p_{\lambda_*}(t)u_0 - 1; \quad u_0(0) = u_0(\omega), \ \ u_0'(0) = u_0'(\omega). \tag{10.24}$$

However, in view of (10.20), (10.21), and (10.23), we have $u_0(t) \ge u_1(t) > 0$ for $t \in [0,\omega]$. Hence, it follows from Theorem 8.3 (with $\gamma \equiv u_0$) that $p_{\lambda_*} \in \mathcal{V}^-(\omega)$ which contradicts (10.18). Thus we have proved that (10.22) is fulfilled.

Let now

$$v_k(t) \overset{\text{def}}{=} \frac{1}{\|u_k\|_C}\, u_k(t), \quad q_k(t) \overset{\text{def}}{=} -\frac{1}{\|u_k\|_C} \quad \text{for } t \in [0,\omega], \ \ k \in \mathbb{N}.$$

It is clear that,

$$\|v_k\|_C = 1 \quad \text{for } k \in \mathbb{N}, \tag{10.25}$$
$$\lim_{k\to+\infty} \|q_k\|_L = 0, \tag{10.26}$$

and v_k is a solution of the problem

$$v_k'' = p_{\lambda_k}(t)v_k + q_k(t); \quad v_k(0) = v_k(\omega), \ \ v_k'(0) = v_k'(\omega).$$

By virtue of Proposition 3.3 and (10.26) we can assume without loss of generality that

$$\lim_{k\to+\infty} v_k^{(i)}(t) = v_0^{(i)} \quad \text{uniformly on } [0,\omega], \ \ i = 0,1, \tag{10.27}$$

where v_0 is a solution of the problem

$$v_0'' = p_{\lambda_*}(t)v_0; \quad v_0(0) = v_0(\omega), \ \ v_0'(0) = v_0'(\omega). \tag{10.28}$$

On the other hand, in view of (10.20), (10.25), and (10.27), it is clear that

$$v_0(t) \ge 0 \quad \text{for } t \in [0,\omega], \ \ \|v_0\|_C = 1.$$

Hence, $v_0(t) > 0$ for $t \in [0,\omega]$ and, consequently, $p_{\lambda_*} \in \mathcal{V}_0(\omega)$. Thus the assertion of the proposition holds for $\widetilde{p}(t) \overset{\text{def}}{=} p_{\lambda_*}(t)$ for $t \in \mathbb{R}$. □

Proposition 10.11. *Let $p_0 \in \mathcal{V}_0(\omega)$. Then there is an $\varepsilon_0 > 0$ such that for any $\varepsilon \in]0,\varepsilon_0[$, the inclusion $p_0 - \varepsilon \in \operatorname{Int}\mathcal{V}^+(\omega)$ holds.*

Proof. In view of Sturm's (separation) theorem and Proposition 2.2, the inclusion $\mathcal{V}_0(\omega) \subset \operatorname{Int}\mathcal{D}(\omega)$ holds. Then there is an $\varepsilon_0 > 0$ such that $p_0 - \varepsilon \in \operatorname{Int}\mathcal{D}(\omega)$ for every $\varepsilon \in]0,\varepsilon_0[$. Denote by γ a positive solution of the problem

$$\gamma'' = p_0(t)\gamma; \quad \gamma(0) = \gamma(\omega), \ \ \gamma'(0) = \gamma'(\omega).$$

It is clear that, γ satisfies (9.1)–(9.4), where $p \equiv p_0 - \varepsilon$, $\varepsilon \in]0,\varepsilon_0[$. Therefore, by virtue of Theorem 9.1′, we get that $p_0 - \varepsilon \in \operatorname{Int}\mathcal{V}^+(\omega)$ for every $\varepsilon \in]0,\varepsilon_0[$. □

11. Efficient Conditions for the Inclusion $p \in \mathcal{V}^-(\omega)$

Theorem 11.1. *Let $p \in L_\omega$, $p \not\equiv 0$,*

$$\big\|[p]_-\big\|_L < \frac{4}{\omega} + \frac{p^*}{4\omega}\,\big\|[p]_-\big\|_L^2, \tag{11.1}$$

and

$$\big\|[p]_+\big\|_L \geq \big\|[p]_-\big\|_L\Big(1 + \frac{p^*}{16}\,\big\|[p]_-\big\|_L^2 - \frac{\omega}{4}\,\big\|[p]_-\big\|_L\Big)^{-1}, \tag{11.2}$$

where the number p^* is defined by (0.16)*. Then $p \in \mathcal{V}^-(\omega)$.*

Proof. It follows from assumptions of the theorem that $[p]_+ \not\equiv 0$. Then, in view of Remark 8.4, we have $[p]_+ \in \mathcal{V}^-(\omega)$. Assume, moreover, that $[p]_- \not\equiv 0$ because otherwise the theorem is trivial. By virtue of Remark 8.2, the problem

$$\gamma'' = [p(t)]_+\gamma - [p(t)]_-; \quad \gamma(0) = \gamma(\omega), \;\; \gamma'(0) = \gamma'(\omega) \tag{11.3}$$

has a unique solution γ and

$$\gamma(t) > 0 \quad \text{for } \; t \in \mathbb{R}. \tag{11.4}$$

Set

$$m \overset{\text{def}}{=} \min\big\{\gamma(t): \; t \in [0,\omega]\big\} \tag{11.5}$$

and choose $a \in [0,\omega[$ such that

$$\gamma(a) = m. \tag{11.6}$$

It follows from (11.3) that

$$\int\limits_0^\omega [p(s)]_+\gamma(s)\,\mathrm{d}s = \int\limits_0^\omega [p(s)]_-\,\mathrm{d}s.$$

Hence, on account of (11.5), we get

$$m \leq \frac{\big\|[p]_-\big\|_L}{\big\|[p]_+\big\|_L}\,. \tag{11.7}$$

By direct calculations one can easily verify that

$$\begin{aligned}\gamma(t) = m &+ \frac{1}{\omega}(a+\omega-t)\int\limits_a^t (s-a)\Big([p(s)]_- - [p(s)]_+\gamma(s)\Big)\,\mathrm{d}s \\ &+ \frac{1}{\omega}(t-a)\int\limits_t^{a+\omega}(a+\omega-s)\Big([p(s)]_- - [p(s)]_+\gamma(s)\Big)\,\mathrm{d}s \quad \text{for } \; t \in [a, a+\omega].\end{aligned}$$

The latter equality, together with (11.4), imply

$$\gamma(t) < m + \frac{1}{\omega}\,I(t) \quad \text{for } \; t \in [a, a+\omega], \tag{11.8}$$

where

$$I(t) \overset{\text{def}}{=} (a+\omega-t)\int\limits_a^t (s-a)[p(s)]_-\,\mathrm{d}s + (t-a)\int\limits_t^{a+\omega}(a+\omega-s)[p(s)]_-\,\mathrm{d}s.$$

Now we estimate the function I. First of all mention that

$$\begin{aligned}I(t) &= (t-a)(a+\omega-t)\big\|[p]_-\big\|_L - \big((a+\omega-t)F_1(t) + (t-a)F_2(t)\big) \\ &\leq (t-a)(a+\omega-t)\Big(\big\|[p]_-\big\|_L - \frac{1}{\omega}\,(F_1(t)+F_2(t))\Big) \quad \text{for } \; t \in [a, a+\omega],\end{aligned}$$

where the functions F_1 and F_2 are defined by (4.1). Hence, by virtue of Proposition 4.1, we get that the inequality

$$I(t) \leq \frac{\omega^2}{4}\left(\big\|[p]_-\big\|_L - \frac{p^*}{4\omega}\,\big\|[p]_-\big\|_L^2\right) \quad \text{for } \; t \in [a, a+\omega]$$

holds. Taking, moreover, into account (11.7), we get from (11.8) that

$$\gamma(t) < \frac{\|[p]_-\|_L}{\|[p]_+\|_L} + \frac{\omega}{4}\|[p]_-\|_L - \frac{p^*}{16}\|[p]_-\|_L^2 \quad \text{for } t \in [a, a+\omega].$$

Hence, on account of (11.1), (11.2), and periodicity of the function γ we get

$$\gamma(t) < 1 \quad \text{for } t \in [0, \omega]. \tag{11.9}$$

Now, it follows from Theorem 8.3, by virtue of (11.3), (11.4), (11.9), and the assumption $[p]_- \not\equiv 0$, that $p \in \mathcal{V}^-(\omega)$. ☐

Remark 11.2. Let $p_0, g \in L_\omega$ and

$$g(t) \geq 0 \quad \text{for } t \in [0, \omega], \quad \operatorname{mes}\big\{t \in [0, \omega] : g(t) = 0\big\} = 0. \tag{11.10}$$

Then there is a $c > 0$ such that $p_0 + cg \in \mathcal{V}^-(\omega)$. Indeed, it is clear that,

$$\lim_{c \to +\infty} \|[p_0 + cg]_+\|_L = +\infty.$$

On the other hand, by virtue of (11.10), one can easily show that

$$\lim_{c \to +\infty} \|[p_0 + cg]_-\|_L = 0.$$

Hence, there is a $c > 0$ such that the function $p(t) \overset{\text{def}}{=} p_0(t) + cg(t)$ for $t \in [0, \omega]$ satisfies (11.1) and (11.2) and, consequently, by virtue of Theorem 11.1, $p \in \mathcal{V}^-(\omega)$.

Theorem 11.3. *Let $p \in L_\omega$, $[p]_-^2 \in L_\omega$, $p \not\equiv 0$,*

$$k^*(\omega)\|[p]_-^2\|_L < 1, \tag{11.11}$$

and

$$\|[p]_+\|_L \geq \|[p]_-\|_L + \frac{\omega}{4}\|[p]_-\|_L^2 \Big(1 - \sqrt{k^*(\omega)\|[p]_-^2\|_L}\Big)^{-1}, \tag{11.12}$$

where $k^*(\omega)$ is the number appearing in Remark 5.1*. Then $p \in \mathcal{V}^-(\omega)$.*

Proof. It follows from assumptions of the theorem that $[p]_+ \not\equiv 0$. Assume, moreover, that $[p]_- \not\equiv 0$ because otherwise the theorem is trivial (see Remark 8.4). By virtue of Proposition 5.2 and (11.11), the inclusion $-[p]_- \in \mathcal{D}(\omega)$ holds. Then, on account of Theorem 9.3, we have $-[p]_- \in \mathcal{V}^+(\omega)$. Thus, in view of Remark 9.2, the problem

$$\gamma'' = -[p(t)]_-\gamma + [p(t)]_+, \tag{11.13}$$

$$\gamma(0) = \gamma(\omega), \quad \gamma'(0) = \gamma'(\omega) \tag{11.14}$$

has a unique solution γ and $\gamma(t) > 0$ for $t \in \mathbb{R}$. Set

$$M \overset{\text{def}}{=} \max\big\{\gamma(t) : t \in [0, \omega]\big\}, \quad m \overset{\text{def}}{=} \min\big\{\gamma(t) : t \in [0, \omega]\big\} \tag{11.15}$$

and choose $a \in [0, \omega[$ and $b \in]a, a+\omega]$ such that

$$\gamma(a) = m, \quad \gamma(b) = M. \tag{11.16}$$

In view of (11.13) and (11.14), it is clear that

$$\int_0^\omega [p(s)]_+ \, ds = \int_0^\omega [p(s)]_-\gamma(s) \, ds. \tag{11.17}$$

Hence,

$$M \geq \frac{\|[p]_+\|_L}{\|[p]_-\|_L} \; (\geq m). \tag{11.18}$$

Multiplying both sides of (11.13) by γ and integrating it on $[a, a+\omega]$ we get

$$\int_a^{a+\omega} \big(\gamma'(s)\big)^2 \, ds = \int_a^{a+\omega} [p(s)]_-\gamma^2(s) \, ds - \int_a^{a+\omega} [p(s)]_+\gamma(s) \, ds. \tag{11.19}$$

Evidently,

$$\int\limits_a^{a+\omega} [p(s)]_-\gamma^2(s)\,\mathrm{d}s - \int\limits_a^{a+\omega} [p(s)]_+\gamma(s)\,\mathrm{d}s$$

$$= \int\limits_a^{a+\omega} [p(s)]_-\big(\gamma(s)-m\big)^2\,\mathrm{d}s - m^2\int\limits_a^{a+\omega} [p(s)]_-\,\mathrm{d}s + 2m\int\limits_a^{a+\omega} [p(s)]_-\gamma(s)\,\mathrm{d}s - \int\limits_a^{a+\omega} [p(s)]_+\gamma(s)\,\mathrm{d}s.$$

Taking, moreover, into account (11.15) and (11.17), we easily conclude from (11.19) that

$$\int\limits_a^{a+\omega} \big(\gamma'(s)\big)^2\,\mathrm{d}s \le \int\limits_a^{a+\omega} [p(s)]_-\big(\gamma(s)-m\big)^2\,\mathrm{d}s + m\int\limits_a^{a+\omega} [p(s)]_+\,\mathrm{d}s - m^2\int\limits_a^{a+\omega} [p(s)]_-\,\mathrm{d}s. \tag{11.20}$$

On the other hand, by virtue of Hölder's inequality and (5.1), we have

$$\begin{aligned}\bigg(\int\limits_a^{a+\omega} [p(s)]_-\big(\gamma(s)-m\big)^2\,\mathrm{d}s\bigg)^2 &\le \int\limits_a^{a+\omega} [p(s)]_-^2\,\mathrm{d}s\int\limits_a^{a+\omega} \big(\gamma(s)-m\big)^4\,\mathrm{d}s\\ &\le k^*(\omega)\int\limits_a^{a+\omega} [p(s)]_-^2\,\mathrm{d}s\bigg(\int\limits_a^{a+\omega} \big(\gamma'(s)\big)^2\,\mathrm{d}s\bigg)^2\end{aligned}$$

which, together with (11.11) and (11.20), results in

$$\int\limits_a^{a+\omega} \big(\gamma'(s)\big)^2\,\mathrm{d}s \le m\Big(\big\|[p]_+\big\|_L - m\big\|[p]_-\big\|_L\Big)\Big(1-\sqrt{k^*(\omega)\big\|[p]_-^2\big\|_L}\Big)^{-1}. \tag{11.21}$$

Since $\gamma' \not\equiv$ *Const.* we get, by virtue of Hölder's inequality, that

$$(M-m)^2 = \bigg(\int\limits_a^b \gamma'(s)\,\mathrm{d}s\bigg)^2 < (b-a)\int\limits_a^b \big(\gamma'(s)\big)^2\,\mathrm{d}s$$

and

$$(M-m)^2 = \bigg(\int\limits_b^{a+\omega} \gamma'(s)\,\mathrm{d}s\bigg)^2 < \big(\omega-(b-a)\big)\int\limits_b^{a+\omega} \big(\gamma'(s)\big)^2\,\mathrm{d}s.$$

Therefore,

$$(M-m)^4 < (b-a)\big(\omega-(b-a)\big)\int\limits_a^b \big(\gamma'(s)\big)^2\,\mathrm{d}s\int\limits_b^{a+\omega} \big(\gamma'(s)\big)^2\,\mathrm{d}s \le \frac{\omega^2}{16}\bigg(\int\limits_a^{a+\omega} \big(\gamma'(s)\big)^2\,\mathrm{d}s\bigg)^2.$$

Thus

$$(M-m)^2 < \frac{\omega}{4}\int\limits_a^{a+\omega} (\gamma'(s))^2\,\mathrm{d}s.$$

Hence, in view of (11.18), we get

$$\frac{\big(\big\|[p]_+\big\|_L - m\big\|[p]_-\big\|_L\big)^2}{\big\|[p]_-\big\|_L^2} < \frac{\omega}{4}\int\limits_a^{a+\omega} (\gamma'(s))^2\,\mathrm{d}s.$$

The latter inequality, together with (11.21), implies

$$\big\|[p]_+\big\|_L < m\bigg(\big\|[p]_-\big\|_L + \frac{\omega}{4}\big\|[p]_-\big\|_L^2\Big(1-\sqrt{k^*(\omega)\big\|[p]_-^2\big\|_L}\Big)^{-1}\bigg)$$

which, together with (11.12), yields

$$m > 1.$$

Taking now into account (11.13), (11.14), and the condition $[p]_+ \not\equiv 0$, we get from Theorem 8.3 that $p \in \mathcal{V}^-(\omega)$. $\square$

Theorem 11.4. *Let $p \in L_\omega$ and there exist a $c > 0$ such that*

$$\big\| [p - c^2]_- \big\|_L \leq 2c \, \frac{e^{c\omega} - 1}{e^{c\omega} + 1} \,. \tag{11.22}$$

Then $p \in \mathcal{V}^-(\omega)$.

Proof. Assume that

$$[p(t) - c^2]_- \not\equiv 0 \tag{11.23}$$

because otherwise $p(t) \geq c^2$ for $t \in \mathbb{R}$ and, in view of Remark 8.4 we get $p \in \mathcal{V}^-(\omega)$. By virtue of Remarks 8.2 and 8.4 and (11.23), the problem

$$\gamma'' = c^2\gamma - [p(t) - c^2]_-, \tag{11.24}$$

$$\gamma(0) = \gamma(\omega), \quad \gamma'(0) = \gamma'(\omega) \tag{11.25}$$

has a unique solution γ and

$$\gamma(t) > 0 \quad \text{for } \ t \in \mathbb{R}. \tag{11.26}$$

Set

$$M \stackrel{\text{def}}{=} \max\big\{\gamma(t) : \ t \in [0, \omega]\big\} \tag{11.27}$$

and choose $a \in [0, \omega[$ such that $\gamma(a) = M$. It is clear that the function γ is a unique solution of Dirichlet problem

$$\gamma'' = c^2\gamma - [p(t) - c^2]_-; \quad \gamma(a) = M, \ \gamma(a + \omega) = M$$

as well. Hence, by virtue of Green's formula (for Dirichlet problem), we get

$$\gamma(t) = \frac{M}{u_2(a)} \big(u_1(t) + u_2(t)\big) + \frac{1}{u_2(a)} \left(u_2(t) \int\limits_a^t u_1(s)h(s)\,\mathrm{d}s + u_1(t) \int\limits_t^{a+\omega} u_2(s)h(s)\,\mathrm{d}s \right) \tag{11.28}$$

for $t \in [a, a + \omega]$, where

$$h(t) \stackrel{\text{def}}{=} [p(t) - c^2]_- \quad \text{for } \ t \in \mathbb{R} \tag{11.29}$$

and u_1 and u_2 are solutions of the initial value problems

$$u_1'' = c^2 u_1; \quad u_1(a) = 0, \ \ u_1'(a) = 1, \tag{11.30}$$

$$u_2'' = c^2 u_2; \quad u_2(a + \omega) = 0, \ \ u_2'(a + \omega) = -1. \tag{11.31}$$

It follows from (11.28), in view of (11.30) and (11.31), that

$$\gamma'(a) = \frac{M}{u_2(a)} \big(1 + u_2'(a)\big) + \frac{1}{u_2(a)} \int\limits_a^{a+\omega} u_2(s)h(s)\,\mathrm{d}s,$$

$$\gamma'(a + \omega) = \frac{M}{u_2(a)} \big(u_1'(a + \omega) - 1\big) - \frac{1}{u_2(a)} \int\limits_a^{a+\omega} u_1(s)h(s)\,\mathrm{d}s.$$

However $\gamma'(a) = \gamma'(a + \omega)$ and thus

$$M\big(u_1'(a + \omega) - u_2'(a) - 2\big) = \int\limits_a^{a+\omega} \big(u_1(s) + u_2(s)\big)h(s)\,\mathrm{d}s. \tag{11.32}$$

Solving (11.30) and (11.31) one can easily verify that

$$u_1'(a + \omega) - u_2'(a) - 2 = \frac{(e^{c\omega} - 1)^2}{e^{c\omega}} \tag{11.33}$$

and

$$u_1(a + \omega) = \frac{e^{2c\omega} - 1}{2c \, e^{c\omega}} \,. \tag{11.34}$$

On the other hand, in view of (11.30) and (11.31), the function $v(t) \stackrel{\text{def}}{=} u_1(t) + u_2(t)$ satisfies $v''(t) = c^2 v(t) > 0$ for $t \in [a, a+\omega]$, $v(a) = v(a+\omega)$, and $v(a) = u_1(a+\omega)$. Hence, on account of (11.34), we get

$$u_1(t) + u_2(t) = v(t) < v(a) = \frac{\mathrm{e}^{2c\omega} - 1}{2c\,\mathrm{e}^{c\omega}} \quad \text{for } t \in \,]a, a+\omega[\,. \tag{11.35}$$

Now it follows from (11.32), in view of (11.23), (11.29), (11.33), and (11.35), that

$$M < \frac{\mathrm{e}^{c\omega} + 1}{2c(\mathrm{e}^{c\omega} - 1)} \big\| [p - c^2]_- \big\|_L.$$

Hence, on account of (11.22) and (11.27), we get

$$\gamma(t) < 1 \quad \text{for } t \in [0, \omega].$$

The latter inequality, together with (11.23)–(11.26), implies (8.11)–(8.14). Therefore, by virtue of Theorem 8.3, we get $p \in \mathcal{V}^-(\omega)$. □

Theorem 11.5. *Let $p \in L_\omega$, $\overline{p} > 0$, and*

$$|\ell(p)(t)| \le \sqrt{\overline{p}} \quad \textit{for } t \in [0, \omega], \tag{11.36}$$

where $\overline{p}$ and $\ell(p)$ are defined by (0.11) *and* (0.14)*, respectively. Then $p \in \mathcal{V}^-(\omega)$.*

Proof. Assume that

$$p(t) \not\equiv \overline{p} \tag{11.37}$$

because otherwise the theorem is trivial (see Remark 8.4). Let

$$\rho(t) \stackrel{\text{def}}{=} -\ell(p)(t) \quad \text{for } t \in [0, \omega].$$

It is clear that

$$\rho'(t) = p(t) - \overline{p} \quad \text{for } t \in [0, \omega]. \tag{11.38}$$

In view of (11.37), evidently $\rho'(t) \not\equiv 0$. Hence, $\ell(p)(t) \not\equiv$ *Const.* Thus it follows from (11.36) that

$$\operatorname{mes}\Big\{ t \in [0, \omega] : \ |\ell(p)(t)| < \sqrt{\overline{p}} \Big\} > 0. \tag{11.39}$$

Now we get from (11.38), in view of (11.36) and (11.39), that

$$\rho'(t) \le p(t) - \rho^2(t) \quad \text{for } t \in [0, \omega], \tag{11.40}$$

$$\operatorname{mes}\Big\{ t \in [0, \omega] : \ \rho'(t) < p(t) - \rho^2(t) \Big\} > 0. \tag{11.41}$$

Integrating (11.38) from 0 to t we get

$$\rho(t) = \rho(0) - \int_0^t \big(p(s) - \overline{p}\big)\,\mathrm{d}s \quad \text{for } t \in [0, \omega]. \tag{11.42}$$

In particular,

$$\rho(0) = \rho(\omega). \tag{11.43}$$

Integration of (11.42) over $[0, \omega]$ implies

$$\int_0^\omega \rho(s)\,\mathrm{d}s = \omega\rho(0) - \int_0^\omega \bigg(\int_0^s \big(p(\xi) - \overline{p}\big)\,\mathrm{d}\xi \bigg)\,\mathrm{d}s = 0. \tag{11.44}$$

Let now

$$\gamma(t) \stackrel{\text{def}}{=} \exp\bigg(\int_0^t \rho(s)\,\mathrm{d}s \bigg) \quad \text{for } t \in [0, \omega].$$

Then (11.40), (11.41), (11.43), and (11.44) imply (8.11)–(8.14). Hence, by virtue of Theorem 8.3, we get $p \in \mathcal{V}^-(\omega)$. □

Example 11.6. Let $\omega = 2\pi$, $p(t) = c + \lambda\cos t$, $\lambda \neq 0$. Then $\overline{p} = c$ and $\ell(p)(t) = \lambda\sin t$. It follows from Theorem 11.5 that if $c \ge \lambda^2$ then $p \in \mathcal{V}^-(\omega)$.

12. Efficient Conditions for the Inclusion $p \in \mathcal{V}^+(\omega)$

Next two theorems immediately follows from Theorem 9.3 and Propositions 5.2 and 5.3.

Theorem 12.1. *Let $p \in L_\omega$, $p \not\equiv 0$, $\overline{p} \le 0$, and*

$$k^*(\omega) \int_0^\omega [p(s)]_-^2 \, \mathrm{d}s < 1.$$

Then $p \in \operatorname{Int} \mathcal{V}^+(\omega)$.

Theorem 12.2. *Let $p \in L_\omega$, $p \not\equiv 0$, $\overline{p} \le 0$, and*

$$\big\|[p]_-\big\|_L \le \frac{4}{\omega} + \frac{p^*}{4\omega} \big\|[p]_-\big\|_L^2. \tag{12.1}$$

Then $p \in \operatorname{Int} \mathcal{V}^+(\omega)$.

The next theorem, in spite of previous ones, does not exclude the case when $\overline{p} > 0$.

Theorem 12.3. *Let $p \in L_\omega$, $p \not\equiv$ Const., and*

$$\ell^2\Big(1 - \frac{\pi^2}{(\mathrm{e}^{\omega\ell} - 1)^2}\Big) \le \overline{p} \le \frac{\ell}{\omega(\mathrm{e}^{\omega\ell} - 1)} \bigg(\int_0^\omega |\ell(p)(s)| \, \mathrm{d}s\bigg)^2, \tag{12.2}$$

where the number ℓ and the function $\ell(p)$ are defined by (0.15) *and* (0.14)*, respectively. Then $p \in \operatorname{Int} \mathcal{V}^+(\omega)$.*

Proof. Introduce the notations

$$u_0(t) \stackrel{\text{def}}{=} \exp\bigg(\int_0^t \ell(p)(\xi) \, \mathrm{d}\xi\bigg), \quad \lambda = \pi\bigg(\int_0^\omega \frac{1}{u_0^2(s)} \, \mathrm{d}s\bigg)^{-1}$$

$$\sigma_\alpha(t) \stackrel{\text{def}}{=} \lambda \operatorname{ctg}\bigg(\lambda \int_\alpha^t \frac{1}{u_0^2(s)} \, \mathrm{d}s\bigg) \quad \text{for } t \in \,]\alpha, \alpha + \omega[\,, \ \ \alpha \in [0, \omega[\,,$$

and

$$\rho_\alpha(t) \stackrel{\text{def}}{=} \ell(p)(t) + \frac{\sigma_\alpha(t)}{u_0^2(t)} \quad \text{for } t \in \,]\alpha, \alpha + \omega[\,, \ \ \alpha \in [0, \omega[\,.$$

Since $\sigma_\alpha'(t) = -\frac{1}{u_0^2(t)} \, (\lambda^2 + \sigma_\alpha^2(t))$ one can easily verify that

$$\rho_\alpha'(t) = p(t) - \overline{p} + \ell^2(p)(t) - \frac{\lambda^2}{u_0^4(t)} - \rho_\alpha^2(t) \quad \text{for } t \in \,]\alpha, \alpha + \omega[\,. \tag{12.3}$$

By virtue of Proposition 4.2, we have

$$u_0^2(t) \int_0^\omega \frac{1}{u_0^2(s)} \, \mathrm{d}s \le \frac{\mathrm{e}^{\omega\ell} - 1}{\ell}\,. \tag{12.4}$$

Hence,

$$\frac{\lambda^2}{u_0^4(t)} \ge \frac{\pi^2 \ell^2}{(\omega\ell - 1)^2}\,. \tag{12.5}$$

Since $p \not\equiv$ *Const.* we have $|\ell(p)| \not\equiv$ *Const.* and hence $\ell^2(p)(t) \le \ell^2$ for $t \in \mathbb{R}$ and $\ell^2(p)(t) \not\equiv \ell^2$ on $[\alpha, \alpha + \omega]$. Taking, moreover, into account the first inequality in (12.2), we get

$$\ell^2(p)(t) \le \frac{\lambda^2}{u_0^4(t)} + \overline{p} \quad \text{for } t \in \mathbb{R}$$

and

$$\operatorname{mes}\bigg\{t \in [\alpha, \alpha + \omega] : \ \ell^2(p)(t) < \frac{\lambda^2}{u_0^4(t)} + \overline{p}\bigg\} > 0.$$

Hence, it follows from (12.3) that

$$\rho'_\alpha(t) \le p(t) - \rho^2_\alpha(t) \quad \text{for } t \in \,]\alpha, \alpha+\omega[\,, \quad \alpha \in [0, \omega[\,, \tag{12.6}$$

$$\operatorname{mes}\Big\{t \in [\alpha, \alpha+\omega] : \ \rho'_\alpha(t) < p(t) - \rho^2_\alpha(t)\Big\} > 0. \tag{12.7}$$

Set

$$\gamma_\alpha(t) \overset{\text{def}}{=} \exp\Bigg[\int\limits_{\alpha+\frac{\omega}{2}}^{t} \rho_\alpha(s)\,\mathrm{d}s\Bigg] \quad \text{for } t \in \,]\alpha, \alpha+\omega[\,. \tag{12.8}$$

By direct calculation one can easily verify that

$$\gamma_\alpha(t) = \sin\bigg(\lambda\int\limits_{\alpha}^{t} \frac{1}{u_0^2(s)}\,\mathrm{d}s\bigg) \exp\Bigg[\int\limits_{\alpha+\frac{\omega}{2}}^{t} \ell(p)(s)\,\mathrm{d}s\Bigg] \frac{1}{\sin\big(\lambda \int\limits_{\alpha}^{\alpha+\frac{\omega}{2}} \frac{1}{u_0^2(s)}\,\mathrm{d}s\big)}\,.$$

Hence, $\gamma_\alpha \in AC'([\alpha, \alpha+\omega])$, $\gamma_\alpha(\alpha) = 0$, $\gamma_\alpha(\alpha+\omega) = 0$, and $\gamma_\alpha(t) > 0$ for $t \in \,]\alpha, \alpha+\omega[\,$. On the other hand, in view of (12.6)–(12.8), we get that

$$\gamma''_\alpha(t) \le p(t)\gamma_\alpha(t) \quad \text{for } t \in [\alpha, \alpha+\omega],$$
$$\operatorname{mes}\Big\{t \in [\alpha, \alpha+\omega] : \ \gamma''_\alpha(t) < p(t)\gamma_\alpha(t)\Big\} > 0.$$

Therefore, by virtue of Proposition 2.6, the inclusion

$$p \in \operatorname{Int}\mathcal{D}(\omega) \tag{12.9}$$

holds.

Let now

$$h(t) \overset{\text{def}}{=} u_0^4(t)\big(\overline{p} - \ell^2(p)(t)\big) \quad \text{for } t \in \mathbb{R},$$
$$\overline{h} \overset{\text{def}}{=} \bigg(\int\limits_0^\omega \frac{1}{u_0^2(s)}\,\mathrm{d}s\bigg)^{-1} \int\limits_0^\omega \frac{h(s)}{u_0^2(s)}\,\mathrm{d}s,$$
$$c \overset{\text{def}}{=} -\bigg(\int\limits_0^\omega \frac{1}{u_0^2(s)}\,\mathrm{d}s\bigg)^{-1} \int\limits_0^\omega u_0^{-2}(t) \int\limits_0^t u_0^{-2}(s)\big(h(s) - \overline{h}\big)\,\mathrm{d}s\,\mathrm{d}t,$$

and

$$\rho(t) \overset{\text{def}}{=} \ell(p)(t) + u_0^{-2}(t)\bigg(c + \int\limits_0^t u_0^{-2}(s)\big(h(s) - \overline{h}\big)\,\mathrm{d}s\bigg) \quad \text{for } t \in [0, \omega].$$

Since $\int\limits_0^\omega \ell(p)(\xi)\,\mathrm{d}\xi = 0$ we have that

$$\int\limits_0^\omega \rho(t)\,\mathrm{d}t = 0. \tag{12.10}$$

On the other hand, since $\ell(p)(0) = \ell(p)(\omega)$ and $\int\limits_0^\omega u_0^{-2}(s)(h(s) - \overline{h})\,\mathrm{d}s = 0$, we get

$$\rho(0) = \rho(\omega). \tag{12.11}$$

By direct calculation one can easily verify that

$$\begin{aligned}\rho'(t) &= p(t) - 2\,\frac{\ell(p)(t)}{u_0^2(t)}\left(c + \int_0^t u_0^{-2}(s)\Big(h(s) - \overline{h}\Big)\,\mathrm{d}s\right) - \ell^2(p)(t) - \frac{\overline{h}}{u_0^4(t)} \\ &= p(t) - \rho^2(t) + \frac{1}{u_0^4(t)}\left(c + \int_0^t u_0^{-2}(s)\Big(h(s) - \overline{h}\Big)\,\mathrm{d}s\right)^2 - \frac{\overline{h}}{u_0^4(t)} \end{aligned} \tag{12.12}$$

for $t \in [0,\omega]$. By virtue of (12.5) and the second inequality in (12.2), we get

$$\overline{p}\int_0^\omega u_0^2(s)\,\mathrm{d}s \int_0^\omega \frac{1}{u_0^2(s)}\,\mathrm{d}s \le \left(\int_0^\omega |\ell(p)(s)|\,\mathrm{d}s\right)^2.$$

On the other hand, by virtue of Hölder's inequality

$$\left(\int_0^\omega |\ell(p)(s)|\,\mathrm{d}s\right)^2 \le \int_0^\omega \frac{1}{u_0^2(s)}\,\mathrm{d}s \int_0^\omega u_0^2(s)\big(\ell(p)(s)\big)^2\,\mathrm{d}s.$$

The latter two inequalities yield that

$$\int_0^\omega u_0^2(s)\big(\overline{p} - (\ell(p)(s))^2\big)\,\mathrm{d}s \le 0$$

and therefore $\overline{h} \le 0$. As it was mentioned above $|\ell(p)(t)| \not\equiv$ *Const*. Hence $h \not\equiv 0$ as well. Therefore, either $\overline{h} < 0$ or $\overline{h} = 0$ and

$$c + \int_0^t u_0^{-2}(s)\big(h(s) - \overline{h}\big)\,\mathrm{d}s \not\equiv 0 \quad \text{on } [0,\omega].$$

Now, it follows from (12.12) that

$$\rho'(t) \ge p(t) - \rho^2(t) \quad \text{for } t \in [0,\omega], \tag{12.13}$$

$$\operatorname{mes}\Big\{t \in [0,\omega] : \ \rho'(t) > p(t) - \rho^2(t)\Big\} > 0. \tag{12.14}$$

Let now

$$\gamma(t) \overset{\text{def}}{=} \exp\left(\int_0^t \rho(s)\,\mathrm{d}s\right) \quad \text{for } t \in [0,\omega].$$

In view of (12.10), (12.11), (12.13), and (12.14), one can easily verify that (9.1)–(9.4) are fulfilled. Taking, moreover, into account (12.9), we get from Theorem 9.1′ that $p \in \operatorname{Int}\mathcal{V}^+(\omega)$. ☐

Corollary 12.4. *Let* $p \in L_\omega$, $p \not\equiv$ *Const.*, $\overline{p} \le 0$, *and*

$$\frac{1}{\ell}\,(\mathrm{e}^{\omega\ell} - 1)\sqrt{|\overline{p}| + \ell^2} \le \pi.$$

Then $p \in \operatorname{Int}\mathcal{V}^+(\omega)$.

Corollary 12.5. *Let* $p \in L_\omega$, $p \not\equiv$ *Const.*,

$$\ell \le \frac{1}{\omega}\,\ln(1+\pi) \tag{12.15}$$

and

$$0 \le \overline{p} \le \frac{\ln(1+\pi)}{\omega^2\pi}\left(\int_0^\omega |\ell(p)(\xi)|\,\mathrm{d}\xi\right)^2. \tag{12.16}$$

Then $p \in \operatorname{Int}\mathcal{V}^+(\omega)$.

Proof. In view of (12.15) and the first inequality in (12.16) we get that the first inequality in (12.2) holds. Taking now into account that the function $x \mapsto \frac{1}{x}\,(\mathrm{e}^{\omega x} - 1)$ is increasing on $]0, +\infty[$, we get from the second inequality in (12.16) that the second inequality in (12.2) is fulfilled. □

Example 12.6. Let $\omega = 2\pi$, $p(t) = c + \lambda \cos t$, $\lambda \neq 0$. Then $\overline{p} = c$, $\ell(p)(t) = \lambda \sin t$, $\ell = |\lambda|$, $\int\limits_0^\omega |\ell(p)(s)|\,\mathrm{d}s = 4|\lambda|$. It follows from Corollary 12.4 that if

$$\lambda^2 - \lambda^2\,\frac{\pi^2}{(\mathrm{e}^{2\pi|\lambda|} - 1)^2} \leq c \leq 0$$

then $p \in \operatorname{Int} \mathcal{V}^+(\omega)$. On the other hand, Corollary 12.5 implies that if

$$|\lambda| \leq \frac{1}{2\pi}\,\ln(1+\pi)$$

and

$$0 \leq c \leq \frac{4\lambda^2}{\pi^3}\,\ln(1+\pi)$$

then $p \in \operatorname{Int} \mathcal{V}^+(\omega)$.

13. Connection with Lyapunov Stability

Consider the equation

$$u'' = p(t)u, \tag{13.1}$$

where $p \in L_\omega$.

Definition 13.1. We say that the equation (13.1) is strongly exponentially dichotomic (SED) if there are $\mu > 0$ and linearly independent solutions u and v of the equation (13.1) such that the functions

$$u(t)\,\mathrm{e}^{-\mu t} \quad \text{and} \quad v(t)\,\mathrm{e}^{\mu t}$$

are ω-periodic and do not change their signs.

Remark 13.2. It is clear that if the equation (13.1) is SED then it is unstable.

Theorem 13.3. *Equation* (13.1) *is SED if and only if* $p \in \mathcal{V}^-(\omega)$.

Proof. Let $p \in \mathcal{V}^-(\omega)$. Then, in view of Remark 0.6, the problem

$$\begin{gathered} \beta'' = p(t)\beta - |p(t)| - 1, \\ \beta(0) = \beta(\omega), \quad \beta'(0) = \beta'(\omega) \end{gathered} \tag{13.2}$$

has a unique solution β and $\beta(t) \geq 1$ for $t \in \mathbb{R}$. On account of Lemma 1.2, there is a solution v of the equation (13.1) such that

$$0 < v(t) \leq \beta(t) \quad \text{for } t \geq 0, \tag{13.3}$$

$$v(0) = \beta(0). \tag{13.4}$$

By virtue of (13.3) and Lemma 1.4, we get

$$v(t+\omega) = \lambda v(t) \quad \text{for } t \geq 0, \tag{13.5}$$

where

$$\lambda = \frac{v(\omega)}{v(0)}\,. \tag{13.6}$$

It easily follows from (13.5) that

$$v(k\omega) = \lambda^k v(0) \quad \text{for } k \in \mathbb{N}. \tag{13.7}$$

On the other hand, (13.2), (13.3), and (13.4) imply

$$v(k\omega) \leq \beta(k\omega) = \beta(0) = v(0) \quad \text{for } k \in \mathbb{N}.$$

Hence, in view of (13.7), we get $\lambda \le 1$. However, $\lambda \ne 1$ because otherwise, in view of (13.5) and the first inequality in (13.3), we get $p \in \mathcal{V}_0(\omega)$, which contradicts our assumption. Therefore,

$$0 < \lambda < 1. \tag{13.8}$$

Denote by u a solution of the initial value problem

$$u'' = p(t)u; \quad u(0) = c_1, \ \ u'(0) = c_2,$$

where

$$c_1 = \frac{\lambda^2 v(0)}{1-\lambda^2} \int\limits_0^\omega \frac{1}{v^2(s)}\,\mathrm{d}s, \quad c_2 = \frac{1 + c_1 v'(0)}{v(0)}\,. \tag{13.9}$$

Clearly,

$$u'(t)v(t) - u(t)v'(t) = c_2 v(0) - c_1 v'(0) = 1. \tag{13.10}$$

Therefore, u and v are linearly independent. Moreover, it follows from (13.10) that $(\frac{u(t)}{v(t)})' = \frac{1}{v^2(t)}$ pro $t \ge 0$ and thus

$$\frac{u(t)}{v(t)} = \frac{c_1}{v(0)} + \int\limits_0^t \frac{1}{v^2(s)}\,\mathrm{d}s \quad \text{for } \ t \ge 0.$$

Hence, in view of (13.5) and (13.9), we get

$$\begin{aligned}
\frac{u(t+\omega)}{v(t+\omega)} &= \frac{c_1}{v(0)} + \int\limits_0^\omega \frac{1}{v^2(s)}\,\mathrm{d}s + \int\limits_\omega^{t+\omega} \frac{1}{v^2(s)}\,\mathrm{d}s = \frac{c_1}{v(0)} + \int\limits_0^\omega \frac{1}{v^2(s)}\,\mathrm{d}s + \frac{1}{\lambda^2}\int\limits_0^t \frac{1}{v^2(s)}\,\mathrm{d}s \\
&= \frac{1}{\lambda^2}\bigg(\frac{c_1}{v(0)} + \int\limits_0^t \frac{1}{v^2(s)}\,\mathrm{d}s\bigg) + \frac{\lambda^2-1}{\lambda^2}\,\frac{c_1}{v(0)} + \int\limits_0^\omega \frac{1}{v^2(s)}\,\mathrm{d}s \\
&= \frac{1}{\lambda^2}\bigg(\frac{c_1}{v(0)} + \int\limits_0^t \frac{1}{v^2(s)}\,\mathrm{d}s\bigg) = \frac{u(t)}{\lambda^2 v(t)} = \frac{u(t)}{\lambda v(t+\omega)} \quad \text{for } \ t \ge 0.
\end{aligned}$$

Thus

$$u(t+\omega) = \frac{1}{\lambda}\,u(t) \quad \text{for } \ t \ge 0. \tag{13.11}$$

Now let $\mu \overset{\text{def}}{=} -\frac{1}{\omega}\ln\lambda$. In view of (13.8), clearly $\mu > 0$. Let, moreover,

$$\varphi(t) \overset{\text{def}}{=} v(t)\,\mathrm{e}^{\mu t}, \quad \psi(t) \overset{\text{def}}{=} u(t)\,\mathrm{e}^{-\mu t} \quad \text{for } \ t \in \mathbb{R}.$$

In view of (13.6), we have

$$\varphi(t+\omega) = v(t+\omega)\,\mathrm{e}^{\mu(t+\omega)} = \lambda v(t)\,\mathrm{e}^{\mu t}\,\mathrm{e}^{\mu\omega} = v(t)\,\mathrm{e}^{\mu t} = \varphi(t) \quad \text{for } \ t \in \mathbb{R}.$$

Analogously, on account of (13.11), we get that $\psi(t+\omega) = \psi(t)$ for $t \in \mathbb{R}$. On the other hand, it is clear that the functions φ and ψ are continuously differentiable. Mention also that since $p \in \mathcal{V}^-(\omega)$ we get from Theorem 8.1 that $p \in \mathcal{D}$. Hence, the functions φ and ψ do not change their signs. Therefore, the equation (13.1) is SED.

Suppose now that the equation (13.1) is SED. Let v is a solution of the equation (13.1) such that

$$v(t) = \varphi(t)\,\mathrm{e}^{-\mu t} \quad \text{for } \ t \in \mathbb{R}, \tag{13.12}$$

where $\mu > 0$ and φ is an ω-periodic and sign-constant. Assume without loss of generality that $\varphi(t) > 0$ for $t \in \mathbb{R}$. Clearly, there is a number $a \in [0, \omega[$ such that

$$\varphi'(a) = 0.$$

Then we get from (13.12) that

$$v'(a) < 0. \tag{13.13}$$

On the other hand, (13.12) implies

$$v(t+\omega) = \lambda v(t) \quad \text{for } \ t \in \mathbb{R}, \tag{13.14}$$

where
$$\lambda = \frac{1}{e^{\mu\omega}} < 1. \tag{13.15}$$
Introduce the notations
$$\gamma(t) \stackrel{\text{def}}{=} v(t+a), \quad p_a(t) \stackrel{\text{def}}{=} p(t+a) \quad \text{for } t \in \mathbb{R}.$$
On account of (13.13)–(13.15), we have
$$\gamma(\omega) = v(a+\omega) = \lambda v(a) < v(a) = \gamma(0)$$
and
$$\gamma'(\omega) = v'(a+\omega) = \lambda v'(a) > v'(a) = \gamma'(0).$$
It is clear that $\gamma''(t) = p_a(t)\gamma(t)$ for $t \in \mathbb{R}$. Hence, by virtue of Theorem 8.3, we get $p_a \in \mathcal{V}^-(\omega)$ which, in view of Remark 8.6 implies that $p \in \mathcal{V}^-(\omega)$. □

Theorem 13.4. *Let $p \in \operatorname{Int} \mathcal{V}^+(\omega)$. Then the equation* (13.1) *is stable.*

Proof. In view of Theorem 7.1, it is sufficient to show that Floquet multipliers of equation (13.1) are complex valued. Suppose the contrary, let $\mu \in \mathbb{R}$ be a Floquet multiplier of equation (13.1). Then, by virtue of Theorem 7.2, there is a nontrivial solution u_0 of the equation (13.1) satisfying
$$u_0(t+\omega) = \mu u_0(t) \quad \text{for } t \in \mathbb{R}. \tag{13.16}$$
Since $p \in \operatorname{Int} \mathcal{V}^+(\omega)$, in view of Theorem 8.1, we have $p \notin \mathcal{D}$. Hence, any solution of the equation (13.1) has at least one zero in $\mathbb{R}$. Taking, moreover, into account (13.16) we get that there is an $a \in [0, \omega[$ such that
$$u_0(a) = 0, \quad u_0(a+\omega) = 0.$$
Thus the function u_0 is a nontrivial solution of the problem
$$u'' = p(t)u; \quad u(a) = 0, \ \ u(a+\omega) = 0.$$
On the other hand, by virtue of Theorem 9.1′, the inclusion $p \in \operatorname{Int} \mathcal{D}(\omega)$ holds as well. Hence, in view of Proposition 2.2, we get the contradiction $u_0 \equiv 0$. □

Remark 13.5. The assumption $p \in \operatorname{Int} \mathcal{V}^+(\omega)$ in Theorem 13.4 cannot weakened to the assumption $p \in \mathcal{V}^+(\omega)$. As it was mentioned above (see Proposition 10.5), $\mathcal{V}^+(\omega) \setminus \operatorname{Int} \mathcal{V}^+(\omega) = \partial \mathcal{D}(\omega)$. By virtue of Proposition 14.1 below there is a $p \in \partial \mathcal{D}(\omega)$ such that the equation (13.1) is unstable. Mention also that the constant function $p(t) \stackrel{\text{def}}{=} -\left(\frac{\pi}{\omega}\right)^2$ also belongs to $\partial \mathcal{V}^+(\omega)$, while the corresponding equation (13.1) is stable. Thus if $p \in \mathcal{V}^+(\omega) \setminus \operatorname{Int} \mathcal{V}^+(\omega)$ then the equation (13.1) may be either stable or unstable.

