Memoirs on Differential Equations and Mathematical Physics
Volume 60, 2013, 111–133

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L^p-DISSIPATIVITY OF THE LAMÉ OPERATOR

In Memory of Victor Kupradze

Abstract. We study conditions for the L^p-dissipativity of the classical linear elasticity operator. In the two-dimensional case we show that L^p-dissipativity is equivalent to the inequality

$$\left(\frac{1}{2}-\frac{1}{p}\right)^2 \leqslant \frac{2(\nu-1)(2\nu-1)}{(3-4\nu)^2}\,.$$

Previously [2] this result has been obtained as a consequence of general criteria for elliptic systems, but here we give a direct and simpler proof. We show that this inequality is necessary for the L^p-dissipativity of the three-dimensional elasticity operator with variable Poisson ratio. We give also a more strict sufficient condition for the L^p-dissipativity of this operator. Finally we find a criterion for the n-dimensional Lamé operator to be L^p-negative with respect to the weight $|x|^{-\alpha}$ in the class of rotationally invariant vector functions.

2010 Mathematics Subject Classification. 74B05, 47B44.
Key words and phrases. Elasticity system, L^p-dissipativity.

რეზიუმე. შევისწავლით დრეკადობის კლასიკური თეორიის წრფივი ოპერატორის L^p-დისიპატიურობის პირობებს. ორგანზომილებიან შემთხვევაში ვაჩვენებთ, რომ L^p-დისიპატიურობის პირობა ეკვივალენტურია

$$\left(\frac{1}{2}-\frac{1}{p}\right)^2 \leqslant \frac{2(\nu-1)(2\nu-1)}{(3-4\nu)^2}$$

უტოლობის. ეს შედეგი [2] ნაშრომში მიღებული იყო ზოგადი ელიფსური შემთხვევის კრიტერიუმიდან გამომდინარე. ჩვენ აქ მოვიყვანთ პირდაპირ და უფრო მარტივ დამტკიცებას. ასევე ვაჩვენებთ, რომ დრეკადობის სამგანზომილებიანი ოპერატორის L^p-დისიპატიურობისთვის ეს უტოლობა არის აუცილებელი პირობა ცვლადი პუასონის კოეფიციენტების შემთხვევაში. ბოლოს, ჩამოვაყალიბებთ კრიტერიუმს, რომელიც უზრუნველყოფს n-განზომილებიანი ლამეს ოპერატორის L^p-უარყოფითობას ბრუნვის მიმართ ინვარიანტულ ვექტორ ფუნქციათა კლასში $|x|^{-\alpha}$ წონით.

1. Introduction

It is well known that Victor Kupradze has made seminal contributions to the theory of elasticity, in particular, to the study of BVPs of statics and steady state oscillations, as well as initial BVPs of general dynamics.

His monographs in the field of elasticity testify the great work he made (see, for instance, [6–9]). In particular, his book *Three-dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity* [10–12]) became a must for every mathematician working in this field.

The present paper concerning elasticity theory is dedicated to him.

Let us consider the classical operator of linear elasticity

$$Eu = \Delta u + (1-2\nu)^{-1}\nabla \operatorname{div} u, \tag{1}$$

where ν is the Poisson ratio. Throughout this paper, we assume that either $\nu > 1$ or $\nu < 1/2$. It is well known that E is strongly elliptic if and only if this condition is satisfied (see, for instance, Gurtin [5, p. 86]).

Let $\mathscr{L}$ be the bilinear form associated with operator (1), i.e.

$$\mathscr{L}(u,v) = -\int_\Omega (\langle \nabla u, \nabla v\rangle + (1-2\nu)^{-1} \operatorname{div} u \operatorname{div} v)\, dx, \tag{2}$$

where $\langle\cdot\,,\,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^n$. Here Ω is a domain of $\mathbb{R}^n$.

Following [1], we say that the form $\mathscr{L}$ is L^p-dissipative in Ω if

$$-\int_\Omega \Big(\big\langle \nabla u, \nabla(|u|^{p-2}u)\big\rangle + (1-2\nu)^{-1} \operatorname{div} u \operatorname{div}(|u|^{p-2}u)\Big)\, dx \leqslant 0 \quad \text{if } p \geqslant 2, \tag{3}$$

$$-\int_\Omega \Big(\big\langle \nabla u, \nabla(|u|^{p'-2}u)\big\rangle + (1-2\nu)^{-1} \operatorname{div} u \operatorname{div}(|u|^{p'-2}u)\Big)\, dx \leqslant 0 \quad \text{if } p < 2, \tag{4}$$

for all $u \in (C_0^1(\Omega))^2$ $(p' = p/(p-1))$. We use here that $|u|^{q-2}u \in C_0^1(\Omega)$ for $q \geqslant 2$ and $u \in C_0^1(\Omega)$.

In [1, 2] necessary and sufficient conditions for the L^p-dissipativity of the forms related to partial differential operators have been obtained. In particular, for the planar elasticity it was proved in [2] that the form $\mathscr{L}$ is L^p-dissipative if and only if

$$\Big(\frac{1}{2} - \frac{1}{p}\Big)^2 \leqslant \frac{2(\nu-1)(2\nu-1)}{(3-4\nu)^2}\,. \tag{5}$$

Let us now suppose that Ω is a sufficiently smooth bounded domain and consider the operator (1) defined on $D(E) = (W^{2,p}(\Omega) \cap \mathring{W}^{1,p}(\Omega))^n$. As usual $W^{l,p}(\Omega)$ denotes the Sobolev space of functions which distributional derivatives of order l are in $L^p(\Omega)$. We also use the notation $\mathring{W}^{1,p}(\Omega)$ for the completion of $C_0^\infty(\Omega)$ in the Sobolev $W^{1,p}(\Omega)$ norm. The operator E is said to be L^p-dissipative ($1 < p < \infty$) in the domain $\Omega \subset \mathbb{R}^n$ if

$$\int_\Omega \big(\Delta u + (1-2\nu)^{-1}\nabla \operatorname{div} u\big)|u|^{p-2}u\, dx \leqslant 0 \tag{6}$$

for any real vector-valued function $u \in D(E)$. Here and in the sequel the integrand is extended by zero on the set where u vanishes.

The equivalence between the L^p-dissipativity of the form and the dissipativity of the operator was discussed in [1, Section 5, p. 1086–1093]. It turns out that, if $n = 2$ and a certain smoothness assumption on $\Omega \subset \mathbb{R}^2$ is fulfilled, the operator of planar elasticity is L^p-dissipative (i.e. (6) holds for any $u \in D(E)$) if and only if condition (5) is satisfied.

In [2] these facts have been established as a consequence of results concerning general systems of partial differential equations, but in the present paper we give a direct and simpler proof just for the Lamé system. The result is followed by two Corollaries (obtained for the first time in [2]) concerning the comparison between the Lamé operator and the Laplacian from the point of view of the L^p-dissipativity.

In Section 3 we show that condition (5) is necessary for the L^p-dissipativity of operator (1), even when the Poisson ratio is not constant. For the time being it is not known if condition (5) is also sufficient for the L^p-dissipativity of elasticity operator for $n > 2$, in particular, for $n = 3$. Nevertheless in the same section we give a more strict explicit condition which is sufficient for the L^p-dissipativity of (1).

In Section 4 we give necessary and sufficient conditions for a weighted L^p-negativity of the Dirichlet–Lamé operator, i.e. for the validity of the inequality

$$\int_\Omega \big(\Delta u + (1-2\nu)^{-1}\nabla \operatorname{div} u\big)|u|^{p-2}u\, \frac{dx}{|x|^\alpha} \leqslant 0 \tag{7}$$

under the condition that the vector u is rotationally invariant, i.e. u depends only on $\varrho = |x|$ and u_ϱ is the only nonzero spherical component of u. Namely we show that (7) holds if and only if

$$-(p-1)(n+p'-2) \leqslant \alpha \leqslant n+p-2.$$

2. L^p-dissipativity of planar elasticity

In this section we give a necessary and sufficient condition for the L^p-dissipativity of operator (1) in the case $n = 2$.

First we consider the L^p-dissipativity of form (2).

