Memoirs on Differential Equations and Mathematical Physics
Volume 58, 2013, 1–24

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THE BOUNDARY VALUE PROBLEMS OF STATIONARY OSCILLATIONS IN THE THEORY OF TWO-TEMPERATURE ELASTIC MIXTURES

Abstract. We derive Green's formulas for the system of differential equations of stationary oscillations in the theory of elastic mixtures, which enable us to prove the uniqueness theorems for solutions of the boundary value problems. The jump formulas for single and double-layer potentials are derived. Using the theories of potentials and integral equations the existence of solutions is proved.

2010 Mathematics Subject Classification. 74A15, 75F20, 74H25, 74B10.

Key words and phrases. Composite body, theory of mixtures, fundamental matrix, metaharmonic function.

რეზიუმე. სტატიაში მიღებულია დრეკად ნარევთა თეორიის სტაციონალური რხევის დიფერენციალურ განტოლებათა სისტემისათვის გრინის ფორმულები, რომელთა დახმარებით დამტკიცებულია სასაზღვრო ამოცანების ამონახსნის ერთადერთობის თეორემები. მიღებულია მარტივი და ორმაგი ფენის პოტენციალების წყვეტის ფორმულები. პოტენციალთა და ინტეგრალურ განტოლებათა თეორიის გამოყენებით დამტკიცებულია ამოცანების ამონახსნის არსებობის თეორემები.

1. Introduction

Elastic composite materials with complex structures, as well as with structures composed of substantially differing materials are widely applied in the modern technological processes. Hemitropic elastic materials, mixtures produced from two or more elastic materials, etc., belong to the class of such composite materials and structures. The study of practical problems of mechanical properties of such materials naturally results in the necessity to develop mathematical models, which would allow to get more precise description of actual processes ongoing during the experiments. Mathematical modeling for such materials commenced as early as in the sixties of the past century. The first mathematical model of an elastic mixture (solid with solid), the so-called diffuse model, was developed by A. Green and T. Steel in 1966. In this model, the interaction force between components depends upon the difference of displacement vectors of components. In the same year they have developed the single-temperature thermoelasticity theory diffuse model of the elastic mixtures. Mathematical model of the linear theory of thermoelasticity of two-temperature elastic mixtures for the composites of granular, fibrous and layered structures was developed in 1984 by L. Khoroshun and N. Soltanov. Normally, the study of processes ongoing in the body is reduced, in the relevant mathematical model described by the system of differential equations with partial derivatives, to the study of boundary value problems (BVPs), mixed type BVPs and boundary-contact problems, and also the fundamental matrix for solving the system of differential equations playing a substantial role. For the diffuse and displacement models of the two-component mixtures (single-temperature) thermoelasticity theory, the issue of steadiness and correctness, identification of the asymptotic behavior of problem solution, proving of the uniqueness and existence theorems, solution of the BVPs for the domains bounded by the specific surfaces, as absolutely and uniformly convergent series, are studied by many scientists, among them: Alves, Munoz Rivera, Quintanilla [2], Basheleishvili [3], Basheleishvili, Zazashvili [4], Burchuladze, Svanadze [6], Gales [9], Giorgashvili, Skhvitaridze [13], [12], Giorgashvili, Karseladze, Sadunishvili [11], Iesan [18], Nappa [29], Natroshvili, Jaghmaidze, Svanadze [36], Svanadze [42], Quintanilla [41], Pompei [40], etc.

In this paper we derive Green's formulas for the system of differential equations of stationary oscillations in the theory of elastic mixtures, which enable us to prove the uniqueness theorems for solutions of the boundary value problems. Further, we establish mapping properties and jump formulas for the single and double-layer potentials, and analyse the Fredholm properties of the corresponding boundary operators. Using the potential method and the theory of singular integral equations, the existence of solutions to the basic boundary value problems is proved.

We treat here only the classical setting of basic boundary value problems for smooth domains, however applying the results obtained in the references: Agranovich [1], Buchukuri, Chkadua, Duduchava, Natroshvili [5], Duduchava, Natroshvili [8], Gao [10], Jentsch, Natroshvili [19–21], Jentsch, Natroshvili, Wendland [22, 23], Kupradze, Gegelia, Basheleishvili, Burchuladze [25], Mitrea, Mitrea, Pipher [28], Natroshvili [30–32], Natroshvili, Giorgashvili, Stratis [33], Natroshvili, Giorgashvili, Zazashvili [34], Natroshvili, Kharibegashvili, Tediashvili [37], Natroshvili, Sadunishvili [38], Natroshvili, Stratis [39], and using the same type approaches and reasonings, one can analyze the generalized basic and mixed type boundary value problems, as well as crack type and interface problems in Sobolev–Slobodetskii and Bessel potential spaces for smooth and Lipschitz domains.

2. Basic Differential Equations

The basic dynamical relationships for the two-component elastic mixtures, taking two-temperature thermal field into consideration, are mathematically described by the following system of partial differential equations [24]

$$\begin{gathered}
a_1\Delta u'(x,t)+b_1\operatorname{grad}\operatorname{div}u'(x,t)+c\Delta u''(x,t)+\\
+d\operatorname{grad}\operatorname{div}u''(x,t)-\varkappa\big[u'(x,t)-u''(x,t)\big]-\\
-\eta_1\operatorname{grad}\vartheta_1(x,t)-\eta_2\operatorname{grad}\vartheta_2(x,t)+\rho_1F'(x,t)=\rho_1\partial_{tt}^2u'(x,t),\\
c\Delta u'(x,t)+d\operatorname{grad}\operatorname{div}u'(x,t)+a_2\Delta u''(x,t)+\\
+b_2\operatorname{grad}\operatorname{div}u''(x,t)+\varkappa\big[u'(x,t)-u''(x,t)\big]-\\
-\zeta_1\operatorname{grad}\vartheta_1(x,t)-\zeta_2\operatorname{grad}\vartheta_2(x,t)+\rho_2F''(x,t)=\rho_2\partial_{tt}^2u''(x,t),\\
\varkappa_1\Delta\vartheta_1(x,t)+\varkappa_2\Delta\vartheta_2(x,t)-\alpha\big[\vartheta_1(x,t)-\vartheta_2(x,t)\big]-\\
-\eta_1\operatorname{div}\partial_tu'(x,t)-\zeta_1\operatorname{div}\partial_tu''(x,t)+G'(x,t)=\varkappa'\partial_t\vartheta_1(x,t),\\
\varkappa_2\Delta\vartheta_1(x,t)+\varkappa_3\Delta\vartheta_2(x,t)+\alpha\big[\vartheta_1(x,t)-\vartheta_2(x,t)\big]-\\
-\eta_2\operatorname{div}\partial_tu'(x,t)-\zeta_2\operatorname{div}\partial_tu''(x,t)+G''(x,t)=\varkappa''\partial_t\vartheta_2(x,t),
\end{gathered}\tag{2.1}$$

where Δ is the three-dimensional Laplace operator, $u'=(u_1',u_2',u_3')^\top$, $u''=(u_1'',u_2'',u_3'')^\top$ are partial displacement vectors, ϑ_1 and ϑ_2 are temperatures of each component of the mixture, $F'=(F_1',F_2',F_3')^\top$, $F''=(F_1'',F_2'',F_3'')^\top$ are the mass forces, G', G'' are the thermal sources located in the components, a_j, b_j, c, d are the elasticity coefficients, $\varkappa$, η_j, ζ_j, $\varkappa_j$, $\varkappa_3$, $\varkappa'$, $\varkappa''$, α, $j=1,2$, are the mechanical and thermal constants of the elastic mixture, ρ_1, ρ_2 are the densities of mixture components, t is a time variable, $x=(x_1,x_2,x_3)$ is a point in the three-dimensional Cartesian space, $\top$ denotes transposition.

In the system (2.1), a_j, b_j, c, d, $j=1,2$, are the constants given as follows [15, 17]

$$a_1=\mu_1-\lambda_5,\ \ b_1=\mu_1+\lambda_5+\lambda_1-\frac{\rho_2}{\rho}\,\alpha_0,$$

$$a_2 = \mu_2 - \lambda_5, \ \ b_2 = \mu_2 + \lambda_5 + \lambda_2 + \frac{\rho_1}{\rho}\,\alpha_0,$$
$$c = \mu_3 + \lambda_5, \ \ d = \mu_3 - \lambda_5 + \lambda_3 - \frac{\rho_1}{\rho}\,\alpha_0, \ \ \alpha_0 = \lambda_3 - \lambda_4, \ \ \rho = \rho_1 + \rho_2,$$

where $\lambda_1, \lambda_2, \dots, \lambda_5, \mu_1, \mu_2, \mu_3$ are elastic constants satisfying the conditions

$$\mu_1 > 0, \ \ \lambda_5 < 0, \ \ \mu_1\mu_2 - \mu_3^2 > 0, \ \ \lambda_1 + \frac{2}{3}\,\mu_1 - \frac{\rho_2}{\rho}\,\alpha_0 > 0,$$
$$\Big(\lambda_1 + \frac{2}{3}\,\mu_1 - \frac{\rho_2}{\rho}\,\alpha_0\Big)\Big(\lambda_2 + \frac{2}{3}\,\mu_2 - \frac{\rho_1}{\rho}\,\alpha_0\Big) > \Big(\lambda_3 + \frac{2}{3}\,\mu_3 - \frac{\rho_1}{\rho}\,\alpha_0\Big)^2.$$

From these inequalities it follows that

$$\begin{gathered} a_1 > 0, \quad a_1 + b_1 > 0, \\ d_1 := a_1 a_2 - c^2 > 0, \quad d_2 := (a_1 + b_1)(a_2 + b_2) - (c + d)^2 > 0. \end{gathered} \tag{2.2}$$

In addition, from physical considerations it follows that

$$\begin{gathered} \rho_1 > 0, \ \ \rho_2 > 0, \ \ \alpha > 0, \ \ \varkappa > 0, \ \ \varkappa' > 0, \ \ \varkappa'' > 0, \\ \varkappa_j > 0, \ j = 1, 2, 3, \quad d_3 := \varkappa_1\varkappa_3 - \varkappa_2^2 > 0. \end{gathered} \tag{2.3}$$

