Memoirs on Differential Equations and Mathematical Physics
Volume 56, 2012, 73–88

Vakhtang Paatashvili

ON SOME PROPERTIES OF ANALYTIC FUNCTIONS FROM SMIRNOV CLASS WITH A VARIABLE EXPONENT

Dedicated to the bless memory of my dear friend Givi Khuskivadze

Abstract. Let D be a simply connected domain bounded by a simple closed rectifiable curve Γ and $L^{p(t)}(D)$ denote the Lebesgue space with variable exponent.

The present work reveals different conditions regarding the functions $p(t)$ and the domain D under fulfilment of which the Cauchy type integrals with density from $L^{p(t)}(\Gamma)$ belong to the Smirnov class $E^{p(t)}(D)$.

When the domain D is bounded by the Lavrent'yev curve, the analogue of the well-known Smirnov's theorem is stated: if $\phi \in E^{p_1(\cdot)}(D)$, $\phi^+(t) \in L^{p_2(t)}(\Gamma)$, then $\phi \in E^{\widetilde{p}(t)}(D)$, where $\widetilde{p}(t) = \max(p_1(t), p_2(t))$.

2010 Mathematics Subject Classification. 47B38, 42B20, 45P05.

Key words and phrases. Smirnov classes of analytic functions, variable exponent, Cauchy type integral, regular curves, Lavrent'yev curves.

რეზიუმე. ვთქვათ, D მარტივი, შეკრული, გაწრფევადი Γ წირით შემოსაზღვრული სასრული არეა, ხოლო $L^{p(t)}(D)$ არის ლებეგის ცვლადმაჩვენებლიანი სივრცე.

გამოვლენილია $p(t)$ ფუნქციისა და D არის მიმართ ისეთი პირობები, რომელთა შესრულება იწვევს $L^{p(t)}(\Gamma)$ კლასის სიმკვრივის მქონე კოშის ტიპის ინტეგრალთა მიკუთვნებას სმირნოვის $E^{p(t)}(D)$ კლასისადმი.

როცა D შემოსაზღვრულია ლავრენტიევის Γ წირით, დადგენილია სმირნოვის ცნობილი თეორემის შემდეგი სახის ანალოგი: თუ $\phi \in E^{p_1(\cdot)}(D)$, ხოლო $\phi^+(t) \in L^{p_2(t)}(\Gamma)$, მაშინ $\phi \in E^{\widetilde{p}(t)}(D)$, სადაც $\widetilde{p}(t) = \max(p_1(t), p_2(t))$.

1. Introduction

Quite recently it became clear that for investigation of a number of questions dealing with analysis and in studying the problems of applied character, the Lebesgue spaces $L^{p(t)}$ with a variable exponent are very useful. In particular, in studying boundary value problems of the theory of analytic and harmonic functions it is advisable to consider them in classes of functions representable by the Cauchy type integral with density from $L^{p(t)}$ and their real parts as well as in classes of functions which reasonably generalize Smirnov classes $E^p(D)$ in the case of a variable exponent $p(t)$.

The works [1]–[3] suggest one (of the possible) such generalization under which all significant properties, inherent in these classes for a constant p, remain valid.

In the present paper we continue investigation of these classes. Special attention is attached to the problem of finding different conditions for the domains D and functions $p(t)$ under fulfilment of which the Cauchy type integrals with density from $L^{p(t)}(\Gamma)$ belong to the class $E^{p(t)}(D)$ (Γ is a simple closed curve bounding the domain D).

To achieve the purpose in view, for the domains bounded by piecewise smooth curves we establish one criterion in order for the analytic in D function ϕ to belong to the class $E^{p(t)}(D)$ (depending on the properties of conformal mapping of the unit circle onto D). However, when the domain D is bounded by the Lavrent'yev curve (i.e. the curves with the chord-arc condition), the analogue of the well-known Smirnov's theorem is fully justified; namely, the conditions are revealed under which: if $\phi(x) \in E^{p_1(t)}(D)$ and $\phi^+(x) \in L^{p_2(t)}(\Gamma)$, then $\phi(z) \in E^{\widetilde{p}(t)}(D)$, $\widetilde{p}(t) = \max(p_1(t), p_2(t))$.

2. Some Definitions and Auxiliary Statements

2.1. The Curves.

(i) Let D be a simply connected domain bounded by a simple finite rectifiable curve $\Gamma = \{t \in \mathbb{C} : \ t = t(s), \ 0 \le s \le l < \infty\}$ with arc-length measure $\nu(t) = s$. Let $\Gamma(t,r) = \Gamma \cap B(t,r)$, where $B(t,r) = \{\tau \in \mathbb{C} : |\tau - t| < r\}$, $t \in \Gamma$, $r > 0$.

A curve Γ is called Carleson one (or regular one), if

$$\sup_{t\in\Gamma, r>0} \frac{\nu[\Gamma(t,r)]}{r} < \infty.$$

(ii) By Λ we denote a set of all Lavrent'yev curves, i.e., the curves Γ for which

$$\sup_{t_1,t_2\in\Gamma} \frac{s(t_1,t_2)}{|t_1 - t_2|} < \infty,$$

where $s(t_1,t_2)$ is length of the smallest of the two arcs lying on Γ and connecting the points t_1 and t_2.

(iii) If Γ is a piecewise smooth closed simple curve with angular points A_k, $k = 1, \dots, n$, and it is boundary of the domain D, and $\pi\nu_k$, $0 \le \nu_k \le 2$

are sizes of interior with respect to D angles at these points, we say that

$$\Gamma \in C_D^1(A_1, A_2, \dots, A_n; \nu_1, \nu_2, \dots, \nu_n).$$

The set of piecewise Lyapunov curves with the same properties we denote by

$$C_D^{1,L}(A_1, A_2, \dots, A_n; \nu_1, \nu_2, \dots, \nu_n).$$

(iv) Assume

$$S_\Gamma : f \to S_\Gamma f, \quad (S_\Gamma f)(t) = \frac{1}{\pi i} \int\limits_\Gamma \frac{f(\tau)}{\tau - t}\, d\tau, \quad t \in \Gamma.$$

We write $\Gamma \in R^p$, $p > 1$, if the operator is continuous in $L^p(\Gamma)$.