Remark 13.6. It follows from Theorem 7.1, Theorem 7.2, Proposition 0.8, and Proposition 1.1 that if $p \in \mathcal{V}_0(\omega)$ then the equation (13.1) is unstable.

14. On Mathieu Equation

On $\mathbb{R}$ consider the equation
$$u'' = p_c(t)u, \tag{14.1}$$
where $p_c(t) \stackrel{\text{def}}{=} -c(1-\cos t)$ for $t \in \mathbb{R}$, $c \in \mathbb{R}$. It is clear that $p_0 \in \mathcal{V}_0(2\pi)$ and, for any $c < 0$, the inclusion $p_c \in \mathcal{V}^-(2\pi)$ holds (see Remark 8.4). Hence, we will be interested in the case $c > 0$. Recall that the number $k^*(2\pi)$ is introduced in Remark 5.1.

Proposition 14.1. *There is a $c_* \in \left[\frac{1}{\sqrt{3\pi k^*(2\pi)}}, \frac{1}{6}\right[$ such that $p_c \in \operatorname{Int} \mathcal{V}^+(2\pi)$ if and only if $c \in]0, c_*[$. Moreover, $p_{c_*} \in \partial \mathcal{D}(2\pi)$ (and, consequently, $p_{c_*} \in \mathcal{V}^+(2\pi)$) and the equation* (14.1) *with $c = c_*$ is unstable.*

Proof. For any $a \in [0, 2\pi]$ put
$$A(a) \overset{\text{def}}{=} \Big\{ c > 0 : \; w_a(t) > 0 \text{ for } t \in]a, a+2\pi] \Big\},$$
where w_a is solution of the initial value problem
$$w'' = p_c(t)w; \quad w(a) = 0, \;\; w'(a) = 1.$$
By virtue of Proposition 5.2, if $c > 0$ and
$$k^*(2\pi)\|p_c^2\|_{L_{2\pi}} < 1$$
then $p_c \in \operatorname{Int}\mathcal{D}(2\pi)$. Taking, moreover, into account Proposition 2.2, we get
$$\Big]0, \frac{1}{\sqrt{3\pi k^*(2\pi)}}\Big[\subset A(a) \quad \text{for} \;\; a \in [0, 2\pi]. \tag{14.2}$$
In particular, $A(a) \neq \varnothing$ for $a \in [0, 2\pi]$. On the other hand, by virtue of Corollary 2 of [17], if
$$-\int\limits_a^{a+2\pi} \sin^2 \frac{s-a}{2}\, p_c(s)\, \mathrm{d}s \geq \frac{\pi}{4}$$
then $c \notin A(a)$. Consequently, the sets $A(a)$ are bounded from above.

Put
$$c(a) \overset{\text{def}}{=} \sup A(a). \tag{14.3}$$
In view of (14.2), it is clear that
$$c(a) \geq \frac{1}{\sqrt{3\pi k^*(2\pi)}}\,. \tag{14.4}$$

Now we will show that
$$c(a) \notin A(a). \tag{14.5}$$
Suppose the contrary, let $c(a) \in A(a)$. Then, in view of Fredholm's first theorem (for Dirichlet problem), the problem
$$u_0'' = p_{c(a)}(t)u_0; \quad u_0(a) = 1, \;\; u_0(a+2\pi) = 1$$
possesses a unique solution u_0 and $u_0(t) > 0$ for $t \in [a, a+2\pi]$. It is clear that there are $\varepsilon_0 > 0$ and $\varepsilon_1 > 0$ such that
$$0 < u_0(t) - \varepsilon_0 < \varepsilon_1 \quad \text{for} \;\; t \in [a, a+2\pi].$$
Put $v(t) = u_0(t) - \varepsilon_0$ for $t \in [a, a+2\pi]$ and $c_1 = c(a)\big(1 + \frac{\varepsilon_0}{\varepsilon_1}\big)$. One can easily verify that
$$\begin{aligned} v''(t) &= p_{c(a)}(t)v(t) + \varepsilon_0 p_{c(a)}(t) \\ &\leq p_{c(a)}(t)v(t) + \frac{\varepsilon_0}{\varepsilon_1}\, p_{c(a)}(t)v(t) = p_{c_1}(t)v(t) \quad \text{for} \;\; t \in [a, a+2\pi]. \end{aligned} \tag{14.6}$$
Let now w be a solution of the problem
$$w'' = p_{c_1}(t)w; \quad w(a) = 0, \;\; w'(a) = 1.$$
In view (14.6) and Sturm's comparison theorem one can easily verify that $w(t) > 0$ fro $t \in]a, a+2\pi]$. Consequently, $c_1 \in A(a)$. However, $c_1 > c(a)$ which contradicts (14.3). Therefore, we have proved that (14.5) holds.

Let now $\{c_k\}_{k=1}^{+\infty} \subset A(a)$ be such that
$$\lim_{k \to +\infty} c_k = c(a). \tag{14.7}$$
For any $k \in \mathbb{N}$, denote by w_k the solution of the problem
$$w_k'' = p_{c_k}(t)w_k; \quad w_k(a) = 0, \;\; w_k'(a) = 1.$$
Let, moreover, w_0 be a solution of the problem
$$w_0'' = p_{c(a)}(t)w_0; \quad w_0(a) = 0, \;\; w_0'(a) = 1.$$
In view of (14.7) and well-posedness of the Cauchy problem it is clear that
$$\lim_{k \to +\infty} w_k(t) = w_0(t) \quad \text{uniformly on} \;\; [a, a+2\pi]. \tag{14.8}$$

On the other hand, since $c_k \in A(a)$, we have $w_k(t) > 0$ for $t \in]a, a+2\pi]$, $k \in \mathbb{N}$. Hence, on account of (14.8), we get $w_0(t) \geq 0$ for $t \in [a, a+2\pi]$. Since w_0 is a nontrivial solution of the linear equation we get from the previous inequality that $w_0(t) > 0$ for $t \in]a, a+2\pi[$. Taking now into account (14.5), we conclude that $w_0(a+2\pi) = 0$. Thus we have proved that, for any $a \in [0, 2\pi]$, the problem

$$u'' = p_{c(a)} u; \quad u(a) = 0, \ \ u(a+2\pi) = 0, \tag{14.9}$$

where $c(a)$ is defined by (14.3), possesses a solution u such that

$$u(t) > 0 \quad \text{for} \ \ t \in]a, a+2\pi[\,. \tag{14.10}$$

Mention that, by virtue of Corollary 2 of [17], if

$$-\int_0^{2\pi} \sin^2 \frac{s}{2}\, p_{c(0)}(s)\, \mathrm{d}s \geq \frac{\pi}{4}$$

then the problem (14.9) with $a = 0$ has no solution satisfying (14.10) with $a = 0$. Hence, we get

$$c(0) < \frac{1}{6}\,. \tag{14.11}$$

Now let

$$c_* \overset{\text{def}}{=} \inf\big\{c(a) : a \in [0, 2\pi]\big\}. \tag{14.12}$$

In view of (14.4) and (14.11), clearly $c_* \in \big[\frac{1}{\sqrt{3\pi k^*(2\pi)}}, \frac{1}{6}\big[$. Let $c \in]0, c_*[$. Then, in view of (14.12), we have that $c < c(a)$ for any $a \in [0, 2\pi]$. Hence, by virtue of Sturm's comparison theorem, $c \in A(a)$ for any $a \in [0, 2\pi]$ and, consequently, in view of Proposition 2.2, we get

$$p_c \in \operatorname{Int} \mathcal{D}(2\pi) \quad \text{for} \ \ c \in]0, c_*[\,. \tag{14.13}$$

Hence, by virtue of Theorem 9.3, for every $c \in]0, c_*[$, we have

$$p_c \in \operatorname{Int} \mathcal{V}^+(2\pi). \tag{14.14}$$

Taking, moreover, into account Proposition 2.1, we obtain from (14.13) that $p_{c_*} \in \mathcal{D}(2\pi)$ (and, consequently, by virtue of Theorem 9.3, $p_{c_*} \in \mathcal{V}^+(2\pi)$).

Now we will show that

$$p_{c_*} \in \partial \mathcal{D}(2\pi). \tag{14.15}$$

Since $p_{c_*} \in \mathcal{D}(2\pi)$ and $\overline{\mathcal{D}(2\pi)} = \mathcal{D}(2\pi)$ (see Proposition 2.1), it is sufficient to show that $p_{c_*} \notin \operatorname{Int} \mathcal{D}(2\pi)$. Suppose the contrary, let $p_{c_*} \in \operatorname{Int} \mathcal{D}(2\pi)$. Then there is a $\widetilde{c} > c_*$ such that $p_{\widetilde{c}} \in \operatorname{Int} \mathcal{D}(2\pi)$ and, consequently, by virtue of Proposition 2.2, $\widetilde{c} \in A(a)$ for any $a \in [0, 2\pi]$. Hence, in view of (14.3), we get that $c(a) \geq \widetilde{c} > c_*$ for any $a \in [0, 2\pi]$ which contradicts (14.12). Thus we have proved that (14.15) is fulfilled.

By virtue of Sturm's comparison theorem and (14.15), we have that if $c > c_*$ then $p_c \notin \mathcal{D}(2\pi)$ and, consequently, in view of Theorem 9.3, $p_c \notin \mathcal{V}^+(2\pi)$. Theorem 9.3 and (14.15) yield that $p_{c_*} \notin \operatorname{Int} \mathcal{V}^+(2\pi)$. Thus the inclusion (14.14) holds if and only if $c \in]0, c_*[$.

Now we will show that the equation

$$u'' = p_{c_*}(t) u \tag{14.16}$$

is unstable. First of all we mention that, by virtue of [10, § 11, Theorem 5.1], if

$$\int_\pi^{3\pi} (s-\pi)(3\pi - s)|p_c(t)|\, \mathrm{d}s \leq 2\pi \tag{14.17}$$

then $c \in A(\pi)$. By direct calculations one can easily verify that the inequality (14.17) is equivalent to $c < \frac{3}{2\pi^2 - 6}$. Hence, in view of (14.3), we have

$$c(\pi) \geq \frac{3}{2\pi^2 - 6}\,. \tag{14.18}$$

However, $\frac{3}{2\pi^2-6} > \frac{1}{6}$ and, consequently, (14.11) and (14.18) imply that $c(0) < c(\pi)$. Taking now into account (14.12), we get

$$c_* < c(\pi). \tag{14.19}$$

Hence, by virtue of Sturm's comparison theorem, we obtain that

$$c_* \in A(\pi). \tag{14.20}$$

To prove that the equation (14.16) is unstable we first show that the Floquet multipliers μ_1 and μ_2 of the equation (14.16) are real valued.

Suppose that the Floquet multipliers are complex valued. Then, by virtue of Theorem 7.3, there are $\alpha, \beta \in \mathbb{R}$ and linearly independent solutions u and v of the equation (14.16) such that

$$u(t+2\pi) = \alpha u(t) - \beta v(t), \quad v(t+2\pi) = \beta u(t) + \alpha v(t) \quad \text{for } t \in \mathbb{R}. \tag{14.21}$$

On the other hand, by virtue of (14.15) and Proposition 2.3, there is an $a \in [0, 2\pi[$ such that the problem

$$u_0'' = p_{c_*}(t)u_0; \quad u_0(a) = 0, \ \ u_0(a+2\pi) = 0$$

possesses a nontrivial solution u_0. Hence, there are constants $c_1, c_2 \in \mathbb{R}$ such that $|c_1| + |c_2| \neq 0$ and $u_0(t) = c_1 u(t) + c_2 v(t)$ for $t \in [a, a+2\pi]$. Consequently, the pair (c_1, c_2) is a nontrivial solution of the system of algebraic equations

$$\begin{gathered} c_1 u(a) + c_2 v(a) = 0, \\ c_1 u(a+2\pi) + c_2 v(a+2\pi) = 0. \end{gathered}$$

However, this system possesses a nontrivial solution if and only if $u(a)v(a+2\pi) - u(a+2\pi)v(a) = 0$. Hence, in view of (14.21), we get $\beta\big(u^2(a)+v^2(a)\big) = 0$ and, consequently, $u(a) = 0$ and $v(a) = 0$ which contradicts the linear independence of u and v. Thus we have proved that the Floquet multipliers are real valued.

In this case, by virtue of Floquet theory, the equation (14.16) is stable if and only if $\mu_1 = \mu_2$, $|\mu_1| = 1$ and any solution u of the equation (14.16) satisfies

$$u(t+2\pi) = \mu_1 u(t) \quad \text{for } t \in \mathbb{R}. \tag{14.22}$$

In view of (14.20), the solution u of the problem

$$u'' = p_{c_*}(t)u; \quad u(\pi) = 0, \ \ u'(\pi) = 1$$

satisfies $u(t) > 0$ for $t \in]\pi, 3\pi]$. Hence, (14.22) does not hold for $t = \pi$ and thus, the equation (14.16) is unstable. □

Remark 14.2. Mention that

$$\frac{1}{\sqrt{3\pi k^*(2\pi)}} = \frac{\Gamma(\frac{1}{4})}{12\Gamma^3(\frac{3}{4})} \approx 0.164\,.$$

15. Appendix

Definition 15.1. We say that the function $p \in L_\omega$ belongs to that set $\widehat{\mathcal{V}}^-(\omega)$ (respectively, $\widehat{\mathcal{V}}^+(\omega)$) if for any function $u \in \widetilde{AC}'([0,\omega])$ satisfying

$$u''(t) \geq p(t)u(t) \quad \text{for } t \in [0,\omega], \tag{15.1}$$

$$u(0) = u(\omega), \quad u'(0) \geq u'(\omega), \tag{15.2}$$

the inequality

$$u(t) \leq 0 \quad \text{for } t \in [0,\omega] \quad \big(\text{respectively, } u(t) \geq 0 \quad \text{for } t \in [0,\omega]\big)$$

is fulfilled.

Proposition 15.2. $\widehat{\mathcal{V}}^-(\omega) = \mathcal{V}^-(\omega)$.

Proof. Clearly, $\widehat{\mathcal{V}}^-(\omega) \subseteq \mathcal{V}^-(\omega)$. Show that $\mathcal{V}^-(\omega) \subseteq \widehat{\mathcal{V}}^-(\omega)$. Let $p \in \mathcal{V}^-(\omega)$ and a function $u \in AC'([0,\omega])$ satisfy (15.1) and (15.2). Suppose that

$$u'(0) > u'(\omega) \tag{15.3}$$

because otherwise the inclusion $p \in \mathcal{V}^-(\omega)$ implies

$$u(t) \le 0 \quad \text{for } t \in [0,\omega]. \tag{15.4}$$

In view of (15.1), clearly

$$u''(t) = p(t)u(t) + q(t) \quad \text{for } t \in [0,\omega], \tag{15.5}$$

where

$$q(t) \overset{\text{def}}{=} u''(t) - p(t)u(t) \quad \text{for } t \in [0,\omega], \tag{15.6}$$

$$q(t) \ge 0 \quad \text{for } t \in [0,\omega]. \tag{15.7}$$

Since $p \in \mathcal{V}^-(\omega)$, the problem

$$v'' = p(t)v + q(t); \quad v(0) = v(\omega), \ \ v'(0) = v'(\omega) \tag{15.8}$$

possesses a unique solution v and

$$v(t) \le 0 \quad \text{for } t \in [0,\omega]. \tag{15.9}$$

Put $w(t) \overset{\text{def}}{=} u(t) - v(t)$ for $t \in [0,\omega]$. It follows from (15.1), (15.2), (15.3), and (15.8) that

$$w''(t) = p(t)w(t) \quad \text{for } t \in [0,\omega], \tag{15.10}$$

$$w(0) = w(\omega), \quad w'(0) > w'(\omega). \tag{15.11}$$

In particular, $w \not\equiv 0$. Taking, moreover, into account that $p \in \mathcal{D}$ (see Theorem 8.1), we get that either

$$w(t) > 0 \quad \text{for } t \in [0,\omega], \tag{15.12}$$

or

$$w(t) < 0 \quad \text{for } t \in [0,\omega]. \tag{15.13}$$

If (15.12) holds then, in view of (15.10), (15.11), and the condition $p \in \mathcal{D}$, it follows from Theorem 9.1 (with $\gamma \equiv w$) that $p \in \mathcal{V}^+(\omega)$ which contradicts our assumption. Therefore, we have proved that (15.13) is fulfilled. Inequality (15.4) now follows from (15.9) and (15.13). $\square$

Remark 15.3. Let $p \in \mathcal{V}^-(\omega)$, $q \in L_\omega$, $c \in \mathbb{R}$, and the functions u and v are solutions of the problems

$$u'' = p(t)u + q(t); \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega) + c, \tag{15.14}$$

$$v'' = p(t)v + q(t); \quad v(0) = v(\omega), \ \ v'(0) = v'(\omega). \tag{15.15}$$

During the proof of Proposition 15.2, it was shown that if $c > 0$ then

$$u(t) < v(t) \quad \text{for } t \in [0,\omega].$$

Consequently, if

$$q(t) \ge 0 \quad \text{for } t \in [0,\omega], \ \ c \ge 0, \tag{15.16}$$

$$c + \operatorname{mes}\big\{t \in [0,\omega] : \ q(t) > 0\big\} > 0, \tag{15.17}$$

then the unique solution u of the problem (15.14) satisfies

$$u(t) < 0 \quad \text{for } t \in [0,\omega].$$

Proposition 15.4. $\widehat{\mathcal{V}}^+(\omega) = \mathcal{V}^+(\omega)$.

Proof. Show that $\mathcal{V}^+(\omega) \subseteq \widehat{\mathcal{V}}^+(\omega)$. Let $p \in \mathcal{V}^+(\omega)$ and a function $u \in AC'([0,\omega])$ satisfy (15.1) and (15.2). If $u'(0) = u'(\omega)$ then, in view of the inclusion $p \in \mathcal{V}^+(\omega)$, we have that

$$u(t) \ge 0 \quad \text{for } t \in [0,\omega]. \tag{15.18}$$

Suppose that (15.3) holds. In view of (15.1), clearly (15.5) holds, where the function q is defined by (15.6) and satisfies (15.7). Since $p \in \mathcal{V}^+(\omega)$, the problem (15.8) possesses a unique solution v and

$$v(t) \ge 0 \quad \text{for } t \in [0,\omega]. \tag{15.19}$$

Put $w(t) \stackrel{\text{def}}{=} u(t) - v(t)$ for $t \in [0,\omega]$. In view of (15.1), (15.2), (15.3), and (15.8), we get that (15.10) is fulfilled and

$$w(0) = w(\omega), \quad w'(0) < w'(\omega). \tag{15.20}$$

Taking, moreover, into account that $p \in \mathcal{D}(\omega)$ (see Theorem 9.1), we get that either

$$w(t) > 0 \quad \text{for } t \in]0,\omega[\,, \tag{15.21}$$

or

$$w(t) < 0 \quad \text{for } t \in]0,\omega[\,. \tag{15.22}$$

Now we will show that (15.22) holds. For this first let us show that $w(0) \leq 0$. Indeed, if $w(0) > 0$ then clearly

$$w(t) > 0 \quad \text{for } t \in [0,\omega].$$

Taking, moreover, into account (15.10) and (15.20), it follows from Theorem 8.3 (with $\gamma \equiv w$) that $p \in \mathcal{V}^-(\omega)$ which contradicts our assumption. Thus we have proved that

$$w(0) \leq 0.$$

If $w(0) = 0$ then, in view of (15.20), clearly (15.22) is fulfilled, while if $w(0) < 0$ then the validity of (15.22) is evident. Thus we have proved that (15.22) holds. Inequality (15.18) now follows from (15.19) and (15.22). ☐

Remark 15.5. During the proof of Proposition 15.4 it was shown that if $p \in \mathcal{V}^+(\omega)$, $q \in L_\omega$, $c > 0$, and u and v solutions of the problems (15.14) and (15.15), respectively, then

$$u(t) > v(t) \quad \text{for } t \in [0,\omega].$$

Taking, moreover, into account Remark 9.2, we get that if (15.16) and (15.17) are fulfilled then

$$u(t) > 0 \quad \text{for } t \in]0,\omega[\,.$$

Remark 15.6. Let $p \in \mathcal{V}^-(\omega)$ (respectively, $p \in \mathcal{V}^+(\omega)$), $q \in L_\omega$, and the functions $u, v \in AC'([0,\omega])$ satisfy

$$u''(t) \geq p(t)u(t) + q(t), \quad v''(t) \leq p(t)v(t) + q(t) \quad \text{for } t \in [0,\omega],$$
$$u(0) = u(\omega), \quad u'(0) \geq u'(\omega), \quad v(0) = v(\omega), \quad v'(0) \leq v'(\omega).$$

Then, by virtue of Proposition 15.2 (respectively, Proposition 15.4), the inequality

$$u(t) \leq v(t) \quad \text{for } t \in [0,\omega] \quad \big(\text{respectively, } u(t) \geq v(t) \quad \text{for } t \in [0,\omega]\big)$$

holds.

Chapter 3

Periodic Boundary Value Problem

16. Positive Solutions of Linear Problem

Consider the problem

$$u'' = p(t)u + q(t); \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega), \tag{16.1}$$

where $p, q \in L_\omega$. Recall that under a solution of the problem (16.1) we understand a function $u \in \widetilde{AC}'([0,\omega])$ satisfying given equation almost everywhere in $[0,\omega]$ and boundary conditions. In this chapter, we will deal with the existence of a **positive** solution of the problem (16.1). Introduce the definition

Definition 16.1. We say that the vector function $(p,q)\colon [0,\omega] \to \mathbb{R}^2$ belongs to the set $\mathcal{U}(\omega)$ if the problem (16.1) is uniquely solvable and its solution is **positive**.

As it was mentioned in Remark 8.2 and Remark 9.2 if $q \not\equiv 0$ then each of the conditions

$$q(t) \leq 0 \quad \text{for } \ t \in [0,\omega], \ \ p \in \mathcal{V}^-(\omega)$$

and

$$q(t) \geq 0 \quad \text{for } \ t \in [0,\omega], \ \ p \in \mathcal{V}^+(\omega)$$

guarantee the inclusion $(p,q) \in \mathcal{U}(\omega)$. Results stated below cover also the case when the function q is not of a constant sign.

Recall that the numbers Q_+, Q_- and $\rho(p)$ are defined by (0.13) and (0.12), respectively.

Theorem 16.2. *Let $p \in \mathcal{V}^-(\omega)$, $q \not\equiv 0$, and*

$$Q_- \geq \rho(p) Q_+. \tag{16.2}$$

Then $(p,q) \in \mathcal{U}(\omega)$. Moreover, the unique solution u of the problem (16.1) *satisfies the estimate*

$$u(t) > \big(Q_- - \rho(p)Q_+\big)\Big(\rho(p)\big\|[p]_+\big\|_L - \big\|[p]_-\big\|_L\Big)^{-1} \quad \textit{for } \ t \in [0,\omega]. \tag{16.3}$$

Proof. In view of Proposition 10.8 we have that $\big\|[p]_+\big\|_L > \big\|[p]_-\big\|_L$. Consequently, $\rho(p)\big\|[p]_+\big\|_L - \big\|[p]_-\big\|_L > 0$ and $[p]_+ \not\equiv 0$. Hence, in view of Remark 8.4 we have $[p]_+ \in \mathcal{V}^-(\omega)$. Introduce the notation

$$c \overset{\text{def}}{=} \big(Q_- - \rho(p)Q_+\big)\Big(\rho(p)\big\|[p]_+\big\|_L - \big\|[p]_-\big\|_L\Big)^{-1}. \tag{16.4}$$

Clearly, $c \geq 0$. Since $[p]_+ \in \mathcal{V}^-(\omega)$ the problem

$$\alpha'' = [p(t)]_+\alpha - c[p(t)]_- + q(t); \quad \alpha(0) = \alpha(\omega), \ \ \alpha'(0) = \alpha'(\omega) \tag{16.5}$$

possesses a unique solution α. Suppose that

$$m = \min\big\{\alpha(t) : \ t \in [0,\omega]\big\}$$

and choose $a \in [0,\omega[$ such that

$$\alpha(a) = m. \tag{16.6}$$

Denote by v the solution of the problem

$$v'' = [p(t)]_+ v; \quad v(a) = 1, \ \ v(a+\omega) = 1. \tag{16.7}$$

By virtue of Proposition 6.11, v satisfies the inequalities

$$\frac{\omega}{v_2(a)} < v(t) < \frac{\omega}{v_2(a)}\,\rho(p) \quad \text{for } t \in\,]a, a+\omega[\,, \tag{16.8}$$

where v_2 is a solution of the problem

$$v_2'' = [p(t)]_+ v_2; \quad v_2(a+\omega) = 0, \;\; v_2'(a+\omega) = -1.$$

It follows from (16.7), in view of (16.8), that

$$0 < v'(a+\omega) - v'(a) = \int\limits_a^{a+\omega} [p(s)]_+ v(s)\,\mathrm{d}s < \frac{\omega}{v_2(a)}\,\rho(p)\big\|[p]_+\big\|_L. \tag{16.9}$$

On the other hand, it is clear that

$$\big(v'(t)\alpha(t) - v(t)\alpha'(t)\big)' = c[p(t)]_- v(t) - q(t)v(t) \quad \text{for } t \in [a, a+\omega].$$

Integration of this equality on $[a, a+\omega]$ yields

$$m\big(v'(a+\omega) - v'(a)\big) = c\int\limits_a^{a+\omega} [p(s)]_- v(s)\,\mathrm{d}s - \int\limits_a^{a+\omega} q(s)v(s)\,\mathrm{d}s.$$

Hence, on account of (16.8) and the condition $[q]_- \not\equiv 0$, we get

$$m\big(v'(a+\omega) - v'(a)\big) > \frac{\omega}{v_2(a)}\Big(c\big\|[p]_-\big\|_L + Q_- - \rho(p)Q_+\Big)$$

which, together with (16.9), results in

$$\alpha(t) > c \quad \text{for } t \in [0,\omega]. \tag{16.10}$$

In view of (16.10), it follows from (16.5) that

$$\begin{gathered}\alpha''(t) \geq p(t)\alpha(t) + q(t) \quad \text{for } t \in [0,\omega], \\ \alpha(0) = \alpha(\omega), \quad \alpha'(0) = \alpha'(\omega).\end{gathered} \tag{16.11}$$

Let now u be a solution of the problem (16.1). Since $p \in \mathcal{V}^-(\omega)$ and (16.11) holds we get, by virtue of Remark 0.6, that $u(t) \geq \alpha(t)$ for $t \in [0,\omega]$ which, together with (16.10), implies the desired estimate (16.3). □

Remark 16.3. Condition (16.2) in Theorem 16.2 is optimal and cannot be weaken to the inequality

$$Q_- \geq (1-\varepsilon)\rho(p)Q_+ \tag{16.12}$$

no matter how small $\varepsilon \in\,]0,1[$ is. Indeed, let $\varepsilon \in\,]0,1[$ and $\delta > 0$ be such that

$$\mathrm{e}^{\pi^2\delta} = \frac{1}{1-\varepsilon}\,.$$

Put $\omega = 2\pi$,

$$p(t) \overset{\text{def}}{=} \delta, \quad q(t) \overset{\text{def}}{=} (1+\delta)\cos t - (1-\varepsilon)\delta.$$

Since $\delta > 0$, in view of Remark 8.4, we have $p \in \mathcal{V}^-(\omega)$. By direct calculation one can easily verify that (16.12) holds. On the other hand, the function $u(t) \overset{\text{def}}{=} 1 - \varepsilon - \cos t$ for $t \in [0,\omega]$ is a solution of the problem (16.1) and its minimum is negative. Consequently, $(p,q) \notin \mathcal{U}(\omega)$.

Before the formulation of the next result we mention that if $p \in \operatorname{Int}\mathcal{V}^+(\omega)$ then, in view of Theorem 9.1′, $p \in \operatorname{Int}\mathcal{D}(\omega)$ as well. It allows us to use the number $\nu^*(p)$ defined by (6.22) in formulation of the next result.

Theorem 16.4. *Let $p \in \operatorname{Int}\mathcal{V}^+(\omega)$, $q \not\equiv 0$, and*

$$Q_+ \geq \nu^*(p)\rho(p)Q_-. \tag{16.13}$$

Then $(p,q) \in \mathcal{U}(\omega)$. Moreover,

$$\nu^*(p)\rho(p)\big\|[p]_-\big\|_L > \big\|[p]_+\big\|_L \tag{16.14}$$

and the unique solution u *of the problem* (16.1) *satisfies the estimate*

$$u(t) > \big(Q_+ - \nu^*(p)\rho(p)Q_-\big)\Big(\nu^*(p)\rho(p)\big\|[p]_-\big\|_L - \big\|[p]_+\big\|_L\Big)^{-1} \quad \textit{for } t \in [0,\omega]. \tag{16.15}$$

Proof. Let u be a solution of the problem (16.1) and

$$m \overset{\text{def}}{=} \min\big\{u(t) : \ t \in [0,\omega]\big\}.$$

Choose $a \in [0,\omega[$ such that

$$u(a) = m.$$

Since $p \in \operatorname{Int}\mathcal{V}^+(\omega)$, we have that $[p]_- \not\equiv 0$ (because otherwise, in view of Remark 8.4, we get $p \in \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$). Moreover, in view of Theorem 9.1′, $p \in \operatorname{Int}\mathcal{D}(\omega)$ as well. Hence, by virtue of Proposition 6.12 (and (6.22)), the unique solution v of the problem

$$v'' = p(t)v; \quad v(a) = 1, \ \ v(a+\omega) = 1 \tag{16.16}$$

satisfies the estimates

$$\frac{1}{\rho(p)} < v(t) \le \nu^*(p) \quad \text{for } \ t \in [a, a+\omega]. \tag{16.17}$$

It is clear that $v'(a) \ne v'(a+\omega)$ (because otherwise $p \in \mathcal{V}_0(\omega)$). Moreover,

$$v'(a) > v'(a+\omega), \tag{16.18}$$

because otherwise, in view of Theorem 8.3 (with $\gamma \equiv v$), we get $p \in \mathcal{V}^-(\omega)$. On the other hand, it follows from (16.16) that

$$v'(a) - v'(a+\omega) = \int\limits_a^{a+\omega} [p(s)]_- v(s)\,\mathrm{d}s - \int\limits_a^{a+\omega} [p(s)]_+ v(s)\,\mathrm{d}s$$

which, together with (16.17) and (16.18) imply

$$0 < v'(a) - v'(a+\omega) \le \nu^*(p)\big\|[p]_-\big\|_L - \frac{1}{\rho(p)}\,\big\|[p]_+\big\|_L. \tag{16.19}$$

Consequently, the inequality (16.14) holds.

It follows from (16.1) and (16.16) that

$$\big(u'(t)v(t) - u(t)v'(t)\big)' = q(t)v(t) \quad \text{for } \ t \in [a, a+\omega].$$

The integration of this equality on $[a, a+\omega]$ results in

$$m\big(v'(a) - v'(a+\omega)\big) = \int\limits_a^{a+\omega} [q(s)]_+ v(s)\,\mathrm{d}s - \int\limits_a^{a+\omega} [q(s)]_- v(s)\,\mathrm{d}s. \tag{16.20}$$

Since $q \not\equiv 0$ and (16.13) holds, we have $[q]_+ \not\equiv 0$ as well. Hence, in view of (16.17), it follows from (16.20) that

$$m\big(v'(a) - v'(a+\omega)\big) > \frac{1}{\rho(p)}\,Q_+ - \nu^*(p)Q_-,$$

which, together with (16.19), yields the desired estimate (16.15). □

Remark 16.5. As it is clear from the proof of Theorem 16.4, the inclusion $p \in \operatorname{Int}\mathcal{V}^+(\omega)$ implies the validity of (16.14).

The next assertion follows immediately from Theorem 16.4.

Proposition 16.6. *Let* $p \in \operatorname{Int}\mathcal{V}^+(\omega)$. *Then there is a* $c > 0$ *such that, for any nontrivial function* $q \in L_\omega$ *satisfying* $Q_+ \ge cQ_-$, *the unique solution* u *of the problem* (16.1) *is positive (and, consequently* $(p,q) \in \mathcal{U}(\omega)$*).*

Mention that the assumption $p \in \operatorname{Int}\mathcal{V}^+(\omega)$ in Proposition 16.6 is optimal and cannot be weaken. More precisely

Proposition 16.7. *Let $p \in \mathcal{V}^+(\omega) \setminus \operatorname{Int}\mathcal{V}^+(\omega)$. Then, for any $r > 0$, there exists a function $q \in L_\omega$ such that*
$$Q_+ > rQ_- \tag{16.21}$$
and the unique solution u of the problem (16.1) *satisfies*
$$\min\big\{u(t) : \ t \in [0,\omega]\big\} < 0. \tag{16.22}$$

Proof. In view of Proposition 10.5, the inclusion $p \in \partial\mathcal{D}(\omega)$ holds. Hence, by virtue of Proposition 2.3, there is an $a \in [0,\omega[$ such that the problem
$$v'' = p(t)v; \quad v(a) = 0, \ \ v(a+\omega) = 0 \tag{16.23}$$
possesses a solution v such that
$$v(t) > 0 \quad \text{for} \ \ t \in \,]a, a+\omega[\,. \tag{16.24}$$
Clearly,
$$v'(a) > v'(a+\omega). \tag{16.25}$$

Let $r > 0$ be fixed. Since
$$\lim_{x\to 0+} \frac{1}{x} \int_a^{a+x} v(s)\,\mathrm{d}s = 0, \tag{16.26}$$
there is a $x_0 \in]0,\omega[$ such that
$$\frac{r}{x_0} \int_a^{a+x_0} v(s)\,\mathrm{d}s < \frac{1}{\omega} \int_{a+x_0}^{a+\omega} v(s)\,\mathrm{d}s. \tag{16.27}$$
Set
$$q(t) = \begin{cases} \dfrac{r}{x_0} & \text{for} \ \ t \in [a, a+x_0[\,, \\ -\dfrac{1}{\omega} & \text{for} \ \ t \in \,]a+x_0, a+\omega] \end{cases}$$
and extend it periodically. Clearly, (16.21) holds. Moreover, in view of (16.27), we have
$$\int_a^{a+\omega} q(s)v(s)\,\mathrm{d}s < 0. \tag{16.28}$$

Let now u be a solution of the problem (16.1). Then, in view of (16.1) and (16.23), we get
$$\big(u'(t)v(t) - u(t)v'(t)\big)' = q(t)v(t) \quad \text{for} \ \ t \in [a, a+\omega]. \tag{16.29}$$
The integration of this equality on $[a, a+\omega]$ results in
$$u(a)\big(v'(a) - v'(a+\omega)\big) = \int_a^{a+\omega} q(s)v(s)\,\mathrm{d}s. \tag{16.30}$$
Hence, in view of (16.25) and (16.28) we get that $u(a) < 0$ and, consequently, (16.22) is fulfilled. $\square$

The next assertion also follows from Theorem 16.4.

Proposition 16.8. *Let $p \in \operatorname{Int}\mathcal{V}^+(\omega)$. Then there is a $c_0 > 0$ such that, for any nontrivial function $q \in L_\omega$ satisfying $q(t) \geq 0$ for $t \in [0,\omega]$, the unique solution u of the problem* (16.1) *admits the estimate*
$$u(t) > c_0\|q\|_L \quad \textit{for} \ \ t \in [0,\omega].$$

Mention that the assumption $p \in \operatorname{Int}\mathcal{V}^+(\omega)$ in Proposition 16.8 is optimal and cannot be weaken. More precisely

Proposition 16.9. *Let $p \in \mathcal{V}^+(\omega) \setminus \operatorname{Int}\mathcal{V}^+(\omega)$. Then, for any $\varepsilon > 0$, there exists a function $q \in L_\omega$ such that*
$$q(t) \geq 0 \quad \textit{for} \ \ t \in [0,\omega], \ \ \|q\|_L = 1, \tag{16.31}$$
and the unique solution u of the problem (16.1) *satisfies*
$$\min\big\{u(t) : \ t \in [0,\omega]\big\} < \varepsilon. \tag{16.32}$$

Proof. In view of Proposition 10.5, we have $p \in \partial\mathcal{D}(\omega)$. Hence, there is an $a \in [0,\omega[$ such that the problem (16.23) possesses a solution v satisfying (16.24). Clearly, (16.25) and (16.26) hold. Put

$$\delta = v'(a) - v'(a+\omega) \tag{16.33}$$

and fix $\varepsilon > 0$. In view of (16.26), there is a $x_0 \in]0,\omega[$ such that

$$\frac{1}{x_0}\int_a^{a+x_0} v(s)\,\mathrm{d}s < \varepsilon\delta. \tag{16.34}$$

Set

$$q(t) = \begin{cases} \dfrac{1}{x_0} & \text{for } t \in [a, a+x_0[\,, \\ 0 & \text{for } t \in]a+x_0, a+\omega] \end{cases} \tag{16.35}$$

and extend it periodically. It is clear that (16.31) holds.

Let now u be a solution of the problem (16.1). Then (16.29) is fulfilled and, consequently, (16.30) holds as well. Hence, in view of (16.33)–(16.35) we get

$$\delta u(a) = \frac{1}{x_0}\int_a^{a+x_0} v(s)\,\mathrm{d}s < \varepsilon\delta$$

which implies (16.32). □

Corollary 16.10. *Let $p \in \operatorname{Int}\mathcal{V}^+(\omega)$, $-[p]_- \in \operatorname{Int}\mathcal{D}(\omega)$, $q \not\equiv 0$ and*

$$Q_+ \geq \nu^*(-[p]_-)Q_-. \tag{16.36}$$

Then $(p,q) \in \mathcal{U}(\omega)$. Moreover, the unique solution u of the problem (16.1) *admits the estimate*

$$u(t) > \frac{Q_+ - \nu^*(-[p]_-)\rho(p)Q_-}{\nu^*(-[p]_-)\rho(p)\big\|[p]_-\big\|_L} \quad \textit{for } t \in [0,\omega]. \tag{16.37}$$

Proof. On account of Theorem 9.3, we have $-[p]_- \in \operatorname{Int}\mathcal{V}^+(\omega)$. Hence, by virtue of Theorem 16.4, the unique solution u_0 of the problem

$$u_0'' = -[p(t)]_-u_0 + q(t); \quad u_0(0) = u_0(\omega), \ \ u_0'(0) = u_0'(\omega) \tag{16.38}$$

(is positive and) satisfies the estimate

$$u_0(t) > \frac{Q_+ - \nu^*(-[p]_-)\rho(p)Q_-}{\nu^*(-[p]_-)\rho(p)\big\|[p]_-\big\|_L} \quad \text{for } t \in [0,\omega]. \tag{16.39}$$

Since u_0 is positive, we get from (16.38) that

$$u_0''(t) \leq p(t)u_0(t) + q(t) \quad \text{for } t \in [0,\omega]; \quad u_0(0) = u_0(\omega), \quad u_0'(0) = u_0'(\omega). \tag{16.40}$$

Let now u be a solution of the problem (16.1). By virtue of (16.1) and (16.40), it follows from Remark 0.6 that

$$u(t) \geq u_0(t) \quad \text{for } t \in [0,\omega]$$

which, together with (16.39), yields the desired estimate (16.37). □

Next corollary follows from Corollary 16.10, Proposition 6.5, and Proposition 6.8.

Corollary 16.11. *Let $p \in L_\omega$, $p \not\equiv 0$, $\overline{p} \leq 0$, $q \not\equiv 0$,*

$$\big\|[p]_-\big\|_L < \frac{4}{\omega} + \frac{p^*}{4\omega}\big\|[p]_-\big\|_L^2, \tag{16.41}$$

and

$$Q_+ \geq c_0 Q_-, \tag{16.42}$$

where

$$c_0 = 1 + \big\|[p]_-\big\|_L\Big(\frac{4}{\omega} + \frac{p^*}{4\omega}\big\|[p]_-\big\|_L^2 - \big\|[p]_-\big\|_L\Big)^{-1}.$$

Then $(p,q) \in \mathcal{U}(\omega)$. *Moreover, the unique solution* u *of the problem* (16.1) *admits the estimate*

$$u(t) > \frac{1}{c_0\big\|[p]_-\big\|_L}\,(Q_+ - c_0Q_-) \quad \textit{for } t \in [0,\omega]. \tag{16.43}$$

Next assertion follows from Corollary 16.10, Proposition 6.6, and Proposition 6.8.

Corollary 16.12. *Let* $p \in L_\omega$, $p \not\equiv 0$, $\overline{p} \le 0$, $[p]_-^2 \in L_\omega$, $q \not\equiv 0$,

$$k^*(\omega)\big\|[p]_-^2\big\|_L < 1, \tag{16.44}$$

and

$$Q_+ \ge c_1Q_-, \tag{16.45}$$

where

$$c_1 = 1 + \frac{\omega}{4}\,\big\|[p]_-\big\|_L\Big(1 - \sqrt{k^*(\omega)\big\|[p]_-^2\big\|_L}\,\Big)^{-1}.$$

Then $(p,q) \in \mathcal{U}(\omega)$. *Moreover, the unique solution* u *of the problem* (16.1) *admits the estimate*

$$u(t) > \frac{1}{c_1\big\|[p]_-\big\|_L}\,(Q_+ - c_1Q_-) \quad \textit{for } t \in [0,\omega]. \tag{16.46}$$

Remark 16.13. Conditions (16.42) and (16.45) in Corollaries 16.11 and 16.12 are optimal and cannot be weaken to the conditions

$$Q_+ \ge (1-\varepsilon)c_0Q_-, \ \text{ resp. } \ Q_+ \ge (1-\varepsilon)c_1Q_-, \tag{16.47}$$

no matter how small $\varepsilon \in]0,1[$ is. Indeed, let $\omega > 0$ and $\varepsilon \in]0,1[$ be fixed. Put $p(t) \stackrel{\text{def}}{=} -\delta$ for $t \in \mathbb{R}$. By direct calculation one can easily verify that there is a $\delta > 0$, small enough, such that (16.41) and (16.44) are fulfilled and

$$(1-\varepsilon)c_0 < 1, \quad (1-\varepsilon)c_1 < 1. \tag{16.48}$$

Let now $q \in L_\omega$ be such that $q \not\equiv 0$ and $Q_+ = Q_-$. In view of (16.48), the inequalities (16.47) are fulfilled. Finally, let u be a solution of the problem (16.1). Clearly, $u \not\equiv 0$ (because otherwise $q \equiv 0$) and

$$\delta\int_0^\omega u(s)\,\mathrm{d}s = \int_0^\omega q(s)\,\mathrm{d}s = Q_+ - Q_- = 0.$$

Hence, $\min\{u(t) : t \in [0,\omega]\} < 0$ and, consequently, $(p,q) \notin \mathcal{U}(\omega)$.

Next assertion follows from Proposition 6.9, Theorem 12.3 and Theorem 16.4. Mention that, in contrast to the previous assertions, it does not exclude the case when $\overline{p} > 0$.

Corollary 16.14. *Let* $p, q \in L_\omega$, $p \not\equiv 0$, $q \not\equiv 0$,

$$\ell^2\Big(1 - \frac{\pi^2}{(\mathrm{e}^{\omega\ell}-1)^2}\Big) < \overline{p} \le \frac{\ell}{\omega(\mathrm{e}^{\omega\ell}-1)}\bigg(\int_0^\omega |\ell(p)(s)|\,\mathrm{d}s\bigg)^2,$$

and

$$Q_+ \ge c\rho(p)Q_-,$$

where

$$c \stackrel{\text{def}}{=} \frac{\mathrm{e}^{\omega\ell}}{1-c_0}\,, \quad c_0 \stackrel{\text{def}}{=} \frac{\mathrm{e}^{\omega\ell}-1}{\pi\ell}\,\sqrt{\ell^2 - \overline{p}}\,.$$

Then $(p,q) \in \mathcal{U}(\omega)$.