If all the functions involved in the system (2.1) are harmonic time dependent, i.e., $u'(x,t) = u'(x)\exp(-i\sigma t)$, $u''(x,t) = u''(x)\exp(-i\sigma t)$, $\vartheta_1(x,t) = \vartheta_1(x)\exp(-i\sigma t)$, $\vartheta_2(x,t) = \vartheta_2(x)\exp(-i\sigma t)$, $F'(x,t) = F'(x)\exp(-i\sigma t)$, $F''(x,t) = F''(x)\exp(-i\sigma t)$, $G'(x,t) = G'(x)\exp(-i\sigma t)$, $G''(x,t) = G''(x)\exp(-i\sigma t)$, where $\sigma \in \mathbb{R}$ is oscillation frequency, $i = \sqrt{-1}$, then from the system (2.1) we obtain the following system of differential equations of the theory of stationary oscillations of two-temperature elastic mixture:

$$\begin{gathered}
a_1\Delta u'(x) + b_1 \operatorname{grad}\operatorname{div} u'(x) + c\Delta u''(x) + d \operatorname{grad}\operatorname{div} u''(x) - \\
-\varkappa\big[u'(x) - u''(x)\big] - \eta_1 \operatorname{grad}\vartheta_1(x) - \eta_2 \operatorname{grad}\vartheta_2(x) + \\
+\rho_1\sigma^2 u'(x) = -\rho_1 F'(x), \\
c\Delta u'(x) + d \operatorname{grad}\operatorname{div} u'(x) + a_2\Delta u''(x) + b_2 \operatorname{grad}\operatorname{div} u''(x) + \\
+\varkappa\big[u'(x) - u''(x)\big] - \zeta_1 \operatorname{grad}\vartheta_1(x) - \zeta_2 \operatorname{grad}\vartheta_2(x) + \\
+\rho_2\sigma^2 u''(x) = -\rho_2 F''(x), \\
\varkappa_1\Delta\vartheta_1(x) + \varkappa_2\Delta\vartheta_2(x) - \alpha\big[\vartheta_1(x) - \vartheta_2(x)\big] + i\sigma\eta_1 \operatorname{div} u'(x) + \\
+i\sigma\zeta_1 \operatorname{div} u''(x) + i\sigma\varkappa'\vartheta_1(x) = -G'(x), \\
\varkappa_2\Delta\vartheta_1(x) + \varkappa_3\Delta\vartheta_2(x) + \alpha\big[\vartheta_1(x) - \vartheta_2(x)\big] + i\sigma\eta_2 \operatorname{div} u'(x) + \\
+i\sigma\zeta_2 \operatorname{div} u''(x) + i\sigma\varkappa''\vartheta_2(x) = -G''(x);
\end{gathered} \tag{2.4}$$

here u', u'', F', F'' are the complex vector-functions and ϑ_1, ϑ_2, G', G'', are the complex scalar functions.

If $\sigma = \sigma_1 + i\sigma_2$ is a complex parameter and $\sigma_2 \neq 0$, then (2.4) is called the system of differential equations of pseudooscillations, and if $\sigma = 0$, then (2.4) is the system of differential equations of statics.

Let us introduce the matrix differential operator of order 8×8, generated by the left hand side expressions in system (2.4),

$$L(\partial,\sigma):=\begin{bmatrix} L^{(1)}(\partial,\sigma) & L^{(2)}(\partial,\sigma) & L^{(5)}(\partial,\sigma) & L^{(6)}(\partial,\sigma)\\ L^{(3)}(\partial,\sigma) & L^{(4)}(\partial,\sigma) & L^{(7)}(\partial,\sigma) & L^{(8)}(\partial,\sigma)\\ L^{(9)}(\partial,\sigma) & L^{(10)}(\partial,\sigma) & L^{(13)}(\partial,\sigma) & L^{(14)}(\partial,\sigma)\\ L^{(11)}(\partial,\sigma) & L^{(12)}(\partial,\sigma) & L^{(15)}(\partial,\sigma) & L^{(16)}(\partial,\sigma)\end{bmatrix}_{8\times 8},$$

where

$$\begin{aligned} L^{(1)}(\partial,\sigma) &:= (a_1\Delta+\alpha')I_3+b_1Q(\partial),\\ L^{(2)}(\partial,\sigma)=L^{(3)}(\partial,\sigma) &:= (c\Delta+\varkappa)I_3+dQ(\partial),\\ L^{(4)}(\partial,\sigma) &:= (a_2\Delta+\alpha'')I_3+b_2Q(\partial),\\ L^{(4+j)}(\partial,\sigma) &:= -\eta_j\nabla^\top,\ L^{(6+j)}(\partial,\sigma)=-\zeta_j\nabla^\top,\ \ j=1,2,\\ L^{(9)}(\partial,\sigma) &:= i\sigma\eta_1\nabla,\ L^{(10)}(\partial,\sigma):=i\sigma\zeta_1\nabla,\\ L^{(11)}(\partial,\sigma) &:= i\sigma\eta_2\nabla,\ L^{(12)}(\partial,\sigma):=i\sigma\zeta_2\nabla,\\ L^{(13)}(\partial,\sigma) &:= \varkappa_1\Delta+\alpha_1,\ L^{(16)}(\partial,\sigma):=\varkappa_3\Delta+\alpha_2,\\ L^{(14)}(\partial,\sigma)=L^{(15)}(\partial,\sigma) &:= \varkappa_2\Delta+\alpha; \end{aligned}$$

here $\alpha'=-\varkappa+\rho_1\sigma^2$, $\alpha''=-\varkappa+\rho_2\sigma^2$ $\alpha_1=-\alpha+i\sigma\varkappa'$, $\alpha_2=-\alpha+i\sigma\varkappa''$, $\nabla\equiv\nabla(\partial):=[\partial_1,\partial_2,\partial_3]$, $\partial=(\partial_1,\partial_2,\partial_3)$, $\partial_j=\partial/\partial x_j$, $j=1,2,3$, I_3 is the 3×3 unit matrix, $Q(\partial):=[\partial_k\partial_j]_{3\times 3}$.

Applying these notation, the system (2.4) can be written as

$$L(\partial,\sigma)U(x)=\Phi(x),$$

where $U=(u',u'',\vartheta_1,\vartheta_2)^\top$, $\Phi=(-\rho_1F',-\rho_2F'',-G',-G'')^\top$.

In what follows, we apply the following differential operators:

$$L_0(\partial):=\begin{bmatrix} L_0^{(1)}(\partial) & L_0^{(2)}(\partial) & [0]_{3\times 1} & [0]_{3\times 1}\\ L_0^{(3)}(\partial) & L_0^{(4)}(\partial) & [0]_{3\times 1} & [0]_{3\times 1}\\ [0]_{1\times 3} & [0]_{1\times 3} & \varkappa_1\Delta & \varkappa_2\Delta\\ [0]_{1\times 3} & [0]_{1\times 3} & \varkappa_2\Delta & \varkappa_3\Delta \end{bmatrix}_{8\times 8}, \tag{2.5}$$

$$\widetilde{L}_0(\partial):=\begin{bmatrix} L_0^{(1)}(\partial) & L_0^{(2)}(\partial)\\ L_0^{(3)}(\partial) & L_0^{(4)}(\partial)\end{bmatrix}_{6\times 6},$$

where

$$\begin{aligned} L_0^{(1)}(\partial) &:= a_1I_3\Delta+b_1Q(\partial),\\ L_0^{(2)}(\partial)=L_0^{(3)}(\partial) &:= cI_3\Delta+dQ(\partial),\\ L_0^{(4)}(\partial) &:= a_2I_3\Delta+b_2Q(\partial). \end{aligned}$$

Further let us introduce the operators

$$
\begin{aligned}
T(\partial,n) &:= \begin{bmatrix} T^{(1)}(\partial,n) & T^{(2)}(\partial,n) \\ T^{(3)}(\partial,n) & T^{(4)}(\partial,n) \end{bmatrix}_{6\times 6}, \\
T^{(l)}(\partial,n) &= \Big[T^{(l)}_{kj}(\partial,n)\Big]_{3\times 3}, \quad l=\overline{1,4},
\end{aligned}
\tag{2.6}
$$

where [15, 16]

$$
\begin{aligned}
T^{(1)}_{kj}(\partial,n) &:= (\mu_1-\lambda_5)\delta_{kj}\partial_n + (\mu_1+\lambda_5)n_j\partial_k + \\
&\quad + \Big(\lambda_1 - \frac{\rho_2}{\rho}\,\alpha_0\Big)n_k\partial_j, \\
T^{(2)}_{kj}(\partial,n) = T^{(3)}_{kj}(\partial,n) &:= (\mu_3+\lambda_5)\delta_{kj}\partial_n + (\mu_3-\lambda_5)n_j\partial_k + \\
&\quad + \Big(\lambda_3 - \frac{\rho_1}{\rho}\,\alpha_0\Big)n_k\partial_j, \\
T^{(4)}_{kj}(\partial,n) &:= (\mu_2-\lambda_5)\delta_{kj}\partial_n + (\mu_2+\lambda_5)n_j\partial_k + \\
&\quad + \Big(\lambda_2 + \frac{\rho_1}{\rho}\,\alpha_0\Big)n_k\partial_j,
\end{aligned}
$$

where $\partial_n = \partial/\partial_n$ is the normal derivative, $n=(n_1,n_2,n_3)$;

$$
\widetilde{T}(\partial,n) := \begin{bmatrix} T^{(1)}(\partial,n) & T^{(2)}(\partial,n) & [0]_{3\times 1} & [0]_{3\times 1} \\ T^{(3)}(\partial,n) & T^{(4)}(\partial,n) & [0]_{3\times 1} & [0]_{3\times 1} \\ [0]_{1\times 3} & [0]_{1\times 3} & \varkappa_1\partial_n & \varkappa_2\partial_n \\ [0]_{1\times 3} & [0]_{1\times 3} & \varkappa_2\partial_n & \varkappa_3\partial_n \end{bmatrix}_{8\times 8},
$$

$$
\mathcal{P}(\partial,n) := \begin{bmatrix} T^{(1)}(\partial,n) & T^{(2)}(\partial,n) & -\eta_1 n^\top & -\eta_2 n^\top \\ T^{(3)}(\partial,n) & T^{(4)}(\partial,n) & -\zeta_1 n^\top & -\zeta_2 n^\top \\ [0]_{1\times 3} & [0]_{1\times 3} & \varkappa_1\partial_n & \varkappa_2\partial_n \\ [0]_{1\times 3} & [0]_{1\times 3} & \varkappa_2\partial_n & \varkappa_3\partial_n \end{bmatrix}_{8\times 8},
$$

$$
\mathcal{P}^*(\partial,n) := \begin{bmatrix} T^{(1)}(\partial,n) & T^{(2)}(\partial,n) & -i\sigma\eta_1 n^\top & -i\sigma\eta_2 n^\top \\ T^{(3)}(\partial,n) & T^{(4)}(\partial,n) & -i\sigma\zeta_1 n^\top & -i\sigma\zeta_2 n^\top \\ [0]_{1\times 3} & [0]_{1\times 3} & \varkappa_1\partial_n & \varkappa_2\partial_n \\ [0]_{1\times 3} & [0]_{1\times 3} & \varkappa_2\partial_n & \varkappa_3\partial_n \end{bmatrix}_{8\times 8}, \tag{2.7}
$$

where $T^{(l)}(\partial,n)$, $l=1,2,3,4$, are given by (2.6), $n^\top = (n_1,n_2,n_3)^\top$.

