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EMPHATIC CONVERGENCE AND SEQUENTIAL SOLUTIONS OF GENERALIZED LINEAR DIFFERENTIAL EQUATIONS

Dedicated to the memory of Temur Chanturia

Abstract. This contribution deals with systems of generalized linear differential equations of the form

$$x_k(t) = \widetilde{x}_k + \int_a^t \mathrm{d}[A_k(s)]\, x_k(s) + f_k(t) - f_k(a), \quad t \in [a,b], \ \ k \in \mathbb{N},$$

where $-\infty < a < b < \infty$, X is a Banach space, $L(X)$ is the Banach space of linear bounded operators on X, $\widetilde{x}_k \in X$, $A_k : [a,b] \to L(X)$ have bounded variations on $[a,b]$, $f_k : [a,b] \to X$ are regulated on $[a,b]$ and the integrals are understood in the Kurzweil–Stieltjes sense.

Our aim is to present new results on continuous dependence of solutions to generalized linear differential equations on the parameter k. We continue our research from [18], where we were assuming that A_k tends uniformly to A and f_k tends uniformly to f on $[a,b]$. Here we are interested in the cases when these assumptions are violated.

Furthermore, we introduce a notion of a sequential solution to generalized linear differential equations as the limit of solutions of a properly chosen sequence of ODE's obtained by piecewise linear approximations of functions A and f. Theorems on the existence and uniqueness of sequential solutions are proved and a comparison of solutions and sequential solutions is given, as well.

The convergence effects occurring in our contribution are, in some sense, very close to those described by Kurzweil and called by him emphatic convergence.

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რეზიუმე. განხილულია განზოგადებულ წრფივ დიფერენციალურ განტოლებათა სისტემა

$$x_k(t) = \widetilde{x}_k + \int_a^t \mathrm{d}[A_k(s)]\, x_k(s) + f_k(t) - f_k(a), \quad t \in [a,b], \ \ k \in \mathbb{N},$$

სადაც $-\infty < a < b < \infty$, X არის ბანახის სივრცე, $L(X)$ არის X-ის შეუღლებული სივრცე, $\widetilde{x}_k \in X$, $A_k : [a,b] \to L(X)$ -ს აქვს შემოსაზღვრული ვარიაცია, $f_k : [a,b] \to X$ რეგულირებადია, ხოლო ინტეგრალი გაიგება კურცვეილ–სტილტიესის აზრით.

მიღებულია ახალი შედეგები ამონახსნების ნატურალური k პარამეტრისაგან უწყვეტად დამოკიდებულების შესახებ, როცა $k \to \infty$. შემოღებულია განზოგადებული წრფივი დიფერენციალური განტოლების სეკვენციალური ამონახსნის ცნება და დამტკიცებულია თეორემები ასეთი ამონახსნის არსებობისა და ერთადერთობის შესახებ.

ნაშრომში დადგენილი ამონახსნთა მიმდევრობის კრებადობა გარკვეული აზრით ახლოს არის ი. კურცვეილის მიერ აღწერილ კრებადობასთან, რომელსაც იგი ემფატიკურს უწოდებს.

1. Introduction

Generalized differential equations were introduced in 1957 by J. Kurzweil in [14]. Since then they were studied by many authors. (See e.g. the monographs by Schwabik, Tvrdý and Vejvoda [29], [25], [32] or the papers by Ashordia [2], [3] or Fraňková [7] and the references therein). Closely related and fundamental is also the contribution by Hildebrandt [10]. Furthermore, during the recent decades, the interest in their special cases like equations with impulses or discrete systems increased considerably (cf. e.g. the monographs [21], [33], [4], [24] or [1]).

Concerning integral equations in a general Banach space, it is worth to highlight the monograph by Hönig [11] having as a background the interior (Dushnik) integral. On the other hand, dealing with the Kurzweil–Stieltjes integral, the contributions by Schwabik in [27] and [28] are essential for this paper. It is well-known that the theory of generalized differential equations in Banach spaces enables the investigation of continuous and discrete systems, including the equations on time scales and the functional differential equations with impulses, from the common standpoint. This fact can be observed in several papers related to special kinds of equations, such as e.g. those by Imaz and Vorel [12], Oliva and Vorel [19], Federson and Schwabik [6].

In this paper we consider linear generalized differential equations of the form

$$x_k(t) = \widetilde{x}_k + \int\limits_a^t \mathrm{d}[A_k(s)]\, x_k(s) + f_k(t) - f_k(a), \quad t \in [a,b], \ \ k \in \mathbb{N}, \tag{1.1}$$

and

$$x(t) = \widetilde{x} + \int\limits_a^t \mathrm{d}[A(s)]\, x(s) + f(t) - f(a), \quad t \in [a,b]. \tag{1.2}$$

In particular, we are interested in finding conditions ensuring the convergence of the solutions x_k of (1.1) to the solution x of (1.2). We continue our research from [9] and [18], where we supposed a.o. that A_k tends uniformly to A and f_k tends uniformly to f on $[a,b]$. Here we will deal, similarly to [31] and [8], with the situation when this assumption is not satisfied.

In the paper we use the following notation:

$\mathbb{N} = \{1, 2, \dots\}$ is the set of natural numbers and $\mathbb{R}$ stands for the space of real numbers. If $-\infty < a < b < \infty$, then $[a,b]$ and (a,b) denote the corresponding closed and open intervals, respectively. Furthermore, $[a,b)$ and $(a,b]$ are the corresponding half-open intervals.

X is a Banach space equipped with the norm $\|\cdot\|_X$ and $L(X)$ is the Banach space of linear bounded operators on X equipped with the usual operator norm. For an arbitrary function $f : [a,b] \to X$, we set

$$\|f\|_\infty = \sup\big\{\|f(t)\|_X;\ t \in [a,b]\big\}.$$

If $f_k : [a,b] \to X$ for $k \in \mathbb{N}$ and $f : [a,b] \to X$ are such that

$$\lim_{k\to\infty} \|f_k - f\|_\infty = 0,$$

we say that f_k tends to f *uniformly* on $[a,b]$ and write $f_k \rightrightarrows f$ on $[a,b]$. If $J \subset \mathbb{R}$ and $f_k \rightrightarrows f$ on $[a,b]$ for each $[a,b]\subset J$, we say that f_k tends to f *locally uniformly* on J and write $f_k \rightrightarrows f$ locally on J.

If for each $t \in [a,b)$ and $s \in (a,b]$, the function $f : [a,b] \to X$ possesses the limits

$$f(t+) := \lim_{\tau\to t+} f(\tau), \quad f(s-) := \lim_{\tau\to s-} f(\tau),$$

we say that f is *regulated* on $[a,b]$. The set of all functions with values in X which are regulated on $[a,b]$ is denoted by $G([a,b],X)$. Furthermore,

$$\Delta^+ f(t) = f(t+) - f(t) \quad \text{for } t \in [a,b), \quad \Delta^+ f(b) = 0,$$
$$\Delta^- f(s) = f(s) - f(s-) \quad \text{for } s \in (a,b], \quad \Delta^- f(a) = 0$$

and

$$\Delta f(t) = f(t+) - f(t-) \quad \text{for } t \in (a,b).$$

Clearly, each function, regulated on $[a,b]$, is bounded on $[a,b]$.