Before the formulation of the next result introduce the notations. Let $p \in \mathcal{D}(\omega)$ and $a \in [0,\omega]$. Denote by v_{1a} and v_{2a} solutions of the problems

$$v_{1a}'' = p(t)v_{1a}; \quad v_{1a}(a) = 0, \ \ v_{1a}'(a) = 1, \tag{16.49}$$

$$v_{2a}'' = p(t)v_{2a}; \quad v_{2a}(a+\omega) = 0, \ \ v_{2a}'(a+\omega) = -1, \tag{16.50}$$

respectively. Since $p \in \mathcal{D}(\omega)$ it is clear that

$$v_{1a}(t) > 0, \quad v_{2a}(t) > 0 \quad \text{for } \ t \in]a, a+\omega[\,. \tag{16.51}$$

Let now $\nu \in]0,1/2[$. Then it is clear that $v_{ia}^{-\frac{\nu}{1-\nu}} \in L([a,a+\omega])$, $i=1,2$. Put

$$\mu_\nu^*(p) \overset{\text{def}}{=} \sup\left\{ \|v_{ia}\|_{C([a,a+\omega])} \left(\int_a^{a+\omega} \frac{1}{[v_{ia}(s)]^{\frac{\nu}{1-\nu}}} \right)^{\frac{1-\nu}{\nu}} : \ a \in [0,\omega], \ \ i=1,2 \right\}. \tag{16.52}$$

It is not difficult to verify that $\mu_\nu^*(p)$ is a finite number.

As it was mentioned above (see Proposition 16.7) if $p \in \mathcal{V}^+(\omega) \setminus \operatorname{Int}\mathcal{V}^+(\omega)$ then the assumption like (16.21) does not guarantee the inclusion $(p,q) \in \mathcal{U}(\omega)$. However, the following theorem is true.

Theorem 16.15. *Let $p \in \mathcal{V}^+(\omega)$, $\nu \in]0,1/2[$, and the function $q \in L_\omega$ satisfy the inequality*

$$\big\|[q]_+\big\|_{L^\nu} > \mu_\nu^*(p)\big\|[q]_-\big\|_L. \tag{16.53}$$

Then $(p,q) \in \mathcal{U}(\omega)$. Moreover, the unique solution u of the problem (16.1) *admits the estimate*

$$u(t) \geq \frac{1}{\mu_\nu^*(p)\big\|[p]_-\big\|_L}\Big(\|[q]_+\|_{L^\nu} - \mu_\nu^*(p)\big\|[q]_-\big\|_L\Big) \quad \textit{for } t \in [0,\omega]. \tag{16.54}$$

Proof. Let $a \in [0,\omega]$. If $v_{1a}(a+\omega)=0$ then $v_{2a}(a+\omega)=0$ as well and, in view of (16.51),

$$v_{1a}'(a+\omega) - v_{1a}'(a) < 0 \ \text{ and } \ v_{2a}'(a+\omega) - v_{2a}'(a) < 0.$$

On the other hand, if $v_{1a}(a+\omega) \neq 0$ then, in view of the equality

$$v_{1a}'(t)v_{2a}(t) - v_{1a}(t)v_{2a}'(t) = Const.$$

we get that $v_{1a}(a+\omega) = v_{2a}(a)$. Put $\gamma(t) \overset{\text{def}}{=} v_{1a}(t) + v_{2a}(t)$ for $t \in [a,a+\omega]$. It is clear that

$$\gamma''(t) = p(t)\gamma(t), \quad \gamma(t) > 0 \quad \text{for } t \in [a,a+\omega], \quad \gamma(a) = \gamma(a+\omega),$$

and $\gamma'(a) \neq \gamma'(a+\omega)$ (since otherwise $p \in \mathcal{V}_0(\omega)$). Hence, in view of Theorem 8.3 and Remark 8.6, we get that $\gamma'(a+\omega) < \gamma'(a)$ (since otherwise $p \in \mathcal{V}^-(\omega)$). Therefore, $v_{1a}'(a+\omega) - v_{1a}'(a) < v_{2a}'(a) - v_{2a}'(a+\omega)$. Consequently, either

$$v_{1a}'(a+\omega) - v_{1a}'(a) < 0, \tag{16.55}$$

or

$$v_{2a}'(a+\omega) - v_{2a}'(a) < 0. \tag{16.56}$$

Thus we have proved that for any $a \in [0,\omega]$, either (16.55) or (16.56) holds.

Let now u be a solution of the problem (16.1), where $q \in L_\omega$ satisfies (16.53). Put $m \overset{\text{def}}{=} \min\{u(t) : t \in [0,\omega]\}$ and choose $a \in [0,\omega[$ such that

$$u(a) = m. \tag{16.57}$$

Clearly,

$$u'(a) = 0. \tag{16.58}$$

Suppose without loss of generality that (16.55) holds. It is clear that,

$$\big(u'(t)v_{1a}(t) - u(t)v_{1a}'(t)\big)' = q(t)v_{1a}(t) \quad \text{for } t \in [a,a+\omega].$$

Integration of this equality on $[a,a+\omega]$, together with (16.57) and (16.58), yields

$$-m\big(v_{1a}'(a+\omega) - v_{1a}'(a)\big) = \int_a^{a+\omega} [q(s)]_+ v_{1a}(s)\,\mathrm{d}s - \int_a^{a+\omega} [q(s)]_- v_{1a}(s)\,\mathrm{d}s. \tag{16.59}$$

By virtue of Hölder's inequality we have

$$\big\|[q]_+\big\|_{L^\nu} \leq A \int_a^{a+\omega} [q(s)]_+ v_{1a}(s)\,\mathrm{d}s,$$

where

$$A \overset{\text{def}}{=} \left(\int_a^{a+\omega} \frac{1}{[v_{1a}(s)]^{\frac{\nu}{1-\nu}}} \right)^{\frac{1-\nu}{\nu}}.$$

On the other hand,

$$\int_a^{a+\omega} [q(s)]_- v_{1a}(s)\,\mathrm{d}s \le \big\|[q]_-\big\|_L \, \|v_{1a}\|_{C([a,a+\omega])}.$$

Taking, moreover, into account (16.55), we get from (16.59) that

$$m\big|v_{1a}'(a+\omega) - v_{1a}'(a)\big| \ge \frac{1}{A}\Big(\big\|[q]_+\big\|_{L^\nu} - A\big\|[q]_-\big\|_L \|v_{1a}\|_{C([a,a+\omega])}\Big). \tag{16.60}$$

On the other hand, in view of (16.51) and (16.55), it follows from (16.49) that

$$\big|v_{1a}'(a+\omega) - v_{1a}'(a)\big| = -\int_a^{a+\omega} p(s)v_{1a}(s)\,\mathrm{d}s \le \int_a^{a+\omega} [p(s)]_- v_{1a}(s)\,\mathrm{d}s \le \big\|[p]_-\big\|_L \, \|v_{1a}\|_{C([a,a+\omega])}.$$

Taking, moreover, into account (16.52) and (16.53), we get from (16.60) that (16.54) holds and, consequently, $(p,q) \in \mathcal{U}(\omega)$. □

Theorem 16.15, together with Proposition 4.3, implies

Corollary 16.16. *Let $p \in \mathcal{V}^+(\omega)$, $\nu \in]0,1/2[$, and*

$$\omega\big\|[p]_+\big\|_L < \frac{(1-\nu)(1-2\nu)}{\nu^2}\,.$$

Let, moreover,

$$\big\|[q]_+\big\|_{L^\nu} > (\omega r)^{\frac{1-\nu}{\nu}} \, \big\|[q]_-\big\|_L,$$

where

$$r \stackrel{\text{def}}{=} (1-\nu)\bigg((1-\nu)^2 - \nu\sqrt{(1-\nu)^2 + (1-\nu)\omega\big\|[p]_+\big\|_L}\bigg)^{-1}.$$

Then $(p,q) \in \mathcal{U}(\omega)$.

Remark 16.17. Let $p(t) \stackrel{\text{def}}{=} -\frac{\pi^2}{\omega^2}$. Then $p \in \mathcal{V}^+(\omega)$, $v_{1a}(t) = v_{2a}(t)$, and $v_{1a}(t) = \frac{\omega}{\pi}\sin(\frac{\pi}{\omega}(t-a))$ for $t \in \mathbb{R}$. One can easily verify that

$$\mu_\nu^*(p) = \pi^{\frac{1-\nu}{2\nu}} \Big(\frac{\omega}{\pi}\Big)^{\frac{1-\nu}{\nu}} \left(\frac{\Gamma\big(\frac{1-2\nu}{2(1-\nu)}\big)}{\Gamma\big(\frac{2-3\nu}{2(1-\nu)}\big)}\right)^{\frac{1-\nu}{\nu}},$$

where Γ is the Gamma function of Euler.

In particular, for $\nu = 1/3$ we get

$$\mu_{\frac{1}{3}}^*(p) = \frac{1}{\pi}\,\omega^2\left(\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}\right)^2.$$

Therefore, Theorem 16.15 implies

Corollary 16.18. *Let*

$$\big\|[q]_+\big\|_{L^{\frac{1}{3}}} > \frac{\omega^2}{\pi}\left(\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}\right)^2 \big\|[q]_-\big\|_L.$$

Then $(-\frac{\pi^2}{\omega^2}, q) \in \mathcal{U}(\omega)$.

Remark 16.19. Consider the problem

$$u'' = p(t)u + q(t); \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega) + c. \tag{16.61}$$

In view of Remark 15.3 (respectively, Remark 15.5), it is clear that if $p \in \mathcal{V}^-(\omega)$ (respectively, $p \in \mathcal{V}^+(\omega)$), $(p,q) \in \mathcal{U}(\omega)$, and $c \le 0$ (respectively, $c \ge 0$), then the unique solution of the problem (16.61) is positive.

Definition 16.20. We say that the vector function $(p,q)\colon [0,\omega] \to \mathbb{R}^2$ belongs to the set $\mathcal{U}_0(\omega)$ if the problem (16.1) is uniquely solvable and its solution is **nonnegative**.

Remark 16.21. It is clear that $\mathcal{U}(\omega) \subset \mathcal{U}_0(\omega)$. Let the problem

$$u'' = p(t)u; \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega), \tag{16.62}$$

has no nontrivial solution, $q_0 \in L_\omega$, $q_0 \not\equiv Const.$ Denote by u_0 and v_0 solutions of the problems

$$u_0'' = p(t)u_0 + 1; \quad u_0(0) = u_0(\omega), \;\; u_0'(0) = u_0'(\omega),$$
$$v_0'' = p(t)v_0 + q_0(t); \quad v_0(0) = v_0(\omega), \;\; v_0'(0) = v_0'(\omega).$$

Put $m \overset{\text{def}}{=} \min\{u_0(t) : t \in [0,\omega]\}$,

$$\lambda \overset{\text{def}}{=} \min\left\{ \frac{v_0(t)}{u_0(t) - m + 1} : \; t \in [0,\omega] \right\},$$

and

$$w(t) \overset{\text{def}}{=} v_0(t) - \lambda\big(u_0(t) - m + 1\big) \in [0,\omega].$$

It is clear that, $w(t) \geq 0$ for $t \in [0,\omega]$, $w \not\equiv 0$, and there is a $t_0 \in [0,\omega]$ such that $w(t_0) = 0$. On the other hand, by direct calculation one can easily verify that the function w is a solution of the problem (16.1) with $q(t) \overset{\text{def}}{=} q_0(t) - \lambda(m-1)p(t) + \lambda$. Clearly, $q \not\equiv 0$ because otherwise we get that $w \equiv 0$. Thus we have shown that if the problem (16.62) has no nontrivial solution then there exists a function $q \in L_\omega$ such that $q \not\equiv 0$ and $(p,q) \in \mathcal{U}_0(\omega) \setminus \mathcal{U}(\omega)$.

Proposition 16.22. *The inclusion $(p,q) \in \mathcal{U}_0(\omega)$ holds if and only if there exists a sequence $\{q_n\}_{n=1}^{+\infty} \subset L_\omega$ such that $(p,q_n) \in \mathcal{U}(\omega)$ for $n \in \mathbb{N}$ and*

$$\lim_{n\to+\infty} \|q_n - q\|_L = 0. \tag{16.63}$$

Proof. Let $(p,q) \in \mathcal{U}_0(\omega)$ and u be a solution of the problem (16.1). Put $q_n(t) \overset{\text{def}}{=} q(t) - \frac{1}{n}\,p(t)$ for $t \in [0,\omega]$, $n \in \mathbb{N}$. Clearly, (16.63) holds. On the other hand, since the function $v(t) \overset{\text{def}}{=} u(t) + \frac{1}{n}$ for $t \in [0,\omega]$ is a solution of the problem

$$v'' = p(t)v + q_n(t); \quad v(0) = v(\omega), \;\; v'(0) = v'(\omega)$$

and $v(t) > 0$ for $t \in [0,\omega]$, we get $(p,q) \in \mathcal{U}(\omega)$ for $n \in \mathbb{N}$ as well.

Let now $\{q_n\}_{n=1}^{+\infty} \subset L_\omega$, $(p,q_n) \in \mathcal{U}(\omega)$ for $n \in \mathbb{N}$, and (16.63) hold. Denote by u and u_n solutions of the problems

$$u'' = p(t)u + q(t); \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega),$$
$$u_n'' = p(t)n_n + q_n(t); \quad u_n(0) = u_n(\omega), \;\; u_n'(0) = u_n'(\omega),$$

respectively. Since $u_n(t) > 0$ for $t \in [0,\omega]$, $n \in \mathbb{N}$, and $\lim\limits_{n\to+\infty} \|u_n - u\|_C = 0$ (see Lemma 3.1) we get that $u(t) \geq 0$ for $t \in [0,\omega]$ and, consequently, $(p,q) \in \mathcal{U}_0(\omega)$. ☐

17. Solvability of Nonlinear Problem

Consider the problem

$$u'' = f(t,u); \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega), \tag{17.1}$$

where $f \in K([0,\omega] \times \mathbb{R}; \mathbb{R})$. Under a solution of the problem (17.1) we understand a function $u \in AC'([0,\omega])$ satisfying given equation for almost all $t \in [0,\omega]$ and boundary conditions.

Theorem 17.1. *Let the inequality*

$$f(t,x)\operatorname{sgn} x \geq p(t)|x| - q(t,|x|) \quad \text{for } \; t \in [0,\omega], \;\; x \in \mathbb{R} \tag{17.2}$$

be fulfilled, where $q \in K_{sl}([0,\omega] \times \mathbb{R}; \mathbb{R}_+)$. If, moreover,

$$p \in \mathcal{V}^-(\omega), \tag{17.3}$$

then the problem (17.1) *has at least one solution.*

Before proving of Theorem 17.1 recall two well-known results from the theory of periodic boundary value problems (see, e.g., [4]).

Proposition 17.2. *Let there exist $\alpha, \beta \in \widetilde{AC}'([0,\omega])$ such that*

$$\alpha(t) \leq \beta(t) \quad \text{for } t \in [0,\omega], \tag{17.4}$$

$$\alpha''(t) \geq f(t, \alpha(t)) \quad \text{for } t \in [0,\omega], \quad \alpha(0) = \alpha(\omega), \ \ \alpha'(0) \geq \alpha'(\omega), \tag{17.5}$$

$$\beta''(t) \leq f(t, \beta(t)) \quad \text{for } t \in [0,\omega], \quad \beta(0) = \beta(\omega), \ \ \beta'(0) \leq \beta'(\omega). \tag{17.6}$$

Then the problem (17.1) *has at least one solution u satisfying*

$$\alpha(t) \leq u(t) \leq \beta(t) \quad \text{for } t \in [0,\omega]. \tag{17.7}$$

Proposition 17.3. *Let the problem*

$$u'' = p(t)u; \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega)$$

have no nontrivial solution and $q \in K_{sl}([0,\omega] \times \mathbb{R}; \mathbb{R})$. Then the problem

$$u'' = p(t)u + q(t,u); \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega)$$

is solvable.

Proof of Theorem 17.1. By virtue of (17.3), Remark 0.5, and Proposition 17.3, the problems

$$\alpha'' = p(t)\alpha + q(t, |\alpha|); \quad \alpha(0) = \alpha(\omega), \ \ \alpha'(0) = \alpha'(\omega), \tag{17.8}$$

$$\beta'' = p(t)\beta - q(t, |\beta|); \quad \beta(0) = \beta(\omega), \ \ \beta'(0) = \beta'(\omega) \tag{17.9}$$

possess solutions α and β, respectively. Since $p \in \mathcal{V}^-(\omega)$ and $q \geq 0$ we have

$$\alpha(t) \leq 0 \leq \beta(t) \quad \text{for } t \in [0,\omega]. \tag{17.10}$$

On the other hand, in view of (17.2) and (17.8)–(17.10), clearly (17.4)–(17.6) are fulfilled. Consequently, by virtue of Proposition 17.2, the problem (17.1) has at least one solution. □

Remark 17.4. Condition (17.3) in Theorem 17.1 is optimal and cannot be weakened (even to $p \in \overline{\mathcal{V}^-(\omega)}$). Indeed, let $p \notin \mathcal{V}^-(\omega)$. Then, by virtue of Proposition 10.10, there is a $\widetilde{p} \in \mathcal{V}_0(\omega)$ such that $\widetilde{p}(t) \geq p(t)$ for $t \in [0,\omega]$. Let $f(t,x) \overset{\text{def}}{=} \widetilde{p}(t)x + 1$ for $t \in [0,\omega]$, $x \in \mathbb{R}$. Clearly, (17.2) is fulfilled (with $q(t,x) \overset{\text{def}}{=} 1$). However, by virtue of Fredholm's third theorem, the problem (17.1) has no solution.

Remark 17.5. Theorem 17.1 together with the results of Section 11 implies several efficient conditions for solvability of the problem (17.1), which generalize and make more complete previously known ones. For example, Theorem 17.1 and Remark 8.4 imply Theorem VII-1.1 in [4] which in its turn improves results of Mawhin [19], while Theorems 17.1 and 11.1 improve results of [8] (see also Theorem VII-1.2 in [4]).

Consider again the problem (17.1) and suppose that

$$f \in K_{loc}([0,\omega] \times]0, +\infty[\,; \mathbb{R}). \tag{17.11}$$

Recall that under a solution of (17.1) now we understand a **positive** function $u \in \widetilde{AC}'([0,\omega])$ satisfying the given equation for almost all $t \in [0,\omega]$ and the boundary conditions. The setting of the problem does not exclude the case when the function f has a singularity in the second variable (for $u = 0$). In this case the problem (17.1) is called phase singular.

Proposition 17.6. *Let* (17.11) *hold and there exist $\alpha, \beta \in \widetilde{AC}'([0,\omega])$ such that $\alpha(t) > 0$ for $t \in [0,\omega]$ and* (17.4)*–*(17.6) *are satisfied. Then the problem* (17.1) *has at least one solution u satisfying* (17.7).

Proof. Let

$$\chi(t,x) \overset{\text{def}}{=} \frac{1}{2}\Big(|x - \alpha(t)| - |x - \beta(t)| + \alpha(t) + \beta(t)\Big) \quad \text{for } t \in [0,\omega], \ \ x \in \mathbb{R}, \tag{17.12}$$

$$\widetilde{f}(t,x) \overset{\text{def}}{=} f(t, \chi(t,x)) \quad \text{for } t \in [0,\omega], \ \ x \in \mathbb{R}. \tag{17.13}$$

Since $0 < \alpha(t) \leq \beta(t)$ for $t \in [0,\omega]$, we have that $\alpha(t) \leq \chi(t,x) \leq \beta(t)$ for $t \in [0,\omega]$ and $x \in \mathbb{R}$. Hence, the function $\widetilde{f}$ is correctly defined and, moreover, $\widetilde{f} \in K([0,\omega] \times \mathbb{R}; \mathbb{R})$. Mention also that, in view of (17.4)–(17.6), (17.12), and (17.13), the inequalities

$$\alpha''(t) \geq \widetilde{f}(t, \alpha(t)), \quad \beta''(t) \leq \widetilde{f}(t, \beta(t)) \quad \text{for } t \in [0,\omega]$$

are fulfilled. Hence, by virtue of Proposition 17.2, the problem

$$u'' = \widetilde{f}(t,u); \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega)$$

has at least one solution u satisfying (17.7). On account of (17.7), (17.12), and (17.13) we get that the function u is a solution of the problem (17.1) as well. □

18. Some Auxiliary Hypotheses

In this chapter we will state some hypotheses guaranteeing the existence of the functions α and β satisfying (17.5) and (17.6), respectively.

Below we suppose that

$$f \in K_{loc}([0,\omega]\times\,]0,+\infty[\,;\mathbb{R}).$$

Proposition 18.1. *Let there exist* $\delta_0 > 0$ *and* $p_0 \in L_\omega$ *such that*

$$f(t,x) \le p_0(t)x \quad \text{for } t\in[0,\omega], \ \ x\in\,]0,\delta_0], \ \ \text{and } p_0 \notin \mathcal{V}^-(\omega). \tag{18.1}$$

Then, for any $c > 0$, *there is* $\alpha \in AC'([0,\omega])$ *satisfying*

$$0 < \alpha(t) \le c \quad \text{for } t \in [0,\omega] \tag{18.2}$$

and (17.5).

Proof. In view of Proposition 10.10, there is a $\widetilde{p}_0 \ge p_0$ such that $\widetilde{p}_0 \in \mathcal{V}_0(\omega)$. Let $u_0(t) > 0$ for $t \in [0,\omega]$ is a solution of the problem

$$u_0'' = \widetilde{p}_0(t)u_0; \quad u_0(0) = u_0(\omega), \ \ u_0'(0) = u_0'(\omega).$$

For given $c > 0$ choose $\varepsilon > 0$ such that $\varepsilon\|u_0\|_C < \min\{c,\delta_0\}$ and set $\alpha(t) \overset{\text{def}}{=} \varepsilon u_0(t)$. In view of (18.1) clearly (17.5) holds. It is also evident that (18.2) is fulfilled as well. □

Let now the function f satisfy the inequality

$$f(t,x) \le p(t)x + h(t,x) \quad \text{for } t \in [0,\omega], \ \ x > 0, \tag{18.3}$$

where

$$p \in L_\omega, \quad h \in K_{loc}([0,\omega]\times\,]0,+\infty[\,;\mathbb{R}). \tag{18.4}$$

Suppose that

$$\begin{cases} h(t,x) \le -h_0(t)\varphi(x) \quad \text{for } t \in [0,\omega], \ \ x \in\,]0,r_0], \\ \qquad h_0 \in L_\omega, \quad \varphi \in C(\,]0,r_0];\,]0,+\infty[\,), \\ \qquad h_0(t) \ge 0 \quad \text{for } t \in [0,\omega], \quad h_0 \not\equiv 0, \\ \qquad \liminf\limits_{x\to 0+} \dfrac{\varphi(x)}{x} > \dfrac{1}{\|h_0\|_L} \displaystyle\int_0^\omega p(s)\,\mathrm{d}s. \end{cases} \tag{H_1}$$

Then there are $\delta_0 > 0$ and $p_0 \in L_\omega$ such that (18.1) holds. Indeed, choose $\delta_0 \in\,]0,r_0]$ and $c > 0$ such that

$$\frac{\varphi(x)}{x} \ge c > \frac{1}{\|h_0\|_L}\int_0^\omega p(s)\,\mathrm{d}s \quad \text{for } x \in\,]0,\delta_0] \tag{18.5}$$

and put $p_0(t) \overset{\text{def}}{=} p(t) - ch_0(t)$ for $t \in [0,\omega]$. Since $\int_0^\omega p_0(s)\,\mathrm{d}s < 0$ we get from Proposition 10.8 that $p_0 \notin \mathcal{V}^-(\omega)$. On the other hand, by virtue of (18.5) and the first inequality in (H_1) we have

$$f(t,x) \le p(t)x - h_0(t)\varphi(x) \le \big(p(t) - ch_0(t)\big)x = p_0(t)x \quad \text{for } t \in [0,\omega], \ \ x \in\,]0,\delta_0].$$

By the same arguments one can easily verify that the hypothesis (H_2) below implies (18.1).

$$\begin{cases} h(t,x) \leq h_0(t,x) \quad \text{for } t \in [0,\omega], \ \ x \in]0,r_0], \ \ h_0 \in K_{loc}([0,\omega]\times]0,+\infty[\,;\mathbb{R}), \\ \text{the function } x \longmapsto \dfrac{1}{x}\, h_0(t,x) \text{ is nondecreasing on }]0,r_0], \\ \lim\limits_{x\to 0+} \dfrac{1}{x} \displaystyle\int_0^\omega h_0(s,x)\,\mathrm{d}s < -\int_0^\omega p(s)\,\mathrm{d}s. \end{cases} \tag{H_2}$$

In particular, if

$$\begin{cases} \text{the function } h(t,\,\cdot\,) \text{ is nondecreasing in }]0,r_0[\text{ for } t\in[0,\omega], \\ h(t,r_0) \leq 0 \quad \text{for } t \in [0,\omega], \ \ \operatorname{mes}\big\{t \in [0,\omega] : \ h(t,r_0) < 0\big\} > 0, \end{cases} \tag{H_3}$$

then (H_2) is fulfilled as well.

Therefore, the following proposition takes place

Proposition 18.2. *Let the function f satisfy* (18.3) *and* (18.4)*. Let, moreover, $k \in \{1,2,3\}$ and the hypothesis (H_k) is fulfilled. Then there are $\delta_0 > 0$ and $p_0 \in L_\omega$ such that* (18.1) *holds. Consequently, for any $c > 0$, there is $\alpha \in AC'([0,\omega])$ satisfying* (18.2) *and* (17.5).

Recall that the set $\mathcal{U}(\omega)$ appearing in the formulation of the next hypothesis is defined in Section 16 (see Definition 16.1).

Introduce the hypothesis

$$\begin{cases} h(t,x) \leq q(t) \quad \text{for } t \in [0,\omega], \ \ x > 0, \\ (p,q) \in \mathcal{U}(\omega). \end{cases} \tag{H_4}$$

Proposition 18.3. *Let the function f satisfy* (18.3) *and* (18.4)*. Let, moreover, the hypothesis (H_4) hold. Then there exists $\alpha \in AC'([0,\omega])$, $\alpha(t) > 0$ for $t \in [0,\omega]$, satisfying* (17.5).

Proof. Let α be a solution of the problem

$$\alpha'' = p(t)\alpha + q(t); \quad \alpha(0) = \alpha(\omega), \ \ \alpha'(0) = \alpha'(\omega).$$

On account of (H_4), we have that $\alpha(t) > 0$ for $t \in [0,\omega]$. Taking, moreover, into account (18.3) clearly α satisfies (17.5). □

Introduce the hypothesis

$$\begin{cases} h(t,x) \leq q(t) \quad \text{for } t \in [0,\omega], \ \ x > 0, \\ [p]_+ \not\equiv 0, \quad q \not\equiv 0, \quad Q_- \geq \rho(p)Q_+, \end{cases} \tag{H_5}$$

where Q_-, Q_+, and $\rho(p)$ are defined by (0.13) and (0.12).

Proposition 18.4. *Let the function f satisfy* (18.3) *and* (18.4)*. Let, moreover, the hypothesis (H_5) hold. Then there exists $\alpha \in AC'([0,\omega])$, $\alpha(t) > 0$ for $t \in [0,\omega]$, satisfying* (17.5).

Proof. By virtue of Theorem 16.2, Remark 8.4, and (H_5), the problem

$$\alpha'' = [p(t)]_+\alpha + q(t); \quad \alpha(0) = \alpha(\omega), \ \ \alpha'(0) = \alpha'(\omega).$$

has a positive solution α. Taking, moreover, into account (18.3), it is clear that (17.5) is fulfilled. □

Recall that the function H is defined by (0.17) and the numbers $\rho(p)$ and Q_+, Q_- are given by (0.12) and (0.13).

Proposition 18.5. *Let $p,q \in L_\omega$, $h \in K_{loc}([0,\omega]\times]0,+\infty[\,;\mathbb{R})$, and*

$$f(t,x) \leq p(t)x + h(t,x) \quad \text{for } t \in [0,\omega], \ \ x > 0. \tag{18.6}$$

Let, moreover, $h(t,\,\cdot\,)$ is nondecreasing, there exists $q \in L_\omega$ such that $[q]_+ \not\equiv 0$,

$$h(t,x) \leq q(t) \quad \text{for } t \in [0,\omega], \ \ x > 0, \tag{18.7}$$

and

$$\lim_{x\to+\infty} H(x) > (1-\rho(p))Q_+, \quad H\Big(\frac{\omega}{4}\,\rho(p)Q_+\Big) \leq (1-\rho(p))Q_+. \tag{18.8}$$

Then there exists $\alpha \in AC'([0,\omega])$, $\alpha(t) > 0$ *for* $t \in [0,\omega]$, *satisfying* (17.5).

Proof. First suppose that $[p]_+ \not\equiv 0$. In view of (18.8) there is a number A such that

$$A \geq \frac{\omega}{4}\,\rho(p)Q_+ \tag{18.9}$$

and

$$H(A) = (1-\rho(p))Q_+. \tag{18.10}$$

Denote by α_0 solution of the problem

$$\alpha_0'' = [p(t)]_+\alpha_0 + h(t,A); \quad \alpha_0(0) = \alpha_0(\omega), \ \ \alpha_0'(0) = \alpha_0'(\omega). \tag{18.11}$$

Suppose that

$$m = \min\big\{\alpha_0(t) : \ t \in [0,\omega]\big\}, \quad M = \max\big\{\alpha_0(t) : \ t \in [0,\omega]\big\}, \tag{18.12}$$

and choose $a \in [0,\omega[$ and $b \in]a, a+\omega[$ such that

$$\alpha_0(a) = m, \quad \alpha_0(b) = M. \tag{18.13}$$

In view of (18.7), clearly $\int\limits_0^\omega [h(s,A)]_+\,\mathrm{d}s \leq Q_+$. Taking, moreover, into account (18.10), we get that

$$\begin{aligned}\int\limits_0^\omega [h(s,A)]_-\,\mathrm{d}s &= \int\limits_0^\omega [h(s,A)]_+\,\mathrm{d}s - H(A)\\ &= \int\limits_0^\omega [h(s,A)]_+\,\mathrm{d}s + \big(\rho(p)-1\big)Q_+ \geq \rho(p)\int\limits_0^\omega [h(s,A)]_+\,\mathrm{d}s.\end{aligned}$$

Consequently, by virtue of Theorem 16.2, we have that

$$m > 0.$$

Now we will estimate $M - m$. It is clear that

$$\begin{aligned}M - m &= \int\limits_a^b \alpha_0'(s)\,\mathrm{d}s = -\int\limits_a^b (s-a)\alpha_0''(s)\,\mathrm{d}s \leq (b-a)\int\limits_a^b \big([q(s)]_+ - h(s,A)\big)\,\mathrm{d}s,\\ M - m &= -\int\limits_b^{a+\omega} \alpha_0'(s)\,\mathrm{d}s = -\int\limits_b^{a+\omega} (a+\omega-s)\alpha_0''(s)\,\mathrm{d}s\\ &\leq (a+\omega-b)\int\limits_b^{a+\omega} \big([q(s)]_+ - h(s,A)\big)\,\mathrm{d}s,\end{aligned} \tag{18.14}$$

and at least one of these two inequalities is strict (because otherwise we get $[p(t)]_+\alpha_0(t) + [q(t)]_+ \equiv 0$ and, consequently, $[q]_+ \equiv 0$). Hence, on account of the inequality $4xy \leq (x+y)^2$ we get

$$\begin{aligned}(M-m)^2 &< (b-a)(a+\omega-b)\int\limits_a^b \big([q(s)]_+ - h(s,A)\big)\,\mathrm{d}s \int\limits_b^{a+\omega} \big([q(s)]_+ - h(s,A)\big)\,\mathrm{d}s\\ &\leq \frac{\omega^2}{16}\bigg(\int\limits_a^{a+\omega} \big([q(s)]_+ - h(s,A)\big)\,\mathrm{d}s\bigg)^2 = \frac{\omega^2}{16}\bigg(\int\limits_0^{\omega} \big([q(s)]_+ - h(s,A)\big)\,\mathrm{d}s\bigg)^2.\end{aligned}$$

Therefore, we have proved that

$$M - m < \frac{\omega}{4}\,(Q_+ - H(A)). \tag{18.15}$$

The latter inequality together with (18.9) and (18.10) implies that there is a $\varepsilon \in]0,m[$ such that

$$M - m + \varepsilon < A. \tag{18.16}$$

Let now $\alpha(t) \overset{\text{def}}{=} \alpha_0(t) - m + \varepsilon$ for $t \in [0,\omega]$. Then, it is clear that,

$$0 < \alpha(t) < A \quad \text{for } \ t \in [0,\omega]. \tag{18.17}$$

Taking, moreover, into account that the function $h(t,\cdot)$ is nondecreasing and (18.6) is fulfilled we easily conclude that

$$\begin{aligned}\alpha''(t) &= [p(t)]_+\alpha(t) + h(t,A) + (m-\varepsilon)[p(t)]_+ \\ &\geq p(t)\alpha(t) + h(t,\alpha(t)) \geq f(t,\alpha(t)) \quad \text{for } t\in[0,\omega].\end{aligned} \tag{18.18}$$

Now let $[p]_+ \equiv 0$. Choose $A>0$ such that (18.9) and (18.10) are satisfied. Then, by virtue Fredholm's third theorem, the problem

$$\alpha_0'' = h(t,A); \quad \alpha_0(0)=\alpha_0(\omega), \;\; \alpha_0'(0)=\alpha_0'(\omega) \tag{18.19}$$

has at least one solution α_0. Extend the functions p, h, q, and α_0 periodically and denote it again by the same letters. Introduce the numbers m and M by (18.12) and choose $a\in[0,\omega[$ and $b\in\,]a,a+\omega[$ such that (18.13) holds. Clearly, inequalities (18.14) are fulfilled and at least one of them is strict (because otherwise we get $[q(t)]_+\equiv 0$). By the same arguments as above we get (18.15). Inequalities (18.9), (18.10), and (18.15) imply that there is a $\varepsilon>0$ such that (18.16) is fulfilled. Let $\alpha(t) \overset{\text{def}}{=} \alpha_0(t)-m+\varepsilon$ for $t\in[0,\omega]$. Clearly, (18.17) holds as well. Taking, moreover, into account that $h(t,\cdot)$ is nonincreasing, (18.6) is fulfilled, and $[p]_+\equiv 0$, we get (18.18). □

Introduce the hypothesis

$$\begin{cases} h\in K_{loc}([0,\omega]\times\,]0,+\infty[\,;\mathbb{R}), \;\; h(t,\cdot) \text{ is nonincreasing,} \\ \text{there is a } x_0>0 \text{ such that} \\ \quad \rho(p)H_+(x_0) + \Big(\rho(p)\big\|[p]_+\big\|_L - \big\|[p]_-\big\|_L\Big)x_0 \leq H_-(x_0), \\ \text{where} \\ \quad H_+(x_0) = \displaystyle\int_0^\omega [h(s,x_0)]_+\,\mathrm{d}s, \;\; H_-(x_0) = \int_0^\omega [h(s,x_0)]_-\,\mathrm{d}s \end{cases} \tag{H_6}$$

Proposition 18.6. *Let $p\in\mathcal{V}^-(\omega)$ and (H_6) hold. Let, moreover,*

$$f(t,x) \leq p(t)x + h(t,x) \quad \textit{for } t\in[0,\omega], \;\; x>x_0, \tag{18.20}$$

where x_0 is the number appearing in (H_6). Then there exists $\alpha\in AC'([0,\omega])$, $\alpha(t)>x_0$ for $t\in[0,\omega]$, satisfying (17.5)*.*

Proof. Since $p\in\mathcal{V}^-(\omega)$, the problem

$$\alpha'' = p(t)\alpha + h(t,x_0); \quad \alpha(0)=\alpha(\omega), \;\; \alpha'(0)=\alpha'(\omega) \tag{18.21}$$

possesses a unique solution α. It follows from (H_6) that $H_-(x_0) > \rho(p)H_+(x_0)$. Hence, by virtue of Theorem 16.2, we get

$$\alpha(t) > \big(H_-(x_0)-\rho(p)H_+(x_0)\big)\Big(\rho(p)\big\|[p]_+\big\|_L - \big\|[p]_-\big\|_L\Big)^{-1} \quad \text{for } t\in[0,\omega]$$

which, together with (H_6), implies that

$$\alpha(t) > x_0 \quad \text{for } t\in[0,\omega].$$

Taking now into account that the function $h(t,\cdot)$ is nonincreasing and (18.20) holds, we get from (18.21) that the function α satisfies (17.5). □

Proposition 18.7. *Let there exist $\delta_1>0$ and $p_1\in L_\omega$ such that*

$$f(t,x) \geq p_1(t)x - q(t,x) \quad \textit{for } t\in[0,\omega], \;\; x>\delta_1, \;\; \textit{and } p_1\in\mathcal{V}^-(\omega), \tag{18.22}$$

where $q\in K_{sl}([0,\omega]\times\mathbb{R};\mathbb{R}_+)$. Then, for any $c>0$, there is $\beta\in AC'([0,\omega])$ satisfying inequalities $\beta(t)\geq c$ for $t\in[0,\omega]$ and (17.6)*.*

Proof. By virtue of Remark 8.2, the problem

$$u_1'' = p_1(t)u_1 - 1; \quad u_1(0) = u_1(\omega), \;\; u_1'(0) = u_1'(\omega)$$

has a (unique) solution u_1 and $u_1(t) > 0$ for $t \in [0,\omega]$. For given $c > 0$ choose $\varepsilon > 0$ such that $\varepsilon u_1(t) > \max\{c, \delta_1\}$ for $t \in [0,\omega]$ and consider the problem

$$\beta'' = p_1(t)\beta - q(t,\beta) - \varepsilon; \quad \beta(0) = \beta(\omega), \;\; \beta'(0) = \beta'(\omega). \tag{18.23}$$

By virtue of Proposition 17.3, the problem (18.23) has a solution β. Taking into account that $p_1 \in \mathcal{V}^-(\omega)$, $q \geq 0$, and $\varepsilon > 0$ one can easily verify that $\beta(t) \geq \varepsilon u_1(t)$ for $t \in [0,\omega]$ (see Remark 0.6). Consequently, $\beta(t) > \max\{c, \delta_1\}$ for $t \in [0,\omega]$. On the other hand, in view of (18.22) and (18.23), clearly (17.6) holds as well. □

Let now the function f satisfy

$$f(t,x) \geq p(t)x + h(t,x) \quad \text{for } \; t \in [0,\omega], \;\; x > r_1, \tag{18.24}$$

where $r_1 > 0$ and the functions p and h satisfy (18.4).

Suppose that $p \in \mathcal{V}^-(\omega)$ and

$$\begin{cases} h(t,x) \geq -q_0(t,x) \quad \text{for } \; t \in [0,\omega], \;\; x > r_1, \\ \qquad\qquad q_0 \in K_{sl}([0,\omega] \times \mathbb{R}; \mathbb{R}_+). \end{cases} \tag{H_7}$$

Then it is clear that (18.22) holds with $\delta_1 = r_1$, $p_1(t) \overset{\text{def}}{=} p(t)$, and $q(t,x) \overset{\text{def}}{=} q_0(t,x)$.

In particular, if $p \in \mathcal{V}^-(\omega)$ and

$$\text{the function } \; h(t,\,\cdot\,) \; \text{ is nondecreasing in } \;]r_1, +\infty[\,, \tag{H_8}$$

then (H_7) holds with $q_0(t,x) \overset{\text{def}}{=} |h(t,r_1)|$ for $t \in [0,\omega]$, $x \in \mathbb{R}$.

Observe also that the conditions $p \in \mathcal{V}^-(\omega)$ and

$$\lim_{x \to +\infty} \frac{1}{x} \int_0^\omega [h(s,x)]_- \,\mathrm{d}s = 0 \tag{H_9}$$

also imply (18.22) with $\delta_1 = 1$, $p_1(t) \overset{\text{def}}{=} p(t)$, and $q(t,x) \overset{\text{def}}{=} |h(t,[x-1]_+ + 1)|$.

Suppose now that $p \in \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$, $q \in K_{sl}([0,\omega] \times \mathbb{R}; \mathbb{R}_+)$, and

$$\begin{cases} h(t,x) \geq g_0(t)g(x) - q(t,x) \quad \text{for } \; t \in [0,\omega], \;\; x > r_1, \\ \qquad g_0 \in L_\omega, \;\; g_0(t) \geq 0 \quad \text{for } \; t \in [0,\omega], \;\; g_0 \not\equiv 0, \\ \qquad g \in C([r_1, +\infty[\,; \mathbb{R}_+), \;\; \liminf\limits_{x \to +\infty} \dfrac{g(x)}{x} > 0. \end{cases} \tag{H_{10}}$$

Then there are $\delta_1 > 0$ and $p_1 \in L_\omega$ such that (18.22) holds. Indeed, choose $c > 0$ and $\delta_1 > r_1$ such that $\frac{g(x)}{x} > c$ for $x > \delta_1$ and put $p_1(t) \overset{\text{def}}{=} p(t) + cg_0(t)$ for $t \in [0,\omega]$. In view of Remarks 8.4 and 8.5 clearly $p_1 \in \mathcal{V}^-(\omega)$. It is also evident that

$$\begin{aligned} f(t,x) &\geq \Big(p(t) + g_0(t)\,\frac{g(x)}{x}\Big)x - q(t,x) \\ &\geq \big(p(t) + cg_0(t)\big)x - q(t,x) = p_1(t)x - q(t,x) \quad \text{for } \; t \in [0,\omega], \;\; x > \delta_1. \end{aligned} \tag{18.25}$$

At last suppose that $p \in L_\omega$, $q \in K_{sl}([0,\omega] \times \mathbb{R}; \mathbb{R}_+)$, and

$$\begin{cases} h(t,x) \geq g_0(t)g(x) - q(t,x) \quad \text{for } \; t \in [0,\omega], \;\; x > r_1, \\ \qquad g_0 \in L_\omega, \;\; g_0(t) \geq 0 \quad \text{for } \; t \in [0,\omega], \\ \qquad \operatorname{mes}\big\{t \in [0,\omega] : \; g_0(t) = 0\big\} = 0, \\ \qquad g \in C([r_1, +\infty[\,; \mathbb{R}_+), \;\; \lim\limits_{x \to +\infty} \dfrac{g(x)}{x} = +\infty. \end{cases} \tag{H_{11}}$$

Then there are $\delta_1 > 0$ and $p_1 \in L_\omega$ such that (18.22) holds. Indeed, by virtue of Remark 11.2, there is a $c > 0$ such that $p + cg_0 \in \mathcal{V}^-(\omega)$. Choose $\delta_1 > r_1$ such that $\frac{g(x)}{x} > c$ for $x > \delta_1$ and put $p_1(t) \overset{\text{def}}{=} p(t) + cg_0(t)$ for $t \in [0,\omega]$. Clearly, (18.25) is fulfilled.

Summarizing the above-said we have

Proposition 18.8. *Let the function f satisfy* (18.24)*. Let, moreover, one of the following items be fulfilled:*

(1) $p \in \mathcal{V}^-(\omega)$, $k \in \{7,8,9\}$, *and* (H_k) *holds,*

(2) $p \in \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$ *and* (H_{10}) *holds, where* $q \in K_{sl}([0,\omega] \times \mathbb{R}; \mathbb{R}_+)$,

(3) (H_{11}) *holds, where* $q \in K_{sl}([0,\omega] \times \mathbb{R}; \mathbb{R}_+)$.