3. Green's Formulas

Let Ω^+ be a finite three-dimensional region bounded by the Lyapunov surface $\partial\Omega$; $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega^+}$.

Definition 3.1. *A vector $U=(u',u'',\vartheta_1,\vartheta_2)^\top$ will be called regular in a domain $\Omega\subset\mathbb{R}^3$ if $U\in C^2(\Omega)\cap C^1(\overline{\Omega})$.*

Let

$$U=(u,\vartheta)^\top,\quad V=(v,\vartheta')^\top,\quad u=(u',u'')^\top,\quad v=(v',v'')^\top,$$
$$\vartheta=(\vartheta_1,\vartheta_2)^\top,\quad \vartheta'=(\vartheta'_1,\vartheta'_2)^\top.$$

It can be proved that for regular vectors u and v, the following Green's formula is valid [36]

$$\int\limits_{\Omega^+} v\cdot\widetilde{L}_0(\partial)u\,dx=\int\limits_{\partial\Omega}[v(z)]^+\cdot[T(\partial,n)u(z)]^+\,ds-\int\limits_{\Omega^+}E(u,v)\,dx, \tag{3.1}$$

where the differential operator $T(\partial,n)$ is given by formula (2.6), $n(z)$ is the outward unit normal vector w.r.t. Ω^+ at the point $z\in\partial\Omega$, $a\cdot b=\sum\limits_{j=1}^{3}a_jb_j$ is the scalar product of vectors a and b, and $E(u,v)$ is a quatratic form defined as follows:

$$\begin{aligned}E(u,v)&=\Big(\lambda_1-\frac{\varrho_2}{\varrho}\,\alpha_0\Big)\operatorname{div}v'\operatorname{div}u'+\Big(\lambda_2+\frac{\varrho_1}{\varrho}\,\alpha_0\Big)\operatorname{div}v''\operatorname{div}u''+\\&\quad+\Big(\lambda_3-\frac{\varrho_1}{\varrho}\,\alpha_0\Big)(\operatorname{div}v'\operatorname{div}u''+\operatorname{div}v''\operatorname{div}u')+\\&\quad+\frac{\mu_1}{2}\sum_{k,j=1}^{3}(\partial_jv'_k+\partial_kv'_j)(\partial_ju'_k+\partial_ku'_j)+\frac{\mu_2}{2}\sum_{k,j=1}^{3}(\partial_jv''_k+\partial_kv''_j)(\partial_ju''_k+\partial_ku''_j)+\\&\quad+\frac{\mu_3}{2}\sum_{k,j=1}^{3}\Big[(\partial_jv'_k+\partial_kv'_j)(\partial_ju''_k+\partial_ku''_j)+(\partial_jv''_k+\partial_kv''_j)(\partial_ju'_k+\partial_ku'_j)\Big]-\\&\quad-\frac{\lambda_5}{2}\sum_{k,j=1}^{3}(\partial_jv'_k-\partial_kv'_j-\partial_jv''_k+\partial_kv''_j)(\partial_ju'_k-\partial_ku'_j-\partial_ju''_k+\partial_ku''_j).\end{aligned} \tag{3.2}$$

Rewrite the vector $L(\partial,\sigma)U$ as

$$L(\partial,\sigma)U=L_0(\partial)U+L'_0(\partial,\sigma)U, \tag{3.3}$$

where

$$L'_0(\partial,\sigma)U=\begin{bmatrix}\alpha'u'+\varkappa u''-\eta_1\nabla^\top\vartheta_1-\eta_2\nabla^\top\vartheta_2\\ \varkappa u'+\alpha''u''-\zeta_1\nabla^\top\vartheta_1-\zeta_2\nabla^\top\vartheta_2\\ i\sigma\eta_1\nabla u'+i\sigma\zeta_1\nabla u''+\alpha_1\vartheta_1+\alpha\vartheta_2\\ i\sigma\eta_2\nabla u'+i\sigma\zeta_2\nabla u''+\alpha\vartheta_1+\alpha_2\vartheta_2\end{bmatrix}_{8\times1}. \tag{3.4}$$

Note that

$$V\cdot L_0(\partial)U=v\cdot\widetilde{L}_0(\partial)u+\vartheta'_1(\varkappa_1\Delta\vartheta_1+\varkappa_2\Delta\vartheta_2)+\vartheta'_2(\varkappa_2\Delta\vartheta_1+\varkappa_3\Delta\vartheta_2). \tag{3.5}$$

The following equality is valid [43]

$$\int\limits_{\Omega^+} \vartheta_k' \Delta \vartheta_j \, dx = \\ = \int\limits_{\partial\Omega} \big[\vartheta_k'(z) \partial_n \vartheta_j(z)\big]^+ \, ds - \int\limits_{\Omega^+} \big(\nabla^\top \vartheta_k' \cdot \nabla^\top \vartheta_j\big) \, dx, \quad k,j = 1,2. \tag{3.6}$$

Using equalities (3.1) and (3.6), from (3.5) we have

$$\int\limits_{\Omega^+} V \cdot L_0(\partial) U \, dx = \int\limits_{\partial\Omega} \big[V(z) \cdot \widetilde{T}(\partial, n) U(z)\big]^+ \, ds - \int\limits_{\Omega^+} E(U,V) \, dx, \tag{3.7}$$

where

$$E(U,V) = E(u,v) + \varkappa_1 (\nabla^\top \vartheta_1' \cdot \nabla^\top \vartheta_1) + \\ + \varkappa_2 \big(\nabla^\top \vartheta_1' \cdot \nabla^\top \vartheta_2 + \nabla^\top \vartheta_2' \cdot \nabla^\top \vartheta_1\big) + \varkappa_3 (\nabla^\top \vartheta_2' \cdot \nabla^\top \vartheta_2)$$

and $E(u,v)$ is given by (3.2).

Multiplying both sides of equality (3.4) by vector $V = (v, \vartheta')^\top$ and taking into consideration the equality

$$\int\limits_{\Omega^+} v' \cdot \nabla^\top \vartheta_j \, dx = \int\limits_{\partial\Omega} \big[\vartheta_j(z) (n(z) \cdot v'(z))\big]^+ \, ds - \int\limits_{\Omega^+} \vartheta_j \nabla v' \, dx, \quad j = 1,2, \tag{3.8}$$

we obtain

$$\int\limits_{\Omega^+} V \cdot L_0'(\partial,\sigma) U \, dx = -\int\limits_{\partial\Omega} \Big[(\eta_1 \vartheta_1 + \eta_2 \vartheta_2)(n \cdot v') + (\zeta_1 \vartheta_1 + \zeta_2 \vartheta_2)(n \cdot v'')\Big]^+ \, ds + \\ + \int\limits_{\Omega^+} \Big[v'(\alpha' u' + \varkappa u'') + v''(\varkappa u' + \alpha'' u'') + \\ + i\sigma \big(\eta_1 \vartheta_1' \nabla u' + \zeta_1 \vartheta_1' \nabla u'' + \eta_2 \vartheta_2' \nabla u' + \zeta_2 \vartheta_2' \nabla u''\big) + \\ + \vartheta_1'(\alpha_1 \vartheta_1 + \alpha \vartheta_2) + \vartheta_2'(\alpha \vartheta_1 + \alpha_2 \vartheta_2)\Big] \, dx. \tag{3.9}$$

Combining equalities (3.7) and (3.9) we get

$$\int\limits_{\Omega^+} V \cdot L(\partial,\sigma) U \, dx = \int\limits_{\partial\Omega} \Big[V(z) \cdot \mathcal{P}(\partial, n) U(z)\Big]^+ \, ds - \\ \int\limits_{\Omega^+} \Big[E(U,V) - v' \cdot (\alpha' u' + \varkappa u'') - v'' \cdot (\varkappa u' + \alpha'' u'') - i\sigma \vartheta_1'(\eta_1 \nabla u' + \zeta_1 \nabla u'') - \\ - i\sigma \vartheta_2'(\eta_2 \nabla u' + \zeta_2 \nabla u'') - \vartheta_1'(\alpha_1 \vartheta_1 + \alpha \vartheta_2) - \vartheta_2'(\alpha \vartheta_1 + \alpha_2 \vartheta_2)\Big] \, dx. \tag{3.10}$$

With the help of equality (3.10), we derive

$$\int\limits_{\Omega^+}\Big[V\cdot L(\partial,\sigma)U-U\cdot L^*(\partial,\sigma)V\Big]\,dx=$$
$$=\int\limits_{\partial\Omega}\Big[V(z)\cdot\mathcal{P}(\partial,n)U(z)-U(z)\cdot\mathcal{P}^*(\partial,n)V(z)\Big]^+\,ds,\quad(3.11)$$

where $L^*(\partial,\sigma)=\big[L(-\partial,\sigma)\big]^\top$ and $\mathcal{P}^*(\partial,n)$ is given by (2.7). The formulas (3.10) and (3.11) are Green's formulas.