Lemma 1. *Let Ω be a domain of $\mathbb{R}^2$. Form (2) is L^p-dissipative if and only if*

$$\int_\Omega \Big[C_p\big|\nabla|v|\big|^2 - \sum_{j=1}^{2}|\nabla v_j|^2 + \gamma\, C_p\, |v|^{-2}\big|v_h\partial_h|v|\big|^2 - \gamma\,|\operatorname{div} v|^2\Big]\, dx \leqslant 0 \tag{8}$$

for any $v \in (C_0^1(\Omega))^2$*, where*

$$C_p = (1-2/p)^2, \quad \gamma = (1-2\nu)^{-1}. \tag{9}$$

Proof. Sufficiency. First suppose $p \geqslant 2$. Let $u \in (C_0^1(\Omega))^2$ and set $v = |u|^{p-2}u$. We have $v \in (C_0^1(\Omega))^2$ and $u = |v|^{(2-p)/p}v$. One checks directly that

$$\begin{aligned}\big\langle \nabla u, \nabla(|u|^{p-2}u)\big\rangle + (1-2\nu)^{-1}\operatorname{div} u \operatorname{div}(|u|^{p-2}u) = \\ = \sum_j |\nabla v_j|^2 - C_p\big|\nabla|v|\big|^2 - \gamma\, C_p\big|v_h\partial_h|v|\big|^2 + \gamma|\operatorname{div} v|^2.\end{aligned}$$

The left-hand side of (3) being equal to the left-hand side of (8), inequality (3) is satisfied for any $u \in C_0^1(\Omega)$.

If $1 < p < 2$ we find

$$\begin{aligned}\big\langle \nabla u, \nabla(|u|^{p'-2}u)\big\rangle + (1-2\nu)^{-1}\operatorname{div} u \operatorname{div}(|u|^{p'-2}u) = \\ = \sum_j |\nabla v_j|^2 - C_{p'}\big|\nabla|v|\big|^2 - \gamma\, C_{p'}\big|v_h\partial_h|v|\big|^2 + \gamma|\operatorname{div} v|^2\end{aligned}$$

and since $1 - 2/p' = -1 + 2/p$ (which implies $C_p = C_{p'}$), we get the result also in this case.

Necessity. Let $p \geqslant 2$ and set

$$g_\varepsilon = (|v|^2 + \varepsilon^2)^{1/2}, \quad u_\varepsilon = g_\varepsilon^{2/p-1}v,$$

where $v \in C_0^1(\Omega)$. We have

$$\begin{aligned}\big\langle \nabla u_\varepsilon, \nabla(|u_\varepsilon|^{p-2}u_\varepsilon)\big\rangle = \\ = |u_\varepsilon|^{p-2}\langle\partial_h u_\varepsilon, \partial_h u_\varepsilon\rangle + (p-2)|u_\varepsilon|^{p-3}\langle\partial_h u_\varepsilon, u_\varepsilon\rangle\, \partial_h|u_\varepsilon|.\end{aligned}$$

A direct computation shows that

$$\begin{aligned}\big\langle \nabla u_\varepsilon, \nabla(\,|u_\varepsilon|^{p-2}u_\varepsilon)\big\rangle = \Big[(1-2/p)^2 g_\varepsilon^{-(p+2)}|v|^p - \\ - 2(1-2/p)g_\varepsilon^{-p}|v|^{p-2}\Big]\sum_k |v_j\partial_k v_j|^2 + g_\varepsilon^{2-p}|v|^{p-2}\langle\partial_h v, \partial_h v\rangle,\end{aligned}$$

$$\begin{aligned}|u_\varepsilon|^{p-3}\langle\partial_h u_\varepsilon, u_\varepsilon\rangle\, \partial_h|u_\varepsilon| = \\ = \Big\{(1-2/p)\Big[(1-2/p)g_\varepsilon^{-(p+2)}|v|^p - g_\varepsilon^{-p}|v|^{p-2}\Big] + \\ + \big[g_\varepsilon^{2-p}|v|^{p-4} - (1-2/p)g_\varepsilon^{-p}|v|^{p-2}\big]\Big\}\sum_k |v_j\partial_k v_j|^2\end{aligned}$$

on the set $E = \{x \in \Omega \mid \ |v(x)| > 0\}$. The inequality $g_\varepsilon^a \leqslant |v|^a$ for $a \leqslant 0$, shows that the right-hand sides are dominated by L^1 functions. Since $g_\varepsilon \to$

$|v|$ pointwise as $\varepsilon \to 0^+$, we find

$$\lim_{\varepsilon\to 0^+} \big\langle \nabla u_\varepsilon, \nabla(|u_\varepsilon|^{p-2}u_\varepsilon)\big\rangle =$$
$$= \langle \partial_h v, \partial_h v\rangle + \Big[(1-2/p)^2 - 2(1-2/p) + 4(p-2)/p^2\Big]|v|^{-2}\sum_k |v_j\partial_k v_j|^2 =$$
$$= -(1-2/p)^2\big|\nabla|v|\big|^2 + \sum_j |\nabla v_j|^2$$

and dominated convergence gives

$$\lim_{\varepsilon\to 0^+}\int_E \big\langle \nabla u_\varepsilon, \nabla(|u_\varepsilon|^{p-2}u_\varepsilon)\big\rangle\, dx = \int_E \Big[-C_p\big|\nabla|v|\big|^2 + \sum_j |\nabla v_j|^2\Big]\, dx. \quad (10)$$

Similar arguments show that

$$\lim_{\varepsilon\to o^+}\int_E \operatorname{div} u_\varepsilon \operatorname{div}(|u_\varepsilon|^{p-2}u_\varepsilon)\, dx =$$
$$= \int_E \Big[-C_p|v|^{-2}\big|v_h\partial_h|v|\big|^2 + |\operatorname{div} v|^2\Big]\, dx. \quad (11)$$

Formulas (10) and (11) lead to

$$\lim_{\varepsilon\to o^+}\int_\Omega \big\langle \nabla u_\varepsilon, \nabla(|u_\varepsilon|^{p-2}u_\varepsilon)\big\rangle + \gamma\operatorname{div}(|u_\varepsilon|^{p-2}u_\varepsilon)\, dx =$$
$$= \int_\Omega \Big(-C_p\big|\nabla|v|\big|^2 + \sum_j |\nabla v_j|^2 - \gamma\, C_p|v|^{-2}\big|v_h\partial_h|v|\big|^2 + \gamma\,|\operatorname{div} v|^2\Big)\, dx. \quad (12)$$

The function u_ε being in $(C_0^1(\Omega))^2$, the left-hand side is greater than or equal to zero and (8) follows.

If $1<p<2$, we can write, in view of (12),

$$\lim_{\varepsilon\to o^+}\int_\Omega \big\langle \nabla u_\varepsilon, \nabla(|u_\varepsilon|^{p'-2}u_\varepsilon)\big\rangle + \gamma\operatorname{div}(|u_\varepsilon|^{p'-2}u_\varepsilon)\, dx =$$
$$= \int_\Omega \Big(-C_{p'}\big|\nabla|v|\big|^2 + \sum_j |\nabla v_j|^2 - \gamma\, C_{p'}|v|^{-2}\big|v_h\partial_h|v|\big|^2 + \gamma\,|\operatorname{div} v|^2)\, dx.$$

Since $C_{p'} = C_p$, (4) implies (8). □

Remark 1. The previous Lemma holds in any dimension with the same proof.

The next Lemma provides a necessary algebraic condition for the L^p-dissipativity of form (2).

Lemma 2. *Let* Ω *be a domain of* $\mathbb{R}^2$*. If form* (2) *is* L^p*-dissipative, we have*

$$C_p\big[\,|\xi|^2 + \gamma\,\langle\xi,\omega\rangle^2\big]\langle\lambda,\omega\rangle^2 - |\xi|^2|\lambda|^2 - \gamma\,\langle\xi,\lambda\rangle^2 \leqslant 0 \quad (13)$$

for any $\xi, \lambda, \omega \in \mathbb{R}^2$, $|\omega| = 1$ (*the constants* C_p *and* γ *being given by* (9)).

Proof. Assume first that $\Omega = \mathbb{R}^2$. Let us fix $\omega \in \mathbb{R}^2$ with $|\omega| = 1$ and take $v(x) = w(x)\,\eta(\log|x|/\log R)$, where

$$w(x) = \mu\,\omega + \psi(x),$$

μ, $R \in \mathbb{R}^+$, $\psi \in (C_0^\infty(\mathbb{R}^2))^2$, $\eta \in C^\infty(\mathbb{R}^2)$, $\eta(t) = 1$ if $t \leqslant 1/2$ and $\eta(t) = 0$ if $t \geqslant 1$.