The set $D = \{\alpha_0, \alpha_1, \dots, \alpha_m\} \subset [a,b]$, where $m \in \mathbb{N}$, is called a *division* of the interval $[a,b]$, if $a = \alpha_0 < \alpha_1 < \dots < \alpha_m = b$. The set of all divisions of the interval $[a,b]$ is denoted by $\mathcal{D}[a,b]$. For a function $f : [a,b] \to X$ and a division $D = \{\alpha_0, \alpha_1, \dots, \alpha_m\} \in \mathcal{D}[a,b]$, we put

$$\nu(D) := m, \quad |D| = \max\{\alpha_i - \alpha_{i-1};\ i = 1,2,\dots,m\},$$
$$v(f,D) := \sum_{j=1}^{m} \|f(\alpha_j) - f(\alpha_{j-1})\|_X$$

and

$$\operatorname{var}_a^b f := \sup\{v(f,D);\ D \in \mathcal{D}[a,b]\}$$

is the *variation* of f over $[a,b]$. We say that f has a bounded variation on $[a,b]$ if $\operatorname{var}_a^b f < \infty$. The set of X-valued functions of bounded variation on $[a,b]$ is denoted by $BV([a,b],X)$ and $\|f\|_{BV} = \|f(a)\|_X + \operatorname{var}_a^b f$. Finally, $C([a,b],X)$ is the set of functions $f : [a,b] \to X$ which are continuous on $[a,b]$. Obviously,

$$BV([a,b],X) \subset G([a,b],X) \text{ and } C([a,b],X) \subset G([a,b],X).$$

The integral which occurs in this paper is the abstract Kurzweil–Stieltjes integral (in short the KS-integral) as defined by Schwabik in [26]. (For its further properties see also our previous paper [17]). For the reader's convenience, let us recall the definition of the KS-integral.

Let $-\infty < a < b < \infty$, $m \in \mathbb{N}$,

$$D = \{\alpha_0, \alpha_1, \dots, \alpha_m\} \in \mathcal{D}[a,b] \text{ and } \xi = (\xi_1, \xi_2, \dots, \xi_m) \in [a,b]^m.$$

Then the couple $P = (D, \xi)$ is called a *partition* of $[a,b]$ if

$$\alpha_{j-1} \le \xi_j \le \alpha_j \quad \text{for} \quad j = 1,2,\dots,m.$$

The set of all partitions of the interval $[a,b]$ is denoted by $\mathcal{P}[a,b]$. An arbitrary function $\delta : [a,b] \to (0,\infty)$ is called a *gauge* on $[a,b]$. Given a gauge δ on $[a,b]$, the partition

$$P = (D,\xi) = \big(\{\alpha_0,\alpha_1,\dots,\alpha_m\},(\xi_1,\xi_2,\dots,\xi_m)\big) \in \mathcal{P}[a,b]$$

is said to be δ-fine, if

$$[\alpha_{j-1},\alpha_j] \subset (\xi_j-\delta(\xi_j),\xi_j+\delta(\xi_j)) \quad \text{for} \;\; j=1,2,\dots,m.$$

The set of all δ-fine partitions of $[a,b]$ is denoted by $\mathcal{A}(\delta;[a,b])$.

For the functions $f : [a,b] \to X$, $G : [a,b] \to L(X)$ and a partition $P \in \mathcal{P}[a,b]$,

$$P = (\{\alpha_0,\alpha_1,\dots,\alpha_m\},(\xi_1,\xi_2,\dots,\xi_m)),$$

we define

$$\Sigma(\Delta G f;P) = \sum_{j=1}^{m} \big[G(\alpha_j)-G(\alpha_{j-1})\big] f(\xi_j).$$

We say that $q \in X$ is the KS-integral of f with respect to G from a to b if

$$\begin{cases} \text{for each } \;\varepsilon>0\; \text{ there is a gauge } \delta \text{ on } [a,b] \text{ such that} \\[2ex] \qquad \big\|q-\Sigma(\Delta G f;P)\big\|_X < \varepsilon \;\text{ for all }\; P \in \mathcal{A}(\delta;[a,b]). \end{cases}$$

In such a case we write

$$q = \int_a^b \mathrm{d}[G(t)]f(t) \quad \text{or, more briefly,} \quad q = \int_a^b \mathrm{d}[G]f.$$

Analogously we define the integral $\int_a^b F\,\mathrm{d}[g]$ for $F : [a,b] \to L(X)$ and $g : [a,b] \to X$.

The following assertion summarizes the properties of the KS-integral needed later. (For the proofs, see [26] and [17].)

Theorem 1.1. *Let $f \in G([a,b],X), G \in G([a,b],L(X))$ and let at least one of the functions f,G have a bounded variation on $[a,b]$. Then there exists the integral $\int_a^b \mathrm{d}[G]f$. Furthermore,*

$$\Big\|\int_a^b \mathrm{d}[G]f\Big\|_X \le 2\|G\|_\infty\big(\|f(a)\|_X + \operatorname{var}_a^b f\big) \;\; \text{if} \;\; f \in BV([a,b],X), \tag{1.3}$$

$$\Big\|\int_a^b \mathrm{d}[G]f\Big\|_X \le (\operatorname{var}_a^b G)\|f\|_\infty \;\; \text{if} \;\; G \in BV([a,b],L(X)), \tag{1.4}$$

$$\left.\begin{aligned}&\int_a^t \mathrm{d}[G]f_k \rightrightarrows \int_a^t \mathrm{d}[G]f \quad \textit{on} \;\; [a,b]\\ &\qquad \textit{if}\; G\in BV([a,b],L(X)),\;\; f_k\in G([a,b],X) \;\textit{for}\; k\in\mathbb{N} \;\textit{and}\; f_k\rightrightarrows f,\end{aligned}\right\} \tag{1.5}$$

$$\left.\begin{aligned}&\int_a^t \mathrm{d}[G_k]f \rightrightarrows \int_a^t \mathrm{d}[G]f \quad \textit{on} \;\; [a,b]\\ &\qquad \textit{if}\; f\in BV([a,b],X),\;\; G_k\in G([a,b],L(X)) \;\textit{for}\; k\in\mathbb{N} \;\textit{and}\; g_k\rightrightarrows g,\end{aligned}\right\} \tag{1.6}$$

$$\left.\begin{aligned}&\int_a^t \mathrm{d}[G_k]f_k \rightrightarrows \int_a^t \mathrm{d}[G]f\\ &\qquad \textit{if}\;\; G_k \in BV([a,b],L(X)),\;\;\; f_k \in G([a,b],X) \;\; \textit{for} \;\; k\in\mathbb{N},\\ &\qquad \sup\{\mathrm{var}_a^b\, G_k;\; k\in\mathbb{N}\} < \infty \;\; \textit{and} \;\; f_k \rightrightarrows f,\;\; G_k \rightrightarrows G \;\; \textit{on} \;\; [a,b].\end{aligned}\right\} \tag{1.7}$$

Remark 1.2. An assertion analogous to that of Theorem 1.1 holds also for the integrals

$$\int_a^b F\,\mathrm{d}[g],\; \int_a^b F_k\,\mathrm{d}[g],\; \int_a^b F\,\mathrm{d}[g_k],\; \int_a^b F_k\,\mathrm{d}[g_k], k\in\mathbb{N},$$

where $F,\ F_k : [a,b] \to L(X)$ and $g,\ f_k : [a,b] \to X$.

2. Generalized Differential Equations

Let $A \in BV([a,b],L(X)),\ f \in G([a,b],X)$ and $\widetilde{x} \in X$. Consider the *generalized linear differential equation* (1.2). We say that a function $x : [a,b] \to X$ is a *solution* of (1.2) on the interval $[a,b]$ if the integral $\int_a^b \mathrm{d}[A]x$ has a sense and equality (1.2) is satisfied for all $t \in [a,b]$.

