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RECENT DEVELOPMENT OF TIME SCALES AND RELATED TOPICS ON DYNAMIC EQUATIONS

Abstract. Recently, Wang and Agarwal have introduced a series of new concepts of time scales on which some well-defined functions can be introduced and studied. Based on these new results, various types of solutions for dynamic equations are studied.

რეზიუმე. ბოლო ხანებში ვანგმა და აგარვალმა შემოგვთავაზეს რიგი ახალი ცნება დროითი სკალისა, რომელზეც შეიძლება შემოღებულ იქნას ზოგიერთი კორექტულად განსაზღვრული ფუნქცია. ამ ახალ შედეგებზე დაყრდნობით შესწავლილია დინამიური განტოლებების სხვადასხვა ტიპის ამონახსნები.

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1. Periodic Time Scales and Changing-Periodic Time Scales

As everyone knows, periodic time scales have some very nice properties so that we can study various problems of differential equations on this type of time scales. Traditionally, we define it as follows:

Definition 1.1 ([1]). A time scale $\mathbb{T}$ is called a periodic time scale if

$$\Pi := \big\{\tau \in \mathbb{R} :\ \mathbb{T}^{\tau} \cup \mathbb{T}^{-\tau} \subset \mathbb{T}\big\} \neq \{0\}, \tag{1.1}$$

where $\mathbb{T}^{\tau} = \{t + \tau : t \in \mathbb{T}\}$.

This type of time scales was first introduced by Kaufmann and Raffoul [1], and was widely adopted by many researchers to study periodic and almost periodic problems of differential equations on time scales.

Recently, during 2014–2015, Wang and Agarwal introduced a series of new concepts of discontinuous functions and dynamic equations on time scales under Definition 1.1, for example, uniformly piecewise almost periodic functions, weighted piecewise pseudo almost automorphic functions and exponential dichotomies for discontinuous dynamic equations, etc., and obtained some related properties, then applied them to study discontinuous dynamic systems on time scales. For more details, one may consult our recent publications [2–6]. In the paper [5], through introducing the concept of relatively dense set on time scales, we introduced an accurate concept of almost periodic functions on periodic time scales under Definition 1.1.

However, in the research, we find that Definition 1.1 is limited. In fact, if a time scale satisfies (1.1), $\mathbb{T}$ must fulfill $\sup \mathbb{T} = +\infty$ and $\inf \mathbb{T} = -\infty$. In other words, if we consider Definition 1.1 from a rigorous mathematical angle, then the following time scale is not periodic:

$$\mathbb{T} = \bigcup_{k=1}^{+\infty} [2k, 2k+1], \tag{1.2}$$

only because $\inf \mathbb{T} = 2$. Hence, the new concept of periodic time scales is introduced by Wang, Agarwal and O'Regan. We add a "direction" to a time scale translation. See the following new concept:

Definition 1.2. We say $\mathbb{T}$ is a periodic time scale if

$$\Pi_2 := \big\{\tau \in \mathbb{R} :\ \mathbb{T}^{\tau} \subseteq \mathbb{T}\big\} \neq \{0\}. \tag{1.3}$$

Furthermore, we can describe it in detail as follows:

(a) if for any $p > 0$ there exists a number $P > p$ such that $P \in \Pi_2$, we say $\mathbb{T}$ is a positive-direction periodic time scale;

(b) if for any $q < 0$ there exists a number $Q < q$ such that $Q \in \Pi_2$, we say $\mathbb{T}$ is a negative-direction periodic time scale.

(c) if $\pm\tau \in \Pi_2$, we say $\mathbb{T}$ is a bi-direction periodic time scale;

(d) we say $\mathbb{T}$ is an oriented-direction periodic time scale if $\mathbb{T}$ is a positive-direction periodic time scale or a negative-direction periodic time scale.

The Definition 1.2 is a new concept of periodic time scales which plays an important role in our recent works. Under this definition, we can easily observe that Definition 1.1 is just a particular case in Definition 1.2 because Definition 1.1 is just a bi-direction periodic time scale in Definition 1.2. Hence, under Definition 1.2, (1.2) turns into an oriented-direction (or a positive-direction) periodic time scale. We emphasize the importance of Definition 1.2 not only because Definition 1.1 is not general but also does it mainly contribute to our new concepts and results, for example, changing-periodic time scales, Decomposition Theorem of Time Scales, and Periodic Coverage Theorem of Time Scales, etc.