Then there are $\delta_1 > 0$ and $p_1 \in L_\omega$ such that (18.22) *holds. Consequently, for any $c > 0$, there is $\beta \in A\widetilde{C}'([0,\omega])$ satisfying inequalities $\beta(t) \ge c$ for $t \in [0,\omega]$ and* (17.6)*.*

Introduce the hypothesis

$$\left\{\begin{array}{l} \qquad\qquad p \in \operatorname{Int}\mathcal{V}^+(\omega), \\ \quad h \in K_{loc}([0,\omega]\times\,]0,+\infty[\,;\mathbb{R}), \quad h(t,\,\cdot\,) \ \text{ is nondecreasing,} \\ \text{there is a } x_0 > 0 \text{ such that} \\ \quad \nu^*(p)\rho(p)H_-(x_0) + \Big(\nu^*(p)\rho(p)\big\|[p]_-\big\|_L - \big\|[p]_+\big\|_L\Big)x_0 \le H_+(x_0), \\ \text{where} \\ \quad H_+(x_0) = \displaystyle\int_0^\omega [h(s,x_0)]_+\,\mathrm{d}s, \quad H_-(x_0) = \int_0^\omega [h(s,x_0)]_-\,\mathrm{d}s \\ \text{and } \nu^*(p) \text{ is the number defined by (6.22).} \end{array}\right. \tag{H_{12}}$$

Proposition 18.9. *Let* (H_{12}) *hold and*

$$f(t,x) \ge p(t)x + h(t,x) \quad \text{for } t \in [0,\omega], \ \ x > x_0, \tag{18.26}$$

where x_0 is the number appearing in (H_{12}). Then there exists $\beta \in A\widetilde{C}'([0,\omega])$ such that $\beta(t) > x_0$ for $t \in [0,\omega]$,

$$\begin{gathered} \beta''(t) \le p(t)\beta(t) + h(t,\beta(t)) \quad \text{for } t \in [0,\omega], \\ \beta(0) = \beta(\omega), \quad \beta'(0) = \beta'(\omega), \end{gathered}$$

and β satisfies (17.6)*.*

Proof. Denote by β a solution of the problem

$$\beta'' = p(t)\beta + h(t,x_0); \quad \beta(0) = \beta(\omega), \ \ \beta'(0) = \beta'(\omega). \tag{18.27}$$

It follows from (H_{12}) that $H_+(x_0) > \nu^*(p)\rho(p)H_-(x_0)$. Hence, by virtue of Theorem 16.4, we get that

$$\beta(t) > \big(H_+(x_0) - \nu^*(p)\rho(p)H_-(x_0)\big)\Big(\nu^*(p)\rho(p)\big\|[p]_-\big\|_L - \big\|[p]_+\big\|_L\Big)^{-1} \quad \text{for } t \in [0,\omega]$$

which, together with (H_{12}), implies that

$$\beta(t) > x_0 \quad \text{for } t \in [0,\omega].$$

Taking now into account that the function $h(t,\,\cdot\,)$ is nondecreasing and (18.26) holds, we get from (18.27) that the function β satisfies all assertion of the proposition. ☐

Suppose now that

$$f(t,x) \ge p(t)x + h(t,x) + q(t) \quad \text{for } t \in [0,\omega], \ \ x > r, \tag{18.28}$$

where

$$p \in \mathcal{V}_0(\omega), \quad h \in K_{loc}([0,\omega]\times\,]0,+\infty[\,;\mathbb{R}), \quad q \in L_\omega. \tag{18.29}$$

Below by u_0 we denote a positive solution of the problem

$$u'' = p(t)u; \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega). \tag{18.30}$$

Introduce the hypothesis

$$\left\{\begin{array}{c} h(t,x) \geq h_0(t,x) \quad \text{for } t \in [0,\omega], \ \ x > r, \ \ h_0 \in K_{loc}([0,\omega]\times\,]0,+\infty[\,;\mathbb{R}), \\ \text{the function } h_0(t,\cdot\,) \text{ is nondecreasing on } \ ]0,+\infty[\,, \\ \lim\limits_{x\to+\infty} \int\limits_0^\omega h_0(s,x)u_0(s)\,\mathrm{d}s > -\int\limits_0^\omega q(s)u_0(s)\,\mathrm{d}s \end{array}\right. \tag{H_{13}}$$

and

$$\left\{\begin{array}{l} h(t,x) \geq -h_0(t)g(x) \quad \text{for } t \in [0,\omega], \ \ x > r, \\ h_0 \in L_\omega, \ \ h_0(t) \geq 0 \ \text{ for } t \in [0,\omega], \ \ h_0 \not\equiv 0, \\ \qquad\qquad g \in C(\,]0,+\infty[\,;\,]0,+\infty[\,), \\ \qquad \limsup\limits_{x\to+\infty} g(x) < \dfrac{\int_0^\omega q(s)u_0(s)\,\mathrm{d}s}{\int_0^\omega h_0(s)u_0(s)\,\mathrm{d}s}\,. \end{array}\right. \tag{H_{14}}$$

Proposition 18.10. *Let the function f satisfy* (18.28) *and* (18.29)*. Let, moreover,* (H_{13}) *be fulfilled (where u_0 is a positive solution of the problem* (18.30)*). Then for any $c > 0$, there is a function $\beta \in AC'([0,\omega])$ satisfying $\beta(t) \geq c$ for $t \in [0,\omega]$ and* (17.6)*.*

Proof. On account of the last condition in (H_{13}), there is an $x_0 > 0$ such that

$$\int\limits_0^\omega \big(h_0(s,x_0) + q(s)\big)u_0(s)\,\mathrm{d}s > 0. \tag{18.31}$$

Put

$$A \stackrel{\text{def}}{=} \Bigg(\int\limits_0^\omega u_0(s)\,\mathrm{d}s\Bigg)^{-1} \int\limits_0^\omega \big(h_0(s,x_0) + q(s)\big)u_0(s)\,\mathrm{d}s$$

and consider the problem

$$\begin{gathered} \beta'' = p(t)\beta + h_0(t,x_0) + q(t) - A; \\ \beta(0) = \beta(\omega), \ \ \beta'(0) = \beta'(\omega). \end{gathered} \tag{18.32}$$

By virtue of Fredholm's third theorem, the problem (18.32) possesses at least one solution β_0. For given $c > 0$ choose $\lambda > 0$ such that

$$\beta_0(t) + \lambda u_0(t) > x_0 + r + c \quad \text{for } t \in [0,\omega] \tag{18.33}$$

and put

$$\beta(t) \stackrel{\text{def}}{=} \beta_0(t) + \lambda u_0(t) \quad \text{for } t \in [0,\omega]. \tag{18.34}$$

It is clear that, β is a solution of the problem (18.32) and

$$\beta(t) \geq x_0 + r + c \quad \text{for } t \in [0,\omega]. \tag{18.35}$$

Taking now into account monotonicity of the function $h_0(t,\cdot\,)$, first condition in (H_{13}), (18.28), and (18.31), we get from (18.32) that the function β satisfies the assertion of the proposition. □

Proposition 18.11. *Let the function f satisfy* (18.28) *and* (18.29)*. Let, moreover,* (H_{14}) *holds (where u_0 is a positive solution of the problem* (18.30)*). Then for any $c > 0$, there is a function $\beta \in AC'([0,\omega])$ satisfying $\beta(t) \geq c$ for $t \in [0,\omega]$ and* (17.6)*.*

Proof. Clearly, there is a $x_0 > 0$ such that

$$g(x) < A \quad \text{for } x > x_0, \tag{18.36}$$

where

$$A \stackrel{\text{def}}{=} \frac{\int_0^\omega q(s)u_0(s)\,\mathrm{d}s}{\int_0^\omega h_0(s)u_0(s)\,\mathrm{d}s}\,.$$

By virtue of Fredholm's third theorem, the problem

$$\beta'' = p(t)\beta - Ah_0(t) + q(t); \quad \beta(0) = \beta(\omega), \ \ \beta'(0) = \beta'(\omega) \tag{18.37}$$

possesses at least one solution β_0. For given $c > 0$ choose $\lambda > 0$ such that (18.33) is fulfilled and introduce the function β by (18.34). Taking now into account first condition in (H_{14}), (18.28), (18.35), and (18.36), we get from (18.37) that the function β satisfies the assertion of the proposition. □

19. Existence of Positive Solutions

In this section we consider the problem

$$u'' = f(t, u); \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega), \tag{19.1}$$

where $f \in K_{loc}([0,\omega]\times\,]0,+\infty[\,;\mathbb{R})$. Recall that under a solution of (19.1) we understand a **positive** function $u \in AC'([0,\omega])$ satisfying the given equation for almost all $t \in [0,\omega]$ and boundary conditions. Below we will establish theorems on the solvability of the problem (19.1) and also derive their corollaries for the problem

$$u'' = p(t)u + h(t, u); \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega), \tag{19.2}$$

where $p \in L_\omega$ and $h \in K_{loc}([0,\omega]\times\,]0,+\infty[\,;\mathbb{R})$.

Next theorem immediately follows from Proposition 17.6 and Propositions 18.1 and 18.7 (with $c = 1$).

Theorem 19.1. *Let there exist $\delta_i > 0$ and $p_i \in L_\omega$, $i = 0, 1$, such that* (18.1) *and* (18.22) *are fulfilled, where $q \in K_{sl}([0,\omega] \times \mathbb{R};\mathbb{R})$. Then the problem* (19.1) *has at least one solution.*

Corollaries 19.2, 19.3, 19.7, 19.8, 19.11, and 19.13 below follow from Theorem 19.1 and Propositions 18.2 and 18.8. Recall that the hypotheses (H_k) are introduced in the previous chapter.

Corollary 19.2. *Let $p \in \mathcal{V}^-(\omega)$ and either (H_1) or (H_2) hold. Let, moreover, either (H_7) or (H_9) be fulfilled. Then the problem* (19.2) *has at least one solution.*

Taking into account that (H_3) implies (H_2) and (H_8) yields (H_7), we get from Corollary 19.2 that

Corollary 19.3. *Let the function $h(t,\cdot)$ be nondecreasing in $]0,+\infty[$ and there is a $r_0 > 0$ such that*

$$h(t, r_0) \le 0 \quad \textit{for} \ \ t \in [0,\omega], \quad \operatorname{mes}\big\{t \in [0,\omega] : \ h(t, r_0) < 0\big\} > 0. \tag{19.3}$$

Then the problem (19.2) *is uniquely solvable provided $p \in \mathcal{V}^-(\omega)$. If, moreover,*

$$\begin{gathered} h(t,x) \le 0 \quad \textit{for} \ \ t \in [0,\omega], \ \ x > 0, \\ \operatorname{mes}\big\{t \in [0,\omega] : \ h(t,x) < 0\big\} > 0 \quad \textit{for} \ \ x > 0, \end{gathered} \tag{19.4}$$

then the condition $p \in \mathcal{V}^-(\omega)$ is necessary for the solvability of the problem (19.2).

Proof. Solvability of (19.2) follows immediately from Corollary 19.2. Let us prove the uniqueness. Suppose that u_1 and u_2 are solutions of the problem (19.2) and for a certain $t_0 \in [0,\omega[$, $u_1(t_0) > u_2(t_0)$. Let $v(t) \overset{\text{def}}{=} u_1(t) - u_2(t)$ for $t \in [0,\omega]$. Then either

$$v(t) \ge 0 \quad \text{for} \ \ t \in [0,\omega], \ \ v \not\equiv 0, \tag{19.5}$$

or there are $0 \le a < b \le \omega$ such that $b - a < \omega$ and

$$v(t) > 0 \quad \text{for} \ \ t \in\,]a,b[\,, \quad v(a) = 0, \ \ v(b) = 0. \tag{19.6}$$

If (19.5) is fulfilled then, in view of the monotonicity of the function h we get

$$v''(t) = p(t)v(t) + \big(h(t,u_1(t)) - h(t,u_2(t))\big) \ge p(t)v(t) \quad \text{for} \ \ t \in [0,\omega].$$

However $v^{(i)}(0) = v^{(i)}(\omega)$, $i = 0, 1$, and $p \in \mathcal{V}^-(\omega)$. Hence, the latter inequality implies that $v(t) \le 0$ for $t \in [0,\omega]$ which contradicts (19.5).

Analogously, if (19.6) is fulfilled then we get

$$v''(t) \ge p(t)v(t) \quad \text{for} \ \ t \in [a,b], \quad v(a) = 0, \ \ v(b) = 0. \tag{19.7}$$

Since $p \in \mathcal{V}^-(\omega)$ it follows from Theorems 8.1 and Proposition 0.8 that $p \in \mathcal{D}(\omega)$. Hence, by virtue of (19.7) and Proposition 2.5 we get $v(t) \le 0$ for $t \in [a,b]$, which contradicts (19.6). Thus the problem (19.2) is uniquely solvable.

Now suppose that the function $h(t,\cdot)$ is nonincreasing and (19.4) holds. Let, moreover, u be a solution of the problem (19.2) and $M = \max\{u(t) : t \in [0,\omega]\}$. then we have

$$h(t,u(t)) \le h(t,M) \le 0 \quad \text{for } \ t \in [0,\omega], \quad h(\cdot\,,M) \not\equiv 0.$$

Hence, by virtue of Theorem 8.3 (with $\gamma = u$) we get $p \in \mathcal{V}^-(\omega)$. □

Remark 19.4. Assumption (19.3) in Corollary 19.3 is essential and cannot be omitted. Indeed, let $p \in \mathcal{V}^-(\omega)$ and $h(t,x) \stackrel{\text{def}}{=} 1$. Clearly, all the conditions of Corollary 19.3 are fulfilled except of (19.3). On the other hand, the problem

$$u'' = p(t)u + 1; \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega)$$

has no positive solution (in fact, a unique solution of this problem is negative).

Remark 19.5. Mention also that the assumption about monotonicity of the function h is essential for the second part of Corollary 19.3 and cannot be omitted. Indeed, let $\omega = 2\pi$, $p(t) = -\frac{\sin t}{2+\sin t}$, and $h(t,x) = -\frac{1}{1+x^2}\,|x-2-\sin t|$. Clearly, (19.4) holds and the function $u(t) = 2+\sin t$ is a solution of the problem (19.2). However, $h(t,u(t)) \equiv 0$ and therefore the function u is a solution of the homogeneous problem

$$u'' = p(t)u, \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega)$$

as well. Thus $p \in \mathcal{V}_0(\omega)$ and, consequently, $p \notin \mathcal{V}^-(\omega)$.

Remark 19.6. During the proof of Corollary 19.3 it was shown that if $p \in \mathcal{V}^-(\omega)$ and the function $h(t,\cdot)$ is nondecreasing then the problem (19.2) has at most one solution.

Corollary 19.7. *Let $p \in \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$ and either (H_1) or (H_2) hold. Let, moreover, (H_{10}) is fulfilled, where $q \in K_{sl}([0,\omega]\times\mathbb{R};\mathbb{R}_+)$. Then the problem* (19.2) *has at least one solution.*

Corollary 19.8. *Let (H_{11}) hold, where $q \in K_{sl}([0,\omega]\times\mathbb{R};\mathbb{R}_+)$. Let, moreover, either (H_1) or (H_2) be fulfilled. Then the problem* (19.2) *has at least one solution.*

Return again to the problem (19.1). In the formulation of the next result we will need the following hypothesis

$$\begin{cases} f_0 \in K_{loc}([0,\omega]\times\,]0,+\infty[\,;\mathbb{R}_+), \ \ \nu \ge 1 \\ \text{the function } f_0(t,\cdot) \text{ is nonincreasing in }]0,+\infty[\text{ for } t\in[0,\omega], \\ \displaystyle\int_0^\omega f_0\Big(s, c|s-a|^{\frac{2\nu-1}{\nu}}\Big)\,\mathrm{d}s = +\infty \quad \text{for } \ a \in [0,\omega[\,, \ \ c>0. \end{cases} \tag{H_{15}}$$

Remark 19.9. Hypothesis (H_{15}) implies that for any $a \in [0,\omega[$ and $b \in \,]a,\omega]$,

$$\lim_{x\to 0+}\int_a^b f_0(s,x)\,\mathrm{d}s = +\infty. \tag{19.8}$$

Indeed, let $0 < x < (b-a)^{\frac{2\nu-1}{\nu}}$. Then it is clear that

$$\int_a^b f_0(s,x)\,\mathrm{d}s \ge \int_{a+x^{\frac{2\nu-1}{\nu}}}^b f_0(s,x)\,\mathrm{d}s \ge \int_{a+x^{\frac{2\nu-1}{\nu}}}^b f_0\big(s,|s-a|^{\frac{2\nu-1}{\nu}}\big)\,\mathrm{d}s$$

and, consequently, (19.8) holds.

Theorem 19.10. *Let the inequality*

$$f(t,x) \le p_0(t)x - f_0(t,x) + q_0(t,x) \quad \textit{for } \ t \in [0,\omega], \ \ x > 0 \tag{19.9}$$

hold, where $p_0 \in L_\omega$, $p_0(t) \ge 0$ for $t \in [0,\omega]$, $p_0 \not\equiv 0$, $\nu \ge 1$, $q_0^\nu \in K([0,\omega]\times\mathbb{R};\mathbb{R}_+)$ and the function f_0 satisfies hypothesis (H_{15}). Let, moreover, there exist $\beta \in AC'([0,\omega])$ such that $\beta(t) > 0$ for $t \in [0,\omega]$ and (17.6) *holds. Then the problem* (19.1) *has at least one solution.*

Proof. First of all mention that $p_0 \in \mathcal{V}^-(\omega)$ (see Remark 8.4). Put

$$g(t,x) \stackrel{\text{def}}{=} f(t,x) - p_0(t)x \quad \text{for } t \in [0,\omega], \ \ x > 0. \tag{19.10}$$

Clearly, $g \in K_{loc}([0,\omega]\times\,]0,+\infty[\,;\mathbb{R})$ and, in view of (19.9), we have

$$g(t,x) \le -f_0(t,x) + q_0(t,x) \quad \text{for } t \in [0,\omega], \ \ x > 0. \tag{19.11}$$

Let $n_0 \in \mathbb{N}$ be such that $n_0 > 1/\min\{\beta(t) : t \in [0,\omega]\}$. For any $n > n_0$ introduce the notations

$$\begin{aligned} \chi_n(x) &= \frac{1}{n} + \big[x - 1/n\big]_+ , \\ \overline{\chi}_n(t,x) &= \frac{1}{n} + \big[x - 1/n\big]_+ - \big[x - \beta(t)\big]_+ \end{aligned} \tag{19.12}$$

and consider the problem

$$u'' = p_0(t)u + g\big(t, \overline{\chi}_n(t,u)\big); \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega). \tag{19.13}$$

By virtue of Proposition 17.3, the problem (19.13) has, for any $n > n_0$, at least one solution u_n. Now we will show that

$$u_n(t) \le \beta(t) \quad \text{for } t \in [0,\omega], \ \ n > n_0. \tag{19.14}$$

Indeed, let $w_n(t) \stackrel{\text{def}}{=} u_n(t) - \beta(t)$ for $t \in [0,\omega]$, $n > n_0$, and suppose that (19.14) is violated. Then either

$$w_n(t) \ge 0 \quad \text{for } t \in [0,\omega], \ \ w_n \not\equiv 0, \tag{19.15}$$

or there are $0 \le a < b \le \omega$ such that $b - a < \omega$ and

$$w_n(t) > 0 \quad \text{for } t \in \,]a,b[\,, \quad w_n(a) = 0, \ \ w_n(b) = 0. \tag{19.16}$$

If (19.15) holds then, on account of (17.6), (19.10), and (19.12), we get

$$\begin{gathered} w_n''(t) \ge p_0(t)w_n(t) \quad \text{for } t \in [0,\omega], \\ w_n(0) = w_n(\omega), \ \ w_n'(0) = w_n'(\omega). \end{gathered}$$

Since $p_0 \in \mathcal{V}^-(\omega)$, the latter inequality implies that $w_n(t) \le 0$ for $t \in [0,\omega]$, which contradicts (19.15).

Analogously, if (19.16) holds then

$$w_n''(t) \ge p_0(t)w_n(t) \quad \text{for } t \in [a,b], \quad w_n(a) = 0, \ \ w_n(b) = 0.$$

Taking now into account that $p_0 \in \mathcal{D}(\omega)$ and $b - a < \omega$ we get, in view of Proposition 2.5, the relation $w_n(t) \le 0$ for $t \in [a,b]$, which contradicts (19.16).

Thus we have proved that (19.14) is fulfilled. Taking now into account (19.12) we get that the function u_n satisfies

$$\begin{gathered} u_n''(t) = p_0(t)u_n(t) + g\big(t, \chi_n(u_n(t))\big); \\ u_n(0) = u_n(\omega), \ \ u_n'(0) = u_n'(\omega). \end{gathered} \tag{19.17}$$

To finish the proof it is sufficient to show that, for a certain $n > n_0$, the inequality

$$u_n(t) \ge \frac{1}{n} \quad \text{for } t \in [0,\omega] \tag{19.18}$$

holds. For this effort first we establish certain estimates of the functions $|u_n|$ and $|u_n'|$.

By virtue of (19.11) and (19.12) clearly

$$g\big(t, \chi_n(u_n(t))\big) \le q_0\big(t, \chi_n(u_n(t))\big) \quad \text{for } t \in [0,\omega], \ \ n > n_0. \tag{19.19}$$

On the other hand, in view of (19.12) and (19.14)

$$0 < \chi_n(u_n(t)) \le \beta^* \quad \text{for } t \in [0,\omega], \ \ n > n_0, \tag{19.20}$$

where $\beta^* = \max\{\beta(t) : t \in [0,\omega]\}$.

Since q_0^ν is a Carathéodory function, there exists $q \in L_\omega^\nu$ such that $q(t) \ge 0$ for $t \in [0,\omega]$ and

$$q_0(t,x) \le q(t) \quad \text{for } t \in [0,\omega], \ \ |x| \le \beta^*. \tag{19.21}$$

Now it follows from (19.19)–(19.21) that

$$g\big(t, \chi_n(u_n(t))\big) \le q(t) \quad \text{for } t \in [0,\omega], \ \ n > n_0. \tag{19.22}$$

Let now v be a solution of the problem

$$v'' = p_0(t)v - q(t); \quad v(0) = v(\omega), \;\; v'(0) = v'(\omega) \tag{19.23}$$

and $v^* = \max\{v(t) : t \in [0,\omega]\}$. In view of (19.17), (19.22), and (19.23) clearly

$$\big(u_n(t) + v(t)\big)'' \leq p_0(t)\big(u_n(t) + v(t)\big) \quad \text{for } \; t \in [0,\omega].$$

Hence, in view of $p_0 \in \mathcal{V}^-(\omega)$, we get that $u_n(t) + v(t) \geq 0$ for $t \in [0,\omega]$ and, consequently,

$$u_n(t) \geq -v^* \quad \text{for } \; t \in [0,\omega], \;\; n > n_0.$$

The latter inequality, together with (19.14), results in

$$|u_n(t)| \leq \beta^* + v^* \quad \text{for } \; t \in [0,\omega], \;\; n > n_0. \tag{19.24}$$

It follows from (19.17), by virtue of (19.22), that

$$0 \leq \int_0^\omega \Big(q(s) - g\big(s, \chi_n(u_n(s))\big)\Big)\,\mathrm{d}s = \int_0^\omega \big(p_0(s)u_n(s) + q(s)\big)\,\mathrm{d}s. \tag{19.25}$$

Hence, on account of (19.24), we get

$$0 \leq \int_0^\omega \Big(q(s) - g\big(s, \chi_n(u_n(s))\big)\Big)\,\mathrm{d}s \leq (\beta^* + v^*)\|p_0\|_L + \|q\|_L \quad \text{for } \; n > n_0. \tag{19.26}$$

On account of (19.17), (19.22), (19.24), (19.25), and (19.26) it is clear that

$$\begin{aligned}
\int_0^\omega |u_n''(s)|\,\mathrm{d}s &= \int_0^\omega \Big|p_0(s)u_n(s) + g\big(s, \chi_n(u_n(s))\big)\Big|\,\mathrm{d}s \\
&= \int_0^\omega \Big|p_0(s)u_n(s) + q(s) - \Big(q(s) - g\big(s, \chi_n(u_n(s))\big)\Big)\Big|\,\mathrm{d}s \\
&\leq \int_0^\omega \Big|p_0(s)u_n(s) + q(s)\Big|\,\mathrm{d}s + \int_0^\omega \Big(q(s) - g\big(s, \chi_n(u_n(s))\big)\Big)\,\mathrm{d}s \leq c_0 \quad \text{for } \; n > n_0,
\end{aligned} \tag{19.27}$$

where $c_0 = 2\big[(\beta^* + v^*)\|p_0\|_L + \|q\|_L\big]$.

Since u_n is a periodic function there is a $t_n \in [0,\omega[$ such that $u_n'(t_n) = 0$. Taking now into account (19.27) we get

$$|u_n'(t)| = \bigg|\int_{t_n}^t u_n''(s)\,\mathrm{d}s\bigg| \leq \int_0^\omega |u_n''(s)|\,\mathrm{d}s \leq c_0 \quad \text{for } \; t \in [0,\omega], \;\; n > n_0. \tag{19.28}$$

Thus we have proved that

$$|u_n(t)| + |u_n'(t)| \leq A \quad \text{for } \; t \in [0,\omega], \;\; n \geq n_0, \tag{19.29}$$

where $A \overset{\text{def}}{=} \beta^* + v^* + c_0$.

Now we will show that

$$|u_n'(t)| \leq B\big(|u_n(t)| + |m_n|\big)^{\frac{\nu-1}{2\nu-1}} \quad \text{for } \; t \in [0,\omega], \;\; n \geq n_0, \tag{19.30}$$

where

$$m_n \overset{\text{def}}{=} \min\big\{u_n(t) : \; t \in [0,\omega]\big\}, \quad B \overset{\text{def}}{=} \Big[\frac{2\nu - 1}{\nu}\,\big(A^{\frac{\nu-1}{\nu}}\|p\|_L + \|q\|_{L^\nu}\big)\Big]^{\frac{\nu}{2\nu-1}}.$$

Indeed, let $n > n_0$ be fixed and $t \in [0,\omega[$ be such that $u_n'(t) \neq 0$. If $u_n'(t) > 0$ then there is a $t_* \in]t - \omega, t]$ such that

$$u_n'(s) > 0 \quad \text{for } \; s \in]t_*, t], \quad u_n'(t_*) = 0, \tag{19.31}$$

while if $u_n'(t) < 0$ then there is a $t^* \in]t, t + \omega]$ such that

$$u_n'(s) < 0 \quad \text{for } \; s \in [t, t^*[, \quad u_n'(t^*) = 0. \tag{19.32}$$

Multiplying both sides of equation in (19.17) by $|u_n'(t)|^{\frac{\nu-1}{\nu}}$, integrating it in on $[t_*, t]$ respectively on $[t, t^*]$, and taking into account (19.22), one gets

$$\frac{\nu}{2\nu-1}|u_n'(t)|^{\frac{2\nu-1}{\nu}} \leq \int_{t_1}^{t_2} p_0(s)u_n(s)|u_n'(s)|^{\frac{\nu-1}{\nu}}\,\mathrm{d}s + \int_{t_1}^{t_2} q(s)|u_n'(s)|^{\frac{\nu-1}{\nu}}\,\mathrm{d}s \quad \text{for } n \geq n_0, \tag{19.33}$$

where $t_1 \overset{\text{def}}{=} t_*$, $t_2 \overset{\text{def}}{=} t$ if $u_n'(t) > 0$ and $t_1 \overset{\text{def}}{=} t$, $t_2 \overset{\text{def}}{=} t^*$ if $u_n'(t) < 0$. In view of (19.31), resp. (19.32), and (19.29) it is clear that for $n \geq n_0$ we have

$$\int_{t_1}^{t_2} p_0(s)u_n(s)|u_n'(s)|^{\frac{\nu-1}{\nu}}\,\mathrm{d}s \leq u_n(t)\int_{t_1}^{t_2} p_0(s)|u_n'(s)|^{\frac{\nu-1}{\nu}}\,\mathrm{d}s \leq u_n(t)A^{\frac{\nu-1}{\nu}}\|p\|_L.$$

On the other hand, by virtue of Hölder's inequality, for $n \geq n_0$ we get

$$\int_{t_1}^{t_2} q(s)|u_n'(s)|^{\frac{\nu-1}{\nu}}\,\mathrm{d}s \leq \|q\|_{L^\nu}\bigg(\int_{t_1}^{t_2}|u_n'(s)|\,\mathrm{d}s\bigg)^{\frac{\nu-1}{\nu}} \leq \|q\|_{L^\nu}\Big(|u_n(t)| + |m_n|\Big)^{\frac{\nu-1}{\nu}}.$$

The latter two inequalities together with (19.33) imply (19.30).

Next we will show that there is an $n_1 > n_0$ such that

$$M_n > \frac{1}{n} \quad \text{for } n \geq n_1, \tag{19.34}$$

where

$$M_n \overset{\text{def}}{=} \max\big\{u_n(t) : \ t \in [0,\omega]\big\}.$$

Indeed, let there is an increasing sequence $\{n_k\}_{k=1}^{+\infty} \subset \mathbb{N}$ such that

$$M_{n_k} \leq \frac{1}{n_k} \quad \text{for } k \in \mathbb{N}.$$

Then, in view of (19.11), (19.12), (19.22), and (19.26), we get that

$$A \geq \int_0^\omega \Big(q(s) - g\big(s, \chi_{n_k}(u_{n_k}(s))\big)\Big)\,\mathrm{d}s \geq \int_0^\omega f_0\big(s, \chi_{n_k}(u_{n_k}(s))\big)\,\mathrm{d}s = \int_0^\omega f_0\Big(s, \frac{1}{n_k}\Big)\,\mathrm{d}s \quad \text{for } k \in \mathbb{N}$$

which contradicts (19.8).

Since the sequences $\{u_n\}_{n=1}^{+\infty}$ and $\{u_n'\}_{n=1}^{+\infty}$ are uniformly bounded (see (19.29)), by virtue of Arzelà-Ascoli lemma we can assume without loss of generality that

$$\lim_{n\to+\infty} u_n(t) = v_0(t) \quad \text{uniformly on } [0,\omega], \tag{19.35}$$

where $v_0 \in C([0,\omega];\mathbb{R})$. Show that

$$v_0(t) \geq 0 \quad \text{for } t \in [0,\omega]. \tag{19.36}$$

Indeed, let there is a $t_0 \in]0,\omega[$ such that $v_0(t_0) < 0$. Then, in view of (19.35), there are $n_2 > n_1$, $a_0 \in]0,\omega[$, and $b_0 \in]a,\omega[$ such that

$$u_n(t) < \frac{1}{2}v_0(t) \quad \text{for } t \in [a,b], \ \ n > n_2.$$

It follows from (19.17), by virtue of (19.11), (19.22), and (19.29), that

$$\begin{aligned}
-2A \leq u_n'(b_0) - u_n'(a_0) &\leq \int_{a_0}^{b_0}\Big[p_0(s)u_n(s) - f_0\Big(s, \frac{1}{n}\Big) + q(s)\Big]\,\mathrm{d}s \\
&\leq A\|p_0\|_L + \|q\|_L - \int_{a_0}^{b_0} f_0\Big(s, \frac{1}{n}\Big)\,\mathrm{d}s \quad \text{for } n > n_2
\end{aligned}$$

which contradicts (19.8). Consequently, (19.36) is fulfilled.

Summarizing above-proved we have that (19.29), (1.21), and (19.34) hold, (19.35) is fulfilled, where $v_0 \in C([0,\omega];\mathbb{R})$ satisfies (19.36).

Now we are able to show that for a certain $n > n_0$, the inequality (19.18) is fulfilled. Suppose the contrary, let $m_n < \frac{1}{n}$ for $n > n_0$. Then, in view of (19.34), there is $a_n \in [0,\omega[$ such that

$$u_n(a_n) = \frac{1}{n} \quad \text{for } n > n_0. \tag{19.37}$$

Assume without loss of generality that

$$\lim_{n\to+\infty} a_n = a, \tag{19.38}$$

where $a \in [0,\omega[$. It follows from (19.30) that

$$|u_n(t) - u_n(a_n)| \le \left| \int_{a_n}^{t} |u_n'(s)|\,\mathrm{d}s \right| \le B \left| \int_{a_n}^{t} \big(|u_n(s)| + |m_n|\big)^{\frac{\nu-1}{2\nu-1}}\,\mathrm{d}s \right| \quad \text{for } t \in [0,\omega], \ \ n > n_0.$$

Hence, on account of (19.35), (19.36), and (19.38),

$$0 \le v_0(t) \le B \left| \int_{a_n}^{t} \big(v_0(s)\big)^{\frac{\nu-1}{2\nu-1}}\,\mathrm{d}s \right| \quad \text{for } t \in [0,\omega].$$

Consequently, by virtue of Proposition 4.4, there is a $c > 0$ such that

$$0 \le v_0(t) \le c|t-a|^{\frac{2\nu-1}{\nu}} \quad \text{for } t \in [0,\omega].$$

On the other hand, in view of (19.35), for any $\varepsilon > 0$ there is $n_\varepsilon > \frac{1}{\varepsilon}$ such that $u_n(t) < \varepsilon + v_0(t)$ for $t \in [0,\omega]$, $n > n_\varepsilon$. Therefore,

$$u_n(t) \le \varepsilon + c|t-a|^{\frac{2\nu-1}{\nu}} \quad \text{for } t \in [0,\omega], \ \ n > n_\varepsilon$$

which together with (19.12) and assumption $n_\varepsilon > \frac{1}{\varepsilon}$ imply that

$$\chi_n(u_n(t)) \le \varepsilon + c|t-a|^{\frac{2\nu-1}{\nu}} \quad \text{for } t \in [0,\omega], \ \ n > n_\varepsilon. \tag{19.39}$$

It follows from (19.26), by virtue of (19.11) and (19.22), that

$$A \ge \int_0^\omega \Big(q(s) - g\big(s, \chi_n(u_n(s))\big)\Big)\,\mathrm{d}s \ge \int_0^\omega f_0\big(s, \chi_n(u_n(s))\big)\,\mathrm{d}s \quad \text{for } n > n_\varepsilon.$$

Taking now into account (19.39) and the monotonicity of the function $f_0(t,\cdot)$, we get

$$A \ge \int_0^\omega f_0\big(s, \varepsilon + c|s-a|^{\frac{2\nu-1}{\nu}}\big)\,\mathrm{d}s.$$

However, the latter inequality contradicts (H_{15}). □

For the problem (19.2) we get the following

Corollary 19.11. *Let*

$$h(t,x) \le -f_0(t,x) + q_0(t,x) \quad \text{for } t \in [0,\omega], \ \ x > 0, \tag{19.40}$$

where q_0 is the same as in Theorem 19.10 *and f_0 satisfies hypothesis (H_{15}). Let, moreover, at least one of the items* (1)–(3) *of Proposition* 18.8 *be fulfilled. Then the problem* (19.2) *has at least one solution.*

Proof. Put $f(t,x) \overset{\text{def}}{=} p(t)x + h(t,x)$. By virtue of Proposition 18.8, there is a $\beta \in \widetilde{AC}'([0,\omega])$, $\beta(t) > 0$ for $t \in [0,\omega]$, satisfying (17.6). On the other hand, in view of (19.40), inequality (19.9) holds with $p_0 \overset{\text{def}}{=} |p| + 1$. Hence, all the conditions of Theorem 19.10 are fulfilled. □

Return again to the problem (19.1).

Theorem 19.12. *Let* (18.22) *hold, where $q \in K_{sl}([0,\omega]\times\mathbb{R};\mathbb{R}_+)$. Let, moreover, there exist $\alpha \in \widetilde{AC}'([0,\omega])$, $\alpha(t) > 0$ for $t \in [0,\omega]$, satisfying* (17.5). *Then the problem* (19.1) *has at least one solution.*

Proof. Let $c \stackrel{\text{def}}{=} \max\{\alpha(t) : t \in [0,\omega]\}$. By virtue of Proposition 18.7 there is a $\beta \in \widetilde{AC}'([0,\omega])$, $\beta(t) \geq c$ for $t \in [0,\omega]$, satisfying (17.6). Hence, in view of Proposition 17.6, the problem (19.1) is solvable. □

Corollary 19.13. *Let* (18.22) *holds, where* $q \in K_{sl}([0,\omega]\times;\mathbb{R}_+)$. *If, moreover,* $k \in \{3,4,5,6\}$ *and conditions of Proposition* 18.k *are fulfilled, then the problem* (19.1) *has at least one solution.*

For the problem (19.2), Corollary 19.13 implies

Corollary 19.14. *Let* $p \in \mathcal{V}^-(\omega)$, (H_9) *holds and*

$$h(t,x) \leq q(t) \quad \textit{for } t \in [0,\omega], \ \ x > 0, \tag{19.41}$$

where $q \in L_\omega$ *and* $q \not\equiv 0$. *Let, moreover, at least one of the following items be fulfilled:*

(1) $Q_- \geq \rho(p)Q_+$;

(2) $Q_- < \rho(p)Q_+$, *the function* $h(t,\,\cdot\,)$ *is nondecreasing, and*

$$\lim_{x\to+\infty} H(x) > (1-\rho(p))Q_+, \quad H\Big(\frac{\omega}{4}\,\rho(p)Q_+\Big) \leq (1-\rho(p))Q_+;$$

(3) *the function* $h(t,\,\cdot\,)$ *is nonincreasing and there is an* $x_0 > 0$ *such that*

$$H(x_0) + \Big(\rho(p)\big\|[p]_+\big\|_L - \big\|[p]_-\big\|_L\Big)x_0 \leq (1-\rho(p))Q_+. \tag{19.42}$$

Then the problem (19.2) *is solvable.*

Proof. Put $f(t,x) \stackrel{\text{def}}{=} p(t)x + h(t,x)$ for $t \in [0,\omega]$, $x > 0$. On account of (H_9), clearly (18.22) holds with $\delta_1 = 1$, $p_1(t) \stackrel{\text{def}}{=} p(t)$, and $q(t,x) \stackrel{\text{def}}{=} \big[h\big(t,[x-1]_+ + 1\big)\big]_-$. Suppose that item (1), resp. (2), is fulfilled. Then it is clear that conditions of Proposition 18.4, resp. Proposition 18.5 hold.

Now suppose that item (3) is fulfilled. We will show that the conditions of Proposition 18.6 (i.e., the hypothesis (H_6)) hold. Indeed, in view of (19.41), we have $H_+(x) \leq Q_+$ for $x > 0$. Hence,

$$\rho(p)H_+(x) - H_-(x) \leq H(x) + (\rho(p)-1)Q_+ \quad \text{for } x > 0.$$

In view of (19.42), it is now clear that the hypothesis (H_6) holds.

Thus we have shown that conditions of Corollary 19.14 imply that conditions of Corollary 19.13 are fulfilled. □

Corollary 19.15. *Let* $p \in \mathcal{V}^-(\omega)$, *the function* $h(t,\,\cdot\,)$ *is nonincreasing and*

$$h(t,x) \geq q(t) \quad \textit{for } t \in [0,\omega], \ \ x > 0, \tag{19.43}$$

where $q \in L_\omega$ *and* $q \not\equiv 0$. *Let, moreover, there exists* $x_0 > 0$ *such that*

$$\rho(p)H(x_0) + \Big(\rho(p)\big\|[p]_+\big\|_L - \big\|[p]_-\big\|_L\Big)x_0 \leq (1-\rho(p))Q_-\,. \tag{19.44}$$

Then the problem (19.2) *is solvable.*

Proof. As in the proof of Corollary 19.14 it is sufficient to show that the hypothesis (H_6) holds. In view of (19.43), clearly $H_-(x) \leq Q_-$ for $x > 0$ and, consequently,

$$\rho(p)H_+(x) - H_-(x) \leq \rho(p)H(x) + (\rho(p)-1)Q_- \quad \text{for } x > 0.$$

Now, in view of (19.44), it is clear that the hypothesis (H_6) holds. □

20. Corollaries

In this section we will apply results of Section 19 to some particular types of equation each of them contains either the term "$+\frac{h_0(t)}{u^\lambda}$" or the term "$-\frac{h_0(t)}{u^\lambda}$". So we will suppose that $h_0 \in L_\omega$,

$$h_0(t) \geq 0 \quad \text{for } t \in [0,\omega], \ \ h_0 \not\equiv 0 \ \text{ and } \ \lambda \neq 0.$$

Recall that under a solution we understand a **positive** function $u \in \widetilde{AC}'([0,\omega])$ satisfying given equation.

Consider the problem

$$u'' = p(t)u - \frac{h_0(t)}{u^\lambda}; \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega). \tag{20.1}$$

Theorem 20.1. *Let $\lambda > -1$. Then the problem* (20.1) *is solvable if and only if $p \in \mathcal{V}^-(\omega)$. If $p \in \mathcal{V}^-(\omega)$ and $\lambda > 0$ then the problem* (20.1) *is uniquely solvable.*

Proof. For $\lambda > 0$, the assertion immediately follows from Corollary 19.3. Suppose that $p \in \mathcal{V}^-(\omega)$, $\lambda \in]-1,0[$, and put $h(t,x) \overset{\text{def}}{=} -h_0(t)x^{|\lambda|}$. Then clearly (H_1) holds (with $\varphi(x) = x^{|\lambda|}$). It is also evident that (H_9) is fulfilled as well. Hence, by virtue of Corollary 19.2, the problem (20.1) is solvable. Necessity of the inclusion $p \in \mathcal{V}^-(\omega)$ follows from Theorem 8.3. □

Consider the problem

$$u'' = p(t)u - \frac{h_0(t)}{u^\lambda} + q(t); \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega), \tag{20.2}$$

where $q \in L_\omega$ and $q \not\equiv 0$. In formulation of the next result we use notations (0.13) and (0.12).

Theorem 20.2. *Let $p \in \mathcal{V}^-(\omega)$ and at least one of the following items be fulfilled:*

(i) $\lambda > -1$ *and*

$$\rho(p)Q_+ \leq Q_-; \tag{20.3}$$

(ii) $\lambda \in]-1,0[$ *and*

$$\|h_0\|_L^{\frac{1}{\lambda+1}} \geq \frac{1}{\lambda+1}\Big(\frac{|\lambda|}{\rho(p)\big\|[p]_+\big\|_L - \big\|[p]_-\big\|_L}\Big)^{\frac{|\lambda|}{\lambda+1}}\big(\rho(p)Q_+ - Q_-\big);$$

(iii) $\lambda > 0$ *and*

$$\|h_0\|_L \geq \Big(\frac{\omega}{4}\,\rho(p)Q_+\Big)^\lambda \big(\rho(p)Q_+ - Q_-\big); \tag{20.4}$$

(iv) $\lambda > 0$ *and there is a $c_0 > 0$ such that*

$$h_0(t) \geq c_0 q(t) \quad \text{for } t \in [0,\omega]; \tag{20.5}$$

(v) $\lambda \geq \frac{1}{2}$, $\eta \in [1/\lambda, 2[$, $[q]_+^{\frac{1}{2-\eta}} \in L_\omega$, *and*

$$\int_0^\omega \frac{h_0(s)}{|s-a|^{\lambda\eta}}\,\mathrm{d}s = +\infty \quad \text{for } a \in [0,\omega[\,. \tag{20.6}$$

Then the problem (20.2) *is solvable. Moreover, if $q(t) \leq 0$ for $t \in [0,\omega]$ then the inclusion $p \in \mathcal{V}^-(\omega)$ is necessary for solvability of* (20.2), *while if either* (iii) *or* (iv) *or* (v) *holds then the problem* (20.2) *is uniquely solvable.*

Proof. Put $h(t,x) = -\frac{h_0(t)}{x^\lambda} + q(t)$. If either (i) or (iii) holds then the problem (20.2) is solvable by virtue of Corollary 19.14, while if (iv) holds then the solvability of (20.2) follows from Corollary 19.3. If (ii) holds then one can easily verify that condition (3) of Corollary 19.14 is fulfilled with

$$x_0 = \Big(\frac{|\lambda|\|h_0\|_L}{\rho(p)\big\|[p]_+\big\|_L - \big\|[p]_-\big\|_L}\Big)^{\frac{1}{\lambda+1}}.$$

Let now (v) be fulfilled. Then (19.40) holds with $f_0(t,x) = \frac{h_0(t)}{x^\lambda}$ and $q_0(t,x) = [q(t)]_+$. Clearly, (H_9) is fulfilled as well. Therefore, by virtue of Corollary 19.11, the problem (20.2) is solvable. Necessity of the inclusion $p \in \mathcal{V}^-(\omega)$ follows from Theorem 8.3, while the uniqueness follows from Remark 19.6. $\square$

Remark 20.3. Condition (20.3) is optimal and cannot be weakened neither to $\rho(p)Q_+ \le (1+\delta)Q_-$ nor to $\rho(p)Q_+ \le Q_- + \delta$, no matter how small $\delta > 0$ would be (see Examples 20.9–20.11 below). Similarly, condition (20.4) cannot be weakened to

$$\|h_0\|_L \ge (1-\delta)\Big(\frac{\omega}{4}\,\rho(p)Q_+\Big)^\lambda \big(\rho(p)Q_+ - Q_-\big),$$

no matter how small $\delta \in]0,1[$ would be (see Example 20.12). Mention also that Example 20.11 below shows that the conditions (20.5) and (20.6) are essential and cannot be omitted.

Consider the problem

$$u'' = p(t)u + \frac{h_0(t)}{u^\lambda} + q(t); \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega), \tag{20.7}$$

where

$$\lambda > 0, \quad h_0, q \in L_\omega, \quad h_0(t) \ge 0 \quad \text{for } t \in [0,\omega], \quad h_0 \not\equiv 0. \tag{20.8}$$

Theorem 20.4. *Let $p \in \mathcal{V}^-(\omega)$ and (20.8) hold. Let, moreover,*

$$Q_- > \rho(p)Q_+$$

and

$$\|h_0\|_L \le \frac{\lambda^\lambda}{(\lambda+1)^{\lambda+1}}\,\frac{(Q_- - \rho(p)Q_+)^{\lambda+1}}{\rho(p)\big(\rho(p)\big\|[p]_+\big\|_L - \big\|[p]_-\big\|_L\big)^\lambda}\,. \tag{20.9}$$

Then the problem (20.7) is solvable.