2.2. Conformal Mappings.

2.2.1. If $z = z(w)$ is a conformal mapping of the circle $U = \{w : \ |w| < 1\}$ onto the domain D with the boundary $\Gamma \in C_D^{1,L}(A_1, A_2, \dots, A_n; \nu_1, \nu_2, \dots, \nu_n)$, $0 < \nu_k \le 2$, then

$$z'(w) \sim \prod_{k=1}^n (w - a_k)^{\nu_k - 1}, \quad A_k = z(a_k), \tag{1}$$

where $f \sim g$ denotes that $0 < \inf |\frac{f}{g}| \le \sup |\frac{f}{g}| < \infty$ [4].

2.2.2. If Γ is a simple closed curve bounding the domain D, and $\Gamma \in \Lambda$, then there exist positive numbers η and σ such that

$$z' \in H^{1+\eta}, \quad \frac{1}{z'} \in H^\sigma, \tag{2}$$

where H^σ is the Hardy class of analytic in U functions (see, e.g., [5, p. 170]).

2.2.3. If $\Gamma \in C_D^1(A_1, A_2, \dots, A_n; \nu_1, \nu_2, \dots, \nu_n)$, $0 < \nu_k \le 2$, then

$$z'(w) \sim \prod_{k=1}^n (w - a_k)^{\nu_k - 1} \exp \int\limits_\gamma \frac{\psi(\zeta)}{\zeta - w}\, ds, \tag{3}$$

where $\psi(\zeta)$ is the real continuous function on γ, $\gamma = \{\zeta : \ |\zeta| = 1\}$ ([6], see also [7, p. 144]).

2.2.4. Let D be the bounded domain with a simple rectifiable boundary Γ, and let $z = z(w)$ be the conformal mapping of U onto D. D is said to be Smirnov's domain (and Γ is said to be Smirnov's curve), if the function $\ln |z'(w)|$ is representable by the Poisson integral, i.e.,

$$\ln |z'(re^{i\varphi})| = \frac{1}{2\pi} \int\limits_0^{2\pi} \ln |z'(e^{i\vartheta})| \, \frac{1 - r^2}{1 + r^2 - 2r\cos(\vartheta - \rho)}\, d\vartheta$$

(for these classes see, e.g., [8, pp. 250–252]).

2.3. Some Properties of the Operator S_Γ and of the Cauchy Type Integrals.

(i) If $p > 1$, then $\Gamma \in R^p$ if and only if Γ is a regular curve ([9]).

(ii) If Γ is a simple closed curve bounding the domain D and the operator S_Γ is continuous from $L^p(\Gamma)$ to $L^s(\Gamma)$, $p > 1$, $s \le p$, then:

(a) D is Smirnov's domain and

(b) the Cauchy type integral

$$(K_\Gamma f)(z) = \frac{1}{2\pi i} \int\limits_\Gamma \frac{f(\tau)}{\tau - z}\, d\tau, \quad z \in D, \quad f \in L^p(\Gamma),$$

belongs to the Smirnov class $E^s(D)$.

In particular, if Γ is a regular curve, then $(K_\Gamma f)(z)$ belongs to the class $E^p(D)$ when $f \in L^p(\Gamma)$, $p > 1$ ([10], [11], see also [7, p. 29]).

(c) Smirnov's Theorem: if D is Smirnov's domain and $\phi \in E^{p_1}(D)$, while $\phi \in L^{p_2}(\Gamma)$, $p_2 > p_1$, then $\phi \in E^{p_2}(D)$ ([12], see also [8, p. 260]).

2.4. Spaces $L^{p(t)}(\Gamma;\omega)$. Classes of Exponents $\mathcal{P}[\Gamma]$ and $\widetilde{\mathcal{P}}(\Gamma)$. Let Γ be a simple rectifiable curve with the equation $t = t(s)$, $0 \le s \le l$, with arc-length measure, and let on Γ be assigned measurable functions $p(t)$ and $\omega(t)$, where $p(t)$ is positive and $\omega(t)$ is almost everywhere other than zero finite function.

Consider a set of measurable on Γ functions $f(t)$ for which

$$I_\Gamma^{p(\cdot)}(f\omega) = \int\limits_0^b \big|f(t(s))\omega(t(s))\big|^{p(t(s))}\, ds < \infty.$$

Denote

$$\|f\|_{L^{p(\cdot)}(\Gamma;\omega)} = \inf\Big\{\lambda > 0:\ I^{p(\cdot)}\Big(\frac{f\omega}{\lambda}\Big) \le 1\Big\}.$$

By $L^{p(\cdot)}(\Gamma;\omega)$ we denote a space of measurable functions f such that $\|f\|_{L^{p(\cdot)}(\Gamma;\omega)} < \infty$. Assume $L^{p(\cdot)}(\Gamma) := L^{p(\cdot)}(\Gamma;1)$. (For detailed account on these spaces see, e.g., [13]).

2.4.1. *Classes of Functions $\mathcal{P}(\Gamma)$ and $\widetilde{\mathcal{P}}(\Gamma)$.* The spaces $L^{p(\cdot)}(\Gamma;\omega)$ in which the function $p(t)$ satisfies the conditions below are thoroughly studied and frequently used in applications:

(1) there is the constant A such that for any t_1, t_2 we have

$$|p(t_1) - p(t_2)| < \frac{A}{|\ln|t_1 - t_2||}\,; \tag{4}$$

(2)

$$\min_{t\in\Gamma} p(t) = \underline{p} > 1. \tag{5}$$

The set of all functions $p(t)$ satisfying the conditions (4), (5) we denote by $\mathcal{P}(\Gamma)$.

If $p \in \mathcal{P}(\Gamma)$, then the set $L^{p(\cdot)}(\Gamma;\omega)$ is the Banach space with the norm $\|\cdot\|_{L^{p(\cdot)}(\Gamma;\omega)}$.