Assume that a vector $U=(u,\vartheta)^\top$ is e solution of equation $L(\partial,\sigma)U=0$. According to (3.3) we obtain

$$L_0(\partial)U+L_0'(\partial,\sigma)U=0,\quad(3.12)$$

where $L_0(\partial)$ is given by formula (2.5) and $L_0'(\partial,\sigma)U$ is defined by equality (3.4).

Let us multiply the first equation of (3.12) by the vector $\overline{u}'$, the second one by the vector $\overline{u}''$ and the complex conjugates of the third and fourth equations, respectively, by the functions $\frac{1}{i\overline{\sigma}}\vartheta_1$ and $\frac{1}{i\overline{\sigma}}\vartheta_2$ and sum up. In addition, taking into consideration equalities (3.1) and (3.8), we obtain

$$\int\limits_{\Omega^+}\bigg[-E(u,\overline{u})+\frac{i}{\varkappa_3\overline{\sigma}}\Big(d_3|\nabla^\top\vartheta_1|^2+|\varkappa_2\nabla^\top\vartheta_1+\varkappa_3\nabla^\top\vartheta_2|^2\Big)-\varkappa|u'-u''|^2+$$
$$+\rho_1\sigma^2|u'|^2+\rho_2\sigma^2|u''|^2+\frac{\alpha i}{\overline{\sigma}}\,|\vartheta_1-\vartheta_2|^2-\big(\varkappa'|\vartheta_1|^2+\varkappa''|\vartheta_2|^2\big)\bigg]\,dx+$$
$$+\int\limits_{\partial\Omega}\bigg[\overline{u}(z)T(\partial,n)u(z)-(\eta_1\vartheta_1+\eta_2\vartheta_2)(n\cdot\overline{u}')-(\zeta_1\vartheta_1+\zeta_2\vartheta_2)(n\cdot\overline{u}'')-$$
$$-\frac{i}{\varkappa_3\overline{\sigma}}\Big(d_3\vartheta_1\partial_n\overline{\vartheta}_1+(\varkappa_2\vartheta_1+\varkappa_3\vartheta_2)(\varkappa_2\partial_n\overline{\vartheta}_1+\varkappa_3\partial_n\overline{\vartheta}_2)\Big)\bigg]^+\,ds=0.\quad(3.13)$$

Here $\overline{u}$ is the complex conjugate of u and

$$E(u,\overline{u})=\frac{d_2}{a_1+b_1}\,|\operatorname{div}u''|^2+\frac{1}{a_1+b_1}\,\big|(a_1+b_1)\operatorname{div}u'+(c+d)\operatorname{div}u''\big|^2+$$
$$+\frac{d_4}{2\mu_1}\sum_{k\neq j=1}^3|\partial_ju_k''+\partial_ku_j''|^2+\frac{1}{2\mu_1}\sum_{k\neq j=1}^3\big|\mu_1(\partial_ju_j'+\partial_ku_j')+\mu_3(\partial_ju_k''+\partial_ku_j'')\big|^2-$$
$$-\frac{\lambda_5}{2}\sum_{k,j=1}^3\big|\partial_ju_k'-\partial_ku_j'-\partial_ju_k''+\partial_ku_j''\big|^2>0,\quad(3.14)$$

where $d_4=\mu_1\mu_2-\mu_3^2>0$. The sesquilinear form $E(u,\overline{u})$ is obtained from formula (3.2) by substituting the vectors v' and v'' by the vectors $\overline{u}'$ and $\overline{u}''$, respectively, and taking into consideration that $\lambda_1-\frac{\rho_2}{\rho}\,\alpha_0=a_1+b_1-2\mu_1$, $\lambda_2+\frac{\rho_1}{\rho}\,\alpha_0=a_2+b_2-2\mu_2$, $\lambda_3-\frac{\rho_1}{\rho}\,\alpha_0=c+d-2\mu_3$.

4. Formulation of Problems. Uniqueness Theorems

Problem $(\mathrm{I}^{(\sigma)})^{\pm}$ **(Dirichlet's problem).** Find a regular vector $U = (u', u'', \vartheta_1, \vartheta_2)^\top$ satisfying the system of differential equations

$$L(\partial,\sigma)U(x) = \Phi^{\pm}(x), \quad x \in \Omega^{\pm}, \tag{4.1}$$

and the boundary conditions

$$\{U(z)\}^{\pm} = f(z), \quad z \in \partial\Omega; \tag{4.2}$$

Problem $(\mathrm{II}^{(\sigma)})^{\pm}$ **(Neumann's problem).** Find a regular vector $U = (u', u'', \vartheta_1, \vartheta_2)^\top$ satisfying (4.1) and the boundary conditions

$$\{\mathcal{P}(\partial,n)\,U(z)\}^{\pm} = F(z), \quad z \in \partial\Omega; \tag{4.3}$$

here $\Phi^{\pm}$ are eight-component given vectors in $\Omega^{\pm}$, respectively while

$$f = (f^{(1)}, f^{(2)}, f^{(3)}, f^{(4)})^\top, \quad F = (F^{(1)}, F^{(2)}, F^{(3)}, F^{(4)})^\top,$$
$$f^{(j)} = (f_1^{(j)}, f_2^{(j)}, f_3^{(j)})^\top, \quad F^{(j)} = (F_1^{(j)}, F_2^{(j)}, F_3^{(j)})^\top, \quad j = 1,2,$$

with $f^{(j)}$, $F^{(j)}$, $j = 3,4$, being scalar function are assumed to be given on the boundary $\partial\Omega^{\pm}$; $n(z)$ is the outward unit normal vector w.r.t. Ω^+ at the point $z \in \partial\Omega$.

In the case of the exterior problems for the domain Ω^-, a vector $U(x)$ in a neighbourhood of infinity has to satisfy some sufficient vanishing conditions allowing one to write Green's formula (3.13) for the domain Ω^-.

Theorem 4.1. *If $\sigma = \sigma_1 + i\sigma_2$, where $\sigma_1 \in R$, $\sigma_2 > 0$, then the homogeneous problems $(\mathrm{I}^{(\sigma)})_0^{+}$ and $(\mathrm{II}^{(\sigma)})_0^{+}$ ($\Phi^+ = 0$, $f = 0$, $F = 0$) have only the trivial solution.*

Proof. If in equation (3.13) we take into consideration the homogeneous boundary conditions, we obtain

$$\int\limits_{\Omega^+} \Big[-E(u,\overline{u}) + \frac{i}{\varkappa_3\overline{\sigma}} \Big(d_3|\nabla^\top\vartheta_1|^2 + |\varkappa_2\nabla^\top\vartheta_1 + \varkappa_3\nabla^\top\vartheta_2|^2 \Big) - \varkappa|u'-u''|^2 +$$
$$+ \rho_1\sigma^2|u'|^2 + \rho_2\sigma^2|u''|^2 + \frac{\alpha i}{\overline{\sigma}}\,|\vartheta_1-\vartheta_2|^2 - \big(\varkappa'|\vartheta_1|^2 + \varkappa''|\vartheta_2|^2\big) \Big]\,dx = 0. \tag{4.4}$$

Separating the imaginary part of the equation (4.4), we obtain

$$\sigma_1 \int\limits_{\Omega^+} \Big[\frac{1}{\varkappa_3|\sigma|} \Big(d_3|\nabla^\top\vartheta_1|^2 + |\varkappa_2\nabla^\top\vartheta_1 + \varkappa_3\nabla^\top\vartheta_2|^2 \Big) +$$
$$+ 2\rho_1\sigma_2|u'|^2 + 2\rho_2\sigma_2|u''|^2 + \frac{\alpha}{|\sigma|^2}\,|\vartheta_1-\vartheta_2|^2 \Big]\,dx = 0. \tag{4.5}$$

Assuming that $\sigma_1 \neq 0$, from (4.5) we get $u'(x)=0$, $u''(x)=0$, $\vartheta_1(x)=\vartheta_2(x)=const$, $x\in\Omega^+$. Taking these data into account in (4.4), we obtain $\vartheta_1(x)=\vartheta_2(x)=0$, $x\in\Omega^+$. If $\sigma_1=0$, then from (4.4) we have

$$\int\limits_{\Omega^+}\bigg[E(u,\overline{u})+\frac{1}{\varkappa_3\sigma_2}\Big(d_3|\nabla^\top\vartheta_1|^2+|\varkappa_2\nabla^\top\vartheta_1+\varkappa_3\nabla^\top\vartheta_2|^2\Big)+\varkappa|u'-u''|^2+$$
$$+\rho_1\sigma_2^2|u'|^2+\rho_2\sigma_2^2|u''|^2+\frac{\alpha}{\sigma_2}\,|\vartheta_1-\vartheta_2|^2+\big(\varkappa'|\vartheta_1|^2+\varkappa''|\vartheta_2|^2\big)\bigg]\,dx=0.$$

From this equation we easily deduce $u'(x)=0$, $u''(x)=0$, $\vartheta_1(x)=0$, $\vartheta_2(x)=0$, $x\in\Omega^+$. □