On the set where $v \neq 0$ one has

$$\begin{gathered}\big\langle \nabla|v|, \nabla|v|\big\rangle = \big\langle \nabla|w|, \nabla|w|\big\rangle\, \eta^2(\log|x|/\log R) + \\ +2\,(\log R)^{-1}|w|\,\big\langle \nabla|w|, x\big\rangle\, |x|^{-2}\eta(\log|x|/\log R)\,\eta'(\log|x|/\log R) + \\ +(\log R)^{-2}|w|^2|x|^{-2}\big(\eta'(\log|x|/\log R)\big)^2.\end{gathered}$$

Choose δ such that $\operatorname{spt}\psi \subset B_\delta(0)$ and $R > \delta^2$. If $|x| > \delta$ one has $w(x) = \mu\,\omega$ and then $\nabla|w| = 0$, while if $|x| < \delta$, then $\eta(\log|x|/\log R) = 1$, $\eta'(\log|x|/\log R) = 0$. Therefore

$$\int_{\mathbb{R}^2} \big\langle \nabla|v|, \nabla|v|\big\rangle\, dx =$$

$$= \int_{B_\delta(0)} \big\langle \nabla|w|, \nabla|w|\big\rangle\, dx + \frac{1}{\log^2 R} \int_{B_R(0)\setminus B_{\sqrt{R}}(0)} \frac{|w|^2}{|x|^2}\big(\eta'(\log|x|/\log R)\big)^2\, dx.$$

Since

$$\lim_{R\to+\infty} \frac{1}{\log^2 R} \int_{B_R(0)\setminus B_{\sqrt{R}}(0)} \frac{dx}{|x|^2} = 0,$$

we find

$$\lim_{R\to+\infty} \int_{\mathbb{R}^2} \big\langle \nabla|v|, \nabla|v|\big\rangle\, dx = \int_{B_\delta(0)} \big\langle \nabla|w|, \nabla|w|\big\rangle\, dx.$$

By similar arguments we obtain

$$\lim_{R\to+\infty} \int_{\mathbb{R}^2} \Big[C_p\big|\nabla|v|\big|^2 - \sum_{j=1}^2 |\nabla v_j|^2 + \gamma\, C_p\, |v|^{-2}\big|v_h\partial_h|v|\big|^2 - \gamma\,|\operatorname{div} v|^2\Big]\, dx =$$

$$= \int_{B_\delta(0)} \Big[C_p\big|\nabla|w|\big|^2 - \sum_{j=1}^2 |\nabla w_j|^2 + \gamma\, C_p\, |w|^{-2}\big|w_h\partial_h|w|\big|^2 - \gamma\,|\operatorname{div} w|^2\Big]\, dx.$$

In view of Lemma 1, (8) holds. Putting v in this formula and letting $R \to +\infty$, we find

$$\int_{B_\delta(0)} \Big[C_p\big|\nabla|w|\big|^2 - \sum_{j=1}^2 |\nabla w_j|^2 + \gamma\, C_p\, |w|^{-2}\big|w_h\partial_h|w|\big|^2 - \gamma\,|\operatorname{div} w|^2\Big]\, dx \leqslant 0. \quad (14)$$

From the identities

$$\partial_h w = \partial_h \psi, \quad \operatorname{div} w = \operatorname{div} \psi,$$

$$\big|\nabla |w|\big|^2 = |\mu\, \omega + \psi|^{-2} \sum_{h=1}^{2} \langle \mu\, \omega + \psi, \partial_h \psi\rangle^2,$$

$$|w|^{-2} |w_h \partial_h w|^2 = |\mu\, \omega + \psi|^{-4} \Big|(\mu\, \omega_h + \psi_h)\langle \mu\, \omega + \psi, \partial_h \psi\rangle\Big|^2$$

we infer, letting $\mu \to +\infty$ in (14),

$$\int_{\mathbb{R}^2} \Big[C_p \sum_{h=1}^{2} \langle \omega, \partial_h \psi\rangle^2 - \sum_{j=1}^{2} |\nabla \psi_j|^2 + \gamma\, C_p \big|\omega_h\, \langle \omega, \partial_h \psi\rangle\big|^2 - \gamma\, |\operatorname{div} \psi|^2\Big]\, dx \leqslant 0. \quad (15)$$

Putting in (15)

$$\psi(x) = \lambda\, \varphi(x)\, \cos(\mu\langle \xi, x\rangle) \;\text{ and }\; \psi(x) = \lambda\, \varphi(x)\, \sin(\mu\langle \xi, x\rangle),$$

where $\lambda \in \mathbb{R}^2$, $\varphi \in C_0^\infty(\mathbb{R}^2)$ and μ is a real parameter, by standard arguments (see, e.g, Fichera [4, p. 107–108]) we find (13).

If $\Omega \neq \mathbb{R}^2$, fix $x_0 \in \Omega$ and $0 < \varepsilon < \operatorname{dist}(x_0, \partial\Omega)$. Given $\psi \in (C_0^1(\Omega)^2$, put the function

$$v(x) = \psi((x - x_0)/\varepsilon)$$

in (8). By a change of variables we find

$$\int_{\mathbb{R}^2} \left[C_p \big|\nabla |\psi|\big|^2 - \sum_{j=1}^{2} |\nabla \psi_j|^2 + \gamma\, C_p\, |\psi|^{-2} \big|\psi_h \partial_h |\psi|\big|^2 - \gamma\, |\operatorname{div} \psi|^2\right] dx \leqslant 0.$$

The arbitrariness of $\psi \in (C_0^1(\Omega)^2$ and what we have proved for $\mathbb{R}^2$ gives the result. □

We are now in a position to give a necessary and sufficient condition for the L^p-dissipativity of form (2).

Theorem 1. *Form* (2) *is L^p-dissipative if and only if*

$$\Big(\frac{1}{2} - \frac{1}{p}\Big)^2 \leqslant \frac{2(\nu - 1)(2\nu - 1)}{(3 - 4\nu)^2}\,. \quad (16)$$

Proof. Necessity. In view of Lemma 2, the L^p-dissipativity of $\mathscr{L}$ implies the algebraic inequality (13) for any $\xi, \lambda, \omega \in \mathbb{R}^2$, $|\omega| = 1$.

Without loss of generality we may suppose $\xi = (1, 0)$ and (13) can be written as

$$C_p(1 + \gamma\omega_1^2)(\lambda_j \omega_j)^2 - |\lambda|^2 - \gamma\lambda_1^2 \leqslant 0 \quad (17)$$

for any $\lambda, \omega \in \mathbb{R}^2$, $|\omega| = 1$.

Condition (17) holds if and only if

$$C_p(1 + \gamma\omega_1^2)\omega_1^2 - 1 - \gamma \leqslant 0,$$

$$\big[C_p(1 + \gamma\omega_1^2)\omega_1\omega_2\big]^2 \leqslant \big[-C_p(1 + \gamma\omega_1^2)\omega_1^2 + 1 + \gamma\big]\, \big[-C_p(1 + \gamma\omega_1^2)\omega_2^2 + 1\big]$$

for any $\omega \in \mathbb{R}^2$, $|\omega| = 1$.

In particular, the second condition has to be satisfied. This can be written in the form

$$1+\gamma-C_p(1+\gamma\omega_1^2)(1+\gamma\omega_2^2)\geqslant 0 \tag{18}$$

for any $\omega\in\mathbb{R}^2$, $|\omega|=1$. The minimum of the left-hand side of (18) on the unit sphere is given by

$$1+\gamma-C_p(1+\gamma/2)^2.$$

Hence (18) is satisfied if and only if $1+\gamma-C_p(1+\gamma/2)^2\geqslant 0$. The last inequality means

$$\frac{2(1-\nu)}{1-2\nu}-\Big(\frac{p-2}{p}\Big)^2\Big(\frac{3-4\nu}{2(1-2\nu)}\Big)^2\geqslant 0,$$

i.e. (16). From the identity $4/(p\,p')=1-(1-2/p)^2$ it follows that (16) can be written also as

$$\frac{4}{p\,p'}\geqslant\frac{1}{(3-4\nu)^2}\,. \tag{19}$$

Sufficiency. In view of Lemma 1, $\mathscr{L}$ is L^p-dissipative if and only if (8) holds for any $v\in(C_0^1(\Omega))^2$. Choose $v\in(C_0^1(\Omega))^2$ and define

$$X_1=|v|^{-1}(v_1\partial_1|v|+v_2\partial_2|v|),\quad X_2=|v|^{-1}(v_2\partial_1|v|-v_1\partial_2|v|),$$
$$Y_1=|v|\big[\partial_1(|v|^{-1}v_1)+\partial_2(|v|^{-1}v_2)\big],\quad Y_2=|v|\big[\partial_1(|v|^{-1}v_2)-\partial_2(|v|^{-1}v_1)\big]$$

on the set $E=\{x\in\Omega\mid v\neq 0\}$. From the identities

$$|\nabla|v||^2=X_1^2+X_2^2,$$
$$Y_1=(\partial_1v_1+\partial_2v_2)-X_1,\quad Y_2=(\partial_1v_2-\partial_2v_1)-X_2$$

it follows

$$Y_1^2+Y_2^2=\big|\nabla|v|\big|^2+(\partial_1v_1+\partial_2v_2)^2+(\partial_1v_2-\partial_2v_1)^2-$$
$$-2(\partial_1v_1+\partial_2v_2)X_1-2(\partial_1v_2-\partial_2v_1)X_2.$$