Along with the class $\mathcal{P}(\Gamma)$, we introduce into consideration one more class of functions $\mathcal{P}_{1+\varepsilon}(\Gamma)$, $\varepsilon > 0$. This is a subset of those functions $p(t)$ from $\mathcal{P}(\Gamma)$ for which the condition (4) is replaced by the condition

$$|p(t_1) - p(t_2)| < \frac{A}{|\lambda|t_1 - t_2||^{1+\varepsilon}}\,. \tag{6}$$

Assume

$$\widetilde{\mathcal{P}}(\Gamma) = \bigcup_{\varepsilon>0} \mathcal{P}_{1+\varepsilon}. \tag{7}$$

2.5. The Hardy and Smirnov Classes with a Variable Exponent. Let D be the inner domain bounded by a simple closed curve Γ, and let $p = p(t)$ be the given on Γ measurable positive function. Moreover, let $z = z(w)$ be the conformal mapping of the circle U with boundary γ onto the domain D, and let $\omega = \omega(z)$ be the measurable on D function.

By $E^{p(t)}(D;\omega)$ we denote a set of all those analytic in D functions $\phi(z)$ for which

$$\sup_{0<z<1} \int_0^{2\pi} \Big|\phi(z(re^{i\vartheta}))\omega(z(re^{i\vartheta}))\Big|^{p(z(e^{i\vartheta}))} |z'(re^{i\vartheta})|\,d\vartheta < \infty. \tag{8}$$

Assume

$$H^{p(\cdot)}(\omega) := E^{p(\cdot)}(U;\omega), \quad H^{p(\cdot)} := H^{p(\cdot)}(1).$$

For the constant p, these classes coincide with the well-known Smirnov and Hardy classes.

2.5.1. *On the Continuity of the Operator S_Γ in the Spaces $L^{p(\cdot)}(\Gamma;\omega)$.* In [14], the authors have proved theorems on the continuity of the operator S_Γ in the spaces $L^{p(\cdot)}(\Gamma;\omega)$. (More earlier works relating to this subject-matter can be found therein).

Combining the results of these theorems, we find that the theorem below is valid.

Theorem A. *For the operator S_Γ to be continuous in the space $L^{p(\cdot)}(\Gamma;\omega)$, where $p \in \mathcal{P}(\Gamma)$ and*

$$\omega(t) = \prod_{k=1}^{n} |t - t_k|^{\alpha_k}, \ \ t_k \in \Gamma, \ \ \alpha \in \mathbb{R},$$

it is necessary and sufficient that Γ is a regular curve and α_k satisfy the condition

$$-\frac{1}{p(t_k)} < \alpha_k < \frac{1}{p'(t_k)}\,, \ \ k = 1, \dots, n.$$

3. One Criterion for Belonging of the Analytic Function to the Class $E^{p(\cdot)}(D)$

If $p(t) = p = const$, then when studying the properties of functions from classes $E^p(D)$, the fact that the involution of the function $\phi \in E^p(D)$ is equivalent to the belonging of the function $\Psi(w) = \phi(z(w))[z'(w)]^{1/p}$ to the Hardy class H^p plays an important role. For variable p, the function $\Psi(w)$ is not even analytic.

It is desirable to have a certain analogue of the above-indicated result for a variable exponent, as well. It is particularly desirable to reveal those classes of domains D and functions $p(t)$ for which reasonable generalization of the above property would be possible.

In [2], such aim has been achieved under the assumption that $p \in \mathcal{P}(\Gamma)$ and the domain D is bounded by a piecewise Lyapunov curve, free from external cusps. Relying on the theorem from item 2.2.1, the following theorem is proved.

Theorem B. *If D is the bounded domain with the boundary $\Gamma \in C_D^{1,L}(A_1, A_2, \dots, A_n; \nu_1, \nu_2, \dots, \nu_n)$, $0 < \nu_k \le 2$, and $p \in \mathcal{P}(\Gamma)$, then the analytic in D function $\phi(z)$ belongs to the class $E^{p(\cdot)}(D)$ if and only if*

$$\Psi(w) = \phi(z(w)) \prod_{k=1}^{n} (w - a_k)^{\frac{\nu_k - 1}{l(a_k)}} \in H^{l(\cdot)}, \ \ l(\tau) = p(z(\tau)). \tag{9}$$

3.1. In this section we will show that Theorem B can be generalized to a sufficiently wide class of functions $p(t)$ for arbitrary piecewise smooth curves.

Theorem 1. *Let $\Gamma \in C_D^{1}(A_1, A_2, \dots, A_n; \nu_1, \nu_2, \dots, \nu_n)$, $0 \le \nu_k \le 2$, and $z = z(w)$ be conformal mapping of the circle U onto the domain. Next, let p be the function of the class*

$$Q(\Gamma) = \Big\{ p : \ p \in \widetilde{\mathcal{P}}(\Gamma), \ l(\tau) = p(z(\tau)) \in \widetilde{\mathcal{P}}(\gamma) \Big\}. \tag{10}$$

The analytic in D function $\phi(z)$ belongs to the class $E^{p(\cdot)}(D)$ if and only if

$$\Psi(w) = \Phi(z(w))\rho(w) \in H^{l(\cdot)}, \tag{11}$$

where

$$\rho(w) = \prod_{k=1}^{n} (w - a_k)^{\nu_k - 1} l(a_k) \exp \int_{\gamma} \frac{\psi(\zeta)}{l(\zeta)} \frac{d\zeta}{\zeta - w}\,, \ \ z(a_k) = A_k, \tag{12}$$

in which $\psi(\zeta)$ is the function from the representation (3) of the function $z'(w)$.

When $\Gamma \in C_D^{1,L}(A_1, A_2, \dots, A_n; \nu_1, \nu_2, \dots, \nu_n)$, $0 < \nu_k \le 2$, and $p \in \mathcal{P}(\Gamma)$, the condition (11) is equivalent to the condition (9).