5. Integral Representation Formulas

The fundamental matrix of solutions of the homogeneous system of differential equations of pseudo-oscillations of the two-temperature elastic mixtures theory reads as ([14,42]):

$$\Gamma(x,\sigma)=$$
$$=\frac{1}{4\pi d_1d_2d_3}\begin{bmatrix}\widetilde{\Psi}_1(x,\sigma) & \widetilde{\Psi}_2(x,\sigma) & \nabla^\top\Psi_{13}(x,\sigma) & \nabla^\top\Psi_{14}(x,\sigma)\\ \widetilde{\Psi}_3(x,\sigma) & \widetilde{\Psi}_4(x,\sigma) & \nabla^\top\Psi_{15}(x,\sigma) & \nabla^\top\Psi_{16}(x,\sigma)\\ \nabla\Psi_{17}(x,\sigma) & \nabla\Psi_{18}(x,\sigma) & \Psi_5(x,\sigma) & \Psi_6(x,\sigma)\\ \nabla\Psi_{19}(x,\sigma) & \nabla\Psi_{20}(x,\sigma) & \Psi_7(x,\sigma) & \Psi_8(x,\sigma)\end{bmatrix},\quad(5.1)$$

where d_1, d_2 are given by (2.2) and d_3 is given by (2.3),

$$\begin{aligned}
\widetilde{\Psi}_1(x,\sigma)&=\Psi_1(x,\sigma)I_3+Q(\partial)\Psi_9(x,\sigma),\\
\widetilde{\Psi}_2(x,\sigma)&=\Psi_2(x,\sigma)I_3+Q(\partial)\Psi_{10}(x,\sigma),\\
\widetilde{\Psi}_3(x,\sigma)&=\Psi_3(x,\sigma)I_3+Q(\partial)\Psi_{11}(x,\sigma),\\
\widetilde{\Psi}_4(x,\sigma)&=\Psi_4(x,\sigma)I_3+Q(\partial)\Psi_{12}(x,\sigma),\\
\Psi_l(x,\sigma)&=\sum_{j=1}^{2}p_j\beta^*_{lj}\frac{e^{ik_j|x|}}{|x|},\quad l=1,2,3,4,\\
\Psi_{l-8}(x,\sigma)&=\sum_{j=3}^{6}p_j\beta^*_{lj}\frac{e^{ik_j|x|}}{|x|},\quad l=13,14,15,16,\\
\Psi_{l+8}(x,\sigma)&=-\sum_{j=1}^{6}p_j\gamma^*_{lj}\frac{e^{ik_j|x|}}{|x|},\quad l=1,2,3,4,\\
\Psi_{l+8}(x,\sigma)&=i\sum_{j=3}^{6}p_j\delta^*_{lj}\frac{e^{ik_j|x|}}{|x|},\quad l=5,6,\dots,12.
\end{aligned}\qquad(5.2)$$

k_j^2, $j=1,2$, and k_j^2, $j=3,4,5,6$, are, respectively, the solutions of the following equations

$$a(z):=d_1z^2-(a_1\alpha''+a_2\alpha'-2c\varkappa)z+\alpha'\alpha''-\varkappa^2=0,$$
$$\Lambda(z):=\big[d_3z^2-(\alpha_1\varkappa_3+\alpha_2\varkappa_1-2\alpha\varkappa_2)z+\alpha_1\alpha_2-\alpha^2\big](a(z)+zb(z))-$$
$$-i\sigma z\big[(\varkappa_3\varepsilon_1(z)+\varkappa_1\varepsilon_3(z)-2\varkappa_2\varepsilon_2(z))z+2\alpha\varepsilon_2(z)-\alpha_2\varepsilon_1(z)-$$
$$-\alpha_1\varepsilon_3(z)\big]-\sigma^2(\eta_1\zeta_2-\eta_2\zeta_1)^2z^2=0,$$

where

$$b(z):=(d_2-d_1)z-(b_1\alpha''+b_2\alpha'-2\varkappa d),$$
$$\varepsilon_1(z):=\eta_1\delta_1''(z)+\zeta_1\delta_1'(z),\quad \varepsilon_3(z):=\eta_2\delta_2''(z)+\zeta_2\delta_2'(z),$$
$$\varepsilon_2(z):=\eta_1\delta_2''(z)+\zeta_1\delta_2'(z)=\eta_2\delta_1''(z)+\zeta_2\delta_1'(z),$$
$$\delta_j'(z):=\eta_j\big[\varkappa-(c+d)z\big]+\zeta_j\big[(a_1+b_1)z-\alpha'\big],\quad j=1,2,$$
$$\delta_j''(z):=\zeta_j\big[\varkappa-(c+d)z\big]+\eta_j\big[(a_2+b_2)z-\alpha''\big],\quad j=1,2;$$
$$\beta_{1j}^*:=\Lambda_j^*(\alpha''-a_2k_j^2),\quad \beta_{2j}^*=\beta_{3j}^*:=\Lambda_j^*(ck_j^2-\varkappa),$$
$$\beta_{4j}^*:=\Lambda_j^*(\alpha'-a_1k_j^2),\quad \beta_{13j}^*:=a_j^*\Big[i\sigma k_j^2\varepsilon_{3j}^*+(\alpha_2-\varkappa_3k_j^2)(a_j^*+b_j^*k_j^2)\Big],$$
$$\beta_{14j}^*=\beta_{15j}^*:=-a_j^*\Big[i\sigma k_j^2\varepsilon_{2j}^*+(\alpha-\varkappa_2k_j^2)(a_j^*+b_j^*k_j^2)\Big],$$
$$\beta_{16j}^*:=a_j^*\big[i\sigma k_j^2\varepsilon_{1j}^*+(\alpha_1-\varkappa_1k_j^2)(a_j^*+b_j^*k_j^2)\big],$$
$$\gamma_{1j}^*:=a_2\Lambda_j^*-\big[a_j^*(a_2+b_2)+b_j^*\alpha''\big]H_j^*-\alpha''\sigma^2(\eta_1\zeta_2-\eta_2\zeta_1)^2k_j^2-$$
$$-i\sigma\Big[(a_j^*\zeta_1^2+\alpha''\varepsilon_{1j}^*)(\alpha_2-\varkappa_3k_j^2)+(a_j^*\zeta_2^2+\alpha''\varepsilon_{3j}^*)(\alpha_1-\varkappa_1k_j^2)-$$
$$-2(a_j^*\zeta_1\zeta_2+\alpha''\varepsilon_{2j}^*)(\alpha-\varkappa_2k_j^2)\Big],$$
$$\gamma_{2j}^*=\gamma_{3j}^*:=-c\Lambda_j^*+\big[a_j^*(c+d)+b_j^*\varkappa\big]H_j^*-\varkappa\sigma^2(\eta_1\zeta_2-\eta_2\zeta_1)^2k_j^2+$$
$$+i\sigma\Big[(a_j^*\eta_1\zeta_1+\varkappa\varepsilon_{1j}^*)(\alpha_2-\varkappa_3k_j^2)+(a_j^*\eta_2\zeta_2+\varkappa\varepsilon_{3j}^*)(\alpha_1-\varkappa_1k_j^2)+$$
$$+\big(2\varkappa\varepsilon_{2j}^*+(\eta_1\zeta_2+\eta_2\zeta_1)a_j^*\big)(\alpha-\varkappa_2k_j^2)\Big],$$
$$\gamma_{4j}^*:=a_1\Lambda_j^*-\big[a_j^*(a_1+b_1)+b_j^*\alpha'\big]H_j^*+\alpha'\sigma^2(\eta_1\zeta_2-\eta_2\zeta_1)^2k_j^2-$$
$$-i\sigma\Big[(a_j^*\eta_1^2+\alpha'\varepsilon_{1j}^*)(\alpha_2-\varkappa_3k_j^2)+(a_j^*\eta_2^2+\alpha'\varepsilon_{3j}^*)(\alpha_1-\varkappa_1k_j^2)-$$
$$-2(a_j^*\eta_1\eta_2+\alpha'\varepsilon_{2j}^*)(\alpha-\varkappa_2k_j^2)\Big],$$
$$\delta_{5j}^*:=ia_j^*\Big[i\sigma\zeta_2(\eta_1\zeta_2-\eta_2\zeta_1)k_j^2+\delta_{1j}''(\alpha_2-\varkappa_3k_j^2)-\delta_{2j}''(\alpha-\varkappa_2k_j^2)\Big],$$
$$\delta_{6j}^*:=ia_j^*\Big[-i\sigma\zeta_1(\eta_1\zeta_2-\eta_2\zeta_1)k_j^2-\delta_{1j}''(\alpha-\varkappa_2k_j^2)+\delta_{2j}''(\alpha_1-\varkappa_1k_j^2)\Big],$$
$$\delta_{7j}^*:=ia_j^*\Big[-i\sigma\eta_2(\eta_1\zeta_2-\eta_2\zeta_1)k_j^2+\delta_{1j}'(\alpha_2-\varkappa_3k_j^2)-\delta_{2j}'(\alpha-\varkappa_2k_j^2)\Big],$$
$$\delta_{8j}^*:=ia_j^*\Big[i\sigma\eta_1(\eta_1\zeta_2-\eta_2\zeta_1)k_j^2-\delta_{1j}'(\alpha-\varkappa_2k_j^2)+\delta_{2j}'(\alpha_1-\varkappa_1k_j^2)\Big],$$

$$\delta^*_{9j} = -i\sigma\delta^*_{5j}, \ \ \delta^*_{10j} = -i\sigma\delta^*_{7j}, \ \ \delta^*_{11j} = -i\sigma\delta^*_{6j}, \ \ \delta^*_{12j} = -i\sigma\delta^*_{8j}.$$

$$a^*_j := d_1 \prod_{j\neq q=1}^{2} (k_j^2 - k_q^2), \quad b^*_j := (d_2 - d_1)k_j^2 - b_2\alpha' - b_1\alpha'' + 2\varkappa d,$$

$$\Lambda^*_j := d_2 d_3 \prod_{j\neq q=3}^{6} (k_j^2 - k_q^2), \quad H^*_j := d_3 k_j^4 - (\alpha_1\varkappa_3 + \alpha_2\varkappa_1 - 2\alpha\varkappa_2)k_j^2 + \alpha_1\alpha_2 - \alpha^2;$$

$$\delta'_{lj} := \eta_l\big[\varkappa - (c+d)k_j^2\big] + \zeta_l\big[(a_1+b_1)k_j^2 - \alpha'\big], \ \ l = 1,2,$$

$$\delta''_{lj} := \zeta_l\big[\varkappa - (c+d)k_j^2\big] + \eta_l\big[(a_2+b_2)k_j^2 - \alpha''\big], \ \ l = 1,2,$$

$$\varepsilon^*_{1j} = \eta_1\delta''_{lj} + \zeta_1\delta'_{1j}, \ \ \varepsilon^*_{2j} = \eta_1\delta''_{2j} + \zeta_1\delta'_{2j}, \ \ \varepsilon^*_{3j} = \eta_2\delta''_{2j} + \zeta_2\delta'_{2j},$$

$$p_j = \prod_{j\neq q=1}^{6} \big(k_j^2 - k_q^2\big)^{-1}.$$

Remark 5.1. *Using formulas (5.1) and (5.2), and the equalities*

$$k_1^{2m}p_1 + k_2^{2m}p_2 + \cdots + k_6^{2m}p_6 = 0, \ \ m = \overline{0,\ 4},$$
$$k_1^{10}p_1 + k_2^{10}p_2 + \cdots + k_6^{10}p_6 = 1,$$

we conclude that in a vicinity of the origin the functions $\Psi_j(x,\sigma)$*,* $j = \overline{1,8}$*, and* $\Psi_j(x,\sigma)$*,* $j = \overline{9,20}$*, are, respectively, of order const* $+\, O(|x|^{-1})$ *and* $O(|x|^{-1})$*.*

Hereinafter, we shall always assume that $k_j \neq k_p$, $j \neq p$, $\Im k_j > 0$, $j = \overline{1,6}$. According to these requirements regarding to equalities (5.2), all entries of $\Gamma(x,\sigma)$ exponentially decay at infinity.