Proof. Put $h(t,x) \overset{\text{def}}{=} \frac{h_0(t)}{x^\lambda} + q(t)$ for $t \in [0,\omega]$, $x > 0$. Clearly, $h(t,\cdot\,)$ is nonincreasing and (19.43) holds. Let now

$$x_0 \overset{\text{def}}{=} \bigg(\frac{\lambda\rho(p)\|h_0\|_L}{\rho(p)\big\|[p]_+\big\|_L - \big\|[p]_-\big\|_L}\bigg)^{\frac{1}{\lambda+1}}.$$

In view of (20.9), one can easily verify that (19.44) is fulfilled. Consequently, the problem (20.7) is solvable by virtue of Corollary 19.15. $\square$

Consider the problem

$$u'' = p(t)u - \frac{h_0(t)}{u^\lambda} + g_0(t)u^\mu + q(t); \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega), \tag{20.10}$$

where $g_0 \in L_\omega$, $g_0 \not\equiv 0$, $\mu \ne 0$, and $\mu \ne 1$.

Theorem 20.5. *Let $p \in \mathcal{V}^-(\omega)$, $\mu \in]0,1[$, and at least one of the following items be fulfilled:*

(a) *either item* (i)*, or* (ii)*, or* (iii) *of Theorem* 20.2 *holds and*

$$g_0(t) \le 0 \quad \text{for } t \in [0,\omega];$$

(b) *item* (iv) *of Theorem* 20.2 *holds and there is a $c > 0$ such that*

$$h_0(t) \ge c g_0(t) \quad \text{for } t \in [0,\omega];$$

(c) *item* (v) *of Theorem* 20.2 *holds and $[g_0]_+^{\frac{1}{2-\eta}} \in L_\omega$.*

Then the problem (20.10) has at least one solution. If, moreover,

$$g_0(t) \le 0, \quad q(t) \le 0 \quad \text{for } t \in [0,\omega], \tag{20.11}$$

then the inclusion $p \in \mathcal{V}^-(\omega)$ is necessary for solvability of (20.10).

Proof. Suppose that (a) holds. Put

$$f(t,x) = p(t)x - \frac{h_0(t)}{x^\lambda} + g_0(t)x^\mu + q(t), \quad h(t,x) = -\frac{h_0(t)}{x^\lambda} + q(t).$$

Then clearly (18.22) is fulfilled with

$$p_1(t) = p(t), \quad q(t,x) = \frac{h_0(t)}{(1+[x-1]_+)^\lambda} + |g_0(t)|\,|x|^\mu + |q(t)|$$

and $\delta_1 = 1$. On the other hand, if item (i), resp. item (ii), resp. item (iii) of Theorem 20.2 is fulfilled then the conditions of Proposition 18.3, resp. Proposition 18.6, resp. Proposition 18.5 hold as well. Therefore, by virtue of Corollary 19.13, the problem (20.10) is solvable.

Let now either (b) or (c) holds. Put

$$h(t,x) = -\frac{h_0(t)}{x^\lambda} + g_0(t)x^\mu + q(t).$$

Clearly, (H_9) holds. On the other hand, if (b) is fulfilled then

$$h(t,x) \le h_0(t,x) \quad \text{for } t \in [0,\omega], \ x > 0,$$

where $h_0(t,x) = -h_0(t)(\frac{1}{x^\lambda} - \frac{x^\mu}{c} - \frac{1}{c_0})$. Hence, for sufficiently small $r_0 > 0$ the hypothesis (H_2) holds. Therefore, by virtue of Corollary 19.2, the problem (20.10) has at least one solution. Suppose now that (c) is fulfilled. Then (19.40) holds with

$$f_0(t,x) = \frac{h_0(t)}{x^\lambda} \ \text{ and } \ q_1(t,x) = |g_0(t)|\,|x|^\mu + |q(t)|.$$

Therefore, by virtue of Corollary 19.11, the problem (20.10) is solvable.

If (20.11) holds then necessity of the inclusion $p \in \mathcal{V}^-(\omega)$ follows from Theorem 8.3. □

Theorem 20.6. *Let $p \in \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$, $\mu > 1$, and*

$$g_0(t) \ge 0 \quad \textit{for } t \in [0,\omega]. \tag{20.12}$$

Let, moreover, either

$$\lambda > -1, \quad q(t) \le 0 \quad \textit{for } t \in [0,\omega], \tag{20.13}$$

or item (iv) *of Theorem* 20.2*, or item* (c) *of Theorem* 20.5 *is fulfilled. Then the problem* (20.10) *is solvable. Moreover, if $\lambda > 0$ then the problem* (20.10) *is uniquely solvable.*

Proof. Put

$$h(t,x) = -\frac{h_0(t)}{x^\lambda} + g_0(t)x^\mu + q(t).$$

Then (H_{10}) is fulfilled with $r_1 = 1$,

$$g(x) = x^\mu, \quad q(t,x) = \frac{h_0(t)}{(1+[x-1]_+)^\lambda} + |q(t)|.$$

On the other hand, condition (20.13), resp. item (iv) of Theorem 20.2 implies (H_2). Therefore, by virtue of Corollary 19.7, the problem (20.10) has at least one solution.

Let now item (v) of Theorem 20.2 holds. Then (19.40) holds with

$$f_0(t,x) = \frac{h_0(t)}{x^\lambda} \ \text{ and } \ q_0(t,x) = g_0(t)|x|^\mu + [q(t)]_+.$$

Hence, the problem (20.10) is solvable by virtue of Corollary 19.11. Assertion about uniqueness follows from Remark 19.6. □

Theorem 20.7. *Let $\mu > 1$,* (20.12) *hold, and either* (20.13) *or item* (iv) *of Theorem* 20.2*, or item* (c) *of Theorem* 20.5 *be fulfilled. Let, moreover,*

$$\operatorname{mes}\big\{t \in [0,\omega] : \ g_0(t) = 0\big\} = 0. \tag{20.14}$$

Then the problem (20.10) *has at least one solution* (*for any $p \in L_\omega$*).

Proof. It is analogous as the proof of Theorem 20.6. Only a difference is that (H_{11}) is fulfilled instead of (H_{10}). Therefore, if either (20.13) or item (iv) of Theorem 20.2 holds then solvability follows from Corollary 19.8, while if item (c) of Theorem 20.5 holds then solvability follows from Corollary 19.11. □

Remark 20.8. Condition (20.14) in Theorem 20.7 is essential and cannot be omitted. Indeed, let $0 < a < b < \omega$, $\lambda = 1$, $h_0(t) \equiv 1$, $q, g_0 \in L_\omega$, and $q(t) \le 0$ for $t \in \mathbb{R}$, and

$$g_0(t) = \begin{cases} 1 & \text{for } t \in [0, a[\cup]b, \omega], \\ 0 & \text{for } t \in [a, b]. \end{cases}$$

Let, moreover, $p \in L_\omega$ be such that the equation $u'' = p(t)u$ is conjugate on $[a, b]$, i.e., its every nontrivial solution has at least one zero in $]a, b[$ (for example, let $p(t) = -(1+\varepsilon)(\frac{\pi}{b-a})^2$ for $t \in [a, b]$ with $\varepsilon > 0$ and $p(t) = 0$ for $t \in [0, a[\cup]b, \omega]$). Then it is clear that the conditions Theorem 20.7 are fulfilled except of the condition (20.14). Suppose that u is a solution of the problem (20.10). Then it is clear that $u(t) > 0$ for $t \in [a, b]$ and

$$u''(t) \le p(t)u(t) \quad \text{for } t \in [a, b].$$

Hence, by virtue of Sturm's comparison theorem, any nontrivial solution of the equation $u'' = p(t)u$ has at most one zero in $[a, b]$ which contradicts the setting of the function p.

Example 20.9. Let $\lambda \in]0, 1[$ and $c > 0$. Consider the problem

$$u'' = u - c^{1-\lambda}u^\lambda + c; \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega). \tag{20.15}$$

Then we have

$$\rho(p) = \mathrm{e}^{\frac{1}{4}\,\omega^2}, \quad Q_+ = c\omega, \quad Q_- = 0.$$

For given $\delta > 0$ choose $c > 0$ such that $c < \frac{\delta}{\omega}\,\mathrm{e}^{-\frac{1}{4}\,\omega^2}$. Clearly, the inequality $\rho(p)Q_+ \le Q_- + \delta$ is fulfilled. However, the problem (20.15) has no solution because $x - c^{1-\lambda}x^\lambda + c > 0$ for $x > 0$.

Example 20.10. Let $\lambda \in]0, 1[$ and $\varepsilon \in]0, 1[$. Consider the problem

$$u'' = \varepsilon^2 u - \varepsilon^2 u^\lambda + \varepsilon^2 + \frac{1}{\varepsilon}\cos t; \quad u(0) = u(2\pi), \ \ u'(0) = u'(2\pi). \tag{20.16}$$

Then $\rho(p) = \mathrm{e}^{\varepsilon^2\pi}$, $Q = 2\varepsilon^2\pi$, $Q_+ = 2\varepsilon^2\pi + Q_-$, and

$$Q_- = \frac{2}{\varepsilon}\sqrt{1-\varepsilon^6} - 2\varepsilon^2(\pi - x),$$

where $x \in]\pi/2, \pi[$ is such that $\cos x = -\varepsilon^3$. Since $\lim\limits_{\varepsilon\to 0+} Q_- = +\infty$, for any $\delta > 0$, there is a $\varepsilon > 0$ such that

$$\mathrm{e}^{\varepsilon^2\pi}\Big(1 + \frac{2\varepsilon^2\pi}{Q_-}\Big) \le 1 + \delta,$$

i.e., the inequality $\rho(p)Q_+ \le (1+\delta)Q_-$ is fulfilled. However, the problem (20.16) has no solution (for any $\varepsilon > 0$). Indeed, if u is a solution of this problem then, in view of the inequality $x - x^\lambda + 1 > 0$ for $x > 0$, we get the contradiction

$$0 = \int_0^{2\pi} u''(s)\,\mathrm{d}s = \varepsilon^2 \int_0^{2\pi} \big(u(s) - u^\lambda(s) + 1\big)\,\mathrm{d}s > 0.$$

Example 20.11. Let $\lambda > 0$ and $\varepsilon > 0$. Consider the problem

$$u'' = \varepsilon u - \frac{\varepsilon(1+\cos t)^\lambda}{u^\lambda} - (1+\varepsilon)\cos t; \quad u(0) = u(2\pi), \ \ u'(0) = u'(2\pi). \tag{20.17}$$

Then

$$\rho(p) = \mathrm{e}^{\varepsilon\pi^2}, \quad Q_+ = 2(1+\varepsilon), \quad Q_- = 2(1+\varepsilon).$$

For given $\delta > 0$ choose $\varepsilon > 0$ such that

$$\mathrm{e}^{\varepsilon\pi^2} < 1 + \delta, \quad 2(1+\varepsilon)\big(\mathrm{e}^{\varepsilon\pi^2} - 1\big) < \delta.$$

Then clearly the inequalities $\rho(p)Q_+ \le (1+\delta)Q_-$ and $\rho(p)Q_+ \le Q_- + \delta$ are fulfilled. Mention also that (20.5) and (20.6) are violated. Now we will show that the problem (20.17) has no solution for any $\varepsilon > 0$. Suppose the contrary, let u be a solution of this problem and put $w(t) = u(t) - 1 - \cos t$. By direct calculations one can easily verify that

$$w''(t) = \varepsilon w(t) + \frac{\varepsilon}{u^\lambda(t)}\big(u^\lambda(t) - (1+\cos t)^\lambda\big) \quad \text{for } t \in [0, 2\pi],$$
$$w(0) = w(2\pi), \quad w'(0) = w'(2\pi).$$

Introduce the notation

$$\varphi_\lambda(x) \overset{\text{def}}{=} \begin{cases} \dfrac{x^\lambda - 1}{x-1} & \text{for } x \neq 1, \\ \lambda & \text{for } x = 1. \end{cases} \tag{20.18}$$

It is clear that, $\varphi_\lambda \in C(\mathbb{R})$ and $\varphi_\lambda(x) > 0$ for $x \in \mathbb{R}$. Moreover,

$$u^\lambda(t) - (1+\cos t)^\lambda = u^{\lambda-1}(t)\varphi_\lambda\Big(\frac{1+\cos t}{u(t)}\Big)w(t) \quad \text{for } t \in [0, 2\pi].$$

Therefore, the function w is a solution of the problem

$$w'' = p_0(t)w; \quad w(0) = w(2\pi), \;\; w'(0) = w'(2\pi), \tag{20.19}$$

where $p_0(t) = \varepsilon(1 + \frac{1}{u(t)}\,\varphi_\lambda(\frac{1+\cos t}{u(t)}))$ for $t \in [0, 2\pi]$. Since $p_0(t) > 0$ for $t \in [0, 2\pi]$, we have $p_0 \in \mathcal{V}^-(\omega)$ (see Remark 8.4) and, consequently, $w \equiv 0$. Hence, we get the contradiction $0 = w(\pi) = u(\pi) \neq 0$.

Example 20.12. Let $\varepsilon > 0$ and $\lambda > 0$. Consider the problem

$$u'' = \varepsilon u - \frac{(1+\cos t)^\lambda}{u^\lambda} + 1 - \varepsilon - (1+\varepsilon)\cos t; \quad u(0) = u(2\pi), \;\; u'(0) = u'(2\pi). \tag{20.20}$$

Then $\rho(p) = \mathrm{e}^{\varepsilon\pi^2}$, $Q = 2\pi(1-\varepsilon)$, and $\|h_0\|_L = \int\limits_0^{2\pi}(1+\cos s)^\lambda\,\mathrm{d}s$. Since

$$\lim_{\varepsilon \to 0+}\big(\rho(p)Q_+ - Q_-\big) = 2\pi$$

for given $\delta > 0$ we can choose $\varepsilon > 0$ such that

$$\rho(p)Q_+ - Q_- < 2\pi(1+\delta). \tag{20.21}$$

On the other hand, since

$$\lim_{\lambda \to 0+}\|h_0\|_L = 2\pi \;\text{ and }\; \lim_{\lambda \to 0+}\Big(\frac{\omega}{4}\,\rho(p)Q_+\Big)^\lambda = 1$$

we can choose $\lambda > 0$ such that

$$\|h_0\|_L > 2\pi(1-\delta^2)\Big(\frac{\omega}{4}\,\rho(p)Q_+\Big)^\lambda.$$

The latter inequality, together with (20.21), implies that

$$\|h_0\|_L > (1-\delta)\Big(\frac{\omega}{4}\,\rho(p)Q_+\Big)^\lambda\big(\rho(p)Q_+ - Q_-\big)$$

holds. Now we will show that the problem (20.20) has no solution for any $\lambda > 0$ and $\varepsilon > 0$. Indeed, let u be a solution of this problem and put $w(t) = u(t) - 1 - \cos t$. By the same arguments as in Example 20.11 one can verify that w is a solution of the problem (20.19), where

$$p_0(t) = \varepsilon + \frac{1}{u(t)}\,\varphi_\lambda\Big(\frac{1+\cos t}{u(t)}\Big) \quad \text{for } t \in [0, 2\pi]$$

and the function φ_λ is defined by (20.18). Since $p_0 \in \mathcal{V}^-(\omega)$ we get $w \equiv 0$, which yields the contradiction $0 = w(\pi) = u(\pi) \neq 0$.

21. Resonance Like Case

In this section we will consider the problem

$$u'' = p(t)u + h(t,u) + q(t); \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega), \tag{21.1}$$

where $h \in K_{loc}([0,\omega]\times\,]0,+\infty[\,;\mathbb{R})$ and $p, q \in L_\omega$. Recall that under a solution of the problem (21.1) we understand a **positive** function $u \in AC'([0,\omega])$ satisfying the given equation almost everywhere and boundary conditions.

In some cases (wee, for example, Corollary 19.3), the inclusion $p \in \mathcal{V}^-(\omega)$ is necessary for the solvability of (21.1). However, we deal in what follows with the case $p \in \mathcal{V}_0(\omega)$. Mention that the problem (21.1) with $p \in \mathcal{V}_0(\omega)$ cover by Corollaries 19.7 and 19.8 (see also Corollary 19.11 and Theorems 20.6 and 20.7). In spite of them in the main results of this chapter we does not require neither hypothesis (H_{10}) nor hypothesis (H_{11}). Below we denote by u_0 a positive solution of the problem

$$u_0'' = p(t)u_0; \quad u_0(0) = u_0(\omega), \;\; u_0'(0) = u_0'(\omega).$$

Theorem 21.1. *Let $p \in \mathcal{V}_0(\omega)$ and either (H_{13}) or (H_{14}) hold. Let, moreover, there exist $\alpha \in AC'([0,\omega])$ such that $\alpha(t) > 0$ for $t \in [0,\omega]$ and*

$$\begin{gathered} \alpha''(t) \geq p(t)\alpha(t) + h(t,\alpha(t)) + q(t) \quad \text{for } t \in [0,\omega], \\ \alpha(0) = \alpha(\omega), \;\; \alpha'(0) = \alpha'(\omega). \end{gathered} \tag{21.2}$$

Then the problem (21.1) *is solvable.*

Proof. Put $f(t,x) \overset{\text{def}}{=} p(t)x + h(t,x) + q(t)$ for $t \in [0,\omega]$, $x > 0$ and $\alpha^* = \max\{\alpha(t) : \; t \in [0,\omega]\}$. Suppose that (H_{13}) (resp., (H_{14})) holds. Then, by virtue of Proposition 18.10 (resp., Proposition 18.11), there exists a function $\beta \in AC'([0,\omega])$ such that $\beta(t) \geq \alpha^*$ for $t \in [0,\omega]$ satisfying (17.6). Hence, by virtue of Proposition 17.6, the problem (21.1) has at least one solution. $\square$

Observe that the hypothesis

$$\left\{\begin{gathered} h(t,x) \leq 0 \quad \text{for } \; t \in [0,\omega], \;\; x > 0, \\ \text{the function } h(t,\cdot\,) \text{ is nondecreasing on } \;]0,+\infty[\,, \\ \lim_{x\to+\infty} \int_0^\omega h(s,x)\,\mathrm{d}s = 0 \end{gathered}\right. \tag{H_{16}}$$

together with the condition

$$\int_0^\omega q(s)u_0(s)\,\mathrm{d}s > 0 \tag{21.3}$$

implies (H_{13}) as well as the condition (21.3) together with the hypothesis

$$\left\{\begin{gathered} h(t,x) \geq -h_0(t)g(x) \quad \text{for } \; t \in [0,\omega], \;\; x > r, \\ h_0 \in L_\omega, \; h_0(t) \geq 0 \; \text{ for } \; t \in [0,\omega], \;\; h_0 \not\equiv 0, \\ g \in C(\,]0,+\infty[\,;\,]0,+\infty[\,), \\ \limsup_{x\to+\infty} g(x) = 0 \end{gathered}\right. \tag{H_{17}}$$

implies (H_{14}). Hence, it follows from Theorem 21.1 that

Corollary 21.2. *Let $p \in \mathcal{V}_0(\omega)$, (21.3) hold and either (H_{16}) or (H_{17}) be fulfilled. Let, moreover, there exists $\alpha \in AC'([0,\omega])$ such that $\alpha(t) > 0$ for $t \in [0,\omega]$ satisfying* (21.2)*. Then the problem* (21.1) *is solvable.*

Mention that condition (21.3) is necessary, in some sense, for the solvability of (21.1). More precisely

Proposition 21.3. *Let $p \in \mathcal{V}_0(\omega)$ and the problem* (21.1) *is solvable. Let, moreover, either*

$$h(t,x) \le h_1(t,x) \quad \text{for } t \in [0,\omega], \ \ x > 0,$$

where $h_1 \in K_{loc}([0,\omega]\times\,]0,+\infty[\,;\mathbb{R})$, $h_1(t,x) \le 0$ for $t \in [0,\omega]$, $x > 0$, $h_1(t,\cdot\,)$ is nondecreasing and

$$\operatorname{mes}\{t \in [0,\omega] : h_1(t,x) < 0\} > 0 \quad \text{for } x > 0,$$

or

$$h(t,x) \le -h_1(t)g(x) \quad \text{for } t \in [0,\omega], \ \ x > 0,$$

where $h_1 \in L_\omega$, $h_1(t) \ge 0$ for $t \in [0,\omega]$, $h_1 \not\equiv 0$, and $g \in C(\,]0,+\infty[\,;\,]0,+\infty[\,)$.

Then inequality (21.3) *holds.*

Proof. Let u be a solution of the problem (21.1). Then, by virtue of Fredholm's third theorem, we have that

$$\int_0^\omega q(s)u_0(s)\,\mathrm{d}s = -\int_0^\omega h(s,u(s))u_0(s)\,\mathrm{d}s.$$

One can easily verify that the conditions of the proposition guarantee that

$$\int_0^\omega h(s,u(s))u_0(s)\,\mathrm{d}s < 0.$$

Hence, (21.3) holds as well. □

Corollary 21.4. *Let $p \in \mathcal{V}_0(\omega)$, hypothesis (H_{16}) holds and*

$$\operatorname{mes}\{t \in [0,\omega] : h(t,x) < 0\} > 0 \quad \text{for } x > 0.$$

Let, moreover, at least one of the following items be fulfilled:

(1) *there exists $r_0 > 0$ such that*

$$h(t,r_0) \le -q(t) \quad \text{for } t \in [0,\omega];$$

(2) $H(\frac{\omega}{4}\,\rho(p)Q_+) \le Q_- - \rho(p)Q_+$.

Then the problem (21.1) *is solvable if and only if* (21.3) *holds.*

Proof. In view of Corollary 21.2, it is sufficient to show that there exists a positive function $\alpha \in AC'([0,\omega])$ satisfying (21.2). Let the item (1) hold. Then the existence of the function α follows from Proposition 18.2 with $k = 3$. Let now the item (2) be fulfilled. If $Q_- \ge \rho(p)Q_+$ then the existence of the function α follows from Propositions 18.4 while if $Q_- < \rho(p)Q_+$ then the existence of the function α follows from Proposition 18.5.

Necessity of the condition (21.3) follows from Proposition 21.3. □

Analogously one can prove that

Corollary 21.5. *Let $p \in \mathcal{V}_0(\omega)$, the hypothesis (H_{14}) hold,* (21.3) *be fulfilled, and*

$$h(t,x) \le 0 \quad \text{for } t \in [0,\omega].$$

Let, moreover, either the item (1) *of Corollary* 21.4 *hold, or the function $h(t,\cdot\,)$ is nondecreasing and the item* (2) *of Corollary* 21.4 *hold. Then the problem* (21.1) *is solvable.*

Proof. In view of Corollary 21.2, it is sufficient to show that there exists a positive function $\alpha \in AC'([0,\omega])$ satisfying (21.2). If the item (1) of Corollary 21.4 holds then the existence of the function α follows from Proposition 18.2 with $k = 3$.

Let now the function $h(t,\cdot\,)$ be nondecreasing and the item (2) of Corollary 21.4 be fulfilled. It is clear that, there exist a finite limit

$$\lim_{x\to+\infty} H(x) = H^*.$$

If $H^* > Q_- - \rho(p)Q_+$ then the existence of the function α follows from Proposition 18.5. Suppose that $H^* \le Q_- - \rho(p)Q_+$. Then, clearly,

$$H(x) \le Q_- - \rho(p)Q_+ \quad \text{for } x > 0. \tag{21.4}$$

Denote by v a solution of the problem

$$v'' = [p(t)]_+ v + h(t,1) + q(t); \quad v(0) = v(\omega), \ \ v'(0) = v'(\omega)$$

and choose $n \in \mathbb{N}$ such that $\|v\|_C \leq n$. Let, moreover, α be a solution of the problem

$$\alpha'' = [p(t)]_+ \alpha + h(t,n) + q(t); \quad \alpha(0) = \alpha(\omega), \ \ \alpha'(0) = \alpha'(\omega). \tag{21.5}$$

By virtue of (21.4) and Theorem 16.2, we get that

$$v(t) > 0, \quad \alpha(t) > 0 \quad \text{for } \ t \in [0,\omega].$$

On the other hand, since $[p]_+ \in \mathcal{V}^-(\omega)$ and the function $h(t,\,\cdot\,)$ is nondecreasing we get that $\alpha(t) \leq v(t)$ for $t \in [0,\omega]$ (see Remark 0.6) and, consequently, $\alpha(t) \leq n$ for $t \in [0,\omega]$. Now it follows from (21.5) that the function α satisfies (21.2). □

Now we reformulate Theorem 19.10 in a suitable for us form.

Theorem 21.6. *Let $p \in \mathcal{V}_0(\omega)$ and*

$$h(t,x) \leq h_1(t,x) \quad \textit{for } \ t \in [0,\omega], \ \ x > 0, \tag{21.6}$$

where $h_1 \in K_{loc}([0,\omega]\times\,]0,+\infty[\,;\mathbb{R})$, $h_1(t,\,\cdot\,)$ is nonincreasing on $]0,+\infty[\,$, $h_1(t,x) \leq 0$ for $t \in [0,\omega]$, $x > 0$, $\eta \geq 1$, $[q]_+^{\frac{1}{2-\eta}} \in L_\omega$, and

$$\int_0^\omega \Big| h_1\big(s, c|s-a|^\eta\big) \Big| \,\mathrm{d}s = +\infty \quad \textit{for } \ c > 0, \ a \in [0,\omega[\,.$$

Let, moreover, there exist $\beta \in AC'([0,\omega])$ such that $\beta(t) > 0$ for $t \in [0,\omega]$

$$\begin{gathered} \beta''(t) \leq p(t)\beta(t) + h(t,\beta(t)) + q(t) \quad \textit{for } \ t \in [0,\omega], \\ \beta(0) = \beta(\omega), \ \ \beta'(0) = \beta'(\omega). \end{gathered} \tag{21.7}$$

Then the problem (21.1) *has at least one solution.*

Corollary 21.7. *Let $p \in \mathcal{V}_0(\omega)$ and* (21.6) *hold, where h_1 satisfies conditions stated in Theorem* 21.6*. Let, moreover, either (H_{13}), or (H_{14}) hold. Then the problem* (21.1) *has at least one solution.*

Proof. By virtue of Proposition 18.10, resp. Proposition 18.11, hypothesis (H_{13}), resp. (H_{14}), implies the existence of a positive function $\beta \in AC'([0,\omega])$ satisfying (21.7). Hence, solvability of the problem (21.1) follows from Theorem 21.6. □

As an example, consider the problem

$$u'' = p(t)u - h_0(t)g(u) + q(t); \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega), \tag{21.8}$$

where $h_0, q \in L_\omega$, $h_0(t) \geq 0$ for $t \in [0,\omega]$, $h_0 \not\equiv 0$, and $g \in C(\,]0,+\infty[\,;\,]0,+\infty[\,)$.

Corollary 21.8. *Let $p \in \mathcal{V}_0(\omega)$ and*

$$\limsup_{x\to+\infty} g(x) < \frac{\int_0^\omega q(s)u_0(s)\,\mathrm{d}s}{\int_0^\omega h_0(s)u_0(s)\,\mathrm{d}s}\,. \tag{21.9}$$

Let, moreover, at least one of the following items be fulfilled:

(1) $\lim\limits_{x\to0+} g(x) = +\infty$ *and there is a $c > 0$ such that $h_0(t) \geq cq(t)$ for $t \in [0,\omega]$;*

(2) *g is nonincreasing and there exists $r > 0$ such that $h_0(t)g(r) \geq q(t)$ for $t \in [0,\omega]$;*

(3) *g is nonincreasing, $\eta \geq 1$, $[q]_+^{\frac{1}{2-\eta}} \in L_\omega$, and*

$$\int_0^\omega h_0(s)g\big(c|s-a|^\eta\big)\,\mathrm{d}s = +\infty \quad \textit{for } \ c > 0, \ a \in [0,\omega[\,;$$

(4) *g is nonincreasing and*

$$\|h_0\|_L\, g\Big(\frac{\omega}{4}\,\rho(p)Q_+\Big) \geq \rho(p)Q_+ - Q_-.$$

Then the problem (21.1) *is solvable if and only if* (21.3) *holds.*

Proof. Let (21.3) hold. Put $h_0(t,x) \overset{\text{def}}{=} -h_0(t)g(x)$ for $t \in [0,\omega]$, $x > 0$. Clearly, (H_{14}) is fulfilled.

Suppose that the item (1) holds. Put $f(t,x) \overset{\text{def}}{=} p(t)x - h_0(t)g(x) + q(t)$ and $h(t,x) \overset{\text{def}}{=} -h_0(t)(g(x) - \frac{1}{c})$ for $t \in [0,\omega]$, $x > 0$. Clearly, there is an $r_0 > 0$ such that $g(x) > \frac{2}{c}$ for $x \in]0, r_0]$. Hence, the hypothesis (H_1) holds with $\varphi(x) \overset{\text{def}}{=} g(x) - \frac{1}{c}$ for $x \in]0, r_0]$. Therefore, by virtue of Proposition 18.2, there exists a positive function $\alpha \in \widetilde{AC}'([0,\omega])$ satisfying (21.2). Consequently, the solvability of the problem (21.8) follows from Theorem 21.1.

If either the item (2), or the item (4) is fulfilled that the solvability of the problem (21.8) follows from Corollary 21.5.

Let now the item (3) be fulfilled. Then the solvability of the problem (21.8) follows from Corollary 21.7 (with $h_1(t,x) \overset{\text{def}}{=} -h_0(t)g(x)$).

Necessity of the condition (21.3) follows from Proposition 21.3. □

Remark 21.9. It is clear that Corollary 21.8 remains true if, instead of (21.9), the condition

$$\limsup_{x\to+\infty} g(x) = 0$$

holds.

Proposition 21.10. *Let $p \in \mathcal{V}_0(\omega)$ and for any $c > 0$, $a > 0$, and $b > a$, there exists $\varphi_{abc} \in L_\omega$ such that*

$$\varphi_{abc}(t) \geq 0 \quad \textit{for } t \in [0,\omega], \quad \varphi_{abc} \not\equiv 0, \tag{21.10}$$

$$h(t, x+c) - h(t,x) \geq \varphi_{abc}(t) \quad \textit{for } t \in [0,\omega], \ x \in [a,b]. \tag{21.11}$$

Then the problem (21.1) *has at most one solution.*

Proof. Let u and v be solutions of the problem (21.1) and there is a $t_0 \in [0,\omega[$, such that $u(t_0) > v(t_0)$. Put $w(t) \overset{\text{def}}{=} u(t) - v(t)$ for $t \in [0,\omega]$. It is clear that, either there exist $t_1 \in [0,\omega[$ and $t_2 \in]t_1,\omega]$ such that

$$w(t) > 0 \quad \text{for } t \in]t_1,t_2[, \quad w(t_1) = 0, \quad w(t_2) = 0, \tag{21.12}$$

or

$$w(t) > 0 \quad \text{for } t \in [0,\omega]. \tag{21.13}$$

First, suppose that (21.12) holds. Since the function $h(t,\cdot\,)$ is nondecreasing we get

$$w''(t) \geq p(t)w(t) \quad \text{for } t \in [t_1,t_2], \quad w(t_1) = 0, \quad w(t_2) = 0.$$

Hence, by virtue of Proposition 0.8 and Proposition 2.5, we get the contradiction $w(t) \leq 0$ for $t \in [t_1,t_2]$.

Now let (21.13) hold. It is clear that the function w is a solution of the problem

$$w'' = p(t)w + \psi(t); \quad w(0) = w(\omega), \ \ w'(0) = w'(\omega),$$

where $\psi(t) \overset{\text{def}}{=} h(t,u(t)) - h(t,v(t))$ and $\psi(t) \geq 0$ for $t \in [0,\omega]$. Hence, by virtue of Fredholm's third theorem we get that

$$\psi \equiv 0 \tag{21.14}$$

and there is a $c_0 > 0$ such that $w(t) = c_0 u_0(t)$ for $t \in [0,\omega]$. Put $a = \min\{v(t) : \ t \in [0,\omega]\}$, $b = \max\{v(t) : \ t \in [0,\omega]\}$, and $c = c_0 \min\{u_0(t) : \ t \in [0,\omega]\}$. Clearly,

$$u(t) = v(t) + c_0 u_0(t) \geq v(t) + c, \quad a \leq v(t) \leq b \quad \text{for } t \in [0,\omega]. \tag{21.15}$$

By virtue of the assumption of the proposition, there exists a function $\varphi_{abc} \in L_\omega$ such that (21.10) and (21.11) are fulfilled. Hence, on account of (21.15), we get

$$\psi(t) = h(t,u(t)) - h(t,v(t)) \geq h(t,v(t)+c) - h(t,v(t)) \geq \varphi_{abc}(t) \quad \text{for } t \in [0,\omega]$$

which, together with (21.10), contradicts (21.14). □

Remark 21.11. Let $h(t,x) \overset{\text{def}}{=} -h_0(t)g(x)$ for $t \in [0,\omega]$, $x > 0$, where $h_0 \in L_\omega$, $h_0 \not\equiv 0$, $h_0(t) \geq 0$ for $t \in [0,\omega]$, and $g \in C(\,]0,+\infty[\,;\,]0,+\infty[\,)$ is decreasing. Then it is clear that the function h satisfies assumptions of Proposition 21.10. Consequently, in this case the problem (21.8) possesses at most one solution.

As a particular case of the problem (21.8) consider the problem

$$u'' = p(t)u - \frac{h_0(t)}{u^\lambda} + q(t); \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega), \tag{21.16}$$

where $h_0, q \in L_\omega$, $h_0(t) \geq 0$ for $t \in [0,\omega]$, and $h_0 \not\equiv 0$. Next corollary follows immediately from Corollary 21.8, Remark 21.9, and Remark 21.11.

Corollary 21.12. *Let $p \in \mathcal{V}_0(\omega)$ and the item* (iii)*, or the item* (iv)*, or the item* (v) *of Theorem* 20.2 *be fulfilled. Then the problem* (21.16) *is solvable if and only if* (21.3) *holds. Moreover, if* (21.3) *holds then the problem* (21.16) *is uniquely solvable.*

Remark 21.13. Condition (20.5), resp. (20.6), in Corollary 21.12 is essential and cannot be omitted. Indeed, consider the problem

$$u'' = -\frac{(1+\cos t)^\lambda}{u^\lambda} + 1 - \cos t; \quad u(0) = u(2\pi), \;\; u'(0) = u'(2\pi). \tag{21.17}$$

Then $p \equiv 0$ and we can suppose that $u_0 \equiv 1$. Clearly, $q(t) = 1 - \cos t$ for $t \in [0, 2\pi]$ and that (21.3) holds. By the same arguments as in Example 20.11 one can show that the problem (21.17) has no solution for any $\lambda > 0$.

Mention also that the condition (20.4) is optimal and cannot be weakened to the condition

$$\|h_0\|_L \geq (1-\delta)\Big(\frac{\omega}{4}\,\rho(p)Q_+\Big)^\lambda \big(\rho(p)Q_+ - Q_-\big) \tag{21.18}$$

no matter how small $\delta \in]0,1[$ is. Indeed, consider again the problem (21.17). Clearly, (21.3) holds, $Q_+ = 2\pi$, $Q_- = 0$, and $\|h_0\|_L = \int_0^{2\pi} (1+\cos s)^\lambda \, ds$. Since

$$\lim_{\lambda\to 0+} \int_0^{2\pi} (1+\cos s)^\lambda \, ds = 2\pi, \quad \lim_{\lambda \to 0+} \Big(\frac{\pi}{2}\, Q_+\Big)^\lambda = 1,$$

for given $\delta \in]0,1[$ there is a $\lambda > 0$ such that (21.18) holds. However, as it was mentioned above the problem (21.17) has no solution for any $\lambda > 0$.

As it was mentioned in Introduction, the study of phase singular periodic problems was initiated in [16] by Lazer and Solimini. Theorem 2.1 of [16] concerns the solvability of the problem

$$u'' = -g(u) + q(t), \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega) \tag{21.19}$$

and reads as follows.

Theorem 21.14 (Lazer and Solimini)**.** *Let $q \in C([0,\omega];\mathbb{R})$ and the function $g \in C(\,]0,+\infty[\,;\,]0,+\infty[\,)$ be such that*

$$\lim_{x\to 0+} g(x) = +\infty, \quad \lim_{x\to+\infty} g(x) = 0. \tag{21.20}$$

Then the problem (21.19) *is solvable if and only if the inequality*

$$\int_0^\omega q(s)\, ds > 0 \tag{21.21}$$

holds.

Theorem 21.14 now follows from Corollary 21.8(1) (and Remark 21.9). Mention that the condition $q \in C([0,\omega];\mathbb{R})$ in Theorem 21.14 is essential and cannot be omitted. Indeed, let $\lambda \in]0,1[$ and $\varepsilon \in]-\frac{1}{2}, \frac{1-2\lambda}{2\lambda}[$. Put $v_0(t) \stackrel{\text{def}}{=} (1+\cos t)^{1+\varepsilon}$ for $t \in [0,2\pi]$. It is clear that $v_0' \in AC([0,2\pi])$ and $v_0'', \frac{1}{v_0^\lambda} \in L_{2\pi}$. Let now $q(t) \stackrel{\text{def}}{=} v_0''(t) + \frac{1}{v_0^\lambda(t)}$ for $t \in [0,2\pi]$. Then (21.21) holds because

$$\int_0^{2\pi} q(s)\,\mathrm{d}s = \int_0^{2\pi} \frac{1}{v_0^\lambda(s)}\,\mathrm{d}s.$$

However, in this case the problem (21.19), where $g(x) \stackrel{\text{def}}{=} \frac{1}{x^\lambda}$ for $x > 0$, has no solution. If we suppose that u is a solution of the problem (21.19) and put $w(t) \stackrel{\text{def}}{=} u(t) - v_0(t)$ for $t \in [0,2\pi]$, then by direct calculations we get that

$$w''(t) = p_0(t)w(t) \quad \text{for } t \in [0,2\pi], \quad w(0) = w(\omega), \quad w'(0) = w'(\omega),$$

where

$$p_0(t) \stackrel{\text{def}}{=} \frac{1}{u(t)v_0^\lambda(t)}\,\varphi_\lambda\Big(\frac{v_0(t)}{u(t)}\Big) \quad \text{for } t \in [0,2\pi]$$

and the function φ_λ is defined by (20.18). Since $p_0(t) > 0$ for $t \in [0,2\pi]$ we have that $p_0 \in \mathcal{V}^-(\omega)$ (see Remark 8.4) and, consequently, $w \equiv 0$. However, the latter identity leads to the contradiction $0 = w(\pi) = u(\pi) \neq 0$. Therefore, we have shown that for any $\lambda \in]0,1[$, there is a $q \in L_\omega$ satisfying (21.21) such that the problem

$$u'' = -\frac{1}{u^\lambda} + q(t); \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega),$$

has no solution. In other words, if $q \in L_\omega$ but $q \notin C([0,\omega];\mathbb{R})$, conditions (21.20) and (21.21) does not guarantee solvability of the problem (21.19). However, Corollary 21.8(3) implies

Corollary 21.15. *Let $q \in L_\omega$, $g \in C(]0,+\infty[;]0,+\infty[)$, g is nonincreasing, $\mu \in [0,\frac{1}{2}[$, $[q]_+^{\frac{1-\mu}{1-2\mu}} \in L_\omega$, and*

$$\int_0^1 \frac{g(x)}{x^\mu}\,\mathrm{d}x = +\infty, \qquad \lim_{x\to+\infty} g(x) = 0.$$

Then the problem (21.19) *is solvable if and only if* (21.21) *holds. Moreover, if* (21.21) *holds and the function g is decreasing then the problem* (21.19) *is uniquely solvable.*

Example constructed above shows that the condition $\int_0^1 \frac{g(x)}{x^\mu}\,\mathrm{d}x = +\infty$ in Corollary 21.15 is essential and cannot be weakened to the condition $\lim\limits_{x\to+0} g(x) = +\infty$. However, from Corollary 21.8(4) we get the following

Corollary 21.16. *Let $g \in C(]0,+\infty[;]0,+\infty[)$ is nonincreasing and*

$$\lim_{x\to+0} g(x) = +\infty, \qquad \lim_{x\to+\infty} g(x) = 0.$$

Let, moreover,

$$\omega\, g\Big(\frac{\omega}{4}\,\rho(p)Q_+\Big) \geq \rho(p)Q_+ - Q_-.$$

Then the problem (21.19) *is solvable if and only if* (21.21) *holds. Moreover, if* (21.21) *holds and the function g is decreasing then the problem* (21.19) *is uniquely solvable.*

To be more specific consider the problem

$$u'' = -\frac{1}{u^\lambda} + q(t); \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega). \tag{21.22}$$

It follows from Corollary 21.12 that

Corollary 21.17. *Let at least one of the following items be fulfilled:*

(1) $\lambda > 0$ *and* $\operatorname{ess\,sup}\{q(t) : \ t \in [0,\omega]\} < +\infty$*;*

(2) $\lambda \in]0,1[$ *and* $\omega > (\frac{\omega}{4}\, Q_+)^\lambda (Q_+ - Q_-)$*;*

(3) $\lambda \in]\frac{1}{2},1[$ *and* $[q]_+^{\frac{\lambda}{2\lambda-1}} \in L_\omega$*;*

(4) $\lambda \geq 1$.

Then the problem (21.22) *is solvable if and only if* (21.21) *holds. Moreover, if* (21.21) *holds then the problem* (21.22) *is uniquely solvable.*

22. Existence of Positive Solutions (Continuation)

In this paragraph we again consider the problem

$$u'' = f(t,u); \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega) \tag{22.1}$$

and its particular case

$$u'' = p(t)u + h(t,u); \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega), \tag{22.2}$$

where $p \in L_\omega$ and $f, h \in K_{loc}([0,\omega]\times]0,+\infty[\,;\mathbb{R})$. Recall that under a solution of the problem (22.1), respectively (22.2), we understand a **positive** function $u \in \widetilde{AC}'([0,\omega])$ satisfying given equation almost everywhere and boundary conditions.

Introduce the hypotheses

$$\begin{cases} f(t,x) \geq p(t)x + h_0(t,x) \quad \text{for } t \in [0,\omega], \;\; x > 0, \\ p \in \mathcal{V}^+(\omega), \;\; h_0 \in K_{loc}([0,\omega]\times\mathbb{R}_+;\mathbb{R}), \;\; h_0(t,\cdot\,) \text{ is nondecreasing}, \\ \beta \in \widetilde{AC}'([0,\omega]), \quad \beta(0) = \beta(\omega), \quad \beta'(0) \leq \beta'(\omega), \\ \beta''(t) \leq p(t)\beta(t) + h_0(t,\beta(t)), \quad \beta(t) > 0 \quad \text{for } t \in [0,\omega], \end{cases} \tag{H_{18}}$$

$$\begin{cases} f(t,x) \leq p_0(t)x + q_0(t,x) \quad \text{for } t \in [0,\omega], \;\; x > r_0, \;\; r_0 > 0, \\ p_0 \in \mathcal{V}^+(\omega), \quad q_0 \in K_{sl}([0,\omega]\times\mathbb{R};\mathbb{R}_+), \end{cases} \tag{H_{19}}$$

$$\begin{cases} f(t,x) \leq p_0(t)x + q_0(t) \quad \text{for } t \in [0,\omega], \;\; x > r_0, \;\; r_0 > 0, \\ p_0 \in \mathcal{V}_0(\omega) \;\text{ and } \displaystyle\int_0^\omega q_0(s)u_0(s)\,\mathrm{d}s < 0, \\ \text{where } u_0 \text{ is a positive solution of the problem} \\ u_0'' = p_0(t)u_0; \;\; u_0(0) = u_0(\omega), \;\; u_0'(0) = u_0'(\omega). \end{cases} \tag{H_{20}}$$

Theorem 22.1. *Let* (H_{18}) *hold and either* (H_{19}) *or* (H_{20}) *be satisfied. Let, moreover*

$$p(t) \leq p_0(t) \quad \text{for } t \in [0,\omega]. \tag{22.3}$$

Then the problem (22.1) *has at least one solution.*

Proof. Choose $n_0 \in \mathbb{N}$ such that $\beta(t) < n_0$ for $t \in [0,\omega]$ and for any $n > n_0$ introduce the notations

$$\begin{aligned} \chi_n(t,x) &\stackrel{\text{def}}{=} \beta(t) + [x-\beta(t)]_+ - [x-n]_+ \quad \text{for } t \in [0,\omega], \;\; x \in \mathbb{R}, \\ h_n(t,x) &\stackrel{\text{def}}{=} h\big(t,\chi_n(t,x)\big) \quad \text{for } t \in [0,\omega], \;\; x \in \mathbb{R}, \end{aligned} \tag{22.4}$$

where

$$h(t,x) \stackrel{\text{def}}{=} f(t,x) - p(t)x \quad \text{for } t \in [0,\omega], \;\; x > 0. \tag{22.5}$$

Clearly, $h_n \in K_{sl}([0,\omega]\times\mathbb{R};\mathbb{R})$. For any $n \geq n_0$ consider the problem

$$u'' = p(t)u + h_n(t,u); \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega). \tag{22.6}$$

In view of Proposition 17.3, the problem (22.6) has at least one solution u_n.