Proof. Let $\phi \in E^{p(\cdot)}(D)$. This is equivalent to the fact that

$$\Psi(w) = \phi(z(w)) \in H^{l(\cdot)}(m(w)), \tag{13}$$

where

$$m(w) = m(re^{i\vartheta}) = |z'(re^{i\vartheta})|^{\frac{1}{p(z(e^{i\vartheta}))}}. \tag{14}$$

Thus

$$\phi(z) \in E^{p(\cdot)}(D) \Longleftrightarrow \Psi(w) = \phi(z(w)) \in H^{l(\cdot)}(m(w)). \tag{15}$$

Let us now make use of the result given in [15]:
if $l \in \widetilde{\mathcal{P}}(\gamma)$, then

$$m(w) \sim m_0(w) = m_0(re^{i\vartheta}) = \\ = \prod_{k=1}^{n} (w-a_k)^{\frac{\nu_k-1}{l(a_k)}} \exp\bigg(\frac{1}{l(e^{i\vartheta})} \int\limits_{\gamma} \frac{\psi(\zeta)}{\zeta - re^{i\vartheta}}\, d\zeta\bigg), \tag{16}$$

and

$$m_0(w) \sim \rho(w) = \prod_{k=1}^{n} (w-a_k)^{\frac{\nu_k-1}{l(a_k)}} \exp \int\limits_{\gamma} \frac{\psi(\zeta)}{l(\zeta)}\, \frac{d\zeta}{\zeta - w}\,. \tag{17}$$

It follows from (16), (17) that $m(w) \sim \rho(w)$, and hence by virtue of (15), we conclude that

$$H^{l(\cdot)}(m(w)) = H^{l(\cdot)}(\rho(w)), \tag{18}$$

whence, in view of (13), it follows that the first statement of the theorem is valid.

Let now $\Gamma \in C_D^{1,L}(A_1, A_2, \dots, A_n; \nu_1, \nu_2, \dots, \nu_n)$, $0 < \nu_k \le 2$, and $p \in \mathcal{P}(\Gamma)$. In this case, the function ψ in the representation (3) belongs to the Hölder class ([7, pp. 146] and [16]). Therefore the function $\int\limits_{\gamma} \frac{\psi(\zeta)}{\zeta - w}\, d\zeta$ is bounded in U (see, e.g., [17, pp. 50, 71]). But then in U are bounded likewise the functions

$$\exp\bigg(\pm \frac{1}{l(e^{i\vartheta})} \int\limits_{\gamma} \frac{\psi(\zeta)}{\zeta - re^{i\vartheta}}\, d\zeta\bigg).$$

Thus, on the basis of (16), we find that the second statement of the theorem is also valid. □

3.2. One Condition for Coincidence of the Classes $Q(\Gamma)$ and $\widetilde{\mathcal{P}}(\Gamma)$.

Theorem 2. *If the domain D is such that for conformal mapping $z = z(w)$ of the circle U onto D we have*

$$z'(w) \in \bigcup_{\delta > 0} H^{1+\delta}, \tag{19}$$

then

$$Q(\Gamma) = \widetilde{\mathcal{P}}(\Gamma). \tag{20}$$

Proof. By virtue of the definition of the class of functions $Q(\Gamma)$ (see (10)), it suffices to state that: if $p \in \widetilde{\mathcal{P}}(\Gamma)$, then $l \in \widetilde{\mathcal{P}}(\gamma)$. Towards this end, we shall use the following statement from [3]:

If $p \in \mathcal{P}(\Gamma)$, then under the condition (19), we have

$$|l(\tau_1 - l(\tau_2)| \leq \frac{A}{|\ln|z(\tau_1) - z(\tau_2)||} < \frac{A'}{|\ln|\tau_1 - \tau_2||}\,. \tag{21}$$

If $p \in \widetilde{\mathcal{P}}(\Gamma)$, then there exists the number $\varepsilon > 0$ for which the condition (6) is fulfilled. Then

$$|l(\tau_1 - l(\tau_2)| \leq \frac{A}{|\ln|z(\tau_1) - z(\tau_2)||^{1+\varepsilon}}$$

and (21) yields $|l(\tau_1 - l(\tau_2)| \leq A'|\ln|\tau_1 - \tau_2||^{-(1+\varepsilon)}$. Consequently, $l \in \mathcal{P}_{1+\varepsilon}(\gamma)$, and hence $l \in \widetilde{\mathcal{P}}(\gamma)$. □

Corollary 1. *If* $\Gamma \in \Lambda$*, then the equality* (20) *holds.*

This statement follows immediately from Theorem 2, if we take into account the fact that the inclusions (2) in the case under consideration are valid (see item 2.2.2).

Corollary 2. *If* $\Gamma \in C^1_D(A_1, A_2, \dots, A_n; \nu_1, \nu_2, \dots, \nu_n)$*,* $0 < \nu_k \leq 2$*,* $k = 1, \dots, n$*, then* $Q(\Gamma) = \widetilde{\mathcal{P}}(\Gamma)$*.*

Indeed, since the function $\exp \int\limits_\gamma \frac{\psi(\zeta)}{\zeta - w}\,d\zeta$ for the continuous real ψ belongs to $\bigcap\limits_{\delta>1} H^\delta$ (see [12] and [7, p. 96]), it is not difficult to state that $z' \in H^{1+\delta_0}$ for some $\delta_0 > 0$.

Corollary 3. *In the assumption of Corollary* 2*, the class* $Q(\Gamma)$ *in Theorem* 1 *can be replaced by the class* $\widetilde{\mathcal{P}}(\Gamma)$*.*

3.3. One Subset of the Class $\widetilde{\mathcal{P}}(\Gamma)$ Contained in $Q(\Gamma)$. Note first that according to Corollary 2, for $p \in \widetilde{\mathcal{P}}(\Gamma)$ the curves Γ of the class $\Gamma \in C^1_D(A_1, A_2, \dots, A_n; \nu_1, \nu_2, \dots, \nu_n)$, $0 < \nu_k \leq 2$, belong to $Q(\Gamma)$. However, if for some j we have $\nu_j = 0$, then this statement is, generally speaking, doubtful. Therefore for such curves it is desirable to indicate certain sets of functions $p(t)$ for which the equality (20) remains valid.