Let us introduce the generalized single and double-layer potentials, and the Newton type volume potential

$$V(\varphi)(x) = \int\limits_S \Gamma(x-y,\sigma)\,\varphi(y)\,dS_y, \ \ x \in \mathbb{R}^3 \setminus S, \tag{5.3}$$

$$W(\varphi)(x) = \int\limits_S \big[\mathcal{P}^*(\partial,n)\Gamma^\top(x-y,\sigma)\big]^\top \varphi(y)\,dS_y, \ \ x \in \mathbb{R}^3 \setminus S, \tag{5.4}$$

$$N_{\Omega^\pm}(\psi)(x) = \int\limits_{\Omega^\pm} \Gamma(x-y,\sigma)\,\psi(y)\,dy, \ \ x \in \mathbb{R}^3,$$

where $\mathcal{P}^*(\partial,n)$ is the boundary differential operator defined by (2.7), $\Gamma(\cdot,\sigma)$ is the fundamental matrix given by (5.1), $\varphi = (\varphi_1,\cdots,\varphi_8)^\top$ is a density vector-function defined on S, while a density vector-function $\psi = (\psi_1,\cdots,\psi_8)^\top$ is defined on $\Omega^\pm$, and we assume that in the case of Ω^- the support of the density vector-function ψ of the Newtonian potential is a compact set.

Due to the equality

$$\sum_{j=1}^{8} L_{kj}(\partial_x,\sigma)\big([\mathcal{P}^*(\partial,n)\Gamma^\top(x-y,\sigma)]^\top\big)_{jp} =$$

$$= \sum_{j,\,q=1}^{8} L_{kj}(\partial_x,\sigma)\mathcal{P}^*_{pq}(\partial,n)\Gamma_{jq}(x-y,\sigma) =$$

$$= \sum_{j,\,q=1}^{8} \mathcal{P}^*_{pq}(\partial,n)L_{kj}(\partial_x,\sigma)\Gamma_{jq}(x-y,\sigma) = 0, \quad x \neq y, \ k,p=\overline{1,8},$$

it can be easily checked that the potentials defined by (5.3) and (5.4) are C^∞–smooth in $\mathbb{R}^3\setminus S$ and solve the homogeneous equation $L(\partial,\sigma)U(x)=0$ in $\mathbb{R}^3\setminus S$ for an arbitrary L_p-summable vector-function φ. The Newtonian potential solves the nonhomogeneous equation

$$L(\partial,\sigma)N_{\Omega^\pm}(\psi)=\psi \ \text{ in } \ \Omega^\pm \ \text{ for } \ \psi\in[C^{0,k}(\overline{\Omega^\pm})]^8.$$

This relation holds true for an arbitrary $\psi\in[L_p(\Omega^\pm)]^8$ with $1<p<\infty$. It is easy to show that $\Gamma(-x,\sigma)$ is a fundamental matrix of the formally adjoint operator $L^*(\partial,\sigma)$, i.e.

$$L^*(\partial,\sigma)\big[\Gamma(-x,\sigma)\big]^\top = I_8\delta(x). \tag{5.5}$$

With the help of Green's formulas (3.11) and (5.5) by standard arguments we can prove the following assertions (cf., e.g., [7, 26, 27] and [36, Ch. I, Lemma 2.1; Ch. II, Lemma 8.2]).

Theorem 5.2. *Let $S=\partial\Omega^+$ be $C^{1,k}$-smooth with $0<k\le 1$, either $\sigma=0$ or $\sigma=\sigma_1+i\,\sigma_2$ with $\sigma_2>0$, and let U be a regular vector of the class $[C^2(\overline{\Omega^+})]^8$. Then there holds the integral representation formula*

$$W\big(\{U\}^+\big)(x)-V\big(\{\mathcal{P}U\}^+\big)(x)+N_{\Omega^+}\big(L(\partial,\sigma)U\big)(x)=$$
$$=\begin{cases} U(x) & \text{for } x\in\Omega^+,\\ 0 & \text{for } x\in\Omega^-.\end{cases}$$

Proof. For the smooth case it easily follows from Green's formula (3.11) with the domain of integration $\Omega^+\setminus B(x,\varepsilon')$, where $x\in\Omega^+$ is treated as a fixed parameter, $B(x,\varepsilon')$ is a ball with the centre at the point x and radius $\varepsilon'>0$ and $\overline{B(x,\varepsilon')}\subset\Omega^+$. One needs to take the j-th column of the fundamental matrix $\Gamma^*(y-x,\sigma)$ for $V(y)$, calculate the surface integrals over the sphere $\Sigma(x,\varepsilon'):=\partial B(x,\varepsilon')$ and pass to the limit as $\varepsilon'\to 0$. $\square$

Similar representation formula holds in the exterior domain Ω^- if a vector U and its derivatives possess some asymptotic properties at infinity. In particular, the following assertion holds.

Theorem 5.3. *Let $S=\partial\Omega^-$ be $C^{1,k}$-smooth with $0<k\le 1$ and let U be a regular vector of the class $[C^2(\overline{\Omega^-})]^8$ such that for any multi-index*

$\alpha=(\alpha_1,\alpha_2,\alpha_3)$ with $0\leq|\alpha|=\alpha_1+\alpha_2+\alpha_3\leq 2$, the function $\partial^\alpha U_j$ is polynomially bounded at infinity, i.e., for sufficiently large $|x|$

$$\big|\partial^\alpha U_j(x)\big|\leq C_0\,|x|^m,\ \ j=\overline{1,8}, \tag{5.6}$$

with some constants m and $C_0>0$. Then there holds the integral representation formula

$$\begin{aligned}-W\big(\{U\}^-\big)(x)+V\big(\{\mathcal{P}U\}^-\big)(x)+N_{\Omega^-}\big(L(\partial,\sigma)U\big)(x)=\\ =\begin{cases}0 & \text{for } x\in\Omega^+,\\ U(x) & \text{for } x\in\Omega^-,\end{cases}\end{aligned} \tag{5.7}$$

with $\sigma=\sigma_1+i\,\sigma_2$, where $\sigma_2>0$.

Proof. The proof immediately follows from Theorem 5.2 and Remark 3.1 (cf. [14]). Indeed, one needs to write the integral representation formula (5.2) for the bounded domain $\Omega^-\cap B(0,R)$, and then send R to $+\infty$ and take into consideration that the surface integral over $\Sigma(0,R)$ tends to zero due to the conditions (5.6) and the exponential decay of the fundamental matrix at infinity. □

Corollary 5.4. *Let $\sigma=\sigma_1+i\,\sigma_2$ with $\sigma_1\in\mathbb{R}$ and $\sigma_2>0$, and U be a solution to the homogeneous equation $L(\partial,\sigma)U=0$ in $\Omega^\pm$ satisfying the condition (5.6) and $U\in[C^{1,k}(\overline{\Omega^\pm})]^8$ for some $0<k\leq 1$. Then the representation formula*

$$U(x)=W([U]_S)(x)-V([\mathcal{P}U]_S)(x),\ \ x\in\Omega^\pm,$$

holds, where $[U]_S=\{U\}^+-\{U\}^-$ and $[\mathcal{P}U]_S=\{\mathcal{P}U\}^+-\{\mathcal{P}U\}^-$ on S.

Proof. It Immediately follows from Theorems 5.2 and 5.3. □

Theorem 5.5. *Assume that $S=\partial\Omega\in C^{m,k}$, $m\geq 1$ and $0<k\leq 1$. If $g\in[C^{0,k'}(S)]^8$, $h\in[C^{0,k'}(S)]^8$, $0<k'<k$, then for each $z\in S$,*

$$[V(g)(z)]^\pm=V(g)(z)=\mathcal{H}g(z), \tag{5.8}$$

$$\big[\mathcal{P}(\partial,n)V(g)(z)\big]^\pm=[\mp 2^{-1}I_8+\mathcal{K}]g(z), \tag{5.9}$$

$$[W(h)(z)]^\pm=\big[\pm 2^{-1}I_8+\mathcal{N}\big]h(z), \tag{5.10}$$

$$\big[\mathcal{P}(\partial,n)W(h)(z)\big]^+=\big[\mathcal{P}(\partial,n)W(h)(z)\big]^-=\mathcal{L}h(z), \tag{5.11}$$

where

$$\mathcal{H}g(z):=\int\limits_S\Gamma(z-y,\sigma)g(y)\,dS_y,$$

$$\mathcal{L}h(z):=\lim_{\Omega^\pm\ni x\to z\in S}\mathcal{P}(\partial_x,n(x))\int\limits_S\big[\mathcal{P}^*(\partial_y,n(y))\Gamma^\top(x-y,\sigma)\big]^\top h(y)\,dS_y,$$

$$\mathcal{K}g(z):=\int\limits_S\big[\mathcal{P}(\partial,n)\Gamma(z-y,\sigma)\big]g(y)\,dS_y,$$

$$\mathcal{N}h(z) := \int\limits_S \big[\mathcal{P}^*(\partial,n)\Gamma^\top(z-y,\sigma)\big]^\top h(y)\,dS_y.$$

The prove of this theorem is analogous to that given in [25,35].