Keeping in mind that $\partial_h|v|=|v|^{-1}v_j\partial_hv_j$, one can check that

$$(\partial_1v_1+\partial_2v_2)\big(v_1\partial_1|v|+v_2\partial_2|v|\big)+(\partial_1v_2-\partial_2v_1)\big(v_2\partial_1|v|-v_1\partial_2|v|\big)=$$
$$=|v|\,\big|\nabla|v|\big|^2+|v|(\partial_1v_1\partial_2v_2-\partial_2v_1\partial_1v_2),$$

which implies

$$\sum_j|\nabla v_j|^2=X_1^2+X_2^2+Y_1^2+Y_2^2. \tag{20}$$

Thus (8) can be written as

$$\int_E\Big[\frac{4}{p\,p'}\,(X_1^2+X_2^2)+Y_1^2+Y_2^2-\gamma\,C_pX_1^2+\gamma\,(X_1+Y_1)^2\Big]\,dx\geqslant 0. \tag{21}$$

Let us prove that

$$\int_E X_1Y_1\,dx=-\int_E X_2Y_2\,dx. \tag{22}$$

Since $X_1 + Y_1 = \operatorname{div} v$ and $X_2 + Y_2 = \partial_1 v_2 - \partial_2 v_1$, keeping in mind (20), we may write

$$2\int_E (X_1Y_1 + X_2Y_2)\, dx =$$
$$= \int_E \Big[(X_1+Y_1)^2 + (X_2+Y_2)^2 - (X_1^2+X_2^2+Y_1^2+Y_2^2)\Big]\, dx =$$
$$= \int_E \Big[(\operatorname{div} v)^2 + (\partial_1 v_2 - \partial_2 v_1)^2 - \sum_j |\nabla v_j|^2\Big]\, dx,$$

i.e.

$$\int_E (X_1Y_1 + X_2Y_2)\, dx = \int_E (\partial_1 v_1 \partial_2 v_2 - \partial_1 v_2 \partial_2 v_1)\, dx.$$

The set $\{x \in \Omega \setminus E | \nabla v(x) \neq 0\}$ has zero measure and then

$$\int_E (X_1Y_1 + X_2Y_2)\, dx = \int_\Omega (\partial_1 v_1 \partial_2 v_2 - \partial_1 v_2 \partial_2 v_1)\, dx.$$

There exists a sequence $\{v^{(n)}\} \subset C_0^\infty(\Omega)$ such that $v^{(n)} \to v$, $\nabla v^{(n)} \to \nabla v$ uniformly in Ω and hence

$$\int_\Omega \partial_1 v_1 \partial_2 v_2\, dx = \lim_{n\to\infty} \int_\Omega \partial_1 v_1^{(n)} \partial_2 v_2^{(n)}\, dx =$$
$$= \lim_{n\to\infty} \int_\Omega \partial_1 v_2^{(n)} \partial_2 v_1^{(n)}\, dx = \int_\Omega \partial_1 v_2 \partial_2 v_1\, dx$$

and (22) is proved. In view of this, (21) can be written as

$$\int_E \Big(\frac{4}{p\,p'}\,(1+\gamma)X_1^2 + 2\vartheta\gamma\, X_1Y_1 + (1+\gamma)Y_1^2\Big)\, dx +$$
$$+ \int_E \Big(\frac{4}{p\,p'}\, X_2^2 - 2(1-\vartheta)\gamma\, X_2Y_2 + Y_2^2\Big)\, dx \geqslant 0$$

for any fixed $\vartheta \in \mathbb{R}$.

If we choose

$$\vartheta = \frac{2(1-\nu)}{3-4\nu}$$

we find

$$(1-\vartheta)\gamma = \frac{1}{3-4\nu}\,, \quad \vartheta^2\gamma^2 = \frac{(1+\gamma)^2}{(3-4\nu)^2}\,.$$

Inequality (19) leads to

$$\vartheta^2\gamma^2 \leqslant \frac{4}{p\,p'}\,(1+\gamma)^2, \quad (1-\vartheta)^2\gamma^2 \leqslant \frac{4}{p\,p'}\,.$$

Observing that (16) implies $1+\gamma = 2(1-\nu)(1-2\nu)^{-1} \geqslant 0$, we get

$$\frac{4}{p\,p'}\,(1+\gamma)x_1^2 + 2\vartheta\gamma\, x_1 y_1 + (1+\gamma) y_1^2 \geqslant 0,$$

$$\frac{4}{p\,p'}\, x_2^2 - 2(1-\vartheta)\gamma\, x_2 y_2 + y_2^2 \geqslant 0$$

for any $x_1, x_2, y_1, y_2 \in \mathbb{R}$. This shows that (21) holds. Then (8) is true for any $v \in (C_0^1(\Omega))^2$ and the proof is complete. □

The results we have obtained so far hold for any domain Ω. For the rest of the present section we suppose that Ω is a bounded domain whose boundary is in the class C^2. We could consider more general domains, in the spirit of Maz'ya and Shaposhnikova [14, Ch. 14], but here we prefer to avoid the related technicalities.

Theorem 2. *Let E be the two-dimensional elasticity operator* (1) *with domain $(W^{2,p}(\Omega) \cap \mathring{W}^{1,p}(\Omega))^2$. The operator E is L^p-dissipative if and only if condition* (16) *holds.*

Proof. By means of the same arguments as in [1, Section 5, p. 1086–1093], we have the equivalence between the L^p-dissipativity of form (2) and the L^p-dissipativity of the elasticity operator (1). The result follows from Theorem 1. □

We shall now give two corollaries of this result. They concerns the comparison between E and Δ from the point of view of the L^p-dissipativity.

Corollary 1. *There exists $k > 0$ such that $E - k\Delta$ is L^p-dissipative if and only if*

$$\Big(\frac{1}{2} - \frac{1}{p}\Big)^2 < \frac{2(\nu-1)(2\nu-1)}{(3-4\nu)^2}\,. \tag{23}$$

Proof. Necessity. We remark that if $E - k\Delta$ is L^p-dissipative, then

$$\begin{cases} k \leqslant 1 & \text{if } \ p = 2, \\ k < 1 & \text{if } \ p \neq 2. \end{cases} \tag{24}$$

In fact, in view of Theorem 1, we have the necessary condition

$$-(1-2/p)^2\big[(1-k)|\xi|^2 + (1-2\nu)^{-1}(\xi_j\omega_j)^2\big](\lambda_j\omega_j)^2 + \\ + (1-k)|\xi|^2|\lambda|^2 + (1-2\nu)^{-1}(\xi_j\lambda_j)^2 \geqslant 0 \tag{25}$$

for any $\xi, \lambda, \omega \in \mathbb{R}^2$, $|\omega| = 1$. If we take $\xi = (1,0)$, $\lambda = \omega = (0,1)$ in (25) we find

$$\frac{4}{p\,p'}\,(1-k) \geqslant 0$$

and then $k \leqslant 1$ for any p. If $p \neq 2$ and $k = 1$, taking $\xi = (1,0)$, $\lambda = (0,1)$, $\omega = (1/\sqrt{2}, 1/\sqrt{2})$ in (25), we find $-(1-2/p)^2(1-2\nu)^{-1} \geqslant 0$. On the other hand, taking $\xi = \lambda = (1,0)$, $\omega = (0,1)$ we find $(1-2\nu)^{-1} \geqslant 0$. This is a contradiction and (24) is proved.

It is clear that if $E - k\Delta$ is L^p-dissipative, then $E - k'\Delta$ is L^p-dissipative for any $k' < k$. Therefore it is not restrictive to suppose that $E - k\Delta$ is L^p-dissipative for some $0 < k < 1$. Moreover, E is also L^p-dissipative.