Let $p(t)$ be such a function from $\mathcal{P}(\Gamma)$ ($\widetilde{\mathcal{P}}(\Gamma)$) which is constant in some neighborhoods of the points A_{ν_j}. By virtue of the above-said, there exists the number $\sigma > 0$ such that as soon as $|t_1 - t_2| < \sigma$, the inequality (4) ((6)) will be fulfilled. Since the conformal mapping of the domains of above-mentioned type transfers the arcs of the boundary Γ into those of the circumference γ (see., e.g., [18, p. 46]), there exist neighborhoods of the points a_{ν_j} at which the condition (4) ((6)) is fulfilled. Consequently, there exists the number $\sigma_\gamma > 0$ such that for $|\tau_1 - \tau_2| < \sigma_\gamma$, $\tau_1, \tau_2 \in \gamma$, the inequality (4) ((6)) will be fulfilled. It is easy to verify that (4) ((6)) is valid for any pairs τ_1, τ_2 lying on γ. This implies that $l(\tau) \in \mathcal{P}(\gamma)$.

From the above, in particular, it follows that for the curves and functions $p(t)$ under consideration, we have $\widetilde{\mathcal{P}}(\Gamma) = Q(\Gamma)$. Moreover, in these assumptions, the set $Q(\Gamma)$ in Theorem 1 can be replaced by the set $\widetilde{\mathcal{P}}(\Gamma)$.

4. The Cauchy Type Integrals and Smirnov Classes

It is not difficult to state that if D is a simply connected domain bounded by a simple rectifiable curve Γ, and $p \in \mathcal{P}(\Gamma)$, then the functions of the class $E^{p(\cdot)}(D)$ are representable by the Cauchy type integral with density from $L^{p(\cdot)}(\Gamma)$ (see Theorem 3 below). However, one fails to inverse this statement to a full entent. It is shown in [2] that in the case of piecewise Lyapunov curves this way is quite possible.

In this section we prove that the integrals $(K_\Gamma \varphi)(z)$, $\varphi \in L^{p(\cdot)}(\Gamma)$, belong to $E^{p(\cdot)}(D)$ under some, very important for applications, assumptions regarding Γ and $p(t)$, including the case in which Γ is an arbitrary piecewise smooth curve, and $p(t) \in Q(\Gamma)$.

4.1. The Representability of Functions from $E^{p(\cdot)}(D)$ by the Cauchy Type Integral.

Theorem 3. *If D is the inner domain bounded by a simple rectifiable curve Γ, and $\phi \in E^{p(\cdot)}(D)$, where $p \in \mathcal{P}(\Gamma)$, then ϕ is representable by the Cauchy type integral with density from $L^{p(\cdot)}(\Gamma)$.*

Proof. It follows from the definition of the class $E^{p(\cdot)}(D)$ that $E^{p(\cdot)}(D) \subset E^{\underline{p}}(D)$, and since $p \in \mathcal{P}(\Gamma)$, hence $\underline{p} > 1$. Thus $E^{p(\cdot)}(D) \subset E^1(D)$. This implies that ϕ is representable by the Cauchy type integral, i.e.,

$$\phi(z) = (K_\Gamma \phi^+)(z), \quad z \in D, \tag{22}$$

(see, e.g., [8, p. 205]). Moreover, the function $F(w) = \phi(z(w))[z'(w)]^{1/\underline{p}}$ is of the Hardy class $H^{\underline{p}}$, and hence almost everywhere on γ there exists an angular boundary value $F^+(\tau)$. Since $z' \in H^1$ (see, e.g., [8, p. 405]), there likewise exists $[z'(w)]^+ = z'(\tau)$. Thus the boundary value of the function $\Phi(z(w))$ exists. Relying on this fact, we can conclude that

$$\lim_{r \to 1} \Big(|\Phi(re^{i\vartheta})|^{p(z(e^{i\vartheta}))} |z'(re^{i\vartheta})| \Big) = \big|\phi(z(e^{i\vartheta}))\big|^{p(z(e^{i\vartheta}))} |z'(e^{i\vartheta})|.$$

Using the Fatou lemma, by virtue of (8), we conclude that

$$\int_0^{2\pi} \big|\phi(z(e^{i\vartheta}))\big|^{p(z(e^{i\vartheta}))} |z'(e^{i\vartheta})| \, d\vartheta < \infty.$$

The above-said is equivalent to the fact that $\int_\Gamma |\phi^+(t)|^{p(t)} \, |dt| < \infty$, i.e., $\phi^+ \in L^{p(\cdot)}(\Gamma)$. But then the equality (22) implies that $\phi(z)$ is represented by the Cauchy type integral with density from $L^{p(\cdot)}(\Gamma)$. □

4.2. On the Belonging of the Cauchy Type Integral to the Class $E^{p(\cdot)}(D)$ for Domains with Piecewise Smooth Boundaries.

Theorem 4. *Let D be the simply connected finite domain bounded by the curve Γ of the class $C^1_D(A_1, A_2, \dots, A_n; \nu_1, \nu_2, \dots, \nu_n)$, and $p \in Q(\Gamma)$. Then the Cauchy type integral $\phi(z) = (K_\Gamma \varphi)(z)$, where $\varphi \in L^{p(\cdot)}(\Gamma)$, belongs to the class $E^{p(\cdot)}(D)$.*

Proof. Since $L^{p(\cdot)}(\Gamma) \subset L^{\underline{p}}(\Gamma)$, therefore $\varphi \in L^{\underline{p}}(\Gamma)$, $\underline{p} > 1$. It is easy to verify that the piecewise smooth curve is regular, and according to statement (ii) of item 2.3, we can conclude that $\phi \in E^{\underline{p}}(D)$. Using Theorem 1, we find that the analytic in U function $\Psi(w) = \phi(z(w))\rho(w))$, where ρ, defined by the equality (12), belongs to the class $H^{\underline{p}}$. Let us now show that $\Psi^+ \in L^{l(\cdot)}(\gamma)$.