Below we will discuss a class of time scales called "changing-periodic time scales" that is a new idea initiated by Wang and Agarwal (see [7]). Note that the following "periodic time scales" are under the sense of Definition 1.2, i.e., they are "oriented-direction periodic time scales".

Definition 1.3. Let $\mathbb{T}$ be an infinite time scale. We say $\mathbb{T}$ is a changing-periodic or a piecewise-periodic time scale if the following conditions are fulfilled:

(a) $\mathbb{T} = \big(\bigcup_{i=1}^{\infty} \mathbb{T}_i\big) \cup \mathbb{T}_r$ and $\{\mathbb{T}_i\}_{i\in\mathbb{Z}^+}$ is a well connected timescale sequence, where $\mathbb{T}_r = \bigcup_{i=1}^{k} [\alpha_i, \beta_i]$ and k is some finite number, and $[\alpha_i, \beta_i]$ are closed intervals for $i = 1, 2, \dots, k$ or $\mathbb{T}_r = \varnothing$;

(b) S_i is a nonempty subset of $\mathbb{R}$ with $0 \notin S_i$ for each $i \in \mathbb{Z}^+$ and $\Pi = \big(\bigcup_{i=1}^{\infty} S_i\big) \cup R_0$, where $R_0 = \{0\}$ or $R_0 = \varnothing$;

(c) for all $t \in \mathbb{T}_i$ and all $\omega \in S_i$, we have $t + \omega \in \mathbb{T}_i$, i.e., $\mathbb{T}_i$ is an ω-periodic time scale;

(d) for $i \neq j$, for all $t \in \mathbb{T}_i \setminus \{t_{ij}^k\}$ and all $\omega \in S_j$, we have $t + \omega \notin \mathbb{T}$, where $\{t_{ij}^k\}$ is the connected points set of the timescale sequence $\{\mathbb{T}_i\}_{i\in\mathbb{Z}^+}$;

(e) $R_0 = \{0\}$ if and only if $\mathbb{T}_r$ is a zero-periodic time scale and $R_0 = \varnothing$ if and only if $\mathbb{T}_r = \varnothing$.

The set Π is called a changing-periods set of $\mathbb{T}$; $\mathbb{T}_i$ is called the periodic sub-timescale of $\mathbb{T}$ and S_i is called the periods subset of $\mathbb{T}$ or the periods set of $\mathbb{T}_i$; $\mathbb{T}_r$ is called the remain timescale of $\mathbb{T}$ and R_0 the remain periods set of $\mathbb{T}$.

Through Definition 1.3, we can obtain a nice result which builds a bridge between periodic time scales and arbitrary time scales with the bounded graininess function $\mu(t)$.

Theorem 1.1. *If $\mathbb{T}$ is an infinite time scale and the graininess function $\mu : \mathbb{T} \to \mathbb{R}^+$ is bounded, then $\mathbb{T}$ is a changing-periodic time scale.*

From Theorem 1.1, we can obtain the following two theorems.

Theorem 1.2 (Decomposition Theorem of Time Scales)**.** *Let $\mathbb{T}$ be an infinite time scale and the graininess function $\mu : \mathbb{T} \to \mathbb{R}^+$ be bounded, then $\mathbb{T}$ is a changing-periodic time scale, i.e., there exists a countable periodic decomposition such that $\mathbb{T} = \big(\bigcup_{i=1}^{\infty} \mathbb{T}_i\big) \cup \mathbb{T}_r$ and $\mathbb{T}_i$ is ω-periodic sub-timescale, $\omega \in S_i$, $i \in \mathbb{Z}^+$, where $\mathbb{T}_i$, S_i, $\mathbb{T}_r$ satisfy the conditions of Definition* 1.3.

Theorem 1.3 (Periodic Coverage Theorem of Time Scales)**.** *Let $\mathbb{T}$ be an infinite time scale and the graininess function $\mu : \mathbb{T} \to \mathbb{R}^+$ be bounded, then $\mathbb{T}$ can be covered by countable periodic time scales.*

Note that "periodic time scales" in Theorems 1.2 and 1.3 imply "oriented-direction periodic time scales", i.e., the concept of periodic time scales we adopt is Definition 1.2.