We first show that

$$u_n(t) \geq \beta(t) \quad \text{for } t \in [0,\omega], \;\; n \geq n_0\,. \tag{22.7}$$

Indeed, in view of (H_{18}), (22.4), and (22.5), we have that

$$\chi_n(t,x) \geq \beta(t) \quad \text{for } t \in [0,\omega], \ x \in \mathbb{R}, \ n \geq n_0\,,$$
$$h_n(t,x) \geq h_0\big(t,\chi_n(t,x)\big) \geq h_0(t,\beta(t)) \quad \text{for } t \in [0,\omega], \ x \in \mathbb{R}, \ n \geq n_0\,.$$

Taking, moreover, into account that the function u_n is a solution of the problem (22.6), we get

$$\begin{gathered} u_n''(t) \geq p(t)u_n(t) + h_0(t,\beta(t)) \quad \text{for } t \in [0,\omega], \\ u_n(0) = u_n(\omega), \quad u_n'(0) = u_n'(\omega). \end{gathered} \tag{22.8}$$

Now it follows Remark 0.6, by virtue of (22.8) and (H_{18}), that (22.7) holds.

Introduce the notation

$$m_n = \min\big\{u_n(t): \ t \in [0,\omega]\big\}, \quad M_n = \max\big\{u_n(t): \ t \in [0,\omega]\big\}. \tag{22.9}$$

To finish the proof it is sufficient to show that for some $n \geq n_0$, the inequality

$$M_n \leq n \tag{22.10}$$

holds. Suppose the contrary, let

$$M_n > n \quad \text{for } n \geq n_0\,. \tag{22.11}$$

First assume that (H_{19}) is satisfied. It is clear that without loss of generality we can assume that the function $q_0(t,\cdot)$ is nondecreasing on $]0,+\infty[$.

Put

$$\widetilde{u}_n(t) \overset{\text{def}}{=} \frac{1}{M_n}\, u_n(t) \quad \text{for } t \in [0,\omega], \ n \geq n_0\,.$$

Clearly, for any $n \geq n_0$ the equalities

$$\begin{gathered} \widetilde{u}_n''(t) = p(t)\widetilde{u}_n(t) + \frac{1}{M_n}\, h_n(t,u_n(t)) \quad \text{for } t \in [0,\omega], \\ \widetilde{u}_n(0) = \widetilde{u}_n(\omega), \quad \widetilde{u}_n'(0) = \widetilde{u}_n'(\omega). \end{gathered} \tag{22.12}$$

are fulfilled. Since $h \in K_{loc}([0,\omega]\times\,]0,+\infty[\,;\mathbb{R})$ there is a function $h^* \in L_\omega$ such that

$$|h(t,x)| \leq h^*(t) \quad \text{for } t \in [0,\omega], \ x \in [\beta_*, \beta_* + r_0], \tag{22.13}$$

where

$$\beta_* \overset{\text{def}}{=} \min\big\{\beta(t) : t \in [0,\omega]\big\}. \tag{22.14}$$

Hence, in view of (H_{19}), (22.4), and (22.5), we get that

$$h_n(t,x) \leq (p_0(t) - p(t))x + q^*(t,x) \quad \text{for } t \in [0,\omega], \ x > 0, \tag{22.15}$$

where

$$q^*(t,x) \overset{\text{def}}{=} q_0(t,x) + h^*(t) \quad \text{for } t \in [0,\omega], \ x > 0. \tag{22.16}$$

Taking now into account (22.3), (22.4), (22.7), and (22.15), we get from (22.12) that

$$\begin{aligned} \widetilde{u}_n''(t) &\leq p(t)\widetilde{u}_n(t) + \frac{1}{M_n}\,(p_0(t) - p(t))\chi_n\big(t,u_n(t)\big) + \frac{1}{M_n}\, q^*\big(t,\chi_n(t,u_n(t))\big) \\ &\leq p_0(t)\widetilde{u}_n(t) + \frac{1}{M_n}\, q^*(t,M_n) \quad \text{for } t \in [0,\omega]. \end{aligned} \tag{22.17}$$

Denote by v_n, where $n \geq n_0$, the solution of the problem

$$v_n'' = p_0(t)v_n + \frac{1}{M_n}\, q^*(t,M_n); \quad v_n(0) = v_n(\omega), \ \ v_n'(0) = v_n'(\omega). \tag{22.18}$$

Since $p_0 \in \mathcal{V}^+(\omega)$ it follows from Remark 0.6, in view of (22.17) and (22.18), that

$$\widetilde{u}_n(t) \leq v_n(t) \quad \text{for } t \in [0,\omega], n \geq n_0$$

and thus (since $\widetilde{u}_n(t) > 0$ for $t \in [0,\omega]$) we have

$$\|v_n\|_C \geq 1 \quad \text{for } n \geq n_0\,. \tag{22.19}$$

On the other hand, since $q^* \in K_{sl}([0,\omega] \times \mathbb{R};\mathbb{R}_+)$ it follows from Lemma 3.1 that

$$\lim_{n\to+\infty} \|v_n\|_C = 0,$$

which contradicts (22.19). Therefore, we have proved that if (H_{19}) holds then for some $n \geq n_0$, the inequality (22.10) is fulfilled.

Suppose now that (H_{20}) holds. Extend the functions p, p_0, q_0, h_n, and u_n ω-periodically and denote them by the same letters. In view of (H_{20}), (22.3), (22.7), and (22.13), one can easily verify that

$$h_n(t, u_n(t)) \leq (p_0(t) - p(t))u_n(t) + |q_0(t)| + h^*(t) \quad \text{for } t \in \mathbb{R}, \ \ n \geq n_0\,. \tag{22.20}$$

Let now $n \geq n_0$ be fixed and $t_0 \in [0, \omega[$ be such that $u_n'(t_0) \neq 0$. If $u_n'(t_0) > 0$ then there is a $t_* \in [t_0 - \omega, t_0[$ such that

$$u_n'(t) > 0 \quad \text{for } t \in]t_*, t_0], \ \ u_n'(t_*) = 0. \tag{22.21}$$

In view of (22.20) and (22.21), we obtain

$$\begin{aligned} u_n'(t_0) &= \int\limits_{t_*}^{t_0} \Big(p(s)u_n(s) + h_n(s, u_n(s))\Big)\,\mathrm{d}s \\ &\leq \int\limits_{t_*}^{t_0} \Big(p_0(s)u_n(s) + |q_0(s)| + h^*(s)\Big)\,\mathrm{d}s \leq Au_n(t_0) + B, \end{aligned}$$

where $A \overset{\text{def}}{=} \|p_0\|_L$ and $B \overset{\text{def}}{=} \|q_0\|_L + \|h^*\|_L$. Analogously, if $u_n'(t_0) < 0$ then there is a $t^* \in]t_0, t_0 + \omega]$ such that

$$u_n'(t) < 0 \quad \text{for } t \in [t_0, t^*[\,, \ \ u_n'(t^*) = 0$$

and

$$-u_n'(t_0) \leq \int\limits_{t_0}^{t^*} \Big(p_0(s)u_n(s) + |q_0(s)| + h^*(s)\Big)\,\mathrm{d}s \leq Au_n(t_0) + B.$$

Therefore, we have proved that for any $n \geq n_0$, the inequality

$$|u_n'(t)| \leq Au_n(t) + B \quad \text{for } t \in [0, \omega] \tag{22.22}$$

holds. Taking now into account (22.7), we easily get from (22.22) that

$$M_n \leq m_n \exp\Big[A\omega + \frac{B\omega}{\beta_*}\Big] \quad \text{for } n \geq n_0\,,$$

where β_* is defined by (22.14). Hence, on account of (22.11), there is a $n_1 > n_0$ such that

$$m_{n_1} > r_0.$$

Consequently, by virtue of (H_{20}), (22.3)–(22.5), and (22.7), we get that

$$h_{n_1}(t, u_{n_1}(t)) \leq (p_0(t) - p(t))u_{n_1}(t) + q_0(t) \quad \text{for } t \in [0, \omega].$$

Therefore,

$$u_{n_1}''(t) \leq p_0(t)u_{n_1}(t) + q_0(t) \quad \text{for } t \in [0, \omega].$$

Now it is clear that

$$\begin{gathered} u_{n_1}''(t) = p_0(t)u_{n_1}(t) + q_0(t) - q_1(t) \quad \text{for } t \in [0, \omega], \\ u_{n_1}(0) = u_{n_1}(\omega), \quad u_{n_1}'(0) = u_{n_1}'(\omega), \end{gathered}$$

where

$$q_1(t) \overset{\text{def}}{=} p_0(t)u_{n_1}(t) + q_0(t) - u_{n_1}''(t) \geq 0 \quad \text{for } t \in [0, \omega]. \tag{22.23}$$

However, by virtue of Fredholm's third theorem, we get that

$$\int\limits_0^{\omega} \big(q_0(s) - q_1(s)\big)u_0(s)\,\mathrm{d}s = 0,$$

where u_0 is a function appearing in the hypothesis (H_{20}). The latter equality, together with (H_{20}) and (22.23), yields the contradiction $0 < 0$. Therefore, we have proved that if (H_{20}) holds then for some $n \geq n_0$, the inequality (22.10) is fulfilled. $\square$

Introduce the hypothesis

$$\begin{cases} f(t,x) \geq p(t)x + q(t) \quad \text{for } t \in [0,\omega], \quad x > 0, \\ \qquad p \in \mathcal{V}^+(\omega), \quad (p,q) \in \mathcal{U}(\omega), \end{cases} \tag{H_{21}}$$

where $\mathcal{U}(\omega)$ is defined by Definition 16.1.

Proposition 22.2. *Hypothesis* (H_{21}) *implies* (H_{18}).

Proof. Clearly, (H_{18}) is fulfilled, where $h_0(t,x) \overset{\text{def}}{=} q(t)$ for $t \in [0,\omega]$, $x \in \mathbb{R}$ and β is a positive solution of the problem

$$\beta'' = p(t)\beta + q(t); \quad \beta(0) = \beta(\omega), \ \ \beta'(0) = \beta'(\omega). \qquad \square$$

Introduce the hypothesis

$$\begin{cases} f(t,x) \geq p(t)x + h_0(t)\varphi(x) \quad \text{for } t \in [0,\omega], \quad x > 0, \\ p \in \mathcal{V}^+(\omega), \ \ h_0 \in L_\omega, \ \ h_0(t) \geq 0 \quad \text{for } t \in [0,\omega], \ \ h_0 \not\equiv 0, \\ \varphi \in C(\mathbb{R}_+;\mathbb{R}_+), \ \varphi \text{ is nondecreasing}, \ \ \lim\limits_{x\to 0+} \dfrac{\varphi(x)}{x} = +\infty. \end{cases} \tag{H_{22}}$$

Proposition 22.3. *Hypothesis* (H_{22}) *implies* (H_{18}).

Proof. Let (H_{22}) holds. Denote by v the solution of the problem

$$v'' = p(t)v + h_0(t); \quad v(0) = v(\omega), \ \ v'(0) = v'(\omega). \tag{22.24}$$

By virtue of Remark 9.2, there is a $c > 0$ such that

$$v(t) \geq c \quad \text{for } t \in [0,\omega]. \tag{22.25}$$

Choose $\varepsilon > 0$ such that

$$\frac{\varphi(\varepsilon)}{\varepsilon} > \frac{1}{c} \tag{22.26}$$

and put $\beta(t) \overset{\text{def}}{=} \varphi(\varepsilon)v(t)$ for $t \in [0,\omega]$. It follows from (22.25) and (22.26) that $\beta(t) \geq \varepsilon$ pro $t \in [0,\omega]$. Hence, in view of (22.24) and the monotonicity of the function φ, we get

$$\begin{gathered} \beta''(t) \leq p(t)\beta(t) + h_0(t)\varphi(\beta(t)) \quad \text{for } t \in [0,\omega], \\ \beta(0) = \beta(\omega), \ \ \beta'(0) = \beta'(\omega). \end{gathered}$$

Now it is clear that (H_{18}) holds with $h_0(t,x) \overset{\text{def}}{=} h_0(t)\varphi(x)$. $\square$

Introduce the hypothesis

$$\begin{cases} f(t,x) \geq p(t)x + h_1(t,x) + q(t) \quad \text{for } t \in [0,\omega], \quad x > 0, \\ p \in \operatorname{Int}\mathcal{V}^+(\omega), \ \ (p,q) \in \mathcal{U}_0(\omega), \ \ h_1 \in K_{loc}([0,\omega]\times\,]0,+\infty[\,;\mathbb{R}_+), \\ h_1(t,\cdot) \text{ is nondecreasing on } [0,+\infty[\,, \ \ \lim\limits_{x\to 0+} \dfrac{1}{x}\displaystyle\int_0^\omega h_1(s,x)\,\mathrm{d}s = +\infty. \end{cases} \tag{H_{23}}$$

where $\mathcal{U}_0(\omega)$ is define in Definition 16.20.

Proposition 22.4. *Hypothesis* (H_{23}) *implies* (H_{18}).

Proof. Let the hypothesis (H_{23}) holds. Let, moreover, $c_0 > 0$ be the number appearing in Proposition 16.8. Choose $x_0 > 0$ such that

$$\frac{1}{x_0}\int_0^\omega h_1(s,x_0)\,\mathrm{d}s > \frac{1}{c_0} \tag{22.27}$$

and denote by β_0 the solution of the problem

$$\beta_0'' = p(t)\beta_0 + h_1(t,x_0); \quad \beta_0(0) = \beta_0(\omega), \ \ \beta_0'(0) = \beta_0'(\omega). \tag{22.28}$$

By virtue of (22.27) and Proposition 16.8, we get that

$$\beta_0(t) > x_0 \quad \text{for } t \in [0,\omega]. \tag{22.29}$$

Denote by v the solution of the problem

$$v'' = p(t)v + q(t); \quad v(0) = v(\omega), \;\; v'(0) = v'(\omega). \tag{22.30}$$

Since $(p,q) \in \mathcal{U}_0(\omega)$ we get

$$v(t) \geq 0 \quad \text{for } t \in [0,\omega]. \tag{22.31}$$

Let now $\beta(t) \overset{\text{def}}{=} \beta_0(t) + v(t)$ for $t \in [0,\omega]$. In view of (22.28)–(22.31) and the monotonicity of the function $h_1(t,\cdot\,)$ we get that

$$\beta(t) > 0 \quad \text{for } t \in [0,\omega], \quad \beta(0) = \beta(\omega), \quad \beta'(0) = \beta'(\omega),$$
$$\beta''(t) \leq p(t)\beta(t) + h_1(t,\beta(t)) + q(t) \quad \text{for } t \in [0,\omega].$$

Now it is clear that (H_{18}) holds with $h_0(t,x) \overset{\text{def}}{=} h_1(t,x) + q(t)$. □

Introduce the hypothesis

$$\begin{cases} f(t,x) \geq p_1(t)x \quad \text{for } t \in [0,\omega], \;\; x > 0, \\ f(t,x) \geq p_1(t)x + h_2(t,x) \quad \text{for } t \in [0,\omega], \;\; x \in \,]0,\delta], \;\; \delta > 0, \;\; p_1 \in \operatorname{Int}\mathcal{V}^+(\omega), \\ h_2 \in K_{loc}([0,\omega]\times\,]0,+\infty[\,;\mathbb{R}_+), \;\; h_2(t,\cdot\,) \text{ is nondecreasing}, \\ \displaystyle\lim_{x\to 0+} \frac{1}{x}\int_0^\omega h_2(s,x)\,\mathrm{d}s = +\infty. \end{cases} \tag{H_{24}}$$

Proposition 22.5. *Hypothesis (H_{24}) implies (H_{18}).*

Proof. Let the hypothesis (H_{24}) holds. Since $p_1 \in \operatorname{Int}\mathcal{V}^+(\omega)$ there is an $\varepsilon > 0$ such that the function

$$p(t) \overset{\text{def}}{=} p_1(t) - \frac{\varepsilon}{\delta}\, h_2(t,\delta) \quad \text{for } t \in [0,\omega]$$

satisfies the inclusion $p \in \mathcal{V}^+(\omega)$. It is clear that

$$f(t,x) \geq p_1(t)x = p(t)x + \frac{\varepsilon}{\delta}\, h_2(t,\delta)x \quad \text{for } t \in [0,\omega], \;\; x > 0,$$
$$f(t,x) \geq p_1(t)x + h_2(t,x) = p(t)x + h_2(t,x) + \frac{\varepsilon}{\delta}\, h_2(t,\delta)x \quad \text{for } t \in [0,\omega], \;\; x \in \,]0,\delta].$$

and thus we have

$$f(t,x) \geq p(t)x + h_1(t,x) \quad \text{for } t \in [0,\omega], \;\; x > 0,$$

where

$$h_1(t,x) \overset{\text{def}}{=} \varepsilon h_2\big(t, x - [x-\delta]_+\big) \quad \text{for } t \in [0,\omega], \;\; x > 0.$$

Clearly, $h_1 \in K([0,\omega]\times\mathbb{R}_+;\mathbb{R}_+)$, $h_1(t,\cdot\,)$ is nondecreasing, and

$$\lim_{x\to 0+} \frac{1}{x}\int_0^\omega h_1(s,x)\,\mathrm{d}s = +\infty.$$

Consequently, the hypothesis (H_{23}) holds (with $q \equiv 0$). Taking now into account Proposition 22.4, it is clear that hypothesis (H_{18}) holds as well. □

Introduce the hypothesis

$$\begin{cases} f(t,x) \geq p(t)x + h(t,x) \quad \text{for } t \in [0,\omega], \;\; x > 0, \\ \qquad \text{the hypothesis } (H_{12}) \text{ holds}. \end{cases} \tag{H_{25}}$$

The next assertion follows from Proposition 18.9.

Proposition 22.6. *Hypothesis (H_{25}) implies (H_{18}).*

The next corollary immediately follows from Theorem 22.1 and Propositions 22.2–22.6.

Corollary 22.7. *Let $k \in \{21, 22, 23, 24, 25\}$ and the hypothesis (H_k) hold. Let, moreover, either (H_{19}) or (H_{20}) be fulfilled and $p(t) \le p_0(t)$ for $t \in [0, \omega]$ if $k \in \{21, 22, 23, 25\}$ and $p_1(t) \le p_0(t)$ for $t \in [0, \omega]$ if $k = 24$. Then the problem (22.1) has at least one solution.*

For the problem (22.2), Theorem 22.1 implies the following assertion.

Corollary 22.8. *Let $p \in \mathcal{V}^+(\omega)$, the function $h(t, \cdot)$ is nondecreasing (on $]0, +\infty[$), and*

$$\lim_{x \to +\infty} \frac{1}{x} \int_0^\omega |h(s, x)| \, \mathrm{d}s = 0. \tag{22.32}$$

Then the problem (22.2) is solvable if and only if there is a function $\beta \in A\widetilde{C}'([0, \omega])$ satisfying $\beta(t) > 0$ for $t \in [0, \omega]$, $\beta(0) = \beta(\omega)$, $\beta'(0) = \beta'(\omega)$, and

$$\beta''(t) \le p(t)\beta(t) + h(t, \beta(t)) \quad \text{for } t \in [0, \omega].$$

Proof. If u is a solution of the problem (22.2) then clearly $\beta(t) \stackrel{\text{def}}{=} u(t)$ for $t \in [0, \omega]$ satisfies conditions of the corollary. Suppose that there exists a function β satisfying the conditions of the corollary. Put $f(t, x) \stackrel{\text{def}}{=} p(t)x + h(t, x)$ for $t \in [0, \omega]$, $x > 0$. It is clear that (H_{18}) holds with $h_0(t, x) \stackrel{\text{def}}{=} h(t, x)$. On the other hand, in view of (22.32), (H_{19}) is fulfilled with $p_0(t) \stackrel{\text{def}}{=} p(t)$, $q_0(t, x) \stackrel{\text{def}}{=} |h(t, 1 + [x - 1]_+)|$, and $r_0 = 1$. Therefore, by virtue of Theorem 22.1, the problem (22.2) has at least one solution. ☐

The next assertion follows from Corollary 22.7 with $k = 21$.

Corollary 22.9. *Let $p \in \mathcal{V}^+(\omega)$, $(p, q) \in \mathcal{U}(\omega)$,*

$$h(t, x) \ge q(t) \quad \text{for } t \in [0, \omega], \ \ x > 0,$$

and (22.32) holds. Then the problem (22.2) is solvable.

Remark 22.10. Let $p \in \mathcal{V}^+(\omega)$, $h(t, x) \stackrel{\text{def}}{=} q(t)$, where $q \not\equiv 0$, and $(p, q) \in \mathcal{U}_0(\omega) \setminus \mathcal{U}(\omega)$ (see Remark 16.21). Then it is clear that the problem (22.2) has no (positive) solution. Therefore, the condition $(p, q) \in \mathcal{U}(\omega)$ in Corollary 22.9 is optimal and cannot be weakened to the condition $(p, q) \in \mathcal{U}_0(\omega)$. However, the following assertion holds.

Corollary 22.11. *Let $p \in \operatorname{Int} \mathcal{V}^+(\omega)$, $(p, q) \in \mathcal{U}_0(\omega)$,*

$$h(t, x) \ge q(t) \quad \text{for } t \in [0, \omega], \ \ x > 0,$$

and the function $h(t, \cdot)$ is nondecreasing on $]0, +\infty[$. If, moreover, the condition (22.32) holds then the problem (22.2) is solvable.

Proof. Put $f(t, x) \stackrel{\text{def}}{=} p(t)x + h(t, x)$. It is clear that (H_{23}) holds with $h_1(t, x) \stackrel{\text{def}}{=} h(t, x) - q(t)$. On the other hand, (H_{19}) is fulfilled with $p_0(t) \stackrel{\text{def}}{=} p(t)$, $q_0(t, x) \stackrel{\text{def}}{=} |h(t, 1 + [x - 1]_+)|$, and $r_0 = 1$. Therefore, by virtue of Corollary 22.7, the problem (22.2) is solvable. ☐

The next assertion follows from Corollary 22.8 and Proposition 18.9.

Corollary 22.12. *Let $p \in \operatorname{Int} \mathcal{V}^+(\omega)$, $h(t, \cdot)$ is nondecreasing, and the hypothesis (H_{12}) holds. Then the problem (22.2) has at least one solution.*

Corollary 22.13. *Let $p \in \mathcal{V}^+(\omega)$, $q \in L_\omega$, $q \not\equiv 0$, $q(t) \ge 0$ for $t \in [0, \omega]$, $r > 0$, $\mu \in [0, 1[$, and*

$$h(t, x) \ge q(t)x^\mu \quad \text{for } t \in [0, \omega], \ \ x > r.$$

Let, moreover, the mapping $x \mapsto \frac{1}{x^\mu} h(t, x)$ is nonincreasing in $]0, +\infty[$. Then the problem (22.2) has at least one solution.

Proof. Put $f(t, x) \stackrel{\text{def}}{=} p(t)x + h(t, x)$. It is clear that

$$\frac{1}{x^\mu} h(t, x) \ge \frac{1}{(r+1)^\mu} h(t, r+1) \ge q(t) \quad \text{for } t \in [0, \omega], \ \ x \in]0, r+1]$$

and thus

$$h(t, x) \ge q(t)x^\mu \quad \text{for } t \in [0, \omega], \ \ x > 0.$$

On the other hand,

$$\frac{1}{x}\,h(t,x)=\frac{1}{x^{1-\mu}}\,\frac{h(t,x)}{x^{\mu}}\le\frac{h(t,1)}{x^{1-\mu}}\quad\text{for }\ t\in[0,\omega],\ \ x>1.$$

Now it is clear that (H_{22}) holds with $h_0(t)\overset{\text{def}}{=}q(t)$ and $\varphi(x)\overset{\text{def}}{=}x^{\mu}$ as well as (H_{19}) is fulfilled with $p_0(t)\overset{\text{def}}{=}p(t)$, $q_0(t,x)\overset{\text{def}}{=}h(t,1)|x|^{\mu}$, and $r_0=1$. Therefore, by virtue of Corollary 22.7, the problem (22.2) is solvable. □

Remark 22.14. For $\mu=0$, Corollary 22.13 reads as follows.

Let $p\in\mathcal{V}^{+}(\omega)$, $h(t,\cdot\,)$ is nonincreasing, and

$$h(t,x)\ge q(t)\quad\textit{for }\ t\in[0,\omega],\ \ x>0, \tag{22.33}$$

where $q\in L_{\omega}$, $q\not\equiv0$, and $q(t)\ge0$ for $t\in[0,\omega]$. Then the problem (22.2) *has at least one solution.*

Observe, that the assumption (22.33) is essential and cannot be weakened to the assumption

$$\begin{gathered} h(t,x)\ge0\quad\text{for }\ t\in[0,\omega],\ \ x>0,\\ \operatorname{mes}\big\{t\in[0,\omega]:\ h(t,x)>0\big\}>0\quad\text{for }\ x>0. \end{gathered} \tag{22.34}$$

Indeed, in view of Proposition 14.1 and Proposition 7.4, there is a $p\in\mathcal{V}^{+}(\omega)$ such that the problem

$$u''=p(t)u+\frac{1}{u^{3}}\quad u(0)=u(\omega),\ \ u'(0)=u'(\omega)$$

has no solution. Hence, the problem (22.2) with $h(t,x)\overset{\text{def}}{=}\frac{1}{x^3}$ has no solution while the function $h(t,\cdot\,)$ is nonincreasing and (22.34) holds.

However the following assertion is fulfilled.

Corollary 22.15. *Let $p\in\operatorname{Int}\mathcal{V}^{+}(\omega)$, $h(t,\cdot\,)$ is nonincreasing (on $]0,+\infty[$),*

$$h(t,x)\ge0\quad\textit{for }\ t\in[0,\omega],\ \ x>0,$$

and there is a $\delta>0$ such that

$$\operatorname{mes}\big\{t\in[0,\omega]:\ h(t,\delta)>0\big\}\ne0.$$

Then the problem (22.2) *has at least one solution.*

Proof. Put $f(t,x)\overset{\text{def}}{=}p(t)x+h(t,x)$. It is clear that (H_{24}) holds with $p_1(t)\overset{\text{def}}{=}p(t)$ and $h_2(t,x)\overset{\text{def}}{=}h(t,\delta)$ as well as (H_{19}) is fulfilled with $p_0(t)\overset{\text{def}}{=}p(t)$, $q_0(t,x)\overset{\text{def}}{=}h(t,1+[x-1]_+)$, and $r_0=1$. Therefore, by virtue of Corollary 22.7, the problem (22.2) is solvable. □

Return again to the problem (22.1). Before the formulation of the next theorem introduce the hypothesis

$$\left\{\begin{gathered} p_1(t)x+\frac{h_1(t)}{\psi_1(x)}\le f(t,x)\le p_2(t)x+\frac{h_2(t)}{\psi_2(x)}+q^*(t,x)\ \text{ for }\ t\in[0,\omega],\ \ x>0,\\ h_1,h_2\in L_{\omega},\ \ h_1(t)\ge0,\ \ h_2(t)\ge0\ \text{ for }\ t\in[0,\omega],\ \ h_1\not\equiv0,\\ p_1,p_2\in\mathcal{V}^{+}(\omega),\quad q^*\in K_{sl}([0,\omega]\times\mathbb{R}_+;\mathbb{R}_+),\\ \psi_1,\psi_2\in C(\,]0,+\infty[\,;\,]0,+\infty[\,)\ \text{ are nondecreasing},\\ \limsup_{x\to+\infty}x\psi_2\Big(\frac{c}{\psi_1(x)}\Big)>0\ \text{ for every }\ c>0. \end{gathered}\right. \tag{H_{26}}$$

Theorem 22.16. *Let the hypothesis (H_{26}) hold. Then the problem* (22.1) *has at least one solution.*

Proof. First of all mention that it follows from (H_{26}) that

$$p_1(t)\le p_2(t)\quad\text{for }\ t\in[0,\omega]. \tag{22.35}$$

Further, it is clear that without loss of generality we can assume that the function $q^*(t,\cdot\,)$ is nondecreasing (on $]0,+\infty[$). Denote by v_0 the solution of the problem

$$v_0''=p_1(t)v_0+h_1(t);\quad v_0(0)=v_0(\omega),\ \ v_0'(0)=v_0'(\omega).$$

By virtue of Remark 9.2, there is a $\nu > 0$ such that

$$v_0(t) \geq \nu \quad \text{for } \ t \in [0,\omega]. \tag{22.36}$$

It follows from the assumptions imposed on the functions ψ_1 and ψ_2 in (H_{26}) that there is an increasing sequence $\{x_n\}_{n=1}^{+\infty} \subset [1,+\infty[$ such that $\lim\limits_{n\to+\infty} x_n = +\infty$ and

$$x_n\psi_2\Big(\frac{\nu}{\psi_1(x_n)}\Big) > \lambda_0 \quad \text{for } \ n \in \mathbb{N}, \tag{22.37}$$

where

$$\lambda_0 \stackrel{\text{def}}{=} \begin{cases} \dfrac{1}{2}\limsup\limits_{x\to+\infty} x\psi_2\Big(\dfrac{\nu}{\psi_1(x)}\Big) & \text{if } \ \limsup\limits_{x\to+\infty} x\psi_2\Big(\dfrac{\nu}{\psi_1(x)}\Big) < +\infty, \\ 1 & \text{if } \ \limsup\limits_{x\to+\infty} x\psi_2\Big(\dfrac{\nu}{\psi_1(x)}\Big) = +\infty. \end{cases}$$

Introduce the notations

$$\chi_n(x) \stackrel{\text{def}}{=} x - [x - x_n]_+ \quad \text{for } \ x \in \mathbb{R}, \ \ n \in \mathbb{N}, \tag{22.38}$$

$$f_n(t,x) \stackrel{\text{def}}{=} p_1(t)x + f(t,\chi_n(x)) - p_1(t)\chi_n(x) \quad \text{for } \ t \in [0,\omega], \quad x > 0, \ \ n \in \mathbb{N}, \tag{22.39}$$

and for any $n \in \mathbb{N}$, consider the problem

$$u'' = f_n(t,u); \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega). \tag{22.40}$$

It is clear that, the function f_n satisfies hypotheses (H_{22}) (with $p(t) \stackrel{\text{def}}{=} p_1(t)$, $\varphi(x) \stackrel{\text{def}}{=} 1$ and $h_0(t) \stackrel{\text{def}}{=} \frac{1}{\psi_1(x_n)}\, h_1(t)$) and (H_{19}) (with $p_0(t) \stackrel{\text{def}}{=} p_2(t)$, $q_0(t,x) \stackrel{\text{def}}{=} \frac{1}{\psi_2(x_1)}\, h_2(t) + q^*(t,x_n) + x_n|p_1(t)|$ and $r_0 = 1$). Therefore, by virtue of Corollary 22.7, the problem (22.40) possesses a solution u_n and

$$u_n(t) > 0 \quad \text{for } \ t \in [0,\omega], \ \ n \in \mathbb{N}.$$

Put $M_n \stackrel{\text{def}}{=} \max\{u_n(t) : \ t \in [0,\omega]\}$. In view of (22.38) and (22.39), to finish the proof it is sufficient to show that for some $n \in \mathbb{N}$, the inequality $M_n \leq x_n$ is fulfilled. Suppose the contrary, let

$$M_n > x_n \quad \text{for } \ n \in \mathbb{N}. \tag{22.41}$$

Then, it is clear that,

$$\begin{aligned} u_n''(t) &\geq p_1(t)u_n(t) + \frac{h_1(t)}{\psi_1(\chi_n(u_n(t)))} \\ &\geq p_1(t)u_n(t) + \frac{h_1(t)}{\psi_1(x_n)} \quad \text{for } \ t \in [0,\omega], \ \ n \in \mathbb{N}. \end{aligned} \tag{22.42}$$

In view of (22.42) and the condition $p \in \mathcal{V}^+(\omega)$, it follows from Remark 0.6 that $\psi_1(x_n)u_n(t) \geq v_0(t)$ for $t \in [0,\omega]$. Hence, on account of (22.36), we get

$$u_n(t) \geq \frac{\nu}{\psi_1(x_n)} \quad \text{for } \ t \in [0,\omega], \ \ n \in \mathbb{N}. \tag{22.43}$$

Consequently,

$$\chi_n(u_n(t)) \geq \chi_n\Big(\frac{\nu}{\psi_1(x_n)}\Big) = \frac{\nu}{\psi_1(x_n)} \quad \text{for } \ n \in \mathbb{N} \tag{22.44}$$

and

$$\psi_2\big(\chi_n(u_n(t))\big) \geq \psi_2\Big(\frac{\nu}{\psi_1(x_n)}\Big) \quad \text{for } \ n \in \mathbb{N}. \tag{22.45}$$

Mention also that, in view of (H_{26}) and (22.35), the inequalities

$$\begin{aligned} u_n''(t) &\leq p_2(t)u_n(t) + f_n\big(t,\chi_n(u_n(t))\big) - p_2(t)\chi_n(u_n(t)) + (p_1(t)-p_2(t))[u_n(t)-x_n]_+ \\ &\leq p_2(t)u_n(t) + \frac{h_2(t)}{\psi_2(\chi_n(u_n(t)))} + q^*(t,x_n) \quad \text{for } \ t \in [0,\omega], \ \ n \in \mathbb{N} \end{aligned} \tag{22.46}$$

hold. Introduce the notations

$$\widetilde{u}_n(t) = \frac{1}{M_n}\, u_n(t) \quad \text{for } \ t \in [0,\omega], \ \ n \in \mathbb{N}.$$

Then, in view of (22.41), (22.42), (22.37), and (22.46), we get that

$$p_1(t)\widetilde{u}_n(t) \le \widetilde{u}_n''(t) \le p_2(t)\widetilde{u}_n(t) + \frac{1}{\lambda_0}\, h_2(t) + \frac{1}{x_n}\, q^*(t,x_n) \quad \text{for } t\in[0,\omega],\ \ n\in\mathbb{N}.$$

Consequently,

$$|\widetilde{u}_n''(t)| \le |p_1(t)| + |p_2(t)| + \frac{1}{\lambda_0}\, h_2(t) + \frac{1}{x_n}\, q^*(t,x_n) \quad \text{for } t\in[0,\omega],\ \ n\in\mathbb{N}. \tag{22.47}$$

Taking now into account that $\widetilde{u}_k$ is a periodic function and $q^*\in K_{sl}([0,\omega]\times\mathbb{R};\mathbb{R}_+)$, we easily get that the sequence $\{\widetilde{u}_n'\}_{n=1}^{+\infty}$ is uniformly bounded. On the other hand, (22.47) implies that for any $s\in[0,\omega[$ and $t\in\,]s,\omega]$, the inequality

$$\big|\widetilde{u}_n'(t) - \widetilde{u}_n'(s)\big| \le \int_s^t \Big(|p_1(\xi)| + |p_2(\xi)| + \frac{1}{\lambda_0}\, h_2(\xi)\Big)\,\mathrm{d}\xi + \delta_n \quad \text{for } n\in\mathbb{N} \tag{22.48}$$

holds, where $\delta_n \overset{\text{def}}{=} \frac{1}{x_n}\int_0^\omega q^*(\xi,x_n)\,\mathrm{d}\xi$. Since $q^*\in K_{sl}([0,\omega]\times\mathbb{R};\mathbb{R}_+)$ it follows from (22.48) that the sequence $\{\widetilde{u}_n'\}_{n=1}^{+\infty}$ is equicontinuous.

We have proved that the sequences $\{\widetilde{u}_n\}_{n=1}^{+\infty}$ and $\{\widetilde{u}_n'\}_{n=1}^{+\infty}$ are uniformly bounded and equicontinuous and thus, by virtue of the Arzelá–Ascoli lemma, we can assume without loss of generality that

$$\lim_{n\to+\infty} \widetilde{u}_n^{(i)} = u_0^{(i)} \quad \text{uniformly on } [0,\omega],\ \ i=0,1, \tag{22.49}$$

where $u_0\in AC'([0,\omega])$. It is clear that,

$$\begin{gathered} u_0(t)\ge 0 \quad \text{for } t\in[0,\omega], \quad \|u_0\|_C = 1,\\ u_0(0)=u_0(\omega),\ \ u_0'(0)=u_0'(\omega). \end{gathered} \tag{22.50}$$

Therefore, either

$$u_0(t)>0 \quad \text{for } t\in[0,\omega], \tag{22.51}$$

or there are $\alpha\in[0,\omega[$ and $\beta\in\,]\alpha,\omega]$ such that

$$u_0(t)>0 \quad \text{for } t\in\,]\alpha,\beta], \quad u_0(\alpha)=0, \quad u_0'(\alpha)=0. \tag{22.52}$$

Suppose first that (22.51) holds. Then, in view of (22.49), there are $\mu_0\in\,]0,1[$ and $n_0\in\mathbb{N}$ such that $\widetilde{u}_n(t)>\mu_0$ for $t\in[0,\omega]$, $n\ge n_0$. Taking, moreover, into account (22.41), we get that

$$u_n(t)\ge \mu_0 x_n \quad \text{for } t\in[0,\omega],\ \ n\ge n_0\,.$$

Consequently, $\chi_n(u_n(t))\ge \mu_0 x_n$ and

$$\psi_2\big(\chi_n(u_n(t))\big) \ge \psi_2(\mu_0 x_n) \quad \text{for } t\in[0,\omega],\ \ n\ge n_0\,. \tag{22.53}$$

Hence, in view of (22.41) and (22.46), we get that

$$\widetilde{u}_n''(t) \le p_2(t)\widetilde{u}_n(t) + \frac{h_2(t)}{x_n\psi_2(\mu_0x_n)} + \frac{1}{x_n}\, q^*(t,x_n) \quad \text{for } t\in[0,\omega],\ \ n\ge n_0\,. \tag{22.54}$$

Let now v_n be a solution of the problem

$$v_n'' = p_2(t)v_n + \frac{h_2(t)}{x_n\psi_2(\mu_0x_n)} + \frac{1}{x_n}\, q^*(t,x_n); \quad v_n(0)=v_n(\omega),\ \ v_n'(0)=v_n'(\omega).$$

In view of (22.54) and the condition $p_2\in\mathcal{V}^+(\omega)$, it follows from Remark 0.6 that

$$v_n(t)\ge \widetilde{u}_n(t) \quad \text{for } t\in[0,\omega],\ \ n\ge n_0. \tag{22.55}$$

On the other hand, since $q^*\in K_{sl}([0,\omega]\times\mathbb{R};\mathbb{R}_+)$ and

$$\lim_{n\to+\infty} \frac{1}{x_n\psi_2(\mu_0x_n)} \int_0^\omega h_2(s)\,\mathrm{d}s = 0,$$

we get from Lemma 3.1 that

$$\lim_{n\to+\infty} v_n(t) = 0 \quad \text{uniformly on } [0,\omega]$$

which, together with (22.49) and (22.55), contradicts (22.50).

Now suppose that (22.52) is fulfilled. Let $\tau \in]\alpha,\beta[$ and $t \in]\tau,\beta[$ be arbitrary. On account of (22.49) and (22.52), there are $\mu_1 \in]0,1[$ and $n_1 \in \mathbb{N}$ such that $\widetilde{u}_n(s) > \mu_1$ for $s \in [\tau,t]$, $n \geq n_1$. Consequently, in view of (22.41), we get that $u_n(s) \geq \mu_1 x_n$ for $s \in [\tau,t]$, $n \geq n_1$ and therefore

$$\psi_2\big(\chi_n(u_n(s))\big) \geq \psi_2(\mu_1 x_n) \quad \text{for } \ s \in [\tau,t], \ \ n \geq n_1\,. \tag{22.56}$$

The latter inequality, together with (22.41) and (22.46), implies that

$$\widetilde{u}_n'(t) - \widetilde{u}_n'(\tau) \leq \int\limits_\tau^t p_2(s)\widetilde{u}_n(s)\,\mathrm{d}s + \frac{1}{x_n\psi_2(\mu_1 x_n)} \int\limits_\tau^t h_2(s)\,\mathrm{d}s + \frac{1}{x_n}\int\limits_\tau^t q^*(s,x_n)\,\mathrm{d}s$$

for $n \geq n_1$ and thus, by virtue of (22.49) and the condition $q^* \in K_{sl}([0,\omega]\times\mathbb{R};\mathbb{R}_+)$, we get that

$$u_0'(t) - u_0'(\tau) \leq \int\limits_\tau^t p_2(s)u_0(s)\,\mathrm{d}s. \tag{22.57}$$

We have proved that for any $\tau \in]\alpha,\beta[$ and $t \in]\tau,\beta[$, the inequality (22.57) holds. Therefore, in view of the condition $u_0'(\alpha) = 0$, we get from (22.57) that

$$u_0'(t) \leq \int\limits_\alpha^t p_2(s)u_0(s)\,\mathrm{d}s \quad \text{for } \ t \in [\alpha,\beta]$$

which, together with the condition $u_0(\alpha) = 0$, yields

$$u_0(t) \leq \int\limits_\alpha^t \bigg(\int\limits_\alpha^s p_2(\xi)u_0(\xi)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s \leq \omega \int\limits_\alpha^t |p_2(s)|u_0(s)\,\mathrm{d}s \quad \text{for } \ t \in [\alpha,\beta].$$

Hence, by virtue of the Gronwall–Belman lemma, we get that $u_0(t) \leq 0$ for $t \in [\alpha,\beta]$ which contradicts (22.52). $\square$

As an example consider the problem

$$u'' = p(t)u + \frac{h_0(t)}{\psi(u)}\,; \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega), \tag{22.58}$$

where the functions h_0 and ψ satisfy the following hypothesis

$$\begin{cases} h_0 \in L_\omega, \quad h_0(t) \geq 0 \quad \text{for } \ t \in \mathbb{R}, \quad h_0 \not\equiv 0, \\ \psi \in C(]0,+\infty[\,;\,]0,+\infty[\,) \ \text{ is nondecreasing.} \end{cases} \tag{H_{27}}$$

It follows from Corollary 22.15 that if, in addition, $p \in \operatorname{Int}\mathcal{V}^+(\omega)$ then the problem (22.58) is solvable. Moreover, if $p \in \mathcal{V}^+(\omega) \setminus \operatorname{Int}\mathcal{V}^+(\omega)$, i.e., if $p \in \partial\mathcal{D}(\omega)$ then none of the results stated above can be applied. Moreover, the example given in Remark 22.14 shows that the sole condition (H_{27}) is insufficient for the solvability of the problem (22.58) in that case (i.e., when $p \in \partial\mathcal{D}(\omega)$). However, Theorem 22.16 implies that

Corollary 22.17. *Let $p \in \mathcal{V}^+(\omega)$ and (H_{27}) hold. Let, moreover,*

$$\limsup_{x\to+\infty} x\psi\Big(\frac{c}{\psi(x)}\Big) > 0$$

for any $c > 0$. Then the problem (22.58) *has at least one solution.*

Before formulation the next result introduce the hypothesis

$$\begin{cases} f(t,x) \geq p_1(t)x - q_1(t,x) \quad \text{for } t \in [0,\omega], \ \ x > 0, \\ f(t,x) \geq p(t)x + h_*(t,x) - q(t,x) \quad \text{for } t \in [0,\omega], \ \ x \in \,]0,\delta], \ \ \delta > 0, \\ q, q_1 \in K_{sl}([0,\omega] \times \mathbb{R}; \mathbb{R}_+), \ \ h_* \in K_{loc}([0,\omega] \times \,]0,+\infty[\,; \mathbb{R}_+), \\ \text{the function } h_*(t,\,\cdot\,) \text{ is nonincreasing}, \\ \displaystyle\int_0^\omega h_*(s, c|s-a|)\,\mathrm{d}s = +\infty \quad \text{for } c > 0, \ \ a \in [0,\omega[\,, \\ p_1(t) \leq p(t) \quad \text{for } t \in [0,\omega], \ \ p_1 \in \operatorname{Int}\mathcal{D}(\omega), \quad p \in \operatorname{Int}\mathcal{V}^+(\omega). \end{cases} \tag{H_{28}}$$

Theorem 22.18. *Let the hypothesis (H_{28}) hold. Let moreover, either (H_{19}) or (H_{20}) be fulfilled and $p(t) \leq p_0(t)$ for $t \in [0,\omega]$. Then the problem* (22.1) *has at least one solution.*

Proof. First of all mention that without loss of generality we can assume that $\delta < 1$ and the functions q_0, q_1, and q are nondecreasing in the second variable on $\mathbb{R}_+$. It is easily follows from hypothesis imposed on the function h_* that

$$\lim_{x \to 0+} \int_0^\omega h_*(s,x)\,\mathrm{d}s = +\infty. \tag{22.59}$$

Introduce the notation

$$h(t,x) \overset{\text{def}}{=} f(t,x) - p(t)x + q(t,x),$$

$$\widetilde{h}(t,x) \overset{\text{def}}{=} h(t, x - [x-\delta]_+), \quad \widetilde{h}_*(t,x) \overset{\text{def}}{=} h_*(t, x - [x-\delta]_+),$$

$$\widetilde{q}(t,x) \overset{\text{def}}{=} h(t, \delta + [x-\delta]_+) - h(t,\delta) - q(t,x).$$

It is clear that, $\widetilde{h} \in K_{loc}([0,\omega] \times \,]0,+\infty[\,; \mathbb{R})$, $\widetilde{q} \in K([0,\omega] \times \mathbb{R}; \mathbb{R})$,

$$\widetilde{h}(t,x) \geq 0 \quad \text{for } t \in [0,\omega], \ \ x > 0, \tag{22.60}$$

$$\widetilde{h}(t,x) \geq \widetilde{h}_*(t,x) \quad \text{for } t \in [0,\omega], \ \ x > 0, \tag{22.61}$$

$$\int_0^\omega \widetilde{h}_*(s, c|s-a|)\,\mathrm{d}s = +\infty \quad \text{for } c > 0, \ \ a \in [0,\omega[\,. \tag{22.62}$$

Moreover, one can easily verify that

$$p(t)x + \widetilde{h}(t,x) + \widetilde{q}(t,x) = f(t,x) \quad \text{for } t \in [0,\omega], \ \ x > 0 \tag{22.63}$$

and

$$\widetilde{q}(t,x) \geq \widetilde{p}(t)x - q_2(t,x) \quad \text{for } t \in [0,\omega], \ \ x > 0, \tag{22.64}$$

where

$$\widetilde{p}(t) \overset{\text{def}}{=} p_1(t) - p(t), \quad q_2(t,x) \overset{\text{def}}{=} q(t,x) + q_1(t,x).$$

Clearly, $q_2 \in K_{sl}([0,\omega] \times \mathbb{R}; \mathbb{R}_+)$ and the function q_2 is nondecreasing in the second variable on $\mathbb{R}_+$.