As far as Γ is the regular curve, and from the condition $p \in Q(\Gamma)$ follows $p \in \mathcal{P}(\Gamma)$, the operator S_Γ is continuous in $L^{p(\cdot)}(\Gamma)$ (see statement (i) of item 2.3). Thus the function $\phi^+(t_0) = \frac{1}{2}\varphi(t_0) + \frac{1}{2}(S_\Gamma\varphi)(t_0)$ belongs to $L^{p(\cdot)}(\Gamma)$, i.e., the function $\phi(z(\tau))[z'(\tau)]^{\frac{1}{p(z(\tau))}} \sim \phi(z(\tau))m^+(\tau)$ (see (14)) belongs to $L^{p(\cdot)}(\Gamma)$. This is the same thing as $\Psi^+ \in L^{l(\cdot)}(\gamma)$.

Thus $\Psi \in H^{\underline{p}}$ and $\Psi^+ \in L^{l(\cdot)}(\gamma)$, where $l \in \mathcal{P}(\gamma)$. We now apply the generalized Smirnov's theorem: if $\Psi(z) \in H^{l_1(\cdot)}$ and $\Psi^+(t) \in L^{l_2(\cdot)}(\gamma)$, $l_2 \in \mathcal{P}(\gamma)$, then $\Psi(z) \in H^{\widetilde{l}}(\cdot)$, where $\widetilde{l}(t) = \max(l_1(\tau), l_2(\tau))$ (under such a statement, this theorem has been proven in [2]). In our case, $\widetilde{l}(\tau) = \max(\underline{p}, l(\tau)) = l(\tau)$. Hence $\Psi(w) \in H^{l(\cdot)}$, i.e., $\phi(z(w)) \in H^{l(\cdot)}(\rho) = H^{l(\cdot)}(m(w))$ (see(18)), and this is the same thing as $\phi(z) \in E^{p(\cdot)}(D)$. □

4.3. On the Belonging of the Cauchy Type Integrals with Density from $L^{p(\cdot)}(\Gamma)$ to the Class $E^{p(\cdot)}(D)$ when $p(t)$ is the Hölder Continuous Function.

If we assume that $p(t)$ is the Hölder class function, then the class of piecewise smooth curves in Theorem 4 can be replaced by another wide set of curves.

Upon our investigation we use Theorem 5 proven below. This theorem generalizes Smirnov's theorem (see 2.3.1) to the case of classes $E^{p(\cdot)}(D)$, when D belongs to a rather wide class of functions.

4.3.1. *Generalization of Smirnov's Theorem.*

Theorem 5. *Let Γ be the simple, rectifiable, closed, regular curve bounding the domain D such that*

$$z'(w) \in \bigcup_{\sigma>1} H^\sigma, \quad \frac{1}{z'(w)} \in \bigcup_{\eta>0} H^\eta, \tag{23}$$

where $z = z(w)$ is the conformal mapping of the circle U onto the domain D.

If $\phi(z) \in E^{\mu(\cdot)}(D)$, $\min\limits_{t\in\Gamma} \mu(t) = \delta > 0$ and $\phi^+(t) \in L^{p(t)}(\Gamma)$, where $p(t)$ is the Hölder class function on Γ, then $\phi(z) \in E^{\widetilde{p}(\cdot)}(D)$, $\widetilde{p}(t) = \max(\mu(t), p(t))$.

Proof. Assume $\Psi(w)=\phi(z(w))$ and show that the function $\Psi(w)$ in the adopted assumptions belongs to a certain Hardy class H^ε, $\varepsilon>0$.

Let ε be a number from the interval $(0,\delta)$. We have

$$I_r=\int_0^{2\pi}|\Psi(re^{i\vartheta})|^\varepsilon\,d\vartheta=\int_0^{2\pi}|\phi(re^{i\vartheta})|^\varepsilon|z'(re^{i\vartheta})|^{\frac{\varepsilon}{\delta}}|z'(re^{i\vartheta})|^{-\frac{\varepsilon}{\delta}}\,d\vartheta.$$

Using Hölder's inequality with the exponent $\delta/\varepsilon>1$, we obtain

$$I_r\le\bigg(\int_0^{2\pi}|\phi(re^{i\vartheta})|^\delta|z'(re^{i\vartheta})|\,d\vartheta\bigg)^{\frac{\varepsilon}{\delta}}\bigg(\int_0^{2\pi}|z'(re^{i\vartheta})|^{-\frac{\varepsilon}{\delta-\varepsilon}}\,d\vartheta\bigg)^{\frac{\delta-\varepsilon}{\delta}}\le$$

$$\le[M(r)]^{\frac{\varepsilon}{\delta}}\bigg(\int_0^{2\pi}\frac{d\vartheta}{|z'(re^{i\vartheta})|^{\frac{\varepsilon}{\delta-\varepsilon}}}\bigg)^{\frac{\delta-\varepsilon}{\delta}},\quad M(r)=\int_0^{2\pi}|\phi(re^{i\vartheta})|^\delta|z'(re^{i\vartheta})|\,d\vartheta.\tag{24}$$

It follows from the condition $\phi\in E^{\mu(\cdot)}(D)$ that $\phi\in E^\delta(D)$, and hence

$$\sup_{0<r<1}M(r)=C<\infty.\tag{25}$$

Further, the condition $\frac{1}{z'}\in\bigcup\limits_{\eta>0}H^\eta$ provides us with $\frac{1}{z'}\in H^{\eta_0}$ for some $\eta_0>0$. We choose ε such that $\frac{\varepsilon}{\delta-\varepsilon}=\eta_0$ (i.e., we take $\varepsilon=\varepsilon_0=\frac{\delta\eta_0}{1+\eta_0}$).

Since $\frac{1}{z'}\in H^{\eta_0}$, therefore

$$\sup_{0<r<1}\int_0^{2\pi}\frac{d\vartheta}{|z'(re^{i\vartheta})|^{\eta_0}}<\infty.$$

In view of the above-said and the inequality (25), from (24) it follows that $\sup I_r<\infty$. Thus we have stated that $\Psi\in H^{\varepsilon_0}$, $\varepsilon_0=\frac{\delta\eta_0}{1+\eta_0}$.