The L^p-dissipativity of $E - k\Delta$ $(0 < k < 1)$ is equivalent to the L^p-dissipativity of the operator

$$E'u = \Delta u + (1-k)^{-1}(1-2\nu)^{-1}\nabla \operatorname{div} u. \tag{26}$$

Setting

$$\nu' = \nu(1-k) + k/2, \tag{27}$$

we have $(1-k)(1-2\nu) = 1-2\nu'$. Theorem 1 shows that

$$\frac{4}{p\,p'} \geqslant \frac{1}{(3-4\nu')^2}\,. \tag{28}$$

Since $3-4\nu' = 3-4\nu-2k(1-2\nu)$, condition (28) means $|3-4\nu-2k(1-2\nu)| \geqslant \sqrt{p\,p'}/2$, i.e.

$$\Big|k - \frac{3-4\nu}{2(1-2\nu)}\Big| \geqslant \frac{\sqrt{p\,p'}}{4|1-2\nu|}\,. \tag{29}$$

Note that the L^p-dissipativity of E implies that (16) holds. In particular, we have $(3-4\nu)/(1-2\nu) > 0$. Hence (29) is satisfied if either

$$k \leqslant \frac{1}{2|1-2\nu|}\Big(|3-4\nu| - \frac{\sqrt{p\,p'}}{2}\Big) \tag{30}$$

or

$$k \geqslant \frac{1}{2|1-2\nu|}\Big(|3-4\nu| + \frac{\sqrt{p\,p'}}{2}\Big). \tag{31}$$

Since

$$\frac{|3-4\nu|}{2|1-2\nu|} - 1 = \frac{3-4\nu}{2(1-2\nu)} - 1 = \frac{1}{2(1-2\nu)} \geqslant -\frac{\sqrt{p\,p'}}{4|1-2\nu|},$$

we have

$$\frac{1}{2|1-2\nu|}\Big(|3-4\nu| + \frac{\sqrt{p\,p'}}{2}\Big) \geqslant 1$$

and (31) is impossible. Then (30) holds. Since $k > 0$, we have the strict inequality in (19) and (23) is proved.

Sufficiency. Suppose (23). Since

$$\frac{4}{p\,p'} > \frac{1}{(3-4\nu)^2}\,,$$

we can take k such that

$$0 < k < \frac{1}{2|1-2\nu|}\Big(|3-4\nu| - \frac{\sqrt{p\,p'}}{2}\Big). \tag{32}$$

Note that

$$\frac{|3-4\nu|}{2|1-2\nu|} - 1 = \frac{3-4\nu}{2(1-2\nu)} - 1 = \frac{1}{2(1-2\nu)} \leqslant \frac{\sqrt{p\,p'}}{4|1-2\nu|}\,.$$

This means
$$\frac{1}{2|1-2\nu|}\Big(|3-4\nu|-\frac{\sqrt{p\,p'}}{2}\Big)\leqslant 1$$
and then $k<1$. Let ν' be given by (27). The L^p-dissipativity of $E-k\Delta$ is equivalent to the L^p-dissipativity of the operator E' defined by (26).

Condition (29) (i.e. (28)) follows from (32) and Theorem 1 gives the result. □

Corollary 2. *There exists $k<2$ such that $k\Delta-E$ is L^p-dissipative if and only if*
$$\Big(\frac{1}{2}-\frac{1}{p}\Big)^2<\frac{2\nu(2\nu-1)}{(1-4\nu)^2}\,. \tag{33}$$

Proof. We may write $k\Delta-E=\widetilde{E}-\widetilde{k}\Delta$, where $\widetilde{k}=2-k$, $\widetilde{E}=\Delta+(1-2\widetilde{\nu})^{-1}\nabla\,\mathrm{div}$, $\widetilde{\nu}=1-\nu$. Theorem 1 shows that $\widetilde{E}-\widetilde{k}\Delta$ is L^p-dissipative if and only if
$$\Big(\frac{1}{2}-\frac{1}{p}\Big)^2<\frac{2(\widetilde{\nu}-1)(2\widetilde{\nu}-1)}{(3-4\widetilde{\nu})^2}\,. \tag{34}$$
Condition (34) coincides with (33) and the corollary is proved. □

3. L^p-dissipativity of three-dimensional elasticity

As far as the three-dimensional Lamé system is concerned, necessary and sufficient conditions for the L^p-dissipativity are not known. The next Theorem shows that condition (16) is necessary, even in the case of a non-constant Poisson ratio. Here Ω is a bounded domain in $\mathbb{R}^3$ whose boundary is in the class C^2.

Theorem 3. *Suppose $\nu=\nu(x)$ is a continuos function defined in Ω such that*
$$\inf_{x\in\Omega}|2\nu(x)-1|>0.$$
If (1) *is L^p-dissipative in Ω, then*
$$\Big(\frac{1}{2}-\frac{1}{p}\Big)^2\leqslant\inf_{x\in\Omega}\frac{2(\nu(x)-1)(2\nu(x)-1)}{(3-4\nu(x))^2}\,. \tag{35}$$

Proof. We have
$$\int_\Omega\big(\Delta u+(1-2\nu(x)\big)^{-1}\nabla\,\mathrm{div}\,u|u|^{p-2}u\,dx\leqslant 0 \tag{36}$$
for any $u\in(W^{2,p}(\Omega)\cap\mathring{W}^{1,p}(\Omega))^3$, in particular, for any $u\in(C_0^\infty(\Omega))^3$. Take $v\in(C_0^\infty(\mathbb{R}^2))^2$, $\varphi\in C_0^\infty(\mathbb{R})$, $\varphi\geqslant 0$ and $x^0\in\Omega$; define $v_\varepsilon(x_1,x_2)=v((x_1-x_1^0)/\varepsilon,(x_2-x_2^0)/\varepsilon)$,
$$u(x_1,x_2,x_3)=\big(v_{\varepsilon,1}(x_1,x_2),v_{\varepsilon,2}(x_1,x_2),0\big)\,\varphi(x_3).$$

We suppose that the support of v is contained in the unit ball, $0<\varepsilon<\mathrm{dist}(x^0,\partial\Omega)$ and the support of φ is contained in $(-\varepsilon,\varepsilon)$. In this way the function u belongs to $(C_0^\infty(\Omega))^3$.

Setting $\gamma(x_1,x_2,x_3)=(1-2\,\nu(x_1,x_2,x_3))^{-1}$, we have

$$\Delta u+\gamma\,\nabla\operatorname{div}u=(\Delta v_\varepsilon+\gamma\,\nabla\operatorname{div}v_\varepsilon)\,\varphi+v_\varepsilon\varphi''$$

and then

$$(\Delta u+\gamma\,\nabla\operatorname{div}u)|u|^{p-2}u=(\Delta v_\varepsilon+\gamma\,\nabla\operatorname{div}v_\varepsilon)\,|v_\varepsilon|^{p-2}v_\varepsilon\varphi^p+v_\varepsilon^2\varphi''\varphi^{p-1}.$$

We can write, in view of (36),

$$\int\limits_{\mathbb{R}}\varphi^p\,dx_3\iint\limits_{\mathbb{R}^2}(\Delta v_\varepsilon+\gamma\,\nabla\operatorname{div}v_\varepsilon)\,|v_\varepsilon|^{p-2}v_\varepsilon\,dx_1\,dx_2+ \\ +\int\limits_{\mathbb{R}}\varphi^{p-1}\varphi''\,dx_3\iint\limits_{\mathbb{R}^2}|v_\varepsilon|^p\,dx_1\,dx_2\leqslant 0.$$

Noting that

$$\Delta v_\varepsilon+\gamma\,\nabla\operatorname{div}v_\varepsilon= \\ =\frac{1}{\varepsilon^2}\left[\Delta v\Big(\frac{x_1-x_1^0}{\varepsilon},\frac{x_1-x_1^0}{\varepsilon}\Big)+\gamma(x_1,x_2,x_3)\nabla\operatorname{div}v\Big(\frac{x_1-x_1^0}{\varepsilon},\frac{x_1-x_1^0}{\varepsilon}\Big)\right],$$

a change of variables in the double integral gives

$$\int\limits_{\mathbb{R}}\varphi^p(x_3)\,dx_3\iint\limits_{\mathbb{R}^2}\Big(\Delta v(t_1,t_2)+\gamma(x_1^0+\varepsilon\,t_1,x_2^0+\varepsilon\,t_2,x_3)\nabla\operatorname{div}v(t_1,t_2)\Big)\times \\ \times\big|v(t_1,t_2)\big|^{p-2}v(t_1,t_2)\,dt_1\,dt_2+ \\ +\varepsilon^2\int\limits_{\mathbb{R}}\varphi^{p-1}\varphi''\,dx_3\iint\limits_{\mathbb{R}^2}|v(t_1,t_2)|^p\,dt_1\,dt_2\leqslant 0.$$

Letting $\varepsilon\to 0^+$, we get

$$\int\limits_{\mathbb{R}}\varphi^p(x_3)\,dx_3\iint\limits_{\mathbb{R}^2}\Big(\Delta v(t_1,t_2)+\gamma(x_1^0,x_2^0,x_3)\nabla\operatorname{div}v(t_1,t_2)\Big)\times \\ \times\big|v(t_1,t_2)\big|^{p-2}v(t_1,t_2)\,dt_1\,dt_2\leqslant 0.$$

For the arbitrariness of φ, this implies

$$\iint\limits_{\mathbb{R}^2}\Big(\Delta v(t_1,t_2)+\gamma(x_1^0,x_2^0,x_3^0)\nabla\operatorname{div}v(t_1,t_2)\Big)\times \\ \times\big|v(t_1,t_2)\big|^{p-2}v(t_1,t_2)\,dt_1\,dt_2\leqslant 0$$

for any $v\in(C_0^\infty(B))^2$, B being the unit ball in $\mathbb{R}^2$.