Obviously, the generalized differential equation (1.2) is equivalent to the equation

$$x(t) = \widetilde{x} + \int_a^t \mathrm{d}[B]x + g(t) - g(a)$$

whenever $B - A$ and $g - f$ are constant on $[a,b]$. Therefore, without loss of generality we may assume that

$$A(a) = A_k(a) = 0 \;\text{ and }\; f(a) = f_k(a) = 0 \;\text{ for }\; k \in \mathbb{N}.$$

For our purposes the following property is crucial:

$$\big[I - \Delta^- A(t)\big]^{-1} \in L(X) \;\text{ for each }\; t \in (a,b]. \tag{2.1}$$

Its importance is well illustrated by the following assertion which summarizes some of the basic properties of generalized linear differential equations in abstract spaces. (For the proof see [18, Lemma 3.2].)

Theorem 2.1. *Let $A \in BV([a,b], L(X))$ satisfy* (2.1). *Then for each $\widetilde{x} \in X$ and each $f \in G([a,b], X)$ the equation* (1.2) *has a unique solution x on $[a,b]$ and $x \in G([a,b], X)$. Moreover, $x - f \in BV([a,b], X)$*

$$0 < c_A := \sup\Big\{ \big\|[I - \Delta^- A(t)]^{-1}\big\|_{L(X)}\,;\ t \in (a,b]\Big\} < \infty, \tag{2.2}$$

$$\|x(t)\|_X \le c_A\,(\|\widetilde{x}\|_X + \|f(a)\|_X + \|f\|_\infty)\exp(c_A \operatorname{var}_a^t A) \ \ \textit{for}\ \ t \in [a,b] \tag{2.3}$$

and

$$\operatorname{var}_a^b(x - f) \le c_A(\operatorname{var}_a^b A)\big(\|\widetilde{x}\|_X + 2\|f\|_\infty\big)\exp(c_A \operatorname{var}_a^b A). \tag{2.4}$$

The following result was proved in [18, Theorem 3.4].

Theorem 2.2. *Let $A, A_k \in BV([a,b], L(X))$ $f, f_k \in G([a,b], X)$, $\widetilde{x}, \widetilde{x}_k \in X$ for $k \in \mathbb{N}$. Assume* (2.1),

$$\alpha^* := \sup\{\operatorname{var}_a^b A_k; k \in \mathbb{N}\} < \infty, \tag{2.5}$$

$$A_k \rightrightarrows A \quad \textit{on}\ \ [a,b], \tag{2.6}$$

$$f_k \rightrightarrows f \quad \textit{on}\ \ [a,b] \tag{2.7}$$

and

$$\lim_{k\to\infty} \widetilde{x}_k = \widetilde{x}. \tag{2.8}$$

Then equation (1.2) *has a unique solution x on $[a,b]$. Furthermore, for each $k \in \mathbb{N}$ sufficiently large, there exists a unique solution x_k on $[a,b]$ for the equation* (1.1) *and*

$$x_k \rightrightarrows x \quad \textit{on}\ \ [a,b]. \tag{2.9}$$

Remark 2.3. If (2.5) is not true, but (2.6) is replaced by a stronger notion of convergence in the sense of Opial ([20, Theorem 1]) (cf. [13, Theorem 1.4.1] for extension to functional differential equations), the conclusion of Theorem 2.2 remains true (see [18, Theorem 4.2]). If (2.6) or (2.7) does not hold, the situation becomes rather more difficult (see [7], [8] and [31]). The next section deals with such a case.

3. Emphatic Convergence

The proofs of the next two lemmas follow the ideas of the proof of [8, Theorem 2.2].

Lemma 3.1. *Let $A, A_k \in BV([a,b], L(X))$, $f, f_k \in G([a,b], X)$, $\widetilde{x}, \widetilde{x}_k \in X$ for $k \in \mathbb{N}$. Assume* (2.1), (2.8),

$$\left.\begin{aligned} &[I - \Delta^- A_k(t)]^{-1} \in L(X)\\ &\qquad \textit{for all}\ \ t \in (a,b]\ \ \textit{and}\ \ k \in \mathbb{N}\ \ \textit{sufficiently large,} \end{aligned}\right\} \tag{3.1}$$

$$A_k \rightrightarrows A\ \ \textit{and}\ \ f_k \rightrightarrows f\ \ \textit{locally on}\ \ (a,b]. \tag{3.2}$$

Then there exists a unique solution x of (1.2) *on $[a,b]$ and, for each $k \in \mathbb{N}$, sufficiently large, there exists a unique solution x_k on $[a,b]$ to the equation* (1.1).

Moreover, let (2.5) *and*

$$\left.\begin{array}{l}\forall\,\varepsilon>0\ \ \exists\,\delta>0\ \ \textit{such that}\ \ \forall\,t\in(a,a+\delta)\ \ \exists\,k_0=k_0(t)\in\mathbb{N}\\ \qquad\textit{such that}\ \ \|x_k(t)-\widetilde{x}_k-\Delta^+A(a)\widetilde{x}-\Delta^+f(a)\|_X<\varepsilon\\ \qquad\textit{for all}\ \ \ k\ge k_0\end{array}\right\}\tag{3.3}$$

hold. Then

$$\lim_{k\to\infty} x_k(t)=x(t)\tag{3.4}$$

is true for $t\in[a,b]$, *while* $x_k\rightrightarrows x$ *locally on* $(a,b]$.

Proof. By (3.1), the solutions x_k of (1.1) exist on $[a,b]$ for all k sufficiently large. Let $\varepsilon>0$ be given and let $\delta>0$ and $k_1\in\mathbb{N}$ be such that

$$\|x(t)-x(a+)\|_X<\varepsilon\ \ \text{for}\ \ t\in(a,a+\delta)\ \ \text{and}\ \ \|\widetilde{x}_k-\widetilde{x}\|_X<\varepsilon\ \ \text{for}\ \ k\ge k_1.$$

We may choose δ in such way that (3.3) holds. In view of this, for $t\in(a,a+\delta)$, let $k_0\in\mathbb{N}, k_0\ge k_1$, be such that

$$\|x_k(t)-\widetilde{x}_k-\Delta^+A(a)\widetilde{x}-\Delta^+f(a)\|_X<\varepsilon\ \ \text{for}\ \ k\ge k_0.$$

Then, taking into account the relations

$$x(a+)=x(a)+\Delta^+A(a)x(a)+\Delta^+f(a)\ \ \text{and}\ \ x(a)=\widetilde{x},$$

we get

$$\begin{aligned}&\|x_k(t)-x(t)\|_X=\\ &\quad=\|(x_k(t)-\widetilde{x}_k)+(\widetilde{x}_k-\widetilde{x})+(\widetilde{x}-x(a+))+(x(a+)-x(t))\|_X\le\\ &\quad\le\|x_k(t)-\widetilde{x}_k-x(a+)+\widetilde{x}\|_X+\|\widetilde{x}-\widetilde{x}_k\|_X+\|x(t)-x(a+)\|_X=\\ &\qquad\qquad=\|x_k(t)-\widetilde{x}_k-\Delta^+A(a)\widetilde{x}-\Delta^-f(a)\|_X+\\ &\qquad\qquad\qquad\qquad+\|\widetilde{x}-\widetilde{x}_k\|_X+\|x(t)-x(a+)\|_X<3\varepsilon.\end{aligned}$$

This means that (3.4) holds for $t\in[a,a+\delta)$.

Now, let an arbitrary $c\in(a,a+\delta)$ be given. We can use Theorem 2.2 to show that the solutions x_k to

$$x_k(t)=x_k(c)+\int_c^t \mathrm{d}[A_k]x_k+f_k(t)-f(t)$$

exist on $[c,b]$ and $x_k\rightrightarrows x$ on $[c,b]$. The assertion of the lemma follows easily. □

Lemma 3.2. *Let* $A,A_k\in BV([a,b],L(X))$, $f,f_k\in G([a,b],X)$, $\widetilde{x},\widetilde{x}_k\in X$ *for* $k\in\mathbb{N}$. *Assume* (2.1), (2.8), (3.1) *and*

$$A_k\rightrightarrows A\ \ \textit{and}\ \ f_k\rightrightarrows f\ \ \textit{locally on}\ \ [a,b).\tag{3.5}$$

Then there exists a unique solution x *of* (1.2) *on* $[a,b]$ *and, for each* $k\in\mathbb{N}$ *sufficiently large, there exists a unique solution* x_k *on* $[a,b]$ *to the equation* (1.1).