Theorem 5.6. *Assume that $S=\partial\Omega\in C^{m,k}$, $m\geq 2$, $0<k'<k\leq 1$, $l\leq m-1$, $\sigma=\sigma_1+i\sigma_2$, $\sigma_2>0$. If $g\in[C^{0,k'}(S)]^8$, $h\in[C^{1,k'}(S)]^8$, then*

$$\begin{aligned}
V &: [C^{l,k'}(S)]^8 \longrightarrow [C^{l+1,k'}(\overline{\Omega^\pm})]^8,\\
W &: [C^{l,k'}(S)]^8 \longrightarrow [C^{l,k'}(\overline{\Omega^\pm})]^8,\\
\mathcal{H} &: [C^{l,k'}(S)]^8 \longrightarrow [C^{l+1,k'}(S)]^8,\\
\mathcal{K} &: [C^{l,k'}(S)]^8 \longrightarrow [C^{l,k'}(S)]^8,\\
\mathcal{N} &: [C^{l,k'}(S)]^8 \longrightarrow [C^{l,k'}(S)]^8,\\
\mathcal{L} &: [C^{l,k'}(S)]^8 \longrightarrow [C^{l-1,k'}(S)]^8.
\end{aligned}$$

Remark 5.7. *Assume that $\sigma=\sigma_1+i\sigma_2$, $\sigma_2>0$ and $\Im k_j>0$. From equation (5.7) it follows that if $L(\partial,\sigma)U(x)=0$, $x\in\Omega^-$, then U is exponentially decaying at infinity and therefore in the unbounded domain Ω^- Green's formula (3.13) holds true,:*

$$\begin{aligned}
&\int\limits_{\Omega^-}\bigg[-E(u,\overline{u})+\frac{i}{\varkappa_3\overline{\sigma}}\Big(d_3|\nabla^\top\vartheta_1|^2+|\varkappa_2\nabla^\top\vartheta_1+\varkappa_3\nabla^\top\vartheta_2|^2\Big)-\varkappa|u'-u''|^2+\\
&\quad+\rho_1\sigma^2|u'|^2+\rho_2\sigma^2|u''|^2+\frac{\alpha i}{\overline{\sigma}}\,|\vartheta_1-\vartheta_2|^2-\big(\varkappa'|\vartheta_1|^2+\varkappa''|\vartheta_2|^2\big)\bigg]\,dx-\\
&-\int\limits_{\partial\Omega}\Big[\overline{u}(z)\cdot T(\partial,n)u(z)-(\eta_1\vartheta_1+\eta_2\vartheta_2)(n\cdot\overline{u'})-(\zeta_1\vartheta_1+\zeta_2\vartheta_2)(n\cdot\overline{u''})-\\
&\quad-\frac{i}{\varkappa_3\overline{\sigma}}\Big(d_3\vartheta_1\partial_n\overline{\vartheta_1}+(\varkappa_2\vartheta_1+\varkappa_3\vartheta_2)\big(\varkappa_2\partial_n\overline{\vartheta_1}+\varkappa_3\partial_n\overline{\vartheta_2}\big)\Big)\Big]^-\,ds=0, \qquad (5.12)
\end{aligned}$$

where the sesquilinear form $E(u,\overline{u})$ is given by (3.14) and the operator $T(\partial,n)$ by formula (2.6).

Similarly to Theorem 4.1 in view of formula (5.12) the following theorem takes place.

Theorem 5.8. *If $\sigma=\sigma_1+i\sigma_2$, where $\sigma_1\in\mathbb{R}$, $\sigma_2>0$, then the homogeneous problems $(\mathrm{I}^{(\sigma)})_0^-$ and $(\mathrm{II}^{(\sigma)})_0^-$ ($\Phi^\pm$, $f=0$, $F=0$) have only the trivial solution.*

The following theorem is valid.

Theorem 5.9. *Let $S=\partial\Omega\in C^{m,k}$ with integer $m\geq 2$ and $0<k\leq 1$. Then:*

(a) *The principal homogenous symbol matrices of the singular integral operators $\mp 2^{-1}I_8+\mathcal{K}$ and $\pm 2^{-1}I_8+\mathcal{N}$ are non-degenerate, while* *the principal homogenous symbol matrices of the operators $\mathcal{H}$ and $\mathcal{L}$ are positive definite;*

(b) *the operators $\mathcal{H}$, $\mp 2^{-1}I_8+\mathcal{K}$, $\pm 2^{-1}I_8+\mathcal{N}$ and $\mathcal{L}$ are elliptic pseudo-differential operators* (*of order* -1, 0, 0 *and* 1, *respectively*) *with zero index;*

(c) *the following equalities hold in appropriate function spaces:*

$$\begin{gathered}\mathcal{N}\mathcal{H}=\mathcal{H}\mathcal{K}, \quad \mathcal{L}\mathcal{N}=\mathcal{K}\mathcal{L},\\ \mathcal{H}\mathcal{L}=-4^{-1}I_8+\mathcal{N}^2, \;\; \mathcal{L}\mathcal{H}=-4^{-1}I_8+\mathcal{K}^2.\end{gathered} \tag{5.13}$$

The proof of this theorem is word for word of the proof of its counterparts in [31, 33, 35, 36].

6. Existence of Classical Solutions of the Boundary Value Problems

This section provides the study of problems stated in Section 4 using the theory of potentials and theory of integral equations. We seek solutions of the problems in the form of single or double-layer potentials allowing one to reduce the BVPs to the correspond boundary integral equations. Simultaneously, the question of invertibility of the obtained integral operators will be considered.

6.1. **Investigation of Dirichlet's problem by the double-layer potential.** We seek solutions of problems $(\mathrm{I}^{(\sigma)})^+$ and $(\mathrm{I}^{(\sigma)})^-$ (see (4.1), $\Phi^{\pm}=0$, (4.2)) by means of the double-layer potential $W(h)(x)$ (see (5.4)), where $h\in C^{1,\beta}(S)$ is the sought for vector-function. Taking into consideration the boundary condition (4.2) and the jump formulas (5.10), for the density h we obtain the following integral equations of second kind

$$\text{BVP } (\mathrm{I}^{(\sigma)})^+ : \; \big[2^{-1}I_8+\mathcal{N}\big]h=f \;\text{ on }\; S, \tag{6.1}$$

$$\text{BVP } (\mathrm{I}^{(\sigma)})^- : \; \big[-2^{-1}I_8+\mathcal{N}\big]h=f \;\text{ on }\; S. \tag{6.2}$$

In the left hand side of(6.1) and (6.2) we have singular integral operators of normal type with the index equal to zero (see Theorem 5.9).

Theorem 6.1. *If $S\in C^{2,\alpha}$ and $f\in C^{1,\beta}$, $0<\beta<\alpha\le 1$, then the problem $(\mathrm{I}^{(\sigma)})^+$ has a unique solution representable by the double-layer potential $W(h)$, where h is determined from uniquely solvable integral equation* (6.1).

Proof. Uniqueness follows from Theorem 4.1. Now, let us show that the operator

$$2^{-1}I_8+\mathcal{N} : C^{1,\beta}(S)\longrightarrow C^{1,\beta}(S) \tag{6.3}$$

is invertible. Note that the operator $-2^{-1}I_8+\mathcal{N}$ the arguments are verbatim. By virtue of Theorem 5.9,operator (6.3) is Fredholm with zero index and therefore for proving its invertibility it is sufficient to show that its kernel $ker(2^{-1}I_8+\mathcal{N})$ is trivial, i.e. we have to show that the homogeneous equation

$$\big[2^{-1}I_8+\mathcal{N}\big]h=0 \;\text{ on } S \tag{6.4}$$

has only the trivial solution. Indeed, assume that h is a solution of (6.4) and construct the double-layer potential $W(h)$. In view of the inclusion $h \in C^{1.\beta}(S)$, we have $W(h) \in C^{1,\beta}\overline{(\Omega^{\pm})}$. It easy to see that equation (6.4) corresponds to Dirichlet's interior homogeneous problem $[W(h)(z)]^{+} = 0$, $z \in S$. Since this problem has only the trivial solution, we conclude $W(h)(x) = 0$, $x \in \Omega^{+}$. Therefore we have $[\mathcal{P}(\partial, n)W(h)(z)]^{+} = 0$, $z \in S$, and according to the Lyapunov-Tauber theorem we deduce $[\mathcal{P}(\partial, n)W(h)(z)]^{+} = [\mathcal{P}(\partial, n)W(h)(z)]^{-} = 0$, $z \in S$ (see Theorem 5.6). This means that $W(h)(x)$ is a solution to the homogeneous problem $(\mathrm{II}^{(\sigma)})^{-}$ which possesses only the trivial solution. Thus $W(h)(x) = 0$, $x \in \Omega^{-}$ and by virtue of formula (5.10) we conclude that $[W(h)(z)]^{+} - [W(h)(z)]^{-} = h(z) = 0$, $z \in S$, i.e. integral equation (6.4) has only the trivial solution. Hence, the operator (6.3) is invertible and therefore the equation (6.1) is unique solvable for arbitrary vector-function $f \in C^{1,\beta}(S)$, which proves the theorem. □

The following theorem can be proved similarly.

Theorem 6.2. *If $S \in C^{2,\alpha}$ and $f \in C^{1,\beta}(S)$, $0 < \beta < \alpha \leq 1$, then the problem $(\mathrm{I}^{(\sigma)})^{-}$ has a unique solution, which is representable by the double-layer potential $W(h)$, where h is determined from unique by solvable integral equation* (6.2).

6.2. **Investigation of Neumann's problem by single-layer potential.** Solutions to the problems $(\mathrm{II}^{(\sigma)})^{+}$ and $(\mathrm{II}^{(\sigma)})^{-}$ (see (4.1), $\Phi^{\pm} = 0$, (4.3)) are sought by single-layer potential $V(g)(x)$, where $g \in C^{0,\beta}(S)$ (see (5.3)). Taking into consideration the boundary conditions (4.3) and the jump formulas (5.9) for the density g we obtain, the following integral equations of second kind respectively

$$\text{BVP } (\mathrm{II}^{(\sigma)})^{+} : \ \big[-2^{-1}I_8 + \mathcal{K}\big]g = F \ \text{ on } \ S, \tag{6.5}$$

$$\text{BVP } (\mathrm{II}^{(\sigma)})^{-} : \ \big[2^{-1}I_8 + \mathcal{K}\big]g = F \ \text{ on } \ S. \tag{6.6}$$

The operators in the left hand side of(6.5) and (6.6) are singular integral operators of normal type with the index equal to zero (see Theorem 5.9).