Further, for any $n \in \mathbb{N}$, put

$$\widetilde{q}_n(t,x) \overset{\text{def}}{=} \widetilde{q}\big(t, [x]_+ - \big[[x]_+ - n\big]_+\big). \tag{22.65}$$

One can easily verify that $\widetilde{q}_n \in K([0,\omega] \times \mathbb{R}; \mathbb{R}_+)$ and

$$p(t)x + \widetilde{h}(t,x) + \widetilde{q}_n(t,x) = p(t)[x-n]_+ + f(t, x - [x-n]_+) \quad \text{for } t \in [0,\omega], \ \ x > 0. \tag{22.66}$$

Moreover, for any $n \in \mathbb{N}$, there is a $q_n^* \in K([0,\omega] \times \mathbb{R}; \mathbb{R}_+)$ such that $q_n^*(t,\,\cdot\,)$ is nondecreasing on $\mathbb{R}_+$ and

$$|\widetilde{q}_n(t,x)| \leq q_n^*(t, [x]_+) \quad \text{for } t \in [0,\omega], \ \ x \in \mathbb{R}, \tag{22.67}$$

$$|\widetilde{q}_n(t,x)| \leq q_n^*(t, n) \quad \text{for } t \in [0,\omega], \ \ x \in \mathbb{R}. \tag{22.68}$$

Consider the problem

$$u'' = p(t)u + \widetilde{h}(t,u) + \widetilde{q}_n(t,u); \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega). \tag{22.69}$$

First we will show that for any $n \in \mathbb{N}$, the problem (22.69) possesses at least one (positive) solution. Indeed, let $n \in \mathbb{N}$ be fixed. For any $k \in \mathbb{N}$ put

$$\widetilde{h}_k(t,x) \stackrel{\text{def}}{=} \widetilde{h}\Big(t, \frac{1}{k} + \Big[x - \frac{1}{k}\Big]_+\Big).$$

One can easily verify that $\widetilde{h}_k \in K([0,\omega] \times \mathbb{R}; \mathbb{R})$,

$$\widetilde{h}_k(t,x) \geq 0 \quad \text{for } t \in [0,\omega], \ \ x \in \mathbb{R}, \ \ k \in \mathbb{N} \tag{22.70}$$

and for any $k \in \mathbb{N}$, there is a $h_k^* \in L_\omega$ such that

$$\widetilde{h}_k(t,x) \leq h_k^*(t) \quad \text{for } t \in [0,\omega], \ \ x \in \mathbb{R}. \tag{22.71}$$

By virtue of (22.68), (22.70), (22.71), and the condition $p \in \operatorname{Int} \mathcal{V}^+(\omega)$, it follows from Proposition 17.3 that for any $k \in \mathbb{N}$, the problem

$$v_k'' = p(t)v_k + \widetilde{h}_k(t,v_k) + \widetilde{q}_n(t,v_k); \quad v_k(0) = v_k(\omega), \ \ v_k'(0) = v_k'(\omega) \tag{22.72}$$

possesses a solution v_k. In view of (22.67), (22.68), and (22.70), clearly

$$v_k''(t) \geq p(t)v_k(t) - q_n^*(t, [v_k(t)]_+) \quad \text{for } t \in [0,\omega], \ \ k \in \mathbb{N}, \tag{22.73}$$

$$v_k''(t) \geq p(t)v_k(t) - q_n^*(t,n) \quad \text{for } t \in [0,\omega], \ \ k \in \mathbb{N}. \tag{22.74}$$

Now, we will show that there is a $k_0 \in \mathbb{N}$ such that

$$\max\big\{v_k(t) : t \in [0,\omega]\big\} > \frac{1}{k_0} \quad \text{for } k > k_0\,. \tag{22.75}$$

Indeed, let v be a solution of the problem

$$v'' = p(t)v + q_n^*(t,1); \quad v(0) = v(\omega), \ \ v'(0) = v'(\omega). \tag{22.76}$$

In view of (22.59), there is a $k_0 \in \mathbb{N}$ such that $k_0 > \frac{1}{\delta}$ and

$$\int_0^\omega h_*\Big(s, \frac{1}{k_0}\Big)\,\mathrm{d}s > \|p\|_L(1 + \|v\|_C) + \|q_n^*(\,\cdot\,,1)\|_L. \tag{22.77}$$

Suppose that for a certain $k > k_0$, the inequality (22.75) is violated, i.e.,

$$\max\big\{v_k(t) : \ t \in [0,\omega]\big\} \leq \frac{1}{k_0} \tag{22.78}$$

holds and, consequently,

$$[v_k(t)]_+ \leq 1 \quad \text{for } t \in [0,\omega]. \tag{22.79}$$

In view of (22.73), (22.76), (22.79), the condition $p \in \operatorname{Int} \mathcal{V}^+(\omega)$, and the monotonicity of the function $q_n^*(t,\,\cdot\,)$, it follows from Remark 0.6 that $v_k(t) + v(t) \geq 0$ for $t \in [0,\omega]$. Taking, moreover, into account (22.79), we get that

$$\|v_k\|_C \leq 1 + \|v\|_C. \tag{22.80}$$

Since v_k is a solution of the problem (22.72) we have

$$\int_0^\omega \widetilde{h}_k(s, v_k(s))\,\mathrm{d}s = -\int_0^\omega \big(p(s)v_k(s) + \widetilde{q}_n(s, v_k(s))\big)\,\mathrm{d}s. \tag{22.81}$$

The latter equality, together with (22.67), (22.79), and (22.80), implies that

$$\int_0^\omega \widetilde{h}_k(s, v_k(s))\,\mathrm{d}s \leq \|p\|_L\big(1 + \|v\|_C\big) + \|q_n^*(\,\cdot\,,1)\|_L. \tag{22.82}$$

On the other hand, in view of (22.61), we have

$$\int_0^\omega \widetilde{h}_k(s, v_k(s))\,\mathrm{d}s \ge \int_0^\omega \widetilde{h}_*\Big(s, \frac{1}{k} + \Big[v_k(s) - \frac{1}{k}\Big]_+\Big)\,\mathrm{d}s. \tag{22.83}$$

Taking now into account that the function $\widetilde{h}_*(t, \cdot\,)$ is nondecreasing and

$$\frac{1}{k} + \Big[v_k(t) - \frac{1}{k}\Big]_+ \le \frac{1}{k_0} < \delta \quad \text{for } t \in [0, \omega],$$

we get from (22.83) that

$$\int_0^\omega \widetilde{h}_k(s, v_k(s))\,\mathrm{d}s \ge \int_0^\omega h_*\Big(s, \frac{1}{k_0}\Big)\,\mathrm{d}s$$

which, together with (22.82), contradicts (22.77). Thus, we have proved that for a certain $k_0 \in \mathbb{N}$, the inequality (22.75) holds.

Now, we will show that for a certain $k > k_0$, the inequality

$$v_k(t) \ge \frac{1}{k} \quad \text{for } t \in [0, \omega] \tag{22.84}$$

is satisfied. Extend the functions p, $\widetilde{h}_k$, and $\widetilde{q}_n$, and v_k periodically and denote them by the same letters. Suppose that the relation (22.84) is violated for every $k > k_0$. Then, in view of (22.75), for any $k > k_0$ there is an $a_k \in [0, \omega[$ such that

$$v_k(a_k) = \frac{1}{k}\,, \quad v_k(a_k + \omega) = \frac{1}{k}\,. \tag{22.85}$$

Since $p \in \operatorname{Int}\mathcal{V}^+(\omega)$ we have $p \in \operatorname{Int}\mathcal{D}(\omega)$ as well (see Theorem 9.1′). Denote by w_k and w_0 the solutions of the problems

$$w_k'' = p(t)w_k - q_n^*(t, n); \quad w_k(a_k) = 1, \ \ w_k(a_k + \omega) = 1,$$

and

$$w_0'' = p(t)w_0 + q_n^*(t, n); \quad w_0(0) = w_0(\omega), \ \ w_0'(0) = w_0'(\omega).$$

In view of (22.74), it follows from Proposition 2.5 and Remark 0.6 that

$$\begin{aligned} v_k(t) &\le w_k(t) \quad \text{for } t \in [a_k, a_k + \omega], \ \ k \in \mathbb{N}, \\ v_k(t) &\ge -w_0(t) \quad \text{for } t \in [0, \omega], \ \ k \in \mathbb{N}. \end{aligned}$$

By virtue of Remark 6.4 and Proposition 6.8, there is a $\nu > 0$ such that

$$w_k(t) \le \nu\big(1 + \|q_n^*(\cdot\,, n)\|_L\big) \quad \text{for } t \in [a_k, a_k + \omega], \ \ k \in \mathbb{N}.$$

Consequently,

$$\|v_k\|_C \le c_0 \quad \text{for } k \in \mathbb{N}, \tag{22.86}$$

where $c_0 \overset{\text{def}}{=} \nu(1 + \|q_n^*(\cdot\,, n)\|_L) + \|w_0\|_C$.

In view of (22.68) and (22.86), it follows from (22.81) that

$$\int_0^\omega \widetilde{h}_k(s, v_k(s))\,\mathrm{d}s \le \frac{c}{2} \quad \text{for } k \in \mathbb{N}, \tag{22.87}$$

where $c = 2(c_0\|p\|_L + \|q_n^*(\cdot\,, n)\|_L))$. Taking now into account (22.68), (22.70), (22.86), and (22.87), we get from (22.72) that

$$\int_0^\omega |v_k''(s)|\,\mathrm{d}s \le c \quad \text{for } k \in \mathbb{N}. \tag{22.88}$$

Since $v_k(0) = v_k(\omega)$ there is a $t_k \in [0, \omega[$ such that $v_k'(t_k) = 0$. Hence, in view of (22.88), we get

$$|v_k'(t)| = \Big|\int_{t_k}^t v_k''(s)\,\mathrm{d}s\Big| \le \int_0^\omega |v_k''(s)|\,\mathrm{d}s \le c \quad \text{for } t \in [0, \omega], \ \ k \in \mathbb{N}.$$

Consequently,

$$|v_k(t) - v_k(s)| \le c|t - s| \quad \text{for } t, s \in [0, \omega], \ \ k \in \mathbb{N}. \tag{22.89}$$

In view of (22.86) and (22.89), the sequence $\{v_k\}_{k=1}^{+\infty}$ is uniformly bounded and equicontinuous on $[0, 2\omega]$. Hence, by virtue of the Arzelá–Ascoli lemma, we can assume without loss of generality that

$$\lim_{k\to+\infty} a_k = a_0 \tag{22.90}$$

and

$$\lim_{k\to+\infty} v_k(t) = v_0(t) \quad \text{uniformly on } [0, 2\omega], \tag{22.91}$$

where $a_0 \in [0, \omega]$ and $v_0 \in C([0, 2\omega]; \mathbb{R})$. It follows from (22.89), in view of (22.85), (22.90), and (22.91), that $|v_0(t)| \le c|t - a_0|$ for $t \in [0, 2\omega]$. Since $v_0(t) = v_0(t + \omega)$ for $t \in [0, \omega]$ we get from the latter inequality that

$$|v_0(t)| \le c|t - a| \quad \text{for } t \in [0, \omega], \tag{22.92}$$

where $a \overset{\text{def}}{=} a_0$ if $a_0 \in [0, \omega[$, and $a \overset{\text{def}}{=} 0$ if $a_0 = \omega$.

Moreover, in view of (22.91), for any $\varepsilon > 0$ there is a $k_\varepsilon > \frac{1}{\varepsilon} + \frac{1}{\delta}$ such that

$$|v_k(t)| \le |v_0(t)| + \varepsilon \quad \text{for } t \in [0, \omega], \ \ k > k_\varepsilon\,.$$

Consequently,

$$\frac{1}{k} + \Big[v_k(t) - \frac{1}{k}\Big]_+ \le |v_0(t)| + \varepsilon \quad \text{for } t \in [0, \omega], \ \ k > k_\varepsilon\,. \tag{22.93}$$

In view of (22.61), (22.92), (22.93), and the monotonicity of the function $\widetilde{h}_*(t, \cdot\,)$, we get that

$$\int_0^\omega \widetilde{h}_k(s, v_k(s))\,\mathrm{d}s \ge \int_0^\omega \widetilde{h}_*\Big(s, \frac{1}{k} + \Big[v_k(s) - \frac{1}{k}\Big]_+\Big)\,\mathrm{d}s \ge \int_0^\omega \widetilde{h}_*\big(s, c|s - a| + \varepsilon\big)\,\mathrm{d}s \quad \text{for } k > k_\varepsilon$$

which, together with (22.87), yields

$$\int_0^\omega \widetilde{h}_*\big(s, c|s - a| + \varepsilon\big)\,\mathrm{d}s \le C.$$

However, $\varepsilon > 0$ was arbitrary and thus, the latter inequality contradicts (22.62). Hence, we have proved that there is a $k > k_0$ such that (22.84) holds, whence we get $\widetilde{h}_k(t, v_k(t)) = \widetilde{h}(t, v_k(t))$ for $t \in [0, \omega]$. Consequently, in view of (22.72), the function v_k is a (positive) solution of the problem (22.69) as well.

Therefore, we have proved that for any $n \in \mathbb{N}$ the problem (22.69) possesses a solution u_n and

$$u_n(t) > 0 \quad \text{for } t \in [0, \omega]. \tag{22.94}$$

In view of (22.66), to finish the proof it is sufficient to show that for a certain $n \in \mathbb{N}$, the inequality

$$\|u_n\|_C \le n \tag{22.95}$$

holds. Assume the contrary, let

$$\|u_n\|_C > n \quad \text{for } n \in \mathbb{N}. \tag{22.96}$$

First, suppose that (H_{19}) holds. We will estimate $\|u_n'\|_C$. Extend the functions p, $\widetilde{h}$, $\widetilde{q}_n$, and u_n periodically and denote them by the same letters. Let $n \in \mathbb{N}$ and $t \in [0, \omega[$ be fixed such that $u_n'(t) \ne 0$. Suppose that $u_n'(t) > 0$. Since u_n is a periodic function there is a $t^* \in]t, t + \omega[$ such that $u_n'(t^*) = 0$. In view of (22.60), (22.64), (22.65), and (22.94), we have

$$\begin{aligned} -u_n'(t) &= \int_t^{t^*} \Big(p(s)u_n(s) + \widetilde{h}(s, u_n(s)) + \widetilde{q}_n(s, u_n(s))\Big)\,\mathrm{d}s \\ &\ge -\int_t^{t^*} \big(|p(s)| + |\widetilde{p}(s)|\big)u_n(s)\,\mathrm{d}s - \int_t^{t^*} q_2(s, n)\,\mathrm{d}s \ge -\big(\|p\|_L + \|\widetilde{p}\|_L\big)\|u_n\|_C - \|q_2(\,\cdot\,, n)\|_L\,. \end{aligned}$$

Analogously, if $u_n'(t) < 0$ then there is a $t_* \in]t-\omega, t[$ such that $u_n'(t_*) = 0$ and

$$u_n'(t) = \int_{t_*}^{t} \Big(p(s)u_n(s) + \widetilde{h}(s, u_n(s)) + \widetilde{q}_n(s, u_n(s))\Big)\,\mathrm{d}s \geq -\big(\|p\|_L + \|\widetilde{p}\|_L\big)\|u_n\|_C - \|q_2(\,\cdot\,, n)\|_L\,.$$

Therefore,

$$\|u_n'\|_C \leq \big(\|p\|_L + \|\widetilde{p}\|_L\big)\|u_n\|_C + \|q_2(\,\cdot\,, n)\|_L \quad \text{for } \ n \in \mathbb{N}. \tag{22.97}$$

Introduce the notation

$$\widetilde{u}_n(t) \stackrel{\text{def}}{=} \frac{1}{\|u_n\|_C}\, u_n(t) \quad \text{for } \ t \in [0, 2\omega], \ \ n \in \mathbb{N}.$$

In view of (22.97), (22.96), and the condition $q_2 \in K_{sl}([0,\omega] \times \mathbb{R}; \mathbb{R}_+)$, there is an $\eta > 0$ such that

$$\|\widetilde{u}_n'\|_C \leq \eta \quad \text{for } \ n \in \mathbb{N}.$$

Hence,the sequence $\{\widetilde{u}_n\}_{n=1}^{\infty}$ is uniformly bounded and equicontinuous (on $[0, 2\omega]$). By virtue of the Arzelá–Ascoli lemma we can assume without loss of generality that

$$\lim_{n \to +\infty} \widetilde{u}_n(t) = u_0(t) \quad \text{uniformly on } \ [0, 2\omega], \tag{22.98}$$

where $u_0 \in C([0, 2\omega])$. In view of (22.94), clearly

$$u_0(t) \geq 0 \quad \text{for } \ t \in [0, 2\omega], \quad \|u_0\|_C = 1.$$

Consequently, either

$$u_0(t) > 0 \quad \text{for } \ t \in [0, \omega], \tag{22.99}$$

or there are $\alpha \in [0, \omega[$ and $\beta \in]\alpha, \alpha + \omega]$ such that

$$u_0(t) > 0 \quad \text{for } \ t \in]\alpha, \beta[\,, \quad u_0(\alpha) = 0, \quad u_0(\beta) = 0. \tag{22.100}$$

First, assume that (22.99) is fulfilled. Then, in view of (22.96) and (22.98), there is a $n_0 > r_0$ such that

$$u_n(t) > \max\{r_0, \delta\} \quad \text{for } \ t \in [0, \omega], \ \ n > n_0\,. \tag{22.101}$$

Taking into account (22.66), (22.101), and the hypothesis (H_{19}), we get

$$\begin{aligned} u_n''(t) &= p(t)[u_n(t) - n]_+ + f\big(t, u_n(t) - [u_n(t) - n]_+\big) \\ &\leq p_0(t)u_n(t) + \big(p(t) - p_0(t)\big)[u_n(t) - n]_+ + q_0\big(t, u_n(t) - [u_n(t) - n]_+\big) \\ &\leq p_0(t)u_n(t) + q_0(t, n) \quad \text{for } \ t \in [0, \omega], \ \ n > n_0\,. \end{aligned} \tag{22.102}$$

Denote by w_n the solution of the problem

$$w_n'' = p_0(t)w_n + \frac{1}{n}\, q_0(t, n); \quad w_n(0) = w_n(\omega), \ \ w_n'(0) = w_n'(\omega).$$

In view of (22.96), (22.102), and Remark 0.6, we get that

$$\widetilde{u}_n(t) \leq w_n(t) \quad \text{for } \ t \in [0, \omega], \ \ n > n_0\,. \tag{22.103}$$

However, since $p_0 \in \mathcal{V}^+(\omega)$ and $q_0 \in K_{sl}([0,\omega] \times \mathbb{R}; \mathbb{R}_+)$ it follows from Lemma 3.1 that

$$\lim_{n \to +\infty} \|w_n\|_C = 0$$

which, together with (22.103) and (22.98), contradicts (22.99).

Suppose now that (22.100) holds. In view of (22.66), (22.94), and the first assumption in (H_{28}), we get that

$$\begin{aligned} u_n''(t) &= p(t)[u_n(t) - n]_+ + f\big(t, u_n(t) - [u_n(t) - n]_+\big) \\ &\geq p_1(t)u_n(t) + \big(p(t) - p_1(t)\big)[u_n(t) - n]_+ - q_1\big(t, u_n(t) - [u_n(t) - n]_+\big) \\ &\geq p_1(t)u_n(t) - q_1(t, n) \quad \text{for } \ t \in [0, \omega], \ \ n \in \mathbb{N}. \end{aligned} \tag{22.104}$$

Denote by v_n the solution of the problem

$$v_n'' = p_1(t)v_n - \frac{1}{n}\, q_1(t, n); \quad v_n(\alpha) = \widetilde{u}_n(\alpha), \ \ v_n(\beta) = \widetilde{u}_n(\beta).$$

It follows from Proposition 2.5, in view of (22.96) and (22.104), that

$$\widetilde{u}_n(t) \leq v_n(t) \quad \text{for } t \in [\alpha, \beta], \quad n \in \mathbb{N}. \tag{22.105}$$

On the other hand, since $q_1 \in K_{sl}([0,\omega] \times \mathbb{R}; \mathbb{R}_+)$ and

$$\lim_{n \to +\infty} \widetilde{u}_n(\alpha) = 0, \qquad \lim_{n \to +\infty} \widetilde{u}_n(\beta) = 0$$

we get from Remark 6.4 and Proposition 6.8 that

$$\lim_{n \to +\infty} \|v_n\|_C = 0$$

which, together with (22.98) and (22.105), contradicts (22.100). Therefore, we have proved that for a certain $n \in \mathbb{N}$, the inequality (22.95) is satisfied.

Suppose now that (H_{20}) holds. Extend the functions appearing in (22.69) periodically and denote them by the same letters. Put

$$m_n = \min\big\{u_n(t) : \ t \in [0,\omega]\big\}, \quad n \in \mathbb{N},$$

and choose $t_n \in [0, \omega[$ such that $u_n(t_n) = m_n$. Consequently,

$$u_n(t_n) = m_n, \quad u_n(t_n + \omega) = m_n \quad \text{for } n \in \mathbb{N}. \tag{22.106}$$

In view of (22.66), (22.94), and the first inequality in (H_{28}), we get that

$$u_n''(t) \geq p_1(t) u_n(t) - q_1(t,n) \quad \text{for } t \in [t_n, t_n + \omega], \quad n \in \mathbb{N}. \tag{22.107}$$

Denote by v_n a solution of the problem

$$v_n'' = p_1(t) v_n - q_1(t,n); \quad v_n(t_n) = m_n, \ \ v_n(t_n + \omega) = m_n. \tag{22.108}$$

By virtue of (22.106)–(22.108), it follows from Proposition 2.5 that

$$u_n(t) \leq v_n(t) \quad \text{for } t \in [t_n, t_n + \omega], \quad n \in \mathbb{N}.$$

On the other hand, it is clear (see Definition 6.2, Remark 6.4, and (6.22)) that

$$v_n(t) \leq m_n \nu^*(p_1) + \rho_0(p_1) \int_0^\omega q_1(s,n)\, \mathrm{d}s \quad \text{for } n \in \mathbb{N}.$$

Taking now into account (22.96) and the condition $q_1 \in K_{sl}([0,\omega] \times \mathbb{R}; \mathbb{R}_+)$, we get that there is a $n_1 \in \mathbb{N}$ such that

$$m_n > r_0 \quad \text{for } n > n_1,$$

where r_0 is the number appearing in (H_{20}). Consequently, the inequality

$$u_n(t) - [u_n(t) - n]_+ > r_0 \quad \text{for } t \in [0,\omega], \quad n > n_1 \tag{22.109}$$

holds. On account of (22.66), (22.69), (H_{20}), and (22.109), we have that

$$u_n''(t) \leq p_0(t) u_n(t) + q_0(t) \quad \text{for } t \in [0,\omega], \quad n > n_1.$$

Therefore, for any $n > n_1$, the function u_n is a solution of the problem

$$u'' = p_0(t) u + \widetilde{q}_n(t); \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega),$$

where $\widetilde{q}_n(t) \overset{\text{def}}{=} u_n''(t) - p_0(t) u_n(t)$ pro $t \in [0,\omega]$, $n > n_1$, and

$$\widetilde{q}_n(t) \leq q_0(t) \quad \text{for } t \in [0,\omega], \quad n > n_1. \tag{22.110}$$

However, by virtue of Fredholm's third theorem, (22.110) and (H_{20}), we get the contradiction

$$0 = \int_0^\omega \widetilde{q}_n(s) u_0(s)\, \mathrm{d}s \leq \int_0^\omega q_0(s) u_0(s)\, \mathrm{d}s < 0 \quad \text{for } n > n_1.$$

Thus we have proved that the inequality (22.95) holds for a certain $n \in \mathbb{N}$. □

For the problem (22.2), Theorem 22.18 implies the following assertion.

Corollary 22.19. *Let* $p \in \operatorname{Int}\mathcal{V}^+(\omega)$, $\delta > 0$, *and*

$$h(t,x) \geq h_*(t,x) - q(t,x) \quad \text{for } t \in [0,\omega], \ \ x \in \,]0,\delta],$$

where $q \in K_{sl}([0,\omega]\times\mathbb{R};\mathbb{R}_+)$, $h_* \in K_{loc}([0,\omega]\times\,]0,+\infty[\,;\mathbb{R}_+)$, $h_*(t,\,\cdot\,)$ *is nonincreasing, and*

$$\int\limits_0^\omega h_*(s, c|s-a|)\,\mathrm{d}s = +\infty \quad \text{for } c > 0, \ a \in [0,\omega[\,.$$

If, moreover,

$$\lim_{x\to+\infty}\frac{1}{x}\int\limits_0^\omega |h(s,x)|\,\mathrm{d}s = 0$$

then the problem (22.2) *has at least one solution.*

Proof. It is not difficult to verify that the assumptions of the corollary imply the validity of hypotheses (H_{20}) and (H_{28}) with $f(t,x) \overset{\text{def}}{=} p(t)x + h(t,x)$,

$$p_1(t) \overset{\text{def}}{=} p(t), \quad p_0(t) \overset{\text{def}}{=} p(t), \quad r_0 \overset{\text{def}}{=} \delta,$$

and

$$q_1(t,x) \overset{\text{def}}{=} q(t,x) + \big|h(t,\delta+[x-\delta]_+)\big|, \quad q_0(t,x) \overset{\text{def}}{=} \big|h(t,\delta+[x-\delta]_+)\big|.$$

Therefore, by virtue of Theorem 22.18, the problem (22.2) has at least one solution. □

Recall that the numbers $\rho(p)$ and Q_+, Q_- are defined by (0.12) and (0.13), respectively, and the function H is given by formula (0.17). Recall also that for $p \in \operatorname{Int}\mathcal{V}^+(\omega)$, the numbers $\rho_0(p)$ and $\nu^*(p)$ are defined in Definition 6.2 and by the formula (6.22), respectively.

Introduce the hypothesis

$$\begin{cases} f(t,x) \geq p(t)x + h(t,x) \quad \text{for } t\in[0,\omega], \ \ x>0, \\ h \in K_{loc}([0,\omega]\times\,]0,+\infty[\,;\mathbb{R}), \ \ h(t,\,\cdot\,) \text{ is nonincreasing in } \,]0,+\infty[\,, \\ h(t,x) \geq q(t) \quad \text{for } t\in[0,\omega], \ \ x>0, \\ p \in \operatorname{Int}\mathcal{V}^+(\omega), \ \ q \in L_\omega, \ \ Q_- \neq 0, \\ H\big(\rho_0(p)Q_-\big) > \big(\nu^*(p)\rho(p)-1\big)Q_-\,. \end{cases} \tag{H_{29}}$$

Theorem 22.20. *Let the hypothesis* (H_{29}) *hold and either* (H_{19}) *or* (H_{20}) *be fulfilled. Then the problem* (22.1) *has at least one solution.*

Proof. First of all mention that the inequality

$$p(t) \leq p_0(t) \quad \text{for } t \in [0,\omega] \tag{22.111}$$

holds. Assume without loss of generality that the function $q_0(t,\,\cdot\,)$ is nondecreasing on $]0,+\infty[\,$. Mention also that, by virtue of Remark 16.5, the inequality (16.14) is fulfilled. In view of the last condition in (H_{29}), there is a $0 < \varepsilon < \min\{1, \rho_0(p)Q_-\}$ such that

$$H\big(\varepsilon\nu^*(p) + \rho_0(p)Q_-\big) + \big(1-\nu^*(p)\rho(p)\big)Q_- > \varepsilon\big(\nu^*(p)\big\|[p]_-\big\|_L - \big\|[p]_+\big\|_L\big). \tag{22.112}$$

Introduce the notation

$$\widetilde{h}(t,x) \overset{\text{def}}{=} f(t,x) - p(t)x \quad \text{for } t\in[0,\omega], \ \ x>0, \tag{22.113}$$

and for any $n > n_0$, where $n_0 > \nu^*(p) + \rho_0(p)Q_-$, put

$$\widetilde{h}_n(t,x) \overset{\text{def}}{=} \widetilde{h}\big(t, \varepsilon + [x-\varepsilon]_+ - [x-n]_+\big) \quad \text{for } t\in[0,\omega], \ \ x\in\mathbb{R}. \tag{22.114}$$

It is clear that, $\widetilde{h} \in K_{loc}([0,\omega]\times\,]0,+\infty[\,;\mathbb{R})$, $\widetilde{h}_n \in K_{sl}([0,\omega]\times\mathbb{R};\mathbb{R})$, and

$$\widetilde{h}(t,x) \geq h(t,x) \quad \text{for } t\in[0,\omega], \ \ x>0, \tag{22.115}$$

$$\widetilde{h}(t,x) \geq q(t) \quad \text{for } t\in[0,\omega], \ \ x>0. \tag{22.116}$$

For any $n > n_0$, consider the problem

$$u'' = p(t)u + \widetilde{h}_n(t,u); \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega). \tag{22.117}$$

By virtue of Proposition 17.3, for any $n > n_0$ the problem (22.117) has at least one solution u_n. Extend the functions p, $\widetilde{h}_n$, q, and u_n periodically and denote them by the same letters. Now we will show that

$$u_n(t) > \varepsilon \quad \text{for } t \in [0,\omega], \ \ n > n_0. \tag{22.118}$$

Introduce the notation

$$m_n \overset{\text{def}}{=} \min\bigl\{u_n(t) : \ t \in [0,\omega]\bigr\}, \quad M_n \overset{\text{def}}{=} \max\bigl\{u_n(t) : \ t \in [0,\omega]\bigr\},$$

and choose $t_n \in [0,\omega[$ such that $u_n(t_n) = m_n$, i.e.,

$$u_n(t_n) = m_n, \quad u_n(t_n + \omega) = m_n. \tag{22.119}$$

Denote by α the solution of the problem

$$\begin{gathered} \alpha'' = p(t)\alpha + h\bigl(t, \varepsilon\nu^*(p) + \rho_0(p)Q_-\bigr); \\ \alpha(0) = \alpha(\omega), \ \ \alpha'(0) = \alpha'(\omega), \end{gathered} \tag{22.120}$$

and by β_n the solution of the problem

$$\beta_n'' = p(t)\beta_n - [q(t)]_-; \quad \beta_n(t_n) = [m_n]_+, \ \ \beta_n(t_n + \omega) = [m_n]_+. \tag{22.121}$$

In view of (22.116) and (22.117), clearly

$$u_n''(t) \geq p(t)u_n(t) - [q(t)]_- \quad \text{for } t \in [0,\omega].$$

Since $p \in \operatorname{Int} \mathcal{V}^+(\omega)$, on account of Theorem 9.1′, we have that $p \in \operatorname{Int} \mathcal{D}(\omega)$ as well. Taking, moreover, into account (22.119), (22.121), and Proposition 2.5, we get that

$$u_n(t) \leq \beta_n(t) \quad \text{for } t \in [t_n, t_n + \omega]. \tag{22.122}$$

On the other hand, by virtue of Remark 6.4 and (6.22), the inequality

$$\beta_n(t) \leq [m_n]_+\nu^*(p) + \rho_0(p)Q_- \quad \text{for } t \in [t_n, t_n + \omega]$$

holds which, together with (22.122), results in

$$u_n(t) \leq [m_n]_+\nu^*(p) + \rho_0(p)Q_- \quad \text{for } t \in [t_n, t_n + \omega], \ \ n > n_0. \tag{22.123}$$

Let now for a certain $n > n_0$, the inequality (22.118) be violated, i.e., $[m_n]_+ \leq \varepsilon$. In view of (22.123), one can easily verify that

$$\varepsilon + [u_n(t) - \varepsilon]_+ - [u_n(t) - n]_+ \leq \varepsilon\nu^*(p) + \rho_0(p)Q_- \quad \text{for } t \in [0,\omega].$$

Taking, moreover, into account (22.114), (22.115), and the monotonicity of the function $h(t,\cdot)$, we get from (22.117) that

$$\begin{gathered} u_n''(t) \geq p(t)u_n(t) + h\bigl(t, \varepsilon\nu^*(p) + \rho_0(p)Q_-\bigr) \quad \text{for } t \in [0,\omega], \\ u_n(0) = u_n(\omega), \quad u_n'(0) = u_n'(\omega). \end{gathered}$$

Now, it follows from Remark 0.6, in view of (22.120), that

$$u_n(t) \geq \alpha(t) \quad \text{for } t \in [0,\omega]. \tag{22.124}$$

However, by virtue of (22.112), it follows from Theorem 16.4 that

$$\alpha(t) > \varepsilon \quad \text{for } t \in [0,\omega]$$

which, together with (22.124), contradicts the assumption $[m_n]_+ \leq \varepsilon$. Thus, we have proved that (22.118) is fulfilled.

Therefore, the function u_n is a solution of the problem

$$u_n'' = p(t)u_n + \widetilde{h}(t, u_n - [u_n - n]_+); \quad u_n(0) = u_n(\omega), \ \ u_n'(0) = u_n'(\omega) \tag{22.125}$$

as well. In view of (22.113), to finish the proof it is sufficient to show that for some $n > n_0$, the inequality

$$M_n \leq n \tag{22.126}$$

holds. Suppose the contrary, let

$$M_n > n \quad \text{for } n > n_0\,. \tag{22.127}$$

First, assume that (H_{19}) holds. By virtue of (22.118) and (22.123), clearly

$$M_n \le m_n \nu^*(p) + \rho_0(p) Q_- \quad \text{for } \; n > n_0. \tag{22.128}$$

Hence, in view of (22.127), there is an $n_1 > n_0$ such that

$$m_n > r_0 \quad \text{for } \; n > n_1, \tag{22.129}$$

where r_0 is the number appearing in the hypothesis (H_{19}). On account of the latter inequality, one can easily verify that

$$u_n(t) - [u_n(t) - n]_+ > r_0 \quad \text{for } \; t \in [0,\omega], \;\; n > n_1. \tag{22.130}$$

Hence, by virtue of (H_{19}) and (22.111), we have

$$\begin{aligned} p(t)u_n(t) &+ \widetilde{h}\big(t, u_n(t) - [u_n(t) - n]_+\big) \\ &\le p_0(t)u_n(t) + \big(p(t) - p_0(t)\big)[u_n(t) - n]_+ + q_0\big(t, u_n(t) - [u_n(t) - n]_+\big) \\ &\le p_0(t)u_n(t) + q_0(t,n) \quad \text{for } \; t \in [0,\omega], \;\; n > n_1. \end{aligned}$$

Consequently,

$$u_n''(t) \le p_0(t)u_n(t) + q_0(t,n) \quad \text{for } \; t \in [0,\omega], \;\; n > n_1.$$

Denote by v_n the solution of the problem

$$v_n''(t) = p_0(t)v_n + \frac{1}{n}\, q_0(t,n); \quad v_n(0) = v_n(\omega), \;\; v_n'(0) = v_n'(\omega).$$

Since $p_0 \in \mathcal{V}^+(\omega)$, in view of Remark 0.6, the inequality

$$u_n(t) \le n v_n(t) \quad \text{for } \; t \in [0,\omega], \;\; n > n_1$$

holds which, together with (22.127), implies

$$\|v_n\|_C \ge 1 \quad \text{for } \; n > n_1.$$

On the other hand, since $q_0 \in K_{sl}([0,\omega] \times \mathbb{R}; \mathbb{R}_+)$, it follows from Lemma 3.1 that

$$\lim_{n \to +\infty} \|v_n\|_C = 0$$

which contradicts the previous inequality. Thus we have proved that for a certain $n \in \mathbb{N}$, the inequality (22.126) holds.

Now, suppose that (H_{20}) holds. By virtue of (22.118) and (22.123), clearly (22.128) is fulfilled. Hence, in view of (22.127), there is a $n_1 > n_0$ such that (22.129) holds, where r_0 is the number appearing in (H_{20}). On account of (22.129) one can easily verify that (22.130) is fulfilled as well. Hence, by virtue of (H_{20}) and (22.111), we have

$$p(t)u_n(t) + \widetilde{h}\big(t, u_n(t) - [u_n(t) - n]_+\big) \le p_0(t)u_n(t) + q_0(t)$$

for $t \in [0,\omega]$, $n > n_1$. Consequently,

$$u_n''(t) \le p_0(t)u_n(t) + q_0(t) \quad \text{for } \; t \in [0,\omega], \;\; n > n_1. \tag{22.131}$$

Now, it is clear that, for any $n > n_1$, the function u_n is a solution of the problem

$$u'' = p_0(t)u + \widetilde{q}_n(t); \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega),$$

where $\widetilde{q}_n(t) \overset{\text{def}}{=} u_n''(t) - p_0(t)u_n(t)$ pro $t \in [0,\omega]$, $n > n_1$. In view of (22.131), we get that

$$\widetilde{q}_n(t) \le q_0(t) \quad \text{for } \; t \in [0,\omega], \;\; n > n_1. \tag{22.132}$$

However, by virtue of Fredholm's third theorem, (22.132) and (H_{20}), we get the contradiction

$$0 = \int_0^\omega \widetilde{q}_n(s)u_0(s)\,\mathrm{d}s \le \int_0^\omega q_0(s)u_0(s)\,\mathrm{d}s < 0 \quad \text{for } \; n > n_1.$$

Thus we have proved that for a certain $n \in \mathbb{N}$, the inequality (22.126) holds. $\square$

Corollary 22.21. *Let $p \in \operatorname{Int} \mathcal{V}^+(\omega)$, the function $h(t,\cdot\,)$ is nonincreasing in $]0,+\infty[$, and*

$$h(t,x) \geq -h_0(t) \quad \text{for } t \in [0,\omega], \;\; x > 0,$$

where $h_0 \in L_\omega$, $h_0(t) \geq 0$ for $t \in [0,\omega]$, and $h_0 \not\equiv 0$. Let, moreover,

$$H\big(\rho_0(p)\|h_0\|_L\big) > \big(\nu^*(p)\rho(p) - 1\big)\|h_0\|_L.$$

Then the problem (22.2) has at least one solution.

Theorem 22.22. *Let*

$$f(t,x) \geq p(t)x + g(x) + q(t) \quad \text{for } t \in [0,\omega], \;\; x > 0, \tag{22.133}$$

where $q \in L_\omega$, $g \in C(\,]0,+\infty[\,;\mathbb{R}_+)$, and

$$\int_0^1 g(x)\,\mathrm{d}x = +\infty, \quad \liminf_{x\to 0+} g(x) > -\frac{1}{\omega}\int_0^\omega q(s)\,\mathrm{d}s. \tag{22.134}$$

Let, moreover, $p \in \operatorname{Int} \mathcal{V}^+(\omega)$ and either (H_{19}) or (H_{20}) hold. Then the problem (22.1) has at least one solution.

Proof. Introduce the notations

$$\begin{aligned}
Q &\overset{\text{def}}{=} \int_0^\omega q(s)\,\mathrm{d}s,\\
h(t,x) &\overset{\text{def}}{=} f(t,x) - p(t)x - g(x) - q(t) \quad \text{for } t \in [0,\omega], \;\; x > 0,\\
\chi_\varepsilon(x) &\overset{\text{def}}{=} \varepsilon + [x-\varepsilon]_+ \quad \text{for } x \in \mathbb{R}, \;\; \varepsilon > 0,\\
\widetilde{h}_\varepsilon(t,x) &\overset{\text{def}}{=} h\big(t,\chi_\varepsilon(x)\big) \quad \text{for } t \in [0,\omega], \;\; x \in \mathbb{R}, \;\; \varepsilon > 0,\\
g_\varepsilon(x) &\overset{\text{def}}{=} \Big(1 + \Big[\frac{\varepsilon}{x} - 1\Big]_+\Big) g\big(\chi_\varepsilon(x)\big) \quad \text{for } x > 0, \;\; \varepsilon > 0,\\
A &\overset{\text{def}}{=} \big\{\varepsilon > 0 : \; g(\varepsilon) \neq 0\big\}.
\end{aligned} \tag{22.135}$$

It is clear that $A \neq \varnothing$, $\widetilde{h}_\varepsilon \in K([0,\omega]\times\mathbb{R};\mathbb{R})$,

$$\widetilde{h}_\varepsilon(t,x) \geq 0 \quad \text{for } t \in [0,\omega], \;\; x \in \mathbb{R}, \tag{22.136}$$

$$p(t)x + g_\varepsilon(x) + q(t) + \widetilde{h}_\varepsilon(t,x) = f(t,x) \quad \text{for } t \in [0,\omega], \;\; x \geq \varepsilon. \tag{22.137}$$

For any $\varepsilon \in A$ consider the problem

$$u'' = p(t)u + g_\varepsilon(u) + \widetilde{h}_\varepsilon(t,u) + q(t); \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega). \tag{22.138}$$

By virtue of (22.136), (22.137), and Theorem 22.18 (with $p_1 \equiv p$, $\delta = \varepsilon$, $H(t,x) = \frac{\varepsilon g(\varepsilon)}{x}$, $q_1(t,x) = |q|$, $q(t,x) = |q(t)|$), the problem (22.138) possesses a (positive) solution u_ε. Extend the functions $\widetilde{h}_\varepsilon$ and u_ε periodically and denote them by the letters. Put

$$m_\varepsilon = \min\big\{u_\varepsilon(t) : \; t \in [0,\omega]\big\}, \quad M_\varepsilon = \max\big\{u_\varepsilon(t) : \; t \in [0,\omega]\big\}.$$

In view of (22.137), to prove theorem it is sufficient to show that there is an $\varepsilon \in A$ such that

$$m_\varepsilon \geq \varepsilon. \tag{22.139}$$

To this effort first we will estimate u_ε'. Let $t \in [0,\omega[$ be such that $u_\varepsilon'(t) \neq 0$. Then either $u_\varepsilon'(t) > 0$ or $u_\varepsilon'(t) < 0$. If $u_\varepsilon'(t) > 0$ then there is $t^* \in]t, t+\omega[$ such that $u_\varepsilon'(t^*) = 0$. Integrating (22.138) on $[t,t^*]$

and taking into account (22.136), we get

$$-u'_\varepsilon(t) = \int\limits_t^{t^*} \Big(p(s)u_\varepsilon(s) + g_\varepsilon\big(u_\varepsilon(s)\big) + \widetilde{h}_\varepsilon\big(s, u_\varepsilon(s)\big) + q(s)\Big)\,\mathrm{d}s$$

$$\geq -\int\limits_t^{t^*} \Big([p(s)]_- u_\varepsilon(s) + |q(s)|\Big)\,\mathrm{d}s.$$

Consequently,

$$u'_\varepsilon(t) \leq \big\|[p]_-\big\|_L M_\varepsilon + \|q\|_L.$$

Analogously, if $u'_\varepsilon(t) < 0$ then there is $t_* \in\,]t-\omega, t[$ such that $u'_\varepsilon(t_*) = 0$ and

$$u'_\varepsilon(t) = \int\limits_{t_*}^{t} \Big(p(s)u_\varepsilon(s) + g_\varepsilon\big(u_\varepsilon(s)\big) + \widetilde{h}_\varepsilon\big(s, u_\varepsilon(s)\big) + q(s)\Big)\,\mathrm{d}s$$

$$\geq -\int\limits_{t_*}^{t} \Big([p(s)]_- u_\varepsilon(s) + |q(s)|\Big)\,\mathrm{d}s$$

and, consequently,

$$-u'_\varepsilon(t) \leq \big\|[p]_-\big\|_L M_\varepsilon + \|q\|_L.$$

Thus, we have proved that

$$\|u'_\varepsilon\|_C \leq \big\|[p]_-\big\|_L M_\varepsilon + \|q\|_L \quad \text{for } \varepsilon \in A. \tag{22.140}$$

By virtue of (22.134) and (22.135), there exists $\delta > 0$ such that

$$g_\varepsilon(x) > \frac{\delta\big\|[p]_-\big\|_L - Q}{\omega} \quad \text{for } x \in\,]0,\delta], \ \ \varepsilon \in A. \tag{22.141}$$

Now, we will show that

$$M_\varepsilon > \delta \quad \text{for } \varepsilon \in A. \tag{22.142}$$

Suppose the contrary, let there is an $\varepsilon \in A$ such that

$$M_\varepsilon \leq \delta. \tag{22.143}$$

In view of (22.136), (22.138), (22.141), and (22.143), it is clear that

$$-\int\limits_0^\omega p(s)u_\varepsilon(s)\,\mathrm{d}s = \int\limits_0^\omega \Big(g_\varepsilon\big(u_\varepsilon(s)\big) + \widetilde{h}_\varepsilon\big(s, u_\varepsilon(s)\big) + q(s)\Big)\,\mathrm{d}s > \delta\big\|[p]_-\big\|_L$$

and

$$-\int\limits_0^\omega p(s)u_\varepsilon(s)\,\mathrm{d}s \leq \int\limits_0^\omega [p(s)]_- u_\varepsilon(s)\,\mathrm{d}s \leq M_\varepsilon\big\|[p]_-\big\|_L.$$

However, the latter two relations contradicts (22.143). Therefore, (22.142) holds. Observe also that, in view of the first condition in (22.134), clearly $A\cap\,]0,\delta[\, \neq \varnothing$.