Moreover, let (2.5) *and*

$$\left.\begin{aligned} &\forall\,\varepsilon>0,\ \delta>0\ \exists\,\tau\in(b-\delta,b),\ \ k_0\in\mathbb{N}\ \text{ such that}\\ &\Big|x_k(b)-x_k(\tau)-\Delta^-A(b)\left[I-\Delta^-A(b)\right]^{-1}x(b-)-\\ &\qquad -\left[I-\Delta^-A(b)\right]^{-1}\Delta^-f(b)\Big|<\varepsilon\ \text{ for all }\ k\geq k_0 \end{aligned}\right\}\tag{3.6}$$

hold. Then (3.4) *is true, while* $x_k \rightrightarrows x$ *locally on* $[a,b)$.

Proof. Due to (2.1) and (3.1), there exists a unique solution x of (1.2) on $[a,b]$, there exists $k_1\in\mathbb{N}$ such that (1.1) has a unique solution x_k on $[a,b]$ for each $k\geq k_1$. Furthermore, by Theorem 2.2, $x_k \rightrightarrows x$ locally on $[a,b)$. It remains to show that

$$\lim_{k\to\infty} x_k(b)=x(b)\tag{3.7}$$

is true, as well. Let $\varepsilon>0$, $\delta\in(0,b-a)$ be given and let $\tau\in(b-\delta,b)$ and $k_0\geq k_1$ be such that (3.6) is true. We have

$$\begin{aligned}&\|x_k(b)-x(b)\|_X=\\ &=\|(x_k(b)-x_k(\tau))+(x_k(\tau)-x(\tau))+(x(\tau)-x(b-))+(x(b-)-x(b))\|_X\leq\\ &\leq\|x_k(b)-x_k(\tau)-x(b)+x(b-)\|_X+\|x(\tau)-x(b-)\|_X+\|x_k(\tau)-x(\tau)\|_X,\end{aligned}$$

wherefrom, having in mind that $x(b)=x(b-)+\Delta^-A(b)x(b)+\Delta^-f(b)$, i.e.,

$$x(b)=[I-\Delta^-A(b)]^{-1}x(b-)+[I-\Delta^-A(b)]^{-1}\Delta^-f(b)$$

and

$$\begin{aligned}x(b)-x(b-)&=\Delta^-A(b)[I-\Delta^-A(b)]^{-1}x(b-)+\\ &\qquad+\big[I+\Delta^-A(b)[I-\Delta^-A(b)]^{-1}\big]\Delta^-f(b),\end{aligned}$$

we deduce that

$$\begin{aligned}\|x_k(b)-x(b)\|_X&\leq\|x_k(b)-x(\tau)-\Delta^-A(b)[I-\Delta^-A(b)]^{-1}x(b-)-\\ &\qquad-\big[I+\Delta^-A(b)[I-\Delta^-A(b)]^{-1}\big]\Delta^-f(b)\|_X+\\ &\qquad\qquad+\|x(\tau)-x(b-)\|_X+\|x_k(\tau)-x(\tau)\|_X.\end{aligned}$$

We can choose δ and k_0 in such a way that $\|x(t)-x(b-)\|_X<\varepsilon$ for each $t\in(b-\delta,b)$ and $\|x_k(\tau)-x(\tau)\|_X<\varepsilon$ for $k\geq k_0$, as well. Furthermore, notice that if $B\in L(X)$ is such that $[I-B]^{-1}\in L(X)$, then $[I-B]^{-1}=I+B[I-B]^{-1}$. Thus, using (3.6), we get

$$\begin{aligned}&\|x_k(b)-x(b)\|_X\leq\|x_k(b)-x(\tau)-\Delta^-A(b)[I-\Delta^-A(b)]^{-1}x(b-)-\\ &-[I-\Delta^-A(b)]^{-1}\Delta^-f(b)\|_X+\|x(\tau)-x(b-)\|_X+\|x_k(\tau)-x(\tau)\|_X<3\varepsilon.\end{aligned}$$

It follows that (3.7) is true and this completes the proof. □

The assertion below may be deduced from Lemmas 3.1 and 3.2

Theorem 3.3. *Let $A, A_k \in BV([a,b], L(X))$, $f, f_k \in G([a,b], X)$, $\widetilde{x}, \widetilde{x}_k \in X$ for $k \in \mathbb{N}$. Assume* (2.1), (2.8) *and* (3.1). *Furthermore, let there exist a division $D = \{s_0, s_2, \ldots, s_m\}$ of the interval $[a,b]$ such that*

$$A_k \rightrightarrows A, f_k \rightrightarrows f \ \textit{ locally on each } \ (s_{i-1}, s_i), \ \ i = 1, 2, \ldots, m. \tag{3.8}$$

Then there exists a unique solution x of (1.2) *on $[a,b]$ and, for each $k \in \mathbb{N}$ sufficiently large, there exists a unique solution x_k on $[a,b]$ to the equation* (1.1).

Moreover, assume (2.5) *and let*

$$\left.\begin{aligned} &\forall\, \varepsilon > 0 \ \ \exists\, \delta_i \in (0, s_i - s_{i-1}) \ \textit{ such that } \forall\, t \in (s_{i-1}, s_{i-1} + \delta_i) \\ &\exists\, k_i = k_i(t) \in \mathbb{N} \ \textit{ such that} \\ &\qquad \|x_k(t) - x_k(s_{i-1}) - \Delta^+ A(s_{i-1}) x(s_{i-1}) - \Delta^+ f(s_{i-1})\|_X < \varepsilon \\ &\textit{for all } \ k \geq k_i \end{aligned}\right\} \tag{3.9}$$

and

$$\left.\begin{aligned} &\forall\, \varepsilon > 0, \delta \in (0, s_i - s_{i-1}) \exists\, \tau_i \in (s_i - \delta, s_i), \ell_i \in \mathbb{N} \ \textit{ such that} \\ &\quad \|x_k(s_i) - x_k(\tau_i) - \Delta^- A(s_i) \left[I - \Delta^- A(s_i)\right]^{-1} x(s_i-) - \\ &\qquad - \left[I - \Delta^- A(s_i)\right]^{-1} \Delta^- f(s_i)\|_X < \varepsilon \ \textit{ for all } \ k \geq \ell_i \end{aligned}\right\} \tag{3.10}$$

hold for each $i = 1, 2, \ldots, m$.

Then (3.4) *is true for all $t \in [a,b]$, while $x_k \rightrightarrows x$ locally on each (s_{i-1}, s_i), $i = 1, 2, \ldots, m$.*

Proof. Obviously, there is a division $D = \{\alpha_0, \alpha_1, \ldots, \alpha_r\}$ of $[a,b]$ such that for each subinterval $[\alpha_{j-1}, \alpha_j]$, $j = 1, 2, \ldots, r$, either the assumptions of Lemma 3.1 or the assumptions of Lemma 3.2 are satisfied with α_{j-1} in place of a and α_k in place of b. Hence the proof follows by Lemmas 3.1 and 3.2. $\square$

4. Sequential Solutions

The aim of this section is to disclose the relationship between the solutions of generalized linear differential equation and limits of solutions of approximating sequences of linear ordinary differential equations generated by piecewise linear approximations of the coefficients A, f.

Let us introduce the following notation.