Since $\Psi\in H^{\varepsilon_0}$, we have $\Psi(w)=e^{i\lambda}b(w)\sigma(w)D(w)$, where $b(w)$ is the Blaschke product, $\sigma(w)\ne0$, $|\sigma(w)|\le1$, $\lambda\in\mathbb{R}$, and

$$D(w)=\exp\frac{1}{2\pi}\int_0^{2\pi}\ln|\Psi(e^{i\varphi})|\,\frac{e^{i\varphi}+w}{e^{i\varphi}-w}\,d\varphi,\quad |w|<1$$

(see [8, p. 110]).

Assume $l(\tau):=l(e^{i\vartheta})=p(z(e^{i\vartheta}))=p(z(\tau))$, $\tau=e^{i\vartheta}$. Then since $p(t)$ is the Hölder class function on Γ, there exist numbers M and $\alpha\in(0,1]$ such that $|p(t_1)-p(t_2)|<M|t_1-t_2|^\alpha$. Consequently,

$$|p(t_1)-p(t_2)|=\big|p(z(\tau_1))-p(z(\tau_2))\big|\le$$

$$\le AM|z(\tau_1-z(\tau_2)|^\alpha=AM\bigg|\int_{\tau_1}^{\tau_2}z'(\tau)\,d\tau\bigg|^\alpha.$$

It follows from the inclusion $z' \in \bigcup_{\sigma>1} H^{\sigma}$ (see (23)) that $z' \in H^{\sigma_0}$ for some $\sigma_0 > 1$. Then the last inequality (in view of the fact that on γ we have $s(\tau_1, \tau_2) \sim |\tau_1 - \tau_2|$) yields

$$|l(\tau_1) - l(\tau_2) \leq AM\bigg(\int_{\tau_1}^{\tau_2} |z'(\tau)|^{\sigma_0}\, |d\tau|\bigg)^{\frac{\alpha}{\sigma_0}} |\tau_1 - \tau_2|^{\frac{\sigma_0 - 1}{\sigma_0}\,\alpha}.$$

Thus $l(\tau)$ is the function from the Hölder class on γ. In view of the above, we can apply the inequality proven in [2]:

$$|\Psi(re^{i\vartheta})|^{l(\vartheta)} \leq A(r,\vartheta)B(r,\vartheta), \tag{26}$$

where

$$A(r,\vartheta) = \exp \frac{1}{2\pi} \int_0^{2\pi} l(\varphi) \ln|\widetilde{\Psi}(e^{i\varphi})| P(r, \vartheta - \varphi)\, d\varphi,$$

$$\widetilde{\Psi}(e^{i\varphi}) = \begin{cases} \Psi(e^{i\varphi}), & \text{if } |\Psi(e^{i\varphi})| \geq 1 \\ 1, & \text{if } |\Psi(e^{i\varphi})| < 1 \end{cases}, \quad P(r,x) = \frac{1 - r^2}{1 + r^2 - 2r\cos x}\,,$$

and for $B(r,\vartheta)$, the following estimate is valid:

$$|B(r,\vartheta)| \leq k_1 \exp k_2 \int_0^{2\pi} |\Psi(e^{i\varphi})|\, d\varphi = k_3,$$

where k_1, k_2 does not depend on Ψ.

The inequality (26) results now in

$$\int_0^{2\pi} |\Psi(re^{i\vartheta})|^{l(\vartheta)} |z'(re^{i\vartheta})|\, d\vartheta \leq$$
$$\leq k_3 \int_0^{2\pi} \exp\bigg(\frac{1}{2\pi} \int_0^{2\pi} \ln|\widetilde{\Psi}(e^{i\varphi})|^{l(\varphi)} P(r, \vartheta - \varphi)\, d\varphi\bigg) |z'(re^{i\vartheta})|\, d\vartheta. \tag{27}$$

Since Γ is the regular curve, therefore D is Smirnov's domain (see statement (ii) of item 2.3), and hence

$$|z'(re^{i\vartheta})| = |z'(w)| = \exp\ln|z'(w)| =$$
$$= \exp \frac{1}{2\pi} \int_0^{2\pi} \ln|z'(re^{i\vartheta})| P(r, \vartheta - \varphi)\, d\varphi. \tag{28}$$

Moreover, we have

$$\ln|\widetilde{\Psi}(e^{i\varphi})|^{l(\varphi)} = \ln\Big|\frac{|\widetilde{\Psi}(e^{i\varphi})|^{l(\varphi)}z'(e^{i\varphi})}{z'(e^{i\varphi})}\Big| = \\ = \ln\Big[|\widetilde{\Psi}(e^{i\varphi})|^{l(\varphi)}|z'(e^{i\varphi})| - \ln|z'(e^{i\varphi})|\Big]. \quad (29)$$

From (27), by virtue of (28) and (29), we can conclude that

$$\int_0^{2\pi} |\Psi(re^{i\vartheta})|^{l(\vartheta)}|z'(re^{i\vartheta})|\,d\vartheta \le \\ \le k_3 \int_0^{2\pi} \exp\frac{1}{2\pi}\int_0^{2\pi} \ln|\widetilde{\Psi}(e^{i\varphi})|^{l(\varphi)}|z'(e^{i\varphi})|P(z,\vartheta-\varphi)\,d\varphi\,d\vartheta \le \\ \le k_3 \int_0^{2\pi} |\widetilde{\Psi}(re^{i\varphi})|^{l(\varphi)}|z'(e^{i\varphi})|\,d\varphi \le \\ \le k_3 \int_0^{2\pi} |\Psi(e^{i\varphi})|^{l(\varphi)}|z'(e^{i\varphi})|\,d\varphi + \int_0^{2\pi} |z'(e^{i\varphi})|\,d\varphi \le \\ \le k_3 \int_0^{2\pi} |\Psi(e^{i\varphi})|^{l(\varphi)}|z'(e^{i\varphi})|\,d\varphi + k_4. \quad (30)$$

By the assumption of the theorem, $\phi^+ \in L^{p(\cdot)}(\Gamma)$. But

$$\int_0^{2\pi} |\Psi(e^{i\varphi})|^{l(\varphi)}|z'(e^{i\varphi})|\,d\varphi = \int_\Gamma |\phi^+(t)|^{p(t)}\,|dt|$$

and from (30) follows

$$\sup_{r<1} \int_0^{2\pi} |\Psi(e^{i\varphi})|^{l(\varphi)}|z'(e^{i\varphi})|\,d\varphi < \infty.$$