It is well known that periodic time scales have some very nice properties, for example, for any $\tau \in \Pi_2$, we have $t + \tau \in \mathbb{T}$, it reflects that periodic time scales have a very good closedness for addition operation, which will contribute a lot to functions theory on time scales because we know that periodic functions, almost periodic functions and almost automorphic functions are described by their translations. Hence, it is very necessary to introduce the concept of changing-periodic time scales because we can introduce some well-defined functions on an arbitrary time scale with the bounded graininess function μ. We should first introduce the proper definitions of these functions on time scales, then some related problems of differential equations on time scales can be proposed and solved. For example, under changing-periodic time scales, we can introduce a new concept of local-almost periodic functions.

Definition 1.4. Let $\mathbb{T}$ be a changing-periodic time scale, i.e., $\mathbb{T}$ satisfies Definition 1.3. A function $f \in C(\mathbb{T} \times D, \mathbb{E}^n)$ is called a local-almost periodic function in $t \in \mathbb{T}$ uniformly for $x \in D$ if the ε-translation numbers set of f

$$E\{\varepsilon, f, S\} = \Big\{\widetilde{\tau} \in S_{\tau_t} : \ \big|f(t+\widetilde{\tau}, x) - f(t,x)\big| < \varepsilon \text{ for all } (t,x) \in \mathbb{T} \times S\Big\}$$

is a relatively dense set for all $\varepsilon > 0$ and for each compact subset S of D; that is, for any given $\varepsilon > 0$ and each compact subset S of D, there exists a constant $l(\varepsilon, S) > 0$ such that each interval of length $l(\varepsilon, S)$ contains a $\widetilde{\tau}(\varepsilon, S) \in E\{\varepsilon, f, S\}$ such that

$$\big|f(t+\widetilde{\tau}, x) - f(t,x)\big| < \varepsilon \quad \text{for all} \quad (t,x) \in \mathbb{T} \times S;$$

here, $\widetilde{\tau}$ is called the ε-local translation number of f and $l(\varepsilon, S)$ is called the local inclusion length of $E\{\varepsilon, f, S\}$.

Under Definition 1.4, we consider the following linear dynamic equation on a changing-periodic time scale $\mathbb{T}$

$$x^{\Delta}(t) = A(t)x(t) + f(t) \tag{1.4}$$

and its associated homogeneous equation

$$x^{\Delta}(t) = A(t)x(t), \tag{1.5}$$

where $A(t)$ is a local-almost periodic matrix function and $f(t)$ is a local-almost periodic vector function. Further, we assume that $f(t)$ and $A(t)$ are synchronously local-almost periodic functions.

Then, we can obtain a theorem to guarantee that (1.4) has a local-almost periodic solution on an arbitrary time scale with the bounded graininess function μ.

Theorem 1.4. *Let $\mathbb{T}$ be a changing-periodic time scale and τ_t be an index function. If* (1.5) *admits an exponential dichotomy on the local part $\mathbb{T}_{\tau_t}$ and $\mathbb{T}_{\tau_t}$ is a bi-direction periodic time scale for all $t \in \mathbb{T}$, then* (1.4) *has the following unique local-almost periodic solution on $\mathbb{T}_{\tau_t}$*

$$x(t) = \int_{-\infty}^{t} X(t)P_{\tau_t}X^{-1}(\sigma_{\tau_t}(s))\Delta_{\tau_t}s - \int_{t}^{+\infty} X(t)(I - P_{\tau_t})X^{-1}(\sigma_{\tau_t}(s))f(s)\Delta_{\tau_t}s, \tag{1.6}$$

where $X(t)$ is the fundamental solution matrix of (1.5)*, $P_{\tau_t}, I - P_{\tau_t}$ are two projections of exponential dichotomy on $\mathbb{T}_{\tau_t}$, σ_{τ_t} is the forward jump operator on the periodic sub-timescale $\mathbb{T}_{\tau_t}$, Δ_{τ_t} is the Δ-integral on the periodic sub-timescale $\mathbb{T}_{\tau_t}$.*

Similarly, the concept of local-almost automorphic functions can also be introduced on changing-periodic time scales, one may consult the paper [7] for more details. Hence, Theorems 1.2 and 1.3 initiate a new idea to solve the closedness for addition operation on an arbitrary time scale with the bounded graininess function μ, which will open an effective avenue to investigate periodic, almost periodic and almost automorphic problems of differential equations on arbitrary time scales.