Notation 4.1. For $A \in BV([a,b], L(X))$, $f \in G([a,b], X)$ and

$$D = \{\alpha_0, \alpha_1, \ldots, \alpha_m\} \in \mathcal{D}[a,b],$$

we define

$$A_D(t) = \begin{cases} A(t) \quad \text{if } \ t \in D, \\ A(\alpha_{i-1}) + \dfrac{A(\alpha_i) - A(\alpha_{i-1})}{\alpha_i - \alpha_{i-1}}\,(t - \alpha_{i-1}) \\ \qquad\qquad \text{if } \ t \in (\alpha_{i-1}, \alpha_i) \ \text{ for some } \ i \in \{1, 2, \dots, m\}, \end{cases} \tag{4.1}$$

and

$$f_D(t) = \begin{cases} f(t) \quad \text{if } \ t \in D, \\ f(\alpha_{i-1}) + \dfrac{f(\alpha_i) - f(\alpha_{i-1})}{\alpha_i - \alpha_{i-1}}\,(t - \alpha_{i-1}) \\ \qquad\qquad \text{if } \ t \in (\alpha_{i-1}, \alpha_i) \ \text{ for some } \ i \in \{1, 2, \dots, m\}. \end{cases} \tag{4.2}$$

The following lemma presents some direct properties for the functions defined in (4.1) and (4.2).

Lemma 4.2. *Assume that $A \in BV([a,b], L(X))$, $f \in G([a,b], X)$. Furthermore, let $D \in \mathcal{D}[a,b]$, $D = \{\alpha_0, \alpha_1, \dots, \alpha_m\}$, and let A_D and f_D be defined by* (4.1) *and* (4.2)*, respectively. Then A_D and f_D are strongly absolutely continuous on $[a,b]$ and*

$$\operatorname{var}_a^b A_D \le \operatorname{var}_a^b A \ \text{ and } \ \|f_D\|_\infty \le \|f\|_\infty.$$

Proof. It is clear that A_D and f_D are strongly absolutely continuous on (α_{i-1}, α_i), for each $i = 1, \dots, m$. Since both functions are continuous on $[a,b]$, the absolute continuity holds on the closed intervals $[\alpha_{i-1}, \alpha_i], i = 1, \dots, m$ (cf. [30, Theorem 7.1.10]).

Let $\varepsilon > 0$ be given. For each $i = 1, \dots, m$, there exists $\eta_i > 0$ such that

$$\sum_{j=1}^{p} \|A_D(b_j) - A_D(a_j)\|_{L(X)} < \frac{\varepsilon}{m}, \ \text{ whenever } \ \sum_{j=1}^{p} (b_j - a_j) < \eta_i,$$

where $[a_j, b_j]$, $j = 1, \dots, p$, are non-overlapping subintervals of $[\alpha_{i-1}, \alpha_i]$.

Let $\eta < \min\{\eta_i;\ i = 1, \dots, m\}$. Consider $\mathcal{F} = \{[c_j, d_j];\ j = 1, \dots, p\}$, a collection of non-overlapping subintervals of $[a,b]$, such that

$$\sum_{j=1}^{p} (d_j - c_j) < \eta.$$

Without loss of generality, we may assume that for each $j = 1, \dots, p$, $[c_j, d_j] \subset [\alpha_{k_j - 1}, \alpha_{k_j}]$, for some $k_j \in \{1, \dots, m\}$. Thus

$$\mathcal{F} = \bigcup_{i=1}^{m} \mathcal{F}_i, \ \text{ with } \ \mathcal{F}_i = \Big\{[c,d] \in \mathcal{F}; [c,d] \cap [\alpha_{i-1}, \alpha_i] \ne \varnothing\Big\},$$

and $\sum_{[c,d]\in\mathcal{F}_i}(d-c)<\eta_i$, $i=1,\dots,m$. In view of this, we get

$$\sum_{j=1}^{p}\|A_D(d_j)-A_D(c_j)\|_{L(X)}\le$$
$$\le\sum_{i=1}^{m}\sum_{[c,d]\in\mathcal{F}_i}\|A_D(d)-A_D(c)\|_{L(X)}<\sum_{i=1}^{m}\frac{\varepsilon}{m}=\varepsilon,$$

which shows that A_D is strongly absolutely continuous on $[a,b]$. Similarly we prove for f_D.

Furthermore, for each $\ell=1,2,\dots,m$ and each $t\in[\alpha_{\ell-1},\alpha_\ell]$ we have

$$\operatorname{var}_{\alpha_{\ell-1}}^{\alpha_\ell}A_D=\|A(\alpha_\ell)-A(\alpha_{\ell-1})\|_{L(X)}\le\operatorname{var}_{\alpha_{\ell-1}}^{\alpha_\ell}A$$

and

$$\|f_D(t)\|_X=\Big\|f(\alpha_{\ell-1})+\frac{f(\alpha_\ell)-f(\alpha_{\ell-1})}{\alpha_\ell-\alpha_{\ell-1}}(t-\alpha_{\ell-1})\Big\|_X=$$
$$=\Big\|f(\alpha_{\ell-1})\frac{\alpha_\ell-t}{\alpha_\ell-\alpha_{\ell-1}}+f(\alpha_\ell)\frac{t-\alpha_{\ell-1}}{\alpha_\ell-\alpha_{\ell-1}}\Big\|_X\le\|f\|_\infty.$$

Therefore,

$$\operatorname{var}_a^b A_D=\sum_{\ell=1}^{m}\operatorname{var}_{\alpha_{\ell-1}}^{\alpha_\ell}A_D\le$$
$$\le\sum_{\ell=1}^{m}\operatorname{var}_{\alpha_{\ell-1}}^{\alpha_\ell}A=\operatorname{var}_a^b A\quad\text{and}\quad\|f_D\|_\infty\le\|f\|_\infty.\qquad\square$$

Remark 4.3. Notice that the functions A_D, f_D, defined in (4.1) and (4.2), respectively, are differentiable on $(\alpha_{i-1},\alpha_i), i=1,\dots,m$, and their derivatives are given by

$$A_D'(t)=\frac{A(\alpha_i)-A(\alpha_{i-1})}{\alpha_i-\alpha_{i-1}}\quad\text{if }\ t\in(\alpha_{i-1},\alpha_i)\ \text{ for some }\ i\in\{1,2,\dots,m\},$$

$$f_D'(t)=\frac{f(\alpha_i)-f(\alpha_{i-1})}{\alpha_i-\alpha_{i-1}}\quad\text{if }\ t\in(\alpha_{i-1},\alpha_i)\ \text{ for some }\ i\in\{1,2,\dots,m\}.$$

By Lemma 4.2, recalling that A_D and f_D are strongly absolutely continuous on $[a,b]$, the Bochner integral (cf. [30, Definition 7.4.16]) exists and hence also the strong McShane and the strong Kurzweil-Henstock integrals (cf. [30, Theorem 5.1.4] and [30, Proposition 3.6.3]). Moreover,

$$A_D(t)=\int_a^t A_D'(s)\,\mathrm{d}s,\quad f_D(t)=\int_a^t f_D'(s)\,\mathrm{d}s\quad\text{for }\ t\in[a,b],$$

(cf. [30, Theorem 7.3.10]). Consequently,

$$\int_a^t \mathrm{d}[A_D(s)]x(s) = \int_a^t A_D'(s)x(s)\,\mathrm{d}s$$

holds for each $x \in G([a,b],X)$ and $t \in [a,b]$. Hence, the generalized differential equation

$$x(t) = \widetilde{x} + \int_a^t \mathrm{d}[A_D(s)]x(s) + f_D(t) - f_D(a)$$

is equivalent to the initial value problem for the ordinary differential equation (in the Banach space X)

$$x'(t) = A_D'(t)x + f_D'(t), \quad x(a) = \widetilde{x}.$$

Theorem 4.4. *Let $A \in BV([a,b],L(X)) \cap C([a,b],L(X))$, $f \in C([a,b],X)$ and $\widetilde{x} \in X$. Furthermore, let $\{D_k\}$ be a sequence of divisions of the interval $[a,b]$ such that*

$$D_{k+1} \supset D_k \ \text{ for } \ k \in \mathbb{N} \ \text{ and } \ \lim_{k\to\infty} |D_k| = 0. \tag{4.3}$$

Finally, let the sequences $\{A_k\}$ and $\{f_k\}$ be given by

$$A_k = A_{D_k} \ \text{ and } \ f_k = f_{D_k} \ \text{ for } \ k \in \mathbb{N}, \tag{4.4}$$

where A_{D_k} and f_{D_k} are defined as in (4.1) *and* (4.2).