Theorem 6.3. *If $S \in C^{1,\alpha}$ and $F \in C^{0,\beta}(S)$, $0 < \beta < \alpha \leq 1$, then the problem $(\mathrm{II}^{(\sigma)})^{+}$ has a unique solution, which is representable by the single-layer potential $V(g)(x)$, where g is determined from uniquely solvable integral equation* (6.5).

Proof. Uniqueness follows from Theorem 4.1. Now, let us show that the operator

$$-2^{-1}I_8 + \mathcal{K} : C^{0,\beta}(S) \longrightarrow C^{0,\beta}(S) \tag{6.7}$$

is invertible. Note that the invertibility of the operator $2^{-1}I_8 + \mathcal{K}$ can be performed by word for word arguments. By virtue of Theorem 5.9, the operator (6.7) is Fredholm with zero index and therefore for proving its invertibility it is sufficient to show that its kernel $ker(-2^{-1}I_8+\mathcal{K})$ is trivial, i.e. we have to show that the homogeneous equation

$$\big[-2^{-1}I_8+\mathcal{K}\big]g=0 \ \text{ on } \ S \tag{6.8}$$

has only the trivial solution. Indeed, assume that g is e solution of (6.8). Construct the single-layer potential $V(g)$. Since $g\in C^{0,\beta}(S)$, we have $V(g)\in C^{1,\beta}(\overline{\Omega^{\pm}})$. The equation (6.8) corresponds to Neumann's interior homogeneous problem $[\mathcal{P}(\partial,n)V(g)(z)]^+=0$, $z\in S$. Since this problem has only the trivial solution, we get $V(g)(x)=0$, $x\in\Omega^+$. Since $[V(g)(z)]^-=[V(g)(z)]^+=0$, $z\in S$, we have that $V(g)(x)$ is a solution of Dirichlet's exterior homogeneous problem and hence $V(g)(x)=0$, $x\in\Omega^-$. On the other hand, by virtue of formula (5.9) we obtain that $[\mathcal{P}(\partial_z,n(z))V(g)(z)]^- - [\mathcal{P}(\partial_z,n(z))V(g)(z)]^+=g(z)=0$, $z\in S$, i.e. the integral equation (6.8) has only the trivial solution. Consequently, the operator (6.7) is invertible and therefore equation (6.5) is solvable for arbitrary vector-function $F\in C^{0,\beta}(S)$, which proves the theorem. □

The following theorem can be proved similarly.

Theorem 6.4. *If $S\in C^{1,\alpha}$ and $F\in C^{0,\beta}(S)$, $0<\beta<\alpha\leq 1$, then the problem $(\mathrm{II}^{(\sigma)})^-$ has a unique solution, which is representable by the single-layer potential $V(g)$, where g is determined from unique by solvable integral equation* (6.6)*.*

6.3. **Investigation of Dirichlet's problem by single-layer potential.** We seek solutions of the problems $(\mathrm{I}^{(\sigma)})^+$ and $(\mathrm{I}^{(\sigma)})^-$ (see (4.1), $\Phi^{\pm}=0$, (4.2)) by means of the single-layer potential $V(g)(x)$ (see (5.3)), where $g\in C^{0,\beta}(S)$ is the sought for vector-function. Taking into consideration the boundary condition (4.2) and the jump formula (5.8), for the density g we obtain the following integral equation of the first kind:

$$\mathcal{H}g=f \ \text{ on } \ S. \tag{6.9}$$

Theorem 6.5. *If $S\in C^{2,\alpha}$ and $f\in C^{1,\beta}(S)$, $0<\beta<\alpha\leq 1$, then the problem $(\mathrm{I}^{(\sigma)})^{\pm}$ has a unique solution, which can be represented by the single-layer potential $V(g)$, where g is determined from uniquely solvable integral equation* (6.9)*.*

Proof. Uniqueness follows from Theorems 4.1 and 5.9. Now, let us show that the operator

$$\mathcal{H}: C^{0,\beta}(S)\longrightarrow C^{1,\beta}(S) \tag{6.10}$$

is invertible. Applying the operator $\mathcal{L}$ to both sides of the equation (6.9), we obtain (see (5.13)) the singular integral equation

$$\mathcal{L}\mathcal{H}g=(-4^{-1}I_8+\mathcal{K}^2)g=(-2^{-1}I_8+\mathcal{K})(2^{-1}I_8+\mathcal{K})g=\mathcal{L}f, \tag{6.11}$$

where $\mathcal{L}f\in C^{0,\beta}(S)$ and the operator

$$\mathcal{L}\mathcal{H}=(-2^{-1}I_8+\mathcal{K})(2^{-1}I_8+\mathcal{K}): C^{0,\alpha}(S)\longrightarrow C^{0,\alpha}(S)$$

is a singular operator of normal type with the index equal to zero.By the same arguments applied in [33], it can be shown that the operator (6.11) is invertible. Therefore we can write

$$g = (2^{-1}I_8 + \mathcal{K})^{-1}(-2^{-1}I_8 + \mathcal{K})^{-1}\mathcal{L}f.$$

Let us show that (6.9) and (6.11) are equivalent integral equations. Indeed, if $g \in C^{0,\beta}(S)$ is a solution to the equation (6.9), then it will be a solution to the equation (6.11) as well. Assume now that g is a solution to the equation (6.11). Introduce notation

$$\varphi := (\mathcal{H}g - f) \in C^{1,\beta}(S). \tag{6.12}$$

Then equation (6.11) can be rewritten as

$$\mathcal{L}\varphi = 0 \text{ on } S. \tag{6.13}$$

Construct the double-layer potential $W(\varphi)$ with the density φ determined by equation (6.12). Then it follows that $W(\varphi)$ solves Neumann's homogeneous problem $[\mathcal{P}(\partial_z, n(z))W(\varphi)(z)]^{\pm} = 0$, $z \in S$, in view of equation (6.13). Since this problem has only the trivial solution, we infer $W(\varphi)(x) = 0$, $x \in \Omega^{\pm}$. According to (5.10) we have $[W(\varphi)(z)]^{+} - [W(\varphi)(z)]^{-} = \varphi(z) = 0$, $z \in S$, i.e. g is a solution to equation (6.9). Hence operator (6.10) is invertible. □

Corollary 6.6. *Solution to problem* $(\mathrm{I}^{(\sigma)})^{\pm}$ *is presentable in the following form:*

$$U(x) = V(\mathcal{H}^{-1}f)(x), \quad x \in \Omega^{\pm},$$

where $[U(z)]^{\pm} = f(z)$, $z \in S$.

This representation plays a crucial role in the study of mixed boundary value problems, when on a part of the boundary $\partial\Omega$ the Dirichlet condition is given, while on the remainder part the Neumann condition is prescribed

6.4. **Investigation of Neumann's problem by double-layer potential.** We seek a solution to problem $(\mathrm{II}^{(\sigma)})^{\pm}$ (see (4.1), $\Phi^{\pm} = 0$, (4.3)) in the form of double-layer potential $W(h)$, where $h \in C^{1,\beta}(S)$ is the sought vector (see (5.4)). Taking into consideration the boundary conditions (4.3) and formula (5.11), for the density h we obtain the following integral equation of the "first kind":

$$\mathcal{L}h = F \text{ on } S. \tag{6.14}$$

Theorem 6.7. *If* $S \in C^{1,\alpha}$ *and* $F \in C^{0,\beta}(S)$, $0 < \beta < \alpha \leq 1$, *then the problem* $(\mathrm{II}^{(\sigma)})^{\pm}$ *has a unique solution, which is representable by double-layer potential* $W(h)$, *where* h *is determined from uniquely solvable integral equation* (6.14).

Proof. Uniqueness follows from Theorems 4.1 and 5.9. Now, let us show that the operator

$$\mathcal{L} : C^{1,\beta}(S) \longrightarrow C^{0,\beta}(S) \tag{6.15}$$

is invertible. Apply the operator $\mathcal{H}$ to both sides of equation (6.14) to obtain the singular integral equation

$$\mathcal{H}\mathcal{L}h = (-4^{-1}I_8 + \mathcal{N}^2)h = (-2^{-1}I_8 + \mathcal{N})(2^{-1}I_8 + \mathcal{N})h = \mathcal{H}F, \qquad (6.16)$$

where $\mathcal{H}F \in C^{1,\beta}(S)$ and the operator

$$\mathcal{H}\mathcal{L} = (-2^{-1}I_8 + \mathcal{N})(2^{-1}I_8 + \mathcal{N}) : C^{1,\beta}(S) \longrightarrow C^{1,\beta}(S) \qquad (6.17)$$

is a singular operator of normal type with zero index. Again, applying the arguments as in [33] we can shown that (6.17) is invertible, and therefore we can write

$$h = (2^{-1}I_8 + \mathcal{N})^{-1}(-2^{-1}I_8 + \mathcal{N})^{-1}\mathcal{H}F.$$

Note that the operators $(-2^{-1}I_8 + \mathcal{N})$ and $(2^{-1}I_8 + \mathcal{N})$ commute.

Let us show that (6.14) and (6.16) are equivalent integral equations. Indeed, if $h \in C^{1,\beta}(S)$ is e solution to equation (6.14), then it will be solution to equation (6.16) as well. Introduce notation

$$\psi := (\mathcal{L}h - F) \in C^{0,\beta}(S). \qquad (6.18)$$

Then equation (6.16) can be rewritten as

$$\mathcal{H}\psi = 0 \ \text{ on } \ S. \qquad (6.19)$$

Construct the single-layer potential $V(\psi)$ with the density ψ determined by equation (6.18). Dirichlet's problem $[V(\psi)(z)]^{\pm} = 0$, $z \in S$, corresponds to the equation (6.19). As this problem has only the trivial solution, we have $V(\psi)(x) = 0$, $x \in \Omega^{\pm}$, from which we obtain that $\psi(z) = 0$, $z \in \Omega^{\pm}$, i.e. h is a solution to equation (6.14) and hence the operator (6.15) is invertible. □

Corollary 6.8. *The solution to the problem* $(\mathrm{II}^{(\sigma)})^{\pm}$ *is represented in the following form:*

$$U(x) = W(\mathcal{L}^{-1}F)(x), \quad x \in \Omega^{\pm},$$

where $F(z) = [P(\partial_z, n(z))U(z)]^{\pm}$, $z \in S$.

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(Received 08.12.2011)

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