Now, suppose that

$$m_\varepsilon < \varepsilon \quad \text{for } \varepsilon \in A\cap\,]0,\delta[\,. \tag{22.144}$$

Then, by virtue of (22.142), there is a $t_\varepsilon \in [0,\omega[$ such that

$$u_\varepsilon(t_\varepsilon) = \varepsilon, \quad u_\varepsilon(t_\varepsilon + \omega) = \varepsilon. \tag{22.145}$$

In view of (22.136), it follows from (22.138) that

$$u''_\varepsilon(t) \geq p(t)u_\varepsilon(t) - |q(t)| \quad \text{for } t \in [t_\varepsilon, t_\varepsilon + \omega]. \tag{22.146}$$

By virtue of Theorem 9.1′, we have $p \in \operatorname{Int}\mathcal{D}(\omega)$. Taking, moreover, into account Proposition 2.2, it is clear that the problem

$$\alpha''_\varepsilon = p(t)\alpha_\varepsilon - |q(t)|; \quad \alpha_\varepsilon(t_\varepsilon) = \delta, \ \ \alpha_\varepsilon(t_\varepsilon + \omega) = \delta \tag{22.147}$$

possesses a unique solution $\alpha_\varepsilon \in AC'([t_\varepsilon, t_\varepsilon+\omega])$. Taking together with (22.145)–(22.147) into account Proposition 2.5, we get that

$$u_\varepsilon(t) \le \alpha_\varepsilon(t) \quad \text{for } \ t \in [t_\varepsilon, t_\varepsilon+\omega], \ \ \varepsilon \in A\cap\,]0,\delta[\,. \tag{22.148}$$

On the other hand, by virtue of Remark 6.4 and Proposition 6.8, it is clear that there is a $c_0 > 0$ such that

$$\alpha_\varepsilon(t) \le c_0 \quad \text{for } \ t \in [t_\varepsilon, t_\varepsilon+\omega], \ \ \varepsilon \in A\cap\,]0,\delta[\,.$$

Consequently, it follows from (22.140) and (22.148) that

$$M_\varepsilon \le c_0 \quad \text{for } \ \varepsilon \in A\cap\,]0,\delta[\tag{22.149}$$

$$\|u'_\varepsilon\|_C \le c \quad \text{for } \ \varepsilon \in A\cap\,]0,\delta[\,. \tag{22.150}$$

where $c \stackrel{\text{def}}{=} c_0\big\|[p]_-\big\|_L + \|q\|_L$.

Let now $\tau_\varepsilon \in\,]t_\varepsilon, t_\varepsilon+\omega[$ be such that

$$u_\varepsilon(\tau_\varepsilon) = M_\varepsilon.$$

Then, in view of (22.136), (22.138), (22.142), (22.145), and (22.150), we get that

$$\begin{aligned} -\int\limits_{t_\varepsilon}^{t_\varepsilon+\omega} p(s)u_\varepsilon(s)\,\mathrm{d}s &= \int\limits_{t_\varepsilon}^{t_\varepsilon+\omega} \Big(g_\varepsilon\big(u_\varepsilon(s)\big) + \widetilde{h}_\varepsilon\big(s,u_\varepsilon(s)\big) + q(s)\Big)\,\mathrm{d}s \\ &\ge Q + \int\limits_{t_\varepsilon}^{t_\varepsilon+\omega} g_\varepsilon\big(u_\varepsilon(s)\big)\,\mathrm{d}s \ge Q + \frac{1}{c}\int\limits_{t_\varepsilon}^{t_\varepsilon+\omega} g_\varepsilon\big(u_\varepsilon(s)\big)|u'_\varepsilon(s)|\,\mathrm{d}s \\ &> Q + \frac{1}{c}\int\limits_{t_\varepsilon}^{\tau_\varepsilon} g_\varepsilon\big(u_\varepsilon(s)\big)u'_\varepsilon(s)\,\mathrm{d}s = Q + \frac{1}{c}\int\limits_{\varepsilon}^{M_\varepsilon} g(x)\,\mathrm{d}x \ge Q + \frac{1}{c}\int\limits_{\varepsilon}^{\delta} g(x)\,\mathrm{d}x. \end{aligned}$$

Hence, on account of (22.149), we get that

$$\int\limits_{\varepsilon}^{\delta} g(x)\,\mathrm{d}x \le c\big(Q + c_0\big\|[p]_-\big\|_L\big) \quad \text{for } \ \varepsilon \in A\cap\,]0,\delta[\,. \tag{22.151}$$

In view of the first condition in (22.134), it is clear that $A\cap\,]0,\frac{1}{n}[\, \ne \varnothing$ for $n \in \mathbb{N}$. Therefore, there is a sequence $\{\varepsilon_n\}_{n=1}^{+\infty} \subset A\cap\,]0,\delta[$ such that

$$\lim_{n\to+\infty} \varepsilon_n = 0. \tag{22.152}$$

On the other hand, it follows from (22.151) that

$$\int\limits_{\varepsilon_n}^{\delta} g(x)\,\mathrm{d}x \le c\big(Q + c_0\big\|[p]_-\big\|_L\big) \quad \text{for } \ n \in \mathbb{N}$$

which, together with (22.152), contradicts the first condition in (22.134). Thus, we have proved that (22.139) is fulfilled for some $\varepsilon \in A$. □

23. Corollaries (Continuation)

In this chapter we will apply results of Section 22 to some particular types of equation containing either the term "$+\frac{h_0(t)}{u^\lambda}$" or the term "$-\frac{h_0(t)}{u^\lambda}$", where $h_0 \in L_\omega$, $\lambda \ne 0$, and

$$h_0(t) \ge 0 \quad \text{for } \ t \in [0,\omega], \ \ h_0 \not\equiv 0. \tag{23.1}$$

Recall that under a solution we understand a **positive** function $u \in AC'([0,\omega])$ satisfying given equation.

Consider the problem

$$u'' = p(t)u + \frac{h_0(t)}{u^\lambda}; \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega). \tag{23.2}$$

Theorem 23.1. *Let $\lambda \in]-1,1]$. Then the problem (23.2) is solvable for any h_0 satisfying (23.1) if and only if the inclusion $p \in \mathcal{V}^+(\omega)$ holds.*

Proof. Let $p \in \mathcal{V}^+(\omega)$ and h_0 satisfy (23.1). If $\lambda \in]-1,0]$ then the solvability of the problem (23.2) follows from Corollary 22.13 (with $h(t,x) \overset{\text{def}}{=} h_0(t)x^{|\lambda|}$, $\mu = |\lambda|$, and $q(t) \overset{\text{def}}{=} h_0(t)$). If $\lambda \in]0,1]$ then solvability of the problem (23.2) follows from Corollary 22.17 (with $\psi(x) \overset{\text{def}}{=} x^\lambda$).

Let now $p \in L_\omega$ and the problem (23.2) is solvable for any $h_0 \in L_\omega$ satisfying (23.1). First, we will show that $p \in \mathcal{D}(\omega)$. Suppose the contrary, let $p \notin \mathcal{D}(\omega)$. Then there are $\alpha \in [0,\omega[$ and $\beta \in]\alpha, \alpha+\omega[$ such that the problem

$$u'' = p(t)u, \tag{23.3}$$

$$u(\alpha) = 0, \quad u(\beta) = 0 \tag{23.4}$$

has a nontrivial solution. Let h_0 be an ω-periodic function defined by

$$h_0(t) = \begin{cases} 0 & \text{for } \; t \in [\alpha,\beta], \\ 1 & \text{for } \; t \in]\beta, \alpha+\omega]. \end{cases}$$

Denote by u a solution of the problem of the problem (23.2) and extend it ω-periodically. It is clear that, the restriction of the function u on $[\alpha,\beta]$ is a solution of the equation (23.3) and $u(t) > 0$ for $t \in [\alpha,\beta]$. However, by virtue of Sturm's (separation) theorem, there is a $t_0 \in]\alpha,\beta[$ such that $u(t_0) = 0$ which contradicts previous inequality. Thus we have proved that $p \in \mathcal{D}(\omega)$. Hence, in view of Proposition 10.6, either $p \in \mathcal{V}^+(\omega)$ or $p \in \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$.

Now, we will show that $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$. Indeed, suppose that $p \in \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$ and $h_0 \overset{\text{def}}{=} 1$. Denote by u a solution of the problem (23.2). It is clear that, $u''(t) \geq p(t)u(t)$ for $t \in [0,\omega]$ and $u(t) > 0$ for $t \in [0,\omega]$. Hence, $p \notin \mathcal{V}^-(\omega)$ because otherwise $u(t) \leq 0$ for $t \in [0,\omega]$. Consequently, $p \in \mathcal{V}_0(\omega)$. Let u_0 be a positive solution of the problem

$$u_0'' = p(t)u_0; \quad u_0(0) = u_0(\omega), \;\; u_0'(0) = u_0'(\omega).$$

Then, by virtue of Fredholm's third theorem, we get the contradiction

$$0 < \int_0^\omega \frac{u_0(s)}{u^\lambda(s)}\,\mathrm{d}s = 0.$$

□

Theorem 23.2. *Let $\lambda > 1$, $p \in \operatorname{Int}\mathcal{V}^+(\omega)$, and h_0 satisfy (23.1). Then the problem (23.2) has at least one solution.*

Proof. The validity of the theorem follows immediately from Corollary 22.15. □

Remark 23.3. The assumption $p \in \operatorname{Int}\mathcal{V}^+(\omega)$ in Theorem 23.2 is optimal and cannot be weakened to the assumption $p \in \mathcal{V}^+(\omega)$. Indeed, in view of Proposition 14.1, there is a $p \in \mathcal{V}^+(\omega)$ such that the equation $u'' = p(t)u$ is unstable. Hence, by virtue of Proposition 7.4, the problem

$$u'' = p(t)u + \frac{1}{u^3}; \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega)$$

has no solution.

Consider the problem

$$u'' = p(t)u + \sum_{k=1}^{n} \frac{g_k(t)}{u^{\lambda_k}}; \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega), \tag{23.5}$$

where $g_k \in L_\omega$, $g_k(t) \geq 0$ for $t \in [0,\omega]$, $g_k \not\equiv 0$, $k = 1,\dots n$, and $0 < \lambda_1 < \cdots < \lambda_n$.

Theorem 23.4. *Let $p \in \mathcal{V}^+(\omega)$ and $\lambda_1\lambda_n \leq 1$. Then the problem (23.5) has at least one solution.*

Proof. It is clear that,

$$\frac{g_1(t)}{x^{\lambda_1}} \leq \sum_{k=1}^{n} \frac{g_k(t)}{x^{\lambda_k}} \leq \frac{g(t)}{x^{\lambda_n}} + g(t) \quad \text{for } t \in [0,\omega], \quad x > 0,$$

where

$$g(t) \overset{\text{def}}{=} \sum_{k=1}^{n} g_k(t).$$

Hence, the function $f(t,x) \overset{\text{def}}{=} p(t)x + \sum\limits_{k=1}^{n} \frac{g_k(t)}{x^{\lambda_k}}$ satisfies the hypothesis (H_{26}) and, consequently, by virtue of Theorem 22.16, the problem (23.5) is solvable. $\square$

Consider the problem

$$u'' = p(t)u + \frac{h_0(t)}{u^{\lambda}} + q(t)\,; \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega), \tag{23.6}$$

where $p, q, h_0 \in L_\omega$ and h_0 satisfies (23.1) and $q \not\equiv 0$.

Recall that the numbers $\rho(p)$ and Q_+, Q_- are defined by (0.12) and (0.13), respectively. Recall also that for $p \in \operatorname{Int}\mathcal{V}^+(\omega)$, the numbers $\rho_0(p)$ and $\nu^*(p)$ are defined in Definition 6.2 and by the formula (6.22), respectively.

Theorem 23.5. *Let at least one of the following items be fulfilled:*

(i) $\lambda > -1$, $p \in \mathcal{V}^+(\omega)$, *and* $(p,q) \in \mathcal{U}(\omega)$;

(ii) $\lambda \in]-1,0[$, $p \in \operatorname{Int}\mathcal{V}^+(\omega)$, *and*

$$(1-|\lambda|)\Big(\frac{|\lambda|}{\nu^*(p)\rho(p)\big\|[p]_-\big\|_L - \big\|[p]_+\big\|_L}\Big)^{\frac{|\lambda|}{1-|\lambda|}} \|h_0\|_L^{\frac{1}{1-|\lambda|}} \geq \nu^*(p)\rho(p)Q_- - Q_+;$$

(iii) $\lambda > 0$, $p \in \operatorname{Int}\mathcal{V}^+(\omega)$, *and*

$$\|h_0\|_L > (\rho_0(p)Q_-)^{\lambda}\big(\nu^*(p)\rho(p)Q_- - Q_+\big);$$

(iv) $\lambda \geq 1$, $p \in \operatorname{Int}\mathcal{V}^+(\omega)$, *and*

$$\int_0^\omega \frac{h_0(s)}{|s-a|^{\lambda}}\,\mathrm{d}s = +\infty \quad \textit{for } a \in [0,\omega].$$

Then the problem (23.6) *has at least one solution.*

Proof. If (i) holds then solvability of the problem (23.6) follows from Corollary 22.7 with $k = 21$.

Suppose that (ii) holds. One can easily verify that (H_{12}) is fulfilled with $h(t,x) \overset{\text{def}}{=} \frac{h_0(t)}{x^{\lambda}} + q(t)$ and

$$x_0 = \Big(\frac{\nu^*(p)\rho(p)\big\|[p]_-\big\|_L - \big\|[p]_+\big\|_L}{|\lambda|\|h_0\|_L}\Big)^{\frac{1}{1-|\lambda|}}.$$

Therefore, solvability of the problem (23.6) follows from Corollary 22.7 with $k = 25$.

Let now (iii) is fulfilled. Then solvability of the problem (23.6) follows from Theorem 22.20.

If (iv) is fulfilled then the solvability of the problem (23.6) follows from Corollary 22.19. $\square$

Remark 23.6. Theorem 23.5(i), together with Corollaries 16.11, 16.12, and 16.14, implies efficient conditions of solvability of the problem (23.2). Recall that in Section 6 the estimates of the numbers $\rho_0(p)$ and $\nu^*(p)$ are established. One can easily verify that Proposition 6.6, 6.8, Theorem 12.1, and Theorem 23.5(iii) and (iv) imply the following

Corollary 23.7. *Let* $[p]_-^2 \in L_\omega$, $p \not\equiv 0$, $\overline{p} \leq 0$, *and*

$$k^*(\omega)\big\|[p]_-^2\big\|_L < 1. \tag{23.7}$$

Let, moreover,

$$c_0 \stackrel{\text{def}}{=} \frac{\omega}{4}\Big(1-\sqrt{k^*(\omega)\big\|[p]_-^2\big\|_L}\Big)^{-1}, \tag{23.8}$$

$$c \stackrel{\text{def}}{=} \big(1+c_0\big\|[p]_-\big\|_L\big)\,\mathrm{e}^{\frac{\omega}{4}\,\|[p]_+\|_L}, \tag{23.9}$$

and either

$$\lambda\in\,]-1,0[\,,\quad (1-|\lambda|)\Big(\frac{|\lambda|}{c\big\|[p]_-\big\|_L-\big\|[p]_+\big\|_L}\Big)^{\frac{|\lambda|}{1-|\lambda|}}\|h_0\|_L^{\frac{1}{1-\lambda}}\geq cQ_- - Q_+,$$

or

$$\lambda>0,\quad \|h_0\|_L>(c_0Q_-)^{\lambda}(cQ_- - Q_+).$$

Then the problem (23.6) *has at least one solution.*

As a particular case of the problem (23.6) consider the problem

$$u''=-cu+\frac{1}{u^{\lambda}}+q(t);\quad u(0)=u(\omega),\ \ u'(0)=u'(\omega). \tag{23.10}$$

Theorem 23.5, Remark 6.7, and Remark 6.3 imply

Corollary 23.8. *Let* $c\in\,]0,\frac{\pi^2}{\omega^2}[$, $q\in L_\omega$, *and at least one of the following items be fulfilled:*

(i) $\lambda\geq 1$;

(ii) $\lambda\in\,]0,1[$ *and*

$$\omega>\Big(\frac{\omega^2\sqrt{c}}{4\sin(\omega\sqrt{c})}\,Q_-\Big)^{\lambda}\Big(\frac{1}{\cos\frac{\omega\sqrt{c}}{2}}\,Q_- - Q_+\Big);$$

(iii) $\lambda\in\,]-1,0[$ *and*

$$(1-|\lambda|)\Big(\frac{|\lambda|}{c}\Big)^{\frac{|\lambda|}{1-|\lambda|}}\Big(\cos\frac{\omega\sqrt{c}}{2}\Big)^{\frac{1}{1-|\lambda|}}>Q_- - Q_+\cos\frac{\omega\sqrt{c}}{2}\,.$$

Then the problem (23.10) *is solvable.*

Analogously, Theorem 23.5(i) and Corollary 16.18 imply

Corollary 23.9. *The problem*

$$u''=-\frac{\pi^2}{\omega^2}\,u+\frac{1}{u^{\lambda}}+q(t);\quad u(0)=u(\omega),\ \ u'(0)=u'(\omega)$$

is solvable provided $\lambda>-1$, $q\in L_\omega$, *and*

$$\big\|[q]_+\big\|_{L^{\frac13}}>\frac{\omega^2}{\pi}\left(\frac{\Gamma(\frac14)}{\Gamma(\frac34)}\right)^2\big\|[q]_-\big\|_L.$$

Consider the problem

$$u''=p(t)u-\frac{h_0(t)}{u^{\lambda}}+q(t);\quad u(0)=u(\omega),\ \ u'(0)=u'(\omega), \tag{23.11}$$

where h_0 satisfy (23.1), $q\in L_\omega$, and $q\not\equiv 0$.

Theorem 23.10. *Let* $\lambda>0$, $p\in\operatorname{Int}\mathcal{V}^+(\omega)$, *and*

$$Q_+\geq \nu^*(p)\rho(p)Q_-.$$

If, moreover,

$$\|h_0\|_L\leq\frac{\lambda^{\lambda}}{(\lambda+1)^{\lambda+1}}\,\big(AB^{\lambda}\big)^{-1}\big(Q_+-AQ_-\big)^{\lambda+1},$$

where $A\stackrel{\text{def}}{=}\nu^*(p)\rho(p)$ *and* $B\stackrel{\text{def}}{=}A\big\|[p]_-\big\|_L-\big\|[p]_+\big\|_L$, *then the problem* (23.11) *has at least one solution.*

Proof. One can easily verify that the hypothesis (H_{12}) is fulfilled with $h(t,x)\stackrel{\text{def}}{=}-\frac{h_0(t)}{x^{\lambda}}+q(t)$, $x_0=\big(\frac{\lambda A\|h_0\|_L}{B}\big)^{\frac{1}{1+\lambda}}$. Therefore, solvability of the problem (23.11) follows from Corollary 22.7 with $k=25$. $\square$

Analogously as above (see Remark 23.6 and Corollary 23.7) one can easily verify that Theorem 23.10 implies the following

Corollary 23.11. *Let $\lambda > 0$, $[p]_-^2 \in L_\omega$, $p \not\equiv 0$, $\overline{p} \le 0$, and* (23.7) *hold. Let, moreover,*

$$Q_+ \ge cQ_-$$

and

$$\|h_0\|_L \le \frac{\lambda^\lambda}{(\lambda+1)^{\lambda+1}} \frac{(Q_+ - cQ_-)^{\frac{1}{\lambda+1}}}{c(c\big\|[p]_-\big\|_L - \big\|[p]_+\big\|_L)^\lambda},$$

where the number c is defined by (23.8) *and* (23.9)*. Then the problem* (23.11) *is solvable.*

Consider the problem

$$\begin{gathered} u'' = p(t)u + \frac{h_0(t)}{u^\lambda} + g(t)u^\mu + q(t); \\ u(0) = u(\omega), \quad u'(0) = u'(\omega), \end{gathered} \tag{23.12}$$

where $g \in L_\omega$ and $g \not\equiv 0$.

Theorem 23.12. *Let $\lambda > 0$, $\mu \in]0,1[$, $p \in \mathcal{V}^+(\omega)$, $g(t) \ge 0$ for $t \in [0,\omega]$, and $(p,q) \in \mathcal{U}(\omega)$. Then the problem* (23.12) *is solvable.*

Proof. Theorem 23.12 follows from Corollary 22.9. □

Theorem 23.13. *Let $\lambda > 0$, $\mu \in]0,1[$, $p \in \operatorname{Int}\mathcal{V}^+(\omega)$, $g(t) \ge 0$ for $t \in [0,\omega]$, and*

$$(1-\mu)\Big(\frac{\mu}{\nu^*(p)\rho(p)\big\|[p]_-\big\|_L - \big\|[p]_+\big\|_L}\Big)^{\frac{\mu}{1-\mu}} \|g\|_L^{\frac{1}{1-\mu}} \ge \nu^*(p)\rho(p)Q_- - Q_+. \tag{23.13}$$

Then the problem (23.12) *is solvable.*

Proof. Put $f(t,x) \overset{\text{def}}{=} p(t)x + \frac{h_0(t)}{x^\lambda} + g(t)x^\mu + q(t)$. It is clear that,

$$f(t,x) \ge p(t)x + h_0(t,x) \quad \text{for } \ t \in [0,\omega], \quad x > 0,$$

where $h_0(t,x) \overset{\text{def}}{=} g(t)x^\mu + q(t)$. It is also evident that the function $h_0(t,\cdot\,)$ is nondecreasing (on $]0,+\infty[$). On the other hand, by virtue of (23.13), it follows from Theorem 23.5(iii) (with $\lambda = -\mu$ and $h_0(t) = g(t)$) that the problem

$$\beta'' = p(t)\beta + g(t)\beta^\mu + q(t); \quad \beta(0) = \beta(\omega), \quad \beta'(0) = \beta'(\omega)$$

possesses a solution β. Thus the hypothesis (H_{18}) is fulfilled. Therefore, by virtue of Corollary 22.7, the problem (23.12) is solvable. □

Theorem 23.14. *Let $\lambda > 0$, $\mu \in]0,1[$, $p \in \operatorname{Int}\mathcal{V}^+(\omega)$, and $g(t) \ge 0$ for $t \in [0,\omega]$. Let, moreover,*

$$\|h_0\|_L > \big(\rho_0(p)Q_-\big)^\lambda \big[\nu^*(p)\rho(p)Q_- - Q_+\big].$$

Then the problem (23.12) *is solvable.*

Proof. Theorem 23.14 follows from Theorem 22.20. □

Analogously as above (see Remark 23.6 and Corollary 23.7) one can easily verify that Theorems 23.13 and 23.14 imply the following

Corollary 23.15. *Let $\lambda > 0$, $\mu \in]0,1[$, $g(t) \ge 0$ for $t \in [0,\omega]$, $[p]_-^2 \in L_\omega$, $p \not\equiv 0$, $\overline{p} \le 0$, and*

$$k^*(\omega)\big\|[p]_-^2\big\|_L < 1.$$

Let, moreover, either

$$(1-\mu)\bigg(\frac{\mu}{c\big\|[p]_-\big\|_L - \big\|[p]_+\big\|_L}\bigg)^{\frac{\mu}{1-\mu}} \|g\|_L^{\frac{1}{1-\mu}} \ge cQ_- - Q_+,$$

or

$$\|h_0\|_L > (c_0Q_-)^\lambda(cQ_- - Q_+),$$

where the numbers c_0 and c are defined by (23.8) *and* (23.9)*. Then the problem* (23.12) *has at least one solution.*

Theorem 23.16. *Let $\lambda \geq 1$, $\mu \in]0,1[$, $p \in \operatorname{Int}\mathcal{V}^+(\omega)$, and*

$$\int_0^\omega \frac{h_0(s)}{|s-a|^\lambda}\,\mathrm{d}s = +\infty \quad \text{for } a \in [0,\omega[\,.$$

Then the problem (23.12) *has at least one solution* (*for any $g, q \in L_\omega$*).

Proof. Theorem 23.16 follows from Corollary 22.19 (with $H(t,x) \stackrel{\text{def}}{=} \frac{h_0(t)}{x^\lambda}$, $q(t,x) \stackrel{\text{def}}{=} |g(t)|x^\mu + |q(t)|$, $\delta = 1$). □

24. Resonance Like Case (Continuation)

In this chapter we consider the problem

$$u'' = p_0(t)u + h_0(t,u) + q(t); \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega) \tag{24.1}$$

where $h_0 \in K_{loc}([0,\omega]\times]0,+\infty[\,;\mathbb{R}_+)$ and $q \in L_\omega$. In spite of the assertions stated in Section 22 now we will suppose that

$$p_0 \in \mathcal{V}_0(\omega)$$

and (as above) denote by u_0 a positive solution of the problem

$$u_0'' = p_0(t)u_0; \quad u_0(0) = u_0(\omega), \ \ u_0'(0) = u_0'(\omega).$$

Below we will show that Theorems 22.18, 22.20, and 22.22 imply also the solvability of (24.1).

Before the formulation of main results introduce the hypothesis

$$\begin{cases} \text{for any } \varepsilon > 0 \text{ there are } r > 0 \text{ and } q_r \in L_\omega \text{ such that} \\ \qquad q_r(t) \geq 0 \quad \text{for } t \in [0,\omega], \quad \|q_r\|_L < \varepsilon, \\ \qquad h_0(t,x) \leq q_r(t) \quad \text{for } t \in [0,\omega], \ \ x > r. \end{cases} \tag{H_{30}}$$

It is clear that, each hypotheses (H_{31}) and (H_{32}) below implies (H_{30}), where

$$\begin{cases} h_0(t,\,\cdot\,) \text{ is nonincreasing,} \\ \lim\limits_{x\to+\infty} \int_0^\omega h_0(s,x)\,\mathrm{d}s = 0 \end{cases} \tag{H_{31}}$$

and

$$\begin{cases} h_0(t,x) \leq h_0(t)g_0(x) \quad \text{for } t \in [0,\omega], \ \ x > r_0, \ \ r_0 > 0, \\ h_0 \in L_\omega, \quad g_0 \in C(\mathbb{R}_+;\mathbb{R}_+), \ \text{ and } \lim\limits_{x\to+\infty} g_0(x) = 0. \end{cases} \tag{H_{32}}$$

Further, we will need the following hypotheses

$$\begin{cases} h_0(t,x) \geq h(t,x) \quad \text{for } t \in [0,\omega], \ \ x > 0, \\ h \in K_{loc}([0,\omega]\times]0,+\infty[\,;\mathbb{R}_+), \quad h(t,\,\cdot\,) \ \text{ is nonincreasing,} \\ \int_0^\omega h(s,c|s-a|)\,\mathrm{d}s = +\infty \quad \text{for } c > 0, \ a \in [0,\omega[\end{cases} \tag{H_{33}}$$

and

$$\begin{cases} h_0(t,x) \geq h(t,x) \quad \text{for } t \in [0,\omega], \ \ x > 0, \\ h \in K_{loc}([0,\omega]\times]0,+\infty[\,;\mathbb{R}_+), \quad h(t,\,\cdot\,) \ \text{ is nonincreasing,} \\ H\big(\rho_0(p_0)Q_-\big) > \nu^*(p_0)\rho(p_0)Q_- - Q_+, \end{cases} \tag{H_{34}}$$

where the numbers $\rho(p_0)$, Q_+, Q_+ and the function H are given by (0.12), (0.13), and (0.17), respectively, while the numbers $\nu^*(p_0)$ and $\rho_0(p_0)$ are defined by (6.22) and Definition 6.2.

At last

$$\begin{cases} h_0(t,x) \ge g(x) \quad \text{for } t \in [0,\omega], \ x > 0, \\ g \in C(\,]0,+\infty[\,;\mathbb{R}_+), \quad \displaystyle\int_0^1 g(x)\,\mathrm{d}x = +\infty, \\ \displaystyle\liminf_{x\to 0+} g(x) > -\frac{1}{\omega}\int_0^\omega q(s)\,\mathrm{d}s. \end{cases} \tag{H_{35}}$$

Theorem 24.1. *Let $p_0 \in \mathcal{V}_0(\omega)$, (H_{30}) hold, $k \in \{33, 34, 35\}$, and hypothesis (H_k) be fulfilled. Let, moreover,*

$$\int_0^\omega q(s)u_0(s)\,\mathrm{d}s < 0. \tag{24.2}$$

Then the problem (24.1) *is solvable.*

Proof. Put $f(t,x) \stackrel{\text{def}}{=} p_0(t)x + h_0(t,x) + q(t)$ for $t \in [0,\omega]$, $x > 0$. First we will show that hypothesis (H_{30}) and condition (24.2) imply that the function f satisfies hypothesis (H_{20}). Indeed, by virtue of (H_{30}), there are $r_0 > 0$ and $q_{r_0} \in L_\omega$ such that

$$h_0(t,x) \le q_{r_0}(t) \quad \text{for } t \in [0,\omega], \ x > r_0$$

and

$$\int_0^\omega q_{r_0}(s)\,\mathrm{d}s < -\frac{1}{\|u_0\|_C}\int_0^\omega q(s)u_0(s)\,\mathrm{d}s.$$

Now it is clear that (H_{20}) holds with $q_0(t) \stackrel{\text{def}}{=} q_{r_0}(t) + q(t)$.

Suppose now that $k \in \{33, 35\}$ and (H_k) is fulfilled. By virtue of Proposition 10.11, there is an $\varepsilon > 0$ such that the function $p \stackrel{\text{def}}{=} p_0 - \varepsilon$ satisfies inclusion $p \in \operatorname{Int}\mathcal{V}^+(\omega)$. Thus, if $k = 33$ then (H_{28}) holds (with $H(t,x) \stackrel{\text{def}}{=} h(t,x)$, $q(t,x) \stackrel{\text{def}}{=} -|q(t)|$, and $p_1(t) \stackrel{\text{def}}{=} p(t)$) while if $k = 35$ then (22.133) and (22.134) are fulfilled. Thus solvability of the problem (24.1) follows from Theorem 22.18 if $k = 33$ and from Theorem 22.22 if $k = 35$.

Let now (H_{34}) hold. By virtue of Proposition 10.11 and Proposition 6.14, there is an $\varepsilon > 0$ such that the function $p \stackrel{\text{def}}{=} p_0 - \varepsilon$ satisfies inclusion $p \in \operatorname{Int}\mathcal{V}^+(\omega)$ and

$$H(\rho_0(p)Q_-) > \nu^*(p)\rho(p)Q_- - Q_+.$$

Hence, the hypothesis (H_{29}) is fulfilled. Solvability of the problem (24.1) now follows from Theorem 22.20. □

Condition (24.2) is, in some cases, necessary for the solvability of the problem (24.1). More precisely,

Proposition 24.2. *Let $p_0 \in \mathcal{V}_0(\omega)$ and the problem* (24.1) *is solvable. Let, moreover, either*

$$h_0(t,x) \ge h(t,x) \quad \textit{for } t \in [0,\omega], \ x > 0,$$

where $h(t,\cdot\,)$ is nonincreasing and

$$\operatorname{mes}\big\{t \in [0,\omega] : \ h(t,x) > 0\big\} > 0 \quad \textit{for } x > 0,$$

or

$$h_0(t,x) \ge h_0(t)g(x) \quad \textit{for } t \in [0,\omega], \ x > 0,$$

where $h_0 \in L_\omega$, $h_0(t) \ge 0$ for $t \in [0,\omega]$, $h_0 \not\equiv 0$, and $g \in C(\,]0,+\infty[\,;\,]0,+\infty[\,)$. Then the inequality (24.2) *holds.*

Proof. Let u be a solution of the problem (24.1). Then, by virtue of Fredholm's third theorem, we have

$$\int_0^\omega q(s)u_0(s)\,\mathrm{d}s = -\int_0^\omega h_0(s,u(s))u_0(s)\,\mathrm{d}s. \tag{24.3}$$

One can easily verify that each of the conditions of the proposition implies that there is a $\varphi \in L_\omega$ such that $\varphi(t) \geq 0$ for $t \in [0,\omega]$, $\varphi \not\equiv 0$, and

$$h_0(t,u(t)) \geq \varphi(t) \quad \text{for } \ t \in [0,\omega].$$

However, the latter inequality, together with (24.3), yields (24.2). □

Corollary 24.3. *Let* $p_0 \in \mathcal{V}_0(\omega)$, $h_0(t,\,\cdot\,)$ *is nonincreasing,*

$$\lim_{x\to+\infty} \int_0^\omega h_0(s,x)\,\mathrm{d}s = 0,$$

and

$$\int_0^\omega h_0(s,c|s-a|)\,\mathrm{d}s = +\infty \quad \textit{for } \ c>0, \ a \in [0,\omega[\,.$$

Then the problem (24.1) *is solvable provided* (24.2) *holds. If, moreover,*

$$\operatorname{mes}\big\{t \in [0,\omega]: \ h_0(t,x) > 0\big\} > 0 \quad \textit{for } \ x>0 \tag{24.4}$$

then the condition (24.2) *is necessary for solvability of the problem* (24.1).

Proof. It is clear that (H_{31}) holds as well as (H_{33}) is fulfilled (with $h(t,x) \overset{\text{def}}{=} h_0(t,x)$). However, hypothesis (H_{31}) implies (H_{30}) and, consequently, solvability of the problem (24.1) follows from Theorem 24.1. Second part of Corollary 24.3 follows from Proposition 24.2. □

Remark 24.4. Condition (24.4) is essential for the second part of Corollary 24.3 and cannot be omitted. Indeed, consider the problem

$$u'' = p_0(t)u + \frac{[1-u]_+}{u^3} + q(t); \quad u(0)=u(\omega), \ \ u'(0)=u'(\omega), \tag{24.5}$$

where $p_0 \in \mathcal{V}_0(\omega)$ and

$$\int_0^\omega q(s)u_0(s)\,\mathrm{d}s = 0.$$

By virtue of Fredholm's third theorem, the problem

$$u'' = p_0(t)u + q(t); \quad u(0)=u(\omega), \ \ u'(0)=u'(\omega), \tag{24.6}$$

possesses at least one solution u_1. Choose $c>0$ such that $u_1(t)+cu_0(t) > 1$ for $t \in [0,\omega]$ and put $u(t) = u_1(t)+cu_0(t)$ for $t \in [0,\omega]$. It is clear that the function u is a solution of the problem (24.5) and (24.2) is violated. On the other hand, the function $h_0(t,x) \overset{\text{def}}{=} \frac{[1-x]_+}{x^3}$ satisfies all the conditions of Corollary 24.3 except of the condition (24.4).

Corollary 24.5. *Let* $p_0 \in \mathcal{V}_0(\omega)$, $h_0(t,\,\cdot\,)$ *is nonincreasing,*

$$\lim_{x\to+\infty} \int_0^\omega h_0(s,x)\,\mathrm{d}s = 0,$$

and

$$H_0\Big(\frac{\omega}{4}\,\mathrm{e}^{\omega\sqrt{\overline{p_0}}}\,Q_-\Big) > \mathrm{e}^{\frac{\omega}{2}\sqrt{\overline{p_0}}}\,\rho(p_0)Q_- - Q_+,$$

where

$$H_0(x) \overset{\text{def}}{=} \int_0^\omega h_0(s,x)\,\mathrm{d}s \quad \textit{for } \ x>0.$$

Then the problem (24.1) *is solvable provided* (24.2) *holds. If, moreover,* (24.4) *is fulfilled then the condition* (24.2) *is necessary for solvability of the problem* (24.1).

Proof. Corollary follows from Theorem 24.1 (with $k=34$), Proposition 24.2, and Proposition 6.13. □

Remark 24.6. Condition (24.4) is essential for the second part of Corollary 24.5 and cannot be omitted. Indeed, consider the problem

$$u'' = p_0(t)u + \frac{[1-u]_+}{u^3} + \varepsilon q(t); \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega), \tag{24.7}$$

where $\varepsilon \neq 0$, $q \in L_\omega$, and

$$\int_0^\omega q(s)u_0(s)\,\mathrm{d}s = 0.$$

By the same arguments as in Remark 24.4 one can show that for any $\varepsilon \neq 0$, the problem (24.7) is solvable. Clearly, there is a $\varepsilon > 0$ such that

$$\omega\Big[1-\varepsilon\,\frac{\omega}{4}\,\mathrm{e}^{\omega\sqrt{p_0}}\,Q_-\Big]_+ > \varepsilon^4\Big(\frac{\omega}{4}\,\mathrm{e}^{\omega\sqrt{p_0}}\,Q_-\Big)^3\Big(\mathrm{e}^{\frac{\omega}{2}\sqrt{p_0}}\,\rho(p_0)Q_- - Q_+\Big).$$

Consequently, all the conditions of Corollary 24.5 hold except of (24.4).

Corollary 24.7. *Let $p_0 \in \mathcal{V}_0(\omega)$, $h_0 \in L_\omega$, $g, g_0 \in C(]0,+\infty[\mathbb{R}_+)$, $r_0 > 0$,*

$$\lim_{x\to+\infty} g_0(x) = 0, \quad \liminf_{x\to0+} g(x) > -\frac{1}{\omega}\int_0^\omega q(s)\,\mathrm{d}s, \quad \int_0^1 g(x)\,\mathrm{d}x = +\infty,$$

and

$$\begin{aligned} &h_0(t,x) \leq h_0(t)g_0(x) \quad \textit{for } t \in [0,\omega], \;\; x > r_0, \\ &h_0(t,x) \geq g(x) \quad \textit{for } t \in [0,\omega], \;\; x > 0. \end{aligned}$$

Then the problem (24.1) *is solvable provided* (24.2) *holds. If, moreover,*

$$g(x) > 0 \quad \textit{for } x > 0 \tag{24.8}$$

then the condition (24.2) *is necessary for solvability of the problem* (24.1)*.*

Proof. Clearly, (H_{32}) and (H_{35}) are fulfilled. However, (H_{32}) implies (H_{30}) and, consequently, the solvability of the problem (24.1) follows from Theorem 24.1. As for the necessity of the condition (24.2), it follows from Proposition 24.2. □

Remark 24.8. Example constructed in Remark 24.4 shows that the assumption (24.8) is essential for the second part of Corollary 24.7 and cannot be omitted.

As an example consider the problem

$$u'' = p_0(t)u + \frac{h_0(t)}{u^\lambda} + q(t); \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega), \tag{24.9}$$

where $h_0, q \in L_\omega$, $\lambda > 0$, and

$$h_0(t) \geq 0 \quad \text{for } t \in [0,\omega], \quad h_0 \not\equiv 0. \tag{24.10}$$

Corollary 24.3 implies

Proposition 24.9. *Let $p_0 \in \mathcal{V}_0(\omega)$,* (24.10) *hold, $\lambda \geq 1$, and*

$$\int_0^\omega \frac{h_0(s)}{|s-a|^\lambda}\,\mathrm{d}s = +\infty \quad \textit{for } a \in [0,\omega[\,.$$

Then the problem (24.9) *is solvable if and only if* (24.2) *holds.*

Observe, that Proposition 24.9 does not cover the case when either $\lambda \in]0,1[$ or $\mathrm{mes}\{t \in [0,\omega] : h_0(t) = 0\} > 0$. However, Corollary 24.5 implies

Proposition 24.10. *Let $p_0 \in \mathcal{V}_0(\omega)$,* (24.10) *hold, $\lambda > 0$, and*

$$\|h_0\|_L > \Big(\frac{\omega}{4}\,\mathrm{e}^{\omega\sqrt{p_0}}\,Q_-\Big)^\lambda\Big(\mathrm{e}^{\frac{\omega}{2}\sqrt{p_0}}\,\rho(p_0)Q_- - Q_+\Big).$$

Then the problem (24.9) *is solvable if and only if* (24.2) *holds.*

As another example consider the problem

$$\begin{gathered} u'' = p_0(t)u + h_0(t)g(u) + q(t); \\ u(0) = u(\omega), \quad u'(0) = u'(\omega), \end{gathered} \tag{24.11}$$

where $h_0 \in L_\omega$ satisfies (24.10) and $g \in C(\,]0,+\infty[\,;\,]0,+\infty[\,)$.

It follows from Corollary 24.7 that

Proposition 24.11. *Let $p_0 \in \mathcal{V}_0(\omega)$ and there exist $\delta_0 > 0$ such that*

$$h_0(t) \geq \delta_0 \quad \text{for } t \in [0,\omega]. \tag{24.12}$$

Let, moreover, $g \in C(\,]0,+\infty[\,;\,]0,+\infty[\,)$ and

$$\lim_{x\to 0+} g(x) = +\infty, \quad \lim_{x\to+\infty} g(x) = 0, \quad \int_0^1 g(x)\,\mathrm{d}x = +\infty. \tag{24.13}$$

Then the problem (24.11) *is solvable if and only if* (24.2) *holds.*

The next assertion follows from Corollary 24.5 and covers also the case when (24.12) is violated.

Proposition 24.12. *Let $p_0 \in \mathcal{V}_0(\omega)$,* (24.10) *hold, $g \in C(\,]0,+\infty[\,;\,]0,+\infty[\,)$ be a nonincreasing function, and*

$$\lim_{x\to+\infty} g(x) = 0.$$

Let, moreover,

$$g\Big(\frac{\omega}{4}\,\mathrm{e}^{\omega\sqrt{p_0}}\,Q_-\Big)\|h_0\|_L > \mathrm{e}^{\frac{\omega}{2}\sqrt{p_0}}\,\rho(p_0)Q_- - Q_+.$$

Then the problem (24.11) *is solvable if and only if* (24.2) *holds.*

As it was mentioned in introduction, studies of the phase singular periodic problem was initiated in [16] by Lazer and Solimini. Theorem 3.12 of [16] concerns the solvability of the problem

$$u'' = g(u) + q(t); \quad u(0) = u(\omega), \quad u'(0) = u'(\omega) \tag{24.14}$$

and reads as follows.

Theorem 24.13 (Lazer, Solimini)**.** *Let $g \in C(\,]0,+\infty[\,;\,]0,+\infty[\,)$ and* (24.13) *hold. Then the problem* (24.14) *is solvable if and only if $\int_0^\omega q(s)\,\mathrm{d}s < 0$.*

Theorem 24.13 now follows from Proposition 24.11. In the same paper [16] it is shown that the assumption $\int_0^1 g(x)\,\mathrm{d}x = +\infty$ in Theorem 24.13 is essential and cannot be omitted. More precisely, Theorem 4.1 of [16] states that

Theorem 24.14. *For given $g \in C(\,]0,+\infty[\,;\,]0,+\infty[\,)$ satisfying*

$$\lim_{x\to 0+} g(x) = +\infty, \quad \lim_{x\to+\infty} g(x) = 0, \quad \int_0^1 g(x)\,\mathrm{d}x < +\infty \tag{24.15}$$

there exists $M_0 > 0$ such that for any $M > M_0$, there is a $q \in L_\omega$ such that

$$q(t) \leq 0 \quad \text{for } t \in [0,\omega], \quad \int_0^\omega |q(s)|\,\mathrm{d}s = M$$

and the problem (24.14) *has no solution.*

In other words, if (24.15) holds then the problem (24.14) has no solution for a certain q "large enough". However, if the function q is "small enough" then the problem (24.14) may have a solution. More precisely, it follows from Proposition 24.12 that

Proposition 24.15. *Let* $g \in C(\,]0,+\infty[\,;\,]0,+\infty[\,)$ *be nonincreasing and*

$$\lim_{x \to +\infty} g(x) = 0.$$

Let, moreover,

$$\omega\, g\Big(\frac{\omega}{4}\, Q_-\Big) > Q_- - Q_+.$$

Then the problem (24.14) *is solvable if and only if*

$$\int_0^\omega q(s)\,\mathrm{d}s < 0.$$

To be more specific, consider the particular case of the problem (24.14)

$$u'' = \frac{1}{u^\lambda} + q(t); \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega). \tag{24.16}$$

By virtue of above-mentioned results by Lazer and Solimini, if $\lambda \geq 1$ then the problem (24.16) is solvable if and only if $\int_0^\omega q(s)\,\mathrm{d}s < 0$. Moreover, if $\lambda \in\,]0,1[$ then, in general, the condition $\int_0^\omega q(s)\,\mathrm{d}s < 0$ does not guarantee solvability of (24.16). However, by virtue of Proposition 24.15, if $\lambda \in\,]0,1[$ and

$$\Big(\frac{\omega}{4}\, Q_-\Big)^\lambda (Q_- - Q_+) < \omega$$

then the problem (24.16) is solvable if and only if $\int_0^\omega q(s)\,\mathrm{d}s < 0$.

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