Then equation (1.2) *has a unique solution x on $[a,b]$. Furthermore, for each $k \in \mathbb{N}$, equation* (1.1) *has a solution x_k on $[a,b]$ and* (2.9) *holds.*

Proof. Step 1. Since A is uniformly continuous on $[a,b]$, we have

$$\left.\begin{array}{c} \text{for each } \varepsilon > 0 \text{ there is a } \delta > 0 \text{ such that } \|A(t) - A(s)\|_{L(X)} < \dfrac{\varepsilon}{2} \\ \text{holds for all } \ t,s \in [a,b] \ \text{ such that } \ |t-s| < \delta. \end{array}\right\} \tag{4.5}$$

By (4.3), we can choose $k_0 \in \mathbb{N}$ such that $|D_k| < \delta$, for every $k \geq k_0$.

Given $t \in [a,b]$ and $k \geq k_0$, let $\alpha_{\ell-1}, \alpha_\ell \in \mathcal{D}_k$ be such that $t \in [\alpha_{\ell-1}, \alpha_\ell)$. Notice that $|\alpha_\ell - \alpha_{\ell-1}| < \delta$. So, according to (4.1), (4.4) and (4.5), we get

$$\begin{aligned} \|A_k(t) - A(t)\|_{L(X)} \leq \|A(\alpha_\ell) - A(\alpha_{\ell-1})\|_{L(X)} \Big[\frac{t - \alpha_{\ell-1}}{\alpha_\ell - \alpha_{\ell-1}}\Big] + \\ + \|A(\alpha_{\ell-1}) - A(t)\|_{L(X)} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

As k_0 was chosen independently of t, we can conclude that (2.6) is true.

Step 2. Analogously we can show that (2.7) is true, as well.

Step 3. By Lemma 4.2, (2.5) holds. Moreover, as A and A_k, $k \in \mathbb{N}$, are continuous, the equations (1.2) and (1.1) have unique solutions by Theorem 2.1 and we can complete the proof by using Theorem 2.2. $\square$

Notation 4.5. For the given $f \in G([a,b],X)$ and $k \in \mathbb{N}$, we denote

$$\mathcal{U}_k^+(f) = \Big\{t \in [a,b] : \ \|\Delta^+ f(t)\|_X \geq \frac{1}{k}\Big\},$$
$$\mathcal{U}_k^-(f) = \Big\{t \in [a,b] : \ \|\Delta^- f(t)\|_X \geq \frac{1}{k}\Big\},$$
$$\mathcal{U}_k(f) = \mathcal{U}_k^+(f) \cup \mathcal{U}_k^-(f) \quad \text{and} \quad \mathcal{U}(f) = \bigcup_{k=1}^{\infty} \mathcal{U}_k(f).$$

(Thus $\mathcal{U}(f)$ is a set of points of discontinuity of the function f in $[a,b]$.) Analogous symbols are used also for the operator valued function.

Definition 4.6. Let $A \in BV([a,b],L(X))$, $f \in G([a,b],X)$ and let $\{P_k\}$ be a sequence of divisions of $[a,b]$ such that

$$|P_k| = (1/2)^k \ \text{ for } \ k \in \mathbb{N}. \tag{4.6}$$

We say that the sequence $\{A_k, f_k\}$ is a *piecewise linear approximation* ($\mathcal{PL}$-approximation) of (A,f) if there exists a sequence $\{D_k\} \subset \mathcal{D}[a,b]$ of divisions of the interval $[a,b]$ such that

$$D_k \supset P_k \cup \mathcal{U}_k(A) \cup \mathcal{U}_k(f) \ \text{ for } \ k \in \mathbb{N} \tag{4.7}$$

and A_k, f_k are for $k \in \mathbb{N}$ defined by (4.1), (4.2) and (4.4).

Remark 4.7. Consider the case where $\dim X < \infty$ and let $\{A_k, f_k\}$ be a $\mathcal{PL}$-approximation of (A,f). Then by Lemma 4.2,

$$\operatorname{var}_a^b A_k \leq \operatorname{var}_a^b A \quad \text{and} \quad \|f_k\|_\infty \leq \|f\|_\infty.$$

Furthermore, as A_k are continuous, due to (2.2), we have $c_{A_k} = 1$ for all $k \in \mathbb{N}$. Hence, (2.4) yields

$$\operatorname{var}_a^b(x_k - f_k) \leq \operatorname{var}_a^b A\,(\|\widetilde{x}\|_X + 2\|f\|_\infty) \exp(\operatorname{var}_a^b A) < \infty \quad \text{for all } \ k \in \mathbb{N}$$

and, by Helly's theorem, there is a subsequence $\{k_\ell\}$ of $\mathbb{N}$ and $w \in G([a,b],X)$ such that

$$\lim_{\ell\to\infty} (x_{k_\ell}(t) - f_{k_\ell}(t)) = w(t) - f(t) \quad \text{for } \ t \in [a,b].$$

In particular, $\lim\limits_{\ell\to\infty} x_{k_\ell}(t) = w(t)$ for all $t \in [a,b]$ such that $\lim\limits_{\ell\to\infty} f_{k_\ell}(t) = f(t)$.

In this context, it is worth mentioning that if the set $\mathcal{U}(f)$ has at most a finite number of elements, then

$$\lim_{k\to\infty} f_k(t) = f(t) \quad \text{for all } \ t \in [a,b].$$

Definition 4.8. Let $A \in BV([a,b],L(X))$, $f \in G([a,b],X)$ and $\widetilde{x} \in X$. We say that $x^* : [a,b] \to X$ is a *sequential solution* to equation (1.2) on the interval $[a,b]$ if there is a $\mathcal{PL}$-approximation $\{A_k, f_k\}$ of (A,f) such that

$$\lim_{k\to\infty} x_k(t) = x^*(t) \quad \text{for } \ t \in [a,b] \tag{4.8}$$

holds for solutions $x_k, k \in \mathbb{N}$, of the corresponding approximating initial value problems

$$x_k' = A_k'(t)x_k + f_k'(t), \quad x_k(a) = \widetilde{x}, \;\; k \in \mathbb{N}. \tag{4.9}$$

Remark 4.9. Notice that using the language of Definitions 4.6 and 4.8, we can translate Theorem 4.4 into the following form:
Let $A \in BV([a,b], L(X)) \cap C([a,b], L(X))$, $f \in C([a,b], X)$ and $\widetilde{x} \in X$. Then equation (1.2) *has a unique sequential solution x^* on $[a,b]$ and x^* coincides on $[a,b]$ with the solution of* (1.2)*.*

In the rest of this paper we consider the case where the set $\mathcal{U}(A) \cup \mathcal{U}(f)$ of discontinuities of A, f is non-empty. We will start with the simplest case $\mathcal{U}(A) \cup \mathcal{U}(f) = \{b\}$.

The following natural assertion will be useful for our purposes and, in our opinion, it is not available in literature.