Suppose $p\geqslant 2$. Integrating by parts, we get

$$\mathscr{L}(v,|v|^{p-2}v)\leqslant 0 \tag{37}$$

for any $v\in(C_0^\infty(B))^2$.

Given $v \in (C_0^\infty(B))^2$, define $u_\varepsilon = g_\varepsilon^{2/p-1} v$. Since $u_\varepsilon \in (C_0^\infty(B))^2$, in view of (37) we write

$$\mathscr{L}(u_\varepsilon, |u_\varepsilon|^{p-2} u_\varepsilon) \leqslant 0.$$

By means of the computations we made in the Necessity of Lemma 1, letting $\varepsilon \to 0^+$, we find inequality (8) for any $v \in (C_0^\infty(B))^2$. This implies that (8) holds for any $v \in (C_0^1(B))^2$.

In fact, let $v_m \in (C_0^\infty(B))^2$ such that $v_m \to v$ in C^1-norm. Let us show that

$$\chi_{E_n} |v_m|^{-1} v_m \nabla v_m \to \chi_E |v|^{-1} v \nabla v \ \text{ in } L^2(B), \tag{38}$$

where $E_n = \{x \in B \mid v_m(x) \neq 0\}$, $E = \{x \in \Omega \mid v(x) \neq 0\}$. We see that

$$\chi_{E_n} |v_m|^{-1} v_m \nabla v_m \to \chi_E |v|^{-1} v \nabla v \tag{39}$$

on the set $E \cup \{x \in B \mid \nabla v(x) = 0\}$. The set $\{x \in B \setminus E \mid \nabla v(x) \neq 0\}$ having zero measure, (39) holds almost everywhere. Moreover, since

$$\int\limits_G \chi_{E_n} |v_m|^{-2} |v_m \nabla v_m|^2 \, dx \leqslant \int\limits_G |\nabla v_m|^2 \, dx$$

for any measurable set $G \subset \Omega$ and $\{\nabla v_m\}$ is convergent in $L^2(\Omega)$, the sequence $\{|\chi_{E_n} |v_m|^{-1} v_m \nabla v_m - \chi_E |v|^{-1} v \nabla v|^2\}$ has uniformly absolutely continuous integrals. Now we may appeal to Vitali's Theorem to obtain (38).

Inequality (8) holding for any $v \in (C_0^1(B))^2$, the result follows from Theorem 1.

Let now $1 < p < 2$. From the L^p dissipativity of E it follows that the operator $E - \lambda I$ ($\lambda > 0$) is invertible on $L^p(\Omega)$. This means that for any $f \in L^p(\Omega)$ there exists one and only one $u \in W^{2,p}(\Omega) \cap \mathring{W}^{1,p}(\Omega)$ such that $(E - \lambda I)u = f$. Because of well known regularity results for solutions of elliptic systems [3], we have also that, if f belongs to $L^{p'}(\Omega)$, the solution u belongs to $W^{2,p'}(\Omega) \cap \mathring{W}^{1,p'}(\Omega)$ and there exists the bounded resolvent $(E^* - \lambda I)^{-1} : L^{p'}(\Omega) \to W^{2,p'}(\Omega) \cap \mathring{W}^{1,p'}(\Omega)$.

Since E is L^p-dissipative and $\|(E^* - \lambda I)^{-1}\| = \|(E - \lambda I)^{-1}\|$, we may write

$$\left\|(E^* - \lambda I)^{-1}\right\| \leqslant \frac{1}{\lambda}$$

for any $\lambda > 0$, i.e. we have the $L^{p'}$-dissipativity of E^*, $p' > 2$. We have reduced the proof to the previous case. Therefore (35) holds with p replaced by p'. Since

$$\left(\frac{1}{2} - \frac{1}{p}\right)^2 = \left(\frac{1}{2} - \frac{1}{p'}\right)^2,$$

the proof is complete. □

We do not know if condition (16) is sufficient for the L^p-dissipativity of the three-dimensional elasticity. The next theorem provides a more strict sufficient condition.

Theorem 4. *Let Ω be a domain in $\mathbb{R}^3$. If*

$$(1-2/p)^2 \leqslant \begin{cases} \dfrac{1-2\nu}{2(1-\nu)} & \text{if } \nu < 1/2, \\ \dfrac{2(1-\nu)}{1-2\nu} & \text{if } \nu > 1, \end{cases} \tag{40}$$

operator (1) *is L^p-dissipative.*

Proof. In view of Remark 1, the operator E is L^p-dissipative if and only if inequality (8) holds for any $v \in (C_0^1(\Omega))^3$. This can be written as

$$C_p \int_\Omega \Big[\big|\nabla|v|\big|^2 + \gamma\, |v|^{-2} \big|v_h \partial_h |v|\big|^2 \Big]\, dx \leqslant \leqslant \int_\Omega \Big[\sum_{j=1}^3 |\nabla v_j|^2 + \gamma\, |\operatorname{div} v|^2 \Big]\, dx. \tag{41}$$

Note that the integral on the left-hand side of (41) is nonnegative. In fact, setting $\xi_{hj} = \partial_h v_j$, $\omega_j = |v|^{-1} v_j$, we have

$$\big|\nabla|v|\big|^2 + \gamma\, |v|^{-2} \big|v_h \partial_h |v|\big|^2 = \omega_i \omega_j (\delta_{hk} + \gamma \omega_h \omega_k) \xi_{hi} \xi_{kj}.$$

Then we can write

$$\big|\nabla|v|\big|^2 + \gamma\, |v|^{-2} \big|v_h \partial_h |v|\big|^2 = |\lambda|^2 + \gamma (\lambda \cdot \omega)^2, \tag{42}$$

where λ is the vector whose h-th component is $\omega_i \xi_{hi}$. Since ω is a unit vector and $\gamma > -1$ we have

$$\big|\nabla|v|\big|^2 + \gamma\, |v|^{-2} \big|v_h \partial_h |v|\big|^2 \geqslant 0.$$

Also the right-hand side of (41) is nonengative. In fact, denoting by $\widehat{v_j}$ the Fourier transform of v_j

$$\widehat{v_j}(y) = \int_{\mathbb{R}^3} v_j(x) e^{-iy\cdot x}\, dx,$$

we have

$$\int_\Omega \Big[\sum_{j=1}^3 |\nabla v_j|^2 + \gamma\, |\operatorname{div} v|^2 \Big]\, dx = \int_\Omega (\partial_h v_j \partial_h v_j + \gamma \partial_h v_h \partial_j v_j)\, dx =$$

$$= (2\pi)^{-3} \int_{\mathbb{R}^3} \big(\widehat{\partial_h v_j}\, \overline{\widehat{\partial_h v_j}} + \gamma \widehat{\partial_h v_h}\, \overline{\widehat{\partial_j v_j}}\big)\, dy = (2\pi)^{-3} \int_{\mathbb{R}^3} \big(|y|^2 |\widehat{v}|^2 + \gamma |y \cdot \widehat{v}|^2\big)\, dy \geqslant$$

$$\geqslant \min\{1, 1+\gamma\} (2\pi)^{-3} \int_{\mathbb{R}^3} |y|^2 |\widehat{v}|^2\, dy =$$

$$= \min\{1, 1+\gamma\} \int_\Omega \sum_{j=1}^3 |\nabla v_j|^2\, dx. \tag{43}$$

This implies that (41) holds for any v such that the left-hand side vanishes and that E is L^p-dissipative if and only if

$$C_p \leqslant \inf \frac{\int_\Omega \big[\sum_{j=1}^{3} |\nabla v_j|^2 + \gamma \,|\operatorname{div} v|^2 \big]\, dx}{\int_\Omega \big[|\nabla |v||^2 + \gamma \,|v|^{-2} |v_h \partial_h |v||^2 \big]\, dx}\,, \tag{44}$$

where the infimum is taken over all $v \in (C_0^1(\Omega))^3$ such that the denominator is positive.