Hence $\phi \in E^{p(\cdot)}(D)$; and since $\phi \in E^{\mu(\cdot)}(D)$, then $\phi \in E^{\widetilde{p}(\cdot)}(D)$, $\widetilde{p}(t) = \max(p(t),\mu(t))$. □

4.4. The Cauchy Type Integrals in the Domains with Lavrentiev Boundary.

Theorem 6. *If D is the inner domain bounded by a simple rectifiable curve of the class Λ, and p is the Hölder class function on Γ, then the Cauchy type integral $\phi(z) = (K_\Gamma\varphi)(z)$, where $\varphi \in L^{p(\cdot)}(\Gamma)$, belongs to the class $E^{p(\cdot)}(D)$.*

Proof. In the case under consideration, the both conditions in (23) are fulfilled. Moreover, it can be easily verified that any curve from Λ is regular one. Next, since $\varphi \in L^{\underline{p}}(\Gamma)$, $\underline{p} = \min_{t\in\Gamma} p(t)$, in view of property (ii) in item 2.3, we conclude that $\phi \in E^{\underline{p}}(D)$. Along with the above-said, $\phi^+ = \frac{1}{2}\varphi + \frac{1}{2} S_\Gamma \varphi$, $\varphi \in L^{\underline{p}(\cdot)}(\Gamma)$. Since $p \in \mathcal{P}(\Gamma)$, therefore $S_\Gamma\varphi \in L^{p(\cdot)}(\Gamma)$ (see Theorem A). Consequently, $\phi^+ \in L^{p(\cdot)}(\Gamma)$.

Thus $\phi \in E^{\underline{p}}(D)$ and $\phi^+ \in L^{p(\cdot)}(\Gamma)$, where $p(t)$ is the Hölder class function on Γ. This implies that all requirements of Theorem 5 are fulfilled and hence $\phi \in E^{p(\cdot)}(D)$. □

Acknowledgement

The author is thankful to Vakhtang Kokilashvili for useful discussions of problems related to the present paper.

This work is supported by the Shota Rustaveli National Science Foundation (Project # GNSF/ST09_23_3_100).

References

1. V. Kokilashvili and V. Paatashvili, On Hardy classes of analytic functions with a variable exponent. *Proc. A. Razmadze Math. Inst.* **142** (2006), 134–137.
2. V. Kokilashvili and V. Paatashvili, On variable Hardy and Smisnov classes of analytic functions. *Georgian Internat. J. Sci., Nova Sci. Publ., Inc.* **1** (2008), No. 2, 67–81.
3. V. Kokilashvili and V. Paatashvili, Weighted Hardy and Smirnov classes and the Dirichlet problem for a ring within the framework of variable exponent analysis. *Complex Var. Elliptic Equ.* **56** (2011), No. 10-11, 955–973.
4. S. E. Warschawski, Über das Randverhalten der Ableitung der Abbildungsfunktion bei konformer Abbildung. *Math. Z.* **35** (1932), No. 1, 321–456.
5. Ch. Pommerenke, Boundary behaviour of conformal maps. *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, 299. *Springer-Verlag, Berlin*, 1992.
6. G. Khuskivadze and V. Paatashvili, On the conformal mapping of a circle onto a domain with an arbitrary piecewise smooth boundary. *Proc. A. Razmadze Math. Inst.* **116** (1998), 123–132.
7. G. Khuskivadze, V. Kokilashvili, and V. Paatashvili, Boundary value problems for analytic and harmonic functions in domains with nonsmooth boundaries. Applications to conformal mappings. *Mem. Differential Equations Math. Phys.* **14** (1998), 195 pp.
8. I. I. Privalov, Boundary properties of one-valued analytic functions. (Russian) *Nauka, Moscow*, 1950.
9. G. David, Opérateurs integraux singuliers sur certaines courbes du plan complexe. (French) [Singular integral operators over certain curves in the complex plane] *Ann. Sci. École Norm. Sup. (4)* **17** (1984), No. 1, 157–189.
10. V. P. Havin, Boundary properties of integrals of Cauchy type and of conjugate harmonic functions in regions with rectifiable boundary. (Russian) *Mat. Sb. (N.S.)* **68 (110)** (1965), 499–517.
11. V. Paatashvili, The boundedness of a singular Cauchy operator in L_p spaces. (Russian) *Trudy A. Razmadze Math. Inst.* **58** (1978), 187–196.
12. V. I. Smirnov, Sur les formules de Cauchy et de Green et quelques problḿes qui s'y rattachent. *Bulletin Acad. Sc. Leningrad (7)* (1932), 337–372.
13. O. Kováčik and J. Rákosnic, On spaces $L^{p(x)}$ and $W^{k,p(x)}$. *Czechoslovak Math. J.* **41(116)** (1991), No. 4, 592–618.

14. V. Kokilashvili, V. Paatashvili, and S. Samko, Boundedness in Lebesgue spaces with variable exponent of the Cauchy singular operator on Carleson curves. *Modern operator theory and applications*, 167–186, *Oper. Theory Adv. Appl.*, 170, *Birkhauser, Basel*, 2007.
15. V. Kokilashvili and V. Paatashvili, The Dirichlet problem for harmonic functions from variable exponent Smirnov classes in domains with piecewise smooth boundary. *Problems in mathematical analysis*. No. 52. *J. Math. Sci. (N. Y.)* **172** (2011), No. 3, 401–421.
16. R. Abdulaev, On one theorem of S. Warschawski. *Georgian Math. J.* **10** (2003), No. 1, 1–15.
17. N. I. Muskhelishvili, Singular integral equations. Boundary value problems in the theory of function and some applications of them to mathematical physics. (Russian) Third, corrected and augmented edition. *Izdat. "Nauka", Moscow*, 1968.
18. G. M. Goluzin, Geometrical theory of functions of a complex variable. (Russian) Second edition. *Izdat. "Nauka", Moscow*, 1966.

(Received 15.12.2011)

Author's address:

A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 2 University St., Tbilisi 0186, Georgia.