Lemma 4.10. *Let $A \in BV([a,b], L(X))$. Then*

$$\left.\begin{aligned} \lim_{s \to t-} \frac{1}{t-s} \Big(\int_s^t \exp\Big([A(t) - A(s)]\, \frac{t-r}{t-s} \Big) \mathrm{d}r \Big) = \\ = \int_0^1 \exp\big(\Delta^- A(t)(1-\sigma) \big)\, \mathrm{d}\sigma \quad \text{if } t \in (a,b] \end{aligned}\right\} \tag{4.10}$$

and

$$\left.\begin{aligned} \lim_{s \to t+} \frac{1}{s-t} \Big(\int_t^s \exp\Big([A(s) - A(t)] \frac{s-r}{s-t} \Big)\, \mathrm{d}r \Big) = \\ = \int_0^1 \exp\big(\Delta^+ A(t)(1-\sigma) \big)\, \mathrm{d}\sigma \quad \text{if } t \in [a,b). \end{aligned}\right\} \tag{4.11}$$

where the integrals are the Bochner ones.

Proof. (i) Let $t \in (a,b]$ and $\varepsilon \in (0,1)$ be given. Then there is a $\delta > 0$ such that

$$\|A(t-) - A(s)\|_{L(X)} < \varepsilon \quad \text{whenever } \; t - \delta < s < t.$$

Taking now into account that

$$\big\| \exp(C\tau) - \exp(D\tau) \big\|_{L(X)} \le \|C - D\|_{L(X)} \exp\big((\|C\|_{L(X)} + \|D\|_{L(X)})\tau \big)$$

holds for all $C, D \in L(X)$, $\tau \in \mathbb{R}$, (cf. [22, Corollary 3.1.3]), we get

$$\left\| \frac{1}{t-s} \int_s^t \Big[\exp\Big([A(t) - A(s)]\, \frac{t-r}{t-s} \Big) - \exp\Big(\Delta^- A(t)\, \frac{t-r}{t-s} \Big) \Big] \mathrm{d}r \right\|_X \le$$

$$\le \frac{1}{t-s}\, \|A(t-) - A(s)\|_{L(X)} \int_s^t \exp\big(\varepsilon + 2 \|\Delta^- A(t)\|_{L(X)} \big)\, \mathrm{d}r =$$

$$= \|A(t-) - A(s)\|_{L(X)} \exp\big(\varepsilon + 2\|\Delta^- A(t)\|_{L(X)}\big) \leq$$
$$\leq \varepsilon \exp\big(1 + 2\|\Delta^- A(t)\|_{L(X)}\big) \quad \text{for } \ t-\delta < s < t.$$

Therefore,

$$\lim_{s \to t-} \frac{1}{t-s} \left(\int_s^t \exp\Big([A(t) - A(s)] \, \frac{t-r}{t-s} \Big) \mathrm{d}r \right) =$$
$$= \lim_{s \to t-} \frac{1}{t-s} \left(\int_s^t \exp\Big(\Delta^- A(t) \, \frac{t-r}{t-s} \Big) \mathrm{d}r \right) \quad \text{for } \ t \in (a,b].$$

It is now easy to see that the substitution $\sigma = 1 - \frac{t-r}{t-s}$ into the second integral yields (4.10).
(ii) The relation (4.11) can be justified similarly. $\square$

Lemma 4.11. *Let $A \in BV([a,b], L(X))$ and $f \in G([a,b], X)$ be continuous on $[a,b)$. Let $\widetilde{x} \in X$ and let x be a solution of* (1.2) *on $[a,b)$.*

Then equation (1.2) *has a unique sequential solution x^* on $[a,b]$.*

Moreover, x^* is continuous on $[a,b)$, $x^* = x$ on $[a,b)$ and $x^*(b) = v(1)$, where v is a solution on $[0,1]$ of the initial value problem

$$v' = [\Delta^- A(b)]v + [\Delta^- f(b)], \quad v(0) = x(b-). \tag{4.12}$$

Proof. Let $\{A_k, f_k\}$ be an arbitrary $\mathcal{PL}$-approximation of (A,f) and let $\{D_k\}$ be the corresponding sequence of divisions of $[a,b]$ fulfilling (4.6) and (4.7). Notice that under our assumptions, $D_k = P_k$ for $k \in \mathbb{N}$. For $k \in \mathbb{N}$, we put

$$\tau_k = \max\{t \in P_k;\ t < b\}.$$

By (4.3), we have $b - \frac{b-a}{2^k} \leq \tau_k < b$ for $k \in \mathbb{N}$, and hence

$$\lim_{k \to \infty} \tau_k = b. \tag{4.13}$$

Now, for $k \in \mathbb{N}$ and $t \in [a,b]$, let us define

$$\widetilde{A}_k(t) = \begin{cases} A_k(t) & \text{if } \ t \in [a, \tau_k], \\[2ex] A(\tau_k) + \dfrac{A(b-) - A(\tau_k)}{b - \tau_k}\,(t - \tau_k) & \text{if } \ t \in (\tau_k, b], \end{cases}$$

$$\widetilde{f}_k(t) = \begin{cases} f_k(t) & \text{if } \ t \in [a, \tau_k], \\[2ex] f(\tau_k) + \dfrac{f(b-) - f(\tau_k)}{b - \tau_k}\,(t - \tau_k) & \text{if } \ t \in (\tau_k, b]. \end{cases}$$

Furthermore, let

$$\widetilde{A}(t) = \begin{cases} A(t) & \text{if } \ t \in [a,b), \\ A(b-) & \text{if } \ t = b, \end{cases} \qquad \widetilde{f}(t) = \begin{cases} f(t) & \text{if } \ t \in [a,b), \\ f(b-) & \text{if } \ t = b. \end{cases} \tag{4.14}$$

It is easy to see that for $k \in \mathbb{N}$, $\widetilde{A}_k$ $\widetilde{f}_k$ are strongly absolutely continuous and differentiable a.e. on $[a,b]$, $\widetilde{A} \in BV([a,b],L(X)) \cap C([a,b],L(X))$ and $\widetilde{f} \in C([a,b],X)$.

Step 1. Consider the problems

$$y_k' = \widetilde{A}_k'(t) y_k + \widetilde{f}_k'(t), \quad y_k(a) = \widetilde{x}, \quad k \in \mathbb{N}, \tag{4.15}$$

and

$$y(t) = \widetilde{x} + \int_a^t \mathrm{d}[\widetilde{A}] y + \widetilde{f}(t) - \widetilde{f}(a). \tag{4.16}$$

Taking into account Theorem 4.4 and Remark 4.9, we find that the equation (4.16) possesses a unique solution y on $[a,b]$ and

$$\lim_{k\to\infty} \|y_k - y\|_\infty = 0. \tag{4.17}$$

where for each $k \in \mathbb{N}$, y_k is the solution on $[a,b]$ of (4.15).

Note that y is continuous on $[a,b]$ and $y = x$ on $[a,b)$. Let $\{x_k\}$ be a sequence of solutions of the problems (4.9) on $[a,b]$. We can see that $x_k = y_k$ on $[a,\tau_k]$ for each $k \in \mathbb{N}$, and, due to (4.13), we have

$$\lim_{k\to\infty} x_k(t) = \lim_{k\to\infty} y_k(t) = y(t) = x(t) \quad \text{for } t \in [a,b). \tag{4.18}$$

Step 2. Next, we prove that

$$\lim_{k\to\infty} x_k(\tau_k) = y(b). \tag{4.19}$$

Indeed, let $\varepsilon > 0$ be given and let $\delta > 0$ be such that

$$\|y(t) - y(b)\|_X < \frac{\varepsilon}{2} \quad \text{for } t \in [b-\delta, b].$$

Further, by (4.17), there is a $k_0 \in \mathbb{N}$ such that

$$\tau_k \in [b-\delta, b) \quad \text{and} \quad \|y_k - y\|_\infty < \frac{\varepsilon}{2} \quad \text{whenever } k \geq k_0.$$

Consequently,

$$\begin{aligned} \|x_k(\tau_k) - y(b)\|_X &\leq \|x_k(\tau_k) - y(\tau_k)\|_X + \|y(\tau_k) - y(b)\|_X = \\ &= \|y_k(\tau_k) - y(\tau_k)\|_X + \|y(\tau_k) - y(b)\|_X < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

holds for $k \geq k_0$. This completes the proof of (4.19).