2. Almost Periodic Time Scales

Let

$$\Pi_1 := \big\{\tau \in \mathbb{R} : \ \mathbb{T} \cap \mathbb{T}^{\tau} \neq \varnothing\big\} \neq \{0\},$$

consider a class of time scales satisfying the following definition:

Definition 2.1. Let $\mathbb{T}$ be an oriented-direction intersection time scale. We say $\mathbb{T}$ is an almost periodic time scale if for any given $\varepsilon > 0$ there exists a constant $l(\varepsilon) > 0$ such that each interval of length $l(\varepsilon)$ contains $\tau(\varepsilon) \in \Pi_1$ such that

$$d(\mathbb{T}, \mathbb{T}^{\tau}) < \varepsilon,$$

i.e., for any $\varepsilon > 0$ the set

$$E\{\mathbb{T}, \varepsilon\} = \{\tau \in \Pi_1 : \ d(\mathbb{T}^\tau, \mathbb{T}) < \varepsilon\}$$

is relatively dense in Π_1. Here τ is called the ε-translation number of $\mathbb{T}$ and $l(\varepsilon)$ is called the inclusion length of $E\{\mathbb{T}, \varepsilon\}$, $E\{\mathbb{T}, \varepsilon\}$ is called the ε-translation set of $\mathbb{T}$, and for simplicity, we use the notation $E\{\mathbb{T}, \varepsilon\} := \Pi_\varepsilon$.

The graininess function $\mu(t)$ of this time scale have a very nice property:

Theorem 2.1. *If $\mathbb{T}$ is an almost periodic time scale, then for any $\varepsilon > 0$ there exists a constant $l(\varepsilon) > 0$ such that each interval of length $l(\varepsilon)$ contains $\tau(\varepsilon) \in E\{\varepsilon, \mu\}$ such that*

$$|\mu(t + \tau) - \mu(t)| < \varepsilon \ \text{ for all } \ t \in \mathbb{T} \cap \mathbb{T}^{-\tau}. \tag{2.1}$$

Note that if $\mathbb{T}$ is an oriented-direction periodic time scale, then from Definition 2.1, we have $\sigma(t + \tau) = \sigma(t) + \tau$. Hence, we can see that Definition 2.1 rigorously includes Definition 1.2.

Under Definition 2.1, we introduced a new concept called almost periodic functions on almost periodic time scales (see [3, 8]) as follows:

Definition 2.2. Let $\mathbb{T}$ be an almost periodic time scale, i.e., $\mathbb{T}$ satisfies Definition 2.1. A function $f \in C(\mathbb{T} \times D, \mathbb{E}^n)$ is called an almost periodic function in $t \in \mathbb{T}$ uniformly for $x \in D$ if the ε_2-translation set of f

$$E\{\varepsilon_2, f, S\} = \Big\{\tau \in \Pi_{\varepsilon_1} : \ \big|f(t + \tau, x) - f(t, x)\big| < \varepsilon_2 \text{ for all } (t, x) \in (\mathbb{T} \cap \mathbb{T}^{-\tau}) \times S\Big\}$$

is a relatively dense set in Π_{ε_1} for all $\varepsilon_2 > \varepsilon_1 > 0$ and for each compact subset S of D; that is, for any given $\varepsilon_2 > \varepsilon_1 > 0$ and each compact subset S of D, there exists a constant $l(\varepsilon_2, S) > 0$ such that each interval of length $l(\varepsilon_2, S)$ contains a $\tau(\varepsilon_2, S) \in E\{\varepsilon_2, f, S\}$ such that

$$|f(t + \tau, x) - f(t, x)| < \varepsilon_2 \ \text{ for all } \ (t, x) \in (\mathbb{T} \cap \mathbb{T}^{-\tau}) \times S.$$

This τ is called the ε_2-translation number of f and $l(\varepsilon_2, S)$ is called the inclusion length of $E\{\varepsilon_2, f, S\}$.

Under Definition 2.2, in the paper [3, 8], we obtained some nice properties of almost periodic time scales and almost periodic functions, then we applied them to investigate almost periodic solutions to dynamic equations, and they worked effectively. In the paper [3], we presented five categories of time scales. Then, on each class we introduced and analyzed delays which lead to new types of delay systems on time scales. Finally, some interesting open problems of dynamic equations on almost periodic time scales are proposed.

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