From (42) we get

$$\big|\nabla |v|\big|^2 + \gamma \,|v|^{-2} \big|v_h \partial_h |v|\big|^2 \leqslant$$
$$\leqslant \max\{1, 1+\gamma\}\, |\lambda|^2 \leqslant \max\{1, 1+\gamma\} \sum_{j=1}^{3} |\nabla v_j|^2.$$

Keeping in mind also (43) we find that

$$\frac{\int_\Omega \big[\sum_{j=1}^{3} |\nabla v_j|^2 + \gamma \,|\operatorname{div} v|^2 \big]\, dx}{\int_\Omega \big[|\nabla |v||^2 + \gamma \,|v|^{-2} |v_h \partial_h |v||^2 \big]\, dx} \geqslant \frac{\min\{1, 1+\gamma\}}{\max\{1, 1+\gamma\}}\,.$$

Therefore condition (44) is satisfied if

$$C_p \leqslant \frac{\min\{1, 1+\gamma\}}{\max\{1, 1+\gamma\}}\,.$$

This inequality being equivalent to (40), the proof is complete. □

Remark 2. The Theorems of this section hold in any dimension $n \geqslant 3$ with the same proof.

4. Weighted L^p-negativity of elasticity system defined on rotationally symmetric vector functions

Let Φ be a point on the $(n-2)$-dimensional unit sphere S^{n-2} with spherical coordinates $\{\vartheta_j\}_{j=1,\dots,n-3}$ and φ, where $\vartheta_j \in (0, \pi)$ and $\varphi \in [0, 2\pi)$. A point $x \in \mathbb{R}^n$ is represented as a triple $(\varrho, \vartheta, \Phi)$, where $\varrho > 0$ and $\vartheta \in [0, \pi]$. Correspondingly, a vector u can be written as $u = (u_\varrho, u_\vartheta, u_\Phi)$ with $u_\Phi = (u_{\vartheta_{n-3}}, \dots, u_{\vartheta_1}, u_\varphi)$. We call $u_\varrho, u_\vartheta, u_\Phi$ the spherical components of the vector u.

Theorem 5. *Let the spherical components u_ϑ and u_Φ of the vector u vanish, i.e. $u = (u_\varrho, 0, 0)$, and let u_ϱ depend only on the variable ϱ. Then, if $\alpha \geqslant n-2$, we have*

$$\int_{\mathbb{R}^n} \Big(\Delta u + (1-2\nu)^{-1} \nabla \operatorname{div} u \Big) |u|^{p-2} u \,\frac{dx}{|x|^\alpha} \leqslant 0 \tag{45}$$

for any $u \in (C_0^\infty(\mathbb{R}^n \setminus \{0\}))^n$ *satisfying the aforesaid symmetric conditions, if and only if*

$$-(p-1)(n+p'-2) \leqslant \alpha \leqslant n+p-2. \tag{46}$$

If $\alpha < n-2$ *the same result holds replacing* $(C_0^\infty(\mathbb{R}^n \setminus \{0\}))^n$ *by* $(C_0^\infty(\mathbb{R}^n))^n$.

Proof. Setting

$$g_\varepsilon(s) = (s^2+\varepsilon^2)^{1/2},$$

and denoting by ω_{n-1} the $(n-1)$-dimensional measure of the unit sphere in $\mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n} \Delta u\, g_\varepsilon(|u|)^{p-2} u \frac{dx}{|x|^\alpha} =$$

$$= \omega_{n-1} \int_0^{+\infty} \Big(\frac{1}{\varrho^{n-1}}\, \partial_\varrho(\varrho^{n-1}\partial_\varrho u_\varrho) - \frac{n-1}{\varrho^2} u_\varrho\Big) g_\varepsilon(|u_\varrho|)^{p-2} u_\varrho \varrho^{n-1-\alpha}\, d\varrho.$$

An integration by parts gives

$$\int_0^{+\infty} \partial_\varrho(\varrho^{n-1}\partial_\varrho u_\varrho) g_\varepsilon(|u_\varrho|)^{p-2} u_\varrho \varrho^{-\alpha}\, d\varrho =$$

$$= -\int_0^{+\infty} \varrho^{n-1}\partial_\varrho u_\varrho \partial_\varrho\big(g_\varepsilon(|u_\varrho|)^{p-2} u_\varrho \varrho^{-\alpha}\big)\, d\varrho =$$

$$= -\int_0^{+\infty} \partial_\varrho u_\varrho \partial_\varrho (g_\varepsilon(|u_\varrho|)^{p-2} u_\varrho) \varrho^{n-1-\alpha}\, d\varrho +$$

$$+ \alpha \int_0^{+\infty} g_\varepsilon(|u_\varrho|)^{p-2} u_\varrho \partial_\varrho u_\varrho \varrho^{n-\alpha-2}\, d\varrho. \tag{47}$$

Since

$$\partial_\varrho(g_\varepsilon\big(|u_\varrho|)^p\big) = p\, g_\varepsilon(|u_\varrho|)^{p-2} u_\varrho \partial_\varrho u_\varrho, \tag{48}$$

we have, by means of another integration by parts in the last integral of (47),

$$\alpha \int_0^{+\infty} g_\varepsilon(|u_\varrho|)^{p-2} u_\varrho \partial_\varrho u_\varrho \varrho^{n-\alpha-2}\, d\varrho = \frac{\alpha}{p} \int_0^{+\infty} \partial_\varrho\big(g_\varepsilon(|u_\varrho|)^p\big) \varrho^{n-\alpha-2}\, d\varrho =$$

$$= -\frac{\alpha(n-2-\alpha)}{p} \int_K g_\varepsilon(|u_\varrho|)^p \varrho^{n-3-\alpha}\, d\varrho + \mathcal{O}(\varepsilon^p),$$

where K is the support of u_ϱ.

This proves the identity

$$\int\limits_{\mathbb{R}^n} \Delta u\, g_\varepsilon(|u|)^{p-2} u\, \frac{dx}{|x|^\alpha} = -\omega_{n-1}\Big[(n-1)\int\limits_K g_\varepsilon(|u_\varrho|)^{p-2} u_\varrho^2 \varrho^{n-3-\alpha}\, d\varrho + \\ + \frac{\alpha(n-2-\alpha)}{p}\int\limits_K g_\varepsilon(|u_\varrho|)^p \varrho^{n-3-\alpha}\, d\varrho + \\ + \int\limits_K \partial_\varrho u_\varrho \partial_\varrho\big(g_\varepsilon(|u_\varrho|)^{p-2}u_\varrho\big)\varrho^{n-1-\alpha}\, d\varrho\Big] + \mathcal{O}(\varepsilon^p). \quad (49)$$

We have also

$$\int\limits_{\mathbb{R}^n} \nabla(\operatorname{div} u) g_\varepsilon(|u|)^{p-2} u\, \frac{dx}{|x|^\alpha} = -\int\limits_{\mathbb{R}^n} \operatorname{div} u\ \operatorname{div}\big(g_\varepsilon(|u|)^{p-2}u|x|^{-\alpha}\big)\, dx = \\ = -\omega_{n-1}\int\limits_0^{+\infty} \frac{1}{\varrho^{n-1}}\,\partial_\varrho(\varrho^{n-1}u_\varrho)\partial_\varrho\big(\varrho^{n-1-\alpha}g_\varepsilon(|u_\varrho|)^{p-2}u_\varrho\big)\, d\varrho. \quad (50)$$

Moreover,

$$\frac{1}{\varrho^{n-1}}\,\partial_\varrho(\varrho^{n-1}u_\varrho)\,\partial_\varrho\big(\varrho^{n-1-\alpha}g_\varepsilon(|u_\varrho|)^{p-2}u_\varrho\big) = \\ = (n-1)(n-1-\alpha)\varrho^{n-3-\alpha}g_\varepsilon(|u_\varrho|)^{p-2}u_\varrho^2 + \\ + (n-1)\varrho^{n-2-\alpha}u_\varrho\partial_\varrho\big(g_\varepsilon(|u_\varrho|)^{p-2}u_\varrho\big) + (n-1-\alpha)\varrho^{n-2-\alpha}g_\varepsilon(|u_\varrho|)^{p-2}u_\varrho\partial_\varrho u_\varrho + \\ + \varrho^{n-1-\alpha}\partial_\varrho u_\varrho\partial_\varrho\big(g_\varepsilon(|u_\varrho|)^{p-2}u_\varrho\big). \quad (51)$$

In view of (48) we may write

$$\int\limits_0^{+\infty} \varrho^{n-2-\alpha}g_\varepsilon(|u_\varrho|)^{p-2}u_\varrho\partial_\varrho u_\varrho\, d\varrho = \frac{1}{p}\int\limits_0^{+\infty} \varrho^{n-2-\alpha}\partial_\varrho\big(g_\varepsilon(|u_\varrho|)^p\big)\, d\varrho = \\ = -\frac{n-2-\alpha}{p}\int\limits_K \varrho^{n-3-\alpha}g_\varepsilon(|u_\varrho|)^p\, d\varrho + \mathcal{O}(\varepsilon^p). \quad (52)$$

Since

$$\partial_\varrho\big(g_\varepsilon(|u_\varrho|)^{p-2}u_\varrho^2\big) = u_\varrho\partial_\varrho\big(g_\varepsilon(|u_\varrho|)^{p-2}u_\varrho\big) + g_\varepsilon(|u_\varrho|)^{p-2}u_\varrho\partial_\varrho u_\varrho$$

and using again (48), we find

$$\int\limits_0^{+\infty} \varrho^{n-2-\alpha}u_\varrho\partial_\varrho\big(g_\varepsilon(|u_\varrho|)^{p-2}u_\varrho\big)\, d\varrho = \\ = \int\limits_0^{+\infty} \varrho^{n-2-\alpha}\partial_\varrho\big(g_\varepsilon(|u_\varrho|)^{p-2}u_\varrho^2\big)\, d\varrho - \int\limits_0^{+\infty} \varrho^{n-2-\alpha}g_\varepsilon(|u_\varrho|)^{p-2}u_\varrho\partial_\varrho u_\varrho\, d\varrho =$$

$$= -(n-2-\alpha)\int\limits_K \varrho^{n-3-\alpha} g_\varepsilon(|u_\varrho|)^{p-2} u_\varrho^2\, d\varrho -$$

$$-\frac{1}{p}\int\limits_0^{+\infty} \varrho^{n-2-\alpha}\partial_\varrho\big(g_\varepsilon(|u_\varrho|)^p\big)\, d\varrho + \mathcal{O}(\varepsilon^p) =$$

$$= -(n-2-\alpha)\int\limits_K \varrho^{n-3-\alpha} g_\varepsilon(|u_\varrho|)^{p-2} u_\varrho^2\, d\varrho +$$

$$+\frac{n-2-\alpha}{p}\int\limits_K \varrho^{n-3-\alpha} g_\varepsilon(|u_\varrho|)^p\, d\varrho + \mathcal{O}(\varepsilon^p). \tag{53}$$

By (50), (51), (52) and (53) we obtain

$$\int\limits_{\mathbb{R}^n} \nabla(\operatorname{div} u) g_\varepsilon(|u|)^{p-2} u\, \frac{dx}{|x|^\alpha} =$$

$$= -\omega_{n-1}\Bigg[(n-1)\int\limits_K \varrho^{n-3-\alpha} g_\varepsilon(|u_\varrho|)^{p-2} u_\varrho^2\, d\varrho +$$

$$+\frac{\alpha(n-2-\alpha)}{p}\int\limits_K \varrho^{n-3-\alpha} g_\varepsilon(|u_\varrho|)^p\, d\varrho +$$

$$+\int\limits_K \partial_\varrho u_\varrho \partial_\varrho\big(g_\varepsilon(|u_\varrho|)^{p-2} u_\varrho\big)\varrho^{n-1-\alpha}\, d\varrho\Bigg] + \mathcal{O}(\varepsilon^p). \tag{54}$$

From (49) and (54) it follows that

$$\int\limits_{\mathbb{R}^n} \Big(\Delta u + \frac{1}{1-2\nu}\nabla \operatorname{div} u\Big) g_\varepsilon(|u|)^{p-2} u\, \frac{dx}{|x|^\alpha} =$$

$$= -\omega_{n-1}\frac{2(1-\nu)}{1-2\nu}\Bigg[(n-1)\int\limits_K \varrho^{n-3-\alpha} g_\varepsilon(|u_\varrho|)^{p-2} u_\varrho^2\, d\varrho +$$

$$+\frac{\alpha(n-2-\alpha)}{p}\int\limits_K \varrho^{n-3-\alpha} g_\varepsilon(|u_\varrho|)^p\, d\varrho +$$

$$+\int\limits_K \partial_\varrho u_\varrho \partial_\varrho\big(g_\varepsilon(|u_\varrho|)^{p-2} u_\varrho\big)\varrho^{n-1-\alpha}\, d\varrho\Bigg] + \mathcal{O}(\varepsilon^p).$$

Seeing that, given $a \in \mathbb{R}$, there exists a constant C_α such that $(g_\varepsilon(s))^a \leqslant C_\alpha(s^a + \varepsilon^a)$ $(s \geqslant 0)$, we may apply the dominated convergence theorem and find

$$\int_{\mathbb{R}^n} \Big(\Delta u + \frac{1}{1-2\nu}\nabla \operatorname{div} u\Big)|u|^{p-2}u\, \frac{dx}{|x|^\alpha} =$$

$$= -\omega_{n-1}\frac{2(1-\nu)}{1-2\nu}\Big\{\Big[n-1+\frac{\alpha(n-2-\alpha)}{p}\Big]\int_K \varrho^{n-3-\alpha}|u_\varrho|^p\, d\varrho + \\ + \int_K \partial_\varrho u_\varrho \partial_\varrho(|u_\varrho|^{p-2}u_\varrho)\varrho^{n-1-\alpha}\, d\varrho\Big\}.$$

Keeping in mind that either $\nu > 1$ or $\nu < 1/2$, the last equality shows that (45) holds if and only if

$$\Big[n-1+\frac{\alpha(n-2-\alpha)}{p}\Big]\int_K \varrho^{n-3-\alpha}|u_\varrho|^p\, d\varrho + \\ + \int_K \partial_\varrho u_\varrho \partial_\varrho(|u_\varrho|^{p-2})\varrho^{n-1-\alpha}\, d\varrho \geqslant 0. \quad (55)$$

Setting $v_\varrho = |u_\varrho|^{(p-2)/2}u_\varrho$, we see that (55) is equivalent to

$$\Big[n-1+\frac{\alpha(n-2-\alpha)}{p}\Big]\int_0^{+\infty} |v_\varrho|^2\varrho^{n-3-\alpha}\, d\varrho + \\ + \frac{4}{p\,p'}\int_0^{+\infty} (\partial_\varrho v_\varrho)^2\varrho^{n-1-\alpha}\, d\varrho \geqslant 0. \quad (56)$$

If $\alpha = n-2$ the inequality (56) is obviously satisfied. For $\alpha \neq n-2$, we recall the Hardy inequality (see, for instance, Maz'ya [13, p. 40])

$$\int_0^{+\infty} \frac{v^2(\varrho)}{\varrho^{\alpha-n+3}}\, d\varrho \leqslant \frac{4}{(\alpha-n+2)^2}\int_0^{+\infty} \frac{(\partial_\varrho v(\varrho))^2}{\varrho^{\alpha-n+1}}\, d\varrho, \quad (57)$$

which holds for any $v \in C_0^\infty(\mathbb{R})$ provided $\alpha \neq n-2$, under the condition $v(0) = 0$ when $\alpha > n-2$.

Inequality (56) can be written as

$$-\frac{p\,p'}{4}\Big[n-1+\frac{\alpha(n-2-\alpha)}{p}\Big]\int_0^{+\infty} |v_\varrho|^2\varrho^{n-3-\alpha}\, d\varrho \leqslant \\ \leqslant \int_0^{+\infty} (\partial_\varrho v_\varrho)^2\varrho^{n-1-\alpha}\, d\varrho. \quad (58)$$

Now we see that (58) holds if and only if

$$-\frac{p\,p'}{4}\left[n-1+\frac{\alpha(n-2-\alpha)}{p}\right]\leqslant\frac{(\alpha-n+2)^2}{4}\,. \tag{59}$$

In fact, if (59) holds, then (58) is true, because of (57). Viceversa, if (58) holds, thanks to the arbitrariness of v_ϱ and to the sharpness of the constant in (57), we get (59).

A simple manipulation shows that the latter inequality is equivalent to

$$-\frac{(\alpha-(n+p-2))(\frac{\alpha}{p-1}+(n+p'-2))}{p\,p'}\geqslant 0,$$

which in turn is equivalent to (46). The theorem is proved. □

We remark that the inequalities

$$-(p-1)(n+p'-2)<0<n+p-2$$

are always satisfied and therefore condition (46) is never empty.

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(Received 17.09.2013)

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