Step 3. On the intervals $[\tau_k, b]$, the equations from (4.9) reduce to the equations with constant coefficients

$$x_k' = B_k x_k + e_k, \tag{4.20}$$

where

$$B_k = \frac{A(b) - A(\tau_k)}{b - \tau_k} \quad \text{and} \quad e_k = \frac{f(b) - f(\tau_k)}{b - \tau_k}.$$

Their solutions x_k are on $[\tau_k, b]$ given by

$$x_k(t) = \exp\left(B_k(t-\tau_k)\right) x_k(\tau_k) + \left(\int_{\tau_k}^{t} \exp\left(B_k(t-r)\right) \mathrm{d}r\right) e_k,$$

(cf. [5, Chapter II]). In particular,

$$x_k(b) = \exp\left(A(b) - A(\tau_k)\right) x_k(\tau_k) + \\ + \frac{1}{b-\tau_k}\left(\int_{\tau_k}^{b} \exp\left([A(b) - A(\tau_k)]\,\frac{b-r}{b-\tau_k}\right) \mathrm{d}r\right)[f_k(b) - f_k(\tau_k)].$$

By Lemma 4.10, we have

$$\lim_{k\to\infty} \frac{1}{b-\tau_k}\left(\int_{\tau_k}^{b} \exp\left([A(b) - A(\tau_k)]\,\frac{b-r}{b-\tau_k}\right) \mathrm{d}r\right)[f(b) - f(\tau_k)] =$$

$$= \lim_{k\to\infty} \frac{1}{b-\tau_k}\left(\int_{\tau_k}^{b} \exp\left(\Delta^- A(b)\,\frac{b-r}{b-\tau_k}\right) \mathrm{d}r\right)[f(b) - f(\tau_k)] =$$

$$= \left(\int_0^1 \exp\left(\Delta^- A(b)(1-s)\right) \mathrm{d}s\right)\Delta^- f(b).$$

To summarize,

$$\lim_{k\to\infty} x_k(b) = \exp\left(\Delta^- A(b)\right) y(b) + \left(\int_0^1 \exp\left(\Delta^- A(b)(1-s)\right) \mathrm{d}s\right)\Delta^- f(b),$$

i.e.,

$$\lim_{k\to\infty} x_k(b) = v(1), \tag{4.21}$$

where v is a solution of (4.12) on $[0,1]$.

Step 4. Define

$$x^*(t) = \begin{cases} y(t) & \text{if } t \in [a,b), \\ v(1) & \text{if } t = b. \end{cases}$$

Then $x^*(t) = \lim_{k\to\infty} x_k(t)$ for $t \in [a,b]$ due to (4.19) and (4.21). Therefore, x^* is a sequential solution of (1.2). Since it does not depend on the choice of the approximating sequence $\{A_k, f_k\}$, we can see that x^* is also the unique sequential solution of (1.2). This completes the proof. □

The following assertion concerns a situation, symmetric to that treated by Lemma 4.11. Similarly to the proof of Lemma 4.11, we will deal with the modified equation

$$y(t) = \widetilde{y} + \int_a^t \mathrm{d}[\widetilde{A}]y + \widetilde{f}(t) - \widetilde{f}(a), \tag{4.22}$$

where $\widetilde{y} \in X$ and

$$\widetilde{A}(t) = \begin{cases} A(a+) & \text{if } t = a, \\ A(t) & \text{if } t \in (a,b] \end{cases} \quad \text{and} \quad \widetilde{f}(t) = \begin{cases} f(a+) & \text{if } t = a, \\ f(t) & \text{if } t \in (a,b]. \end{cases} \tag{4.23}$$

Lemma 4.12. *Let $A \in BV([a,b], L(X))$ and $f \in G([a,b], X)$ be continuous on $(a,b]$. Then for each $\widetilde{x} \in X$, equation* (1.2) *has a unique sequential solution x^* on $[a,b]$ which is continuous on $(a,b]$.*

Furthermore, let w be a solution of the initial value problem

$$w' = [\Delta^+ A(a)]w + [\Delta^+ f(a)], \quad w(0) = \widetilde{x} \tag{4.24}$$

and let y be a solution on $[a,b]$ of equation (4.22)*, where $\widetilde{y} = w(1)$. Then x^* coincides with y on $(a,b]$.*

Proof. Let $\{A_k, f_k\}$ be an arbitrary $\mathcal{PL}$-approximation of (A,f) and let $\{D_k\}$ be the corresponding sequence of divisions of $[a,b]$ fulfilling (4.1) and (4.2). Just as in the previous proof, $D_k = P_k$ for $k \in \mathbb{N}$.

For $k \in \mathbb{N}$, we put

$$\tau_k = \min\{t \in P_k : \ t > a\}.$$

By (4.3), we have $a + \frac{b-a}{2^k} \geq \tau_k > a$ for $k \in \mathbb{N}$, and hence

$$\lim_{k \to \infty} \tau_k = a.$$

Let $\{x_k\}$ be a sequence of solutions of the approximating initial value problems (4.9) on $[a,b]$.

Step 1. On the intervals $[a,\tau_k]$, the equations from (4.9) reduce to the equations (4.20) with the coefficients

$$B_k = \frac{A(\tau_k) - A(a)}{\tau_k - a}, \quad e_k = \frac{f(\tau_k) - f(a)}{\tau_k - a}.$$

Their solutions x_k are on $[a,\tau_k]$ given by

$$x_k(t) = \exp(B_k(t-a))\widetilde{x} + \left(\int_a^t \exp\left(B_k(t-r)\right) \mathrm{d}r \right) e_k,$$

(cf. [5, Chapter II]). In particular,

$$x_k(\tau_k) = \exp\left(A(\tau_k) - A(a)\right)\widetilde{x} +$$
$$+ \frac{1}{\tau_k - a} \left(\int_a^{\tau_k} \exp\left([A(\tau_k) - A(a)] \frac{\tau_k - r}{\tau_k - a} \right) \mathrm{d}r \right) [f(\tau_k) - f(\tau_k)].$$

By Lemma 4.10, we have

$$\lim_{k\to\infty} \frac{1}{\tau_k - a}\left(\int_a^{\tau_k} \exp\left([A(\tau_k) - A(a)]\,\frac{\tau_k - r}{\tau_k - a}\right)\mathrm{d}r\right)[f(\tau_k) - f(a)] =$$

$$= \left(\int_0^1 \exp(\Delta^+ A(a)(1-s))\,\mathrm{d}s\right)\Delta^+ f(a).$$

Thus, $\lim\limits_{k\to\infty} x_k(\tau_k) = w(1)$, where w is the solution of (4.24) on $[0,1]$.

Step 2. Consider equation (4.22) with $\widetilde{y} = w(1)$. By Theorem 2.1, it has a unique solution y on $[a,b]$, y is continuous on $[a,b]$ and, by an argument analogous to that used in *Step* 1 of the proof of Lemma 4.11, we can show that the relation

$$\lim_{k\to\infty} x_k(t) = y(t) \quad \text{for } \ t \in (a,b]$$

is true.

Step 3. Analogously to *Step* 4 of the proof of Lemma 4.11, we can complete the proof by showing that the function

$$x^*(t) = \begin{cases} \widetilde{x} & \text{if } \ t = a, \\ y(t) & \text{if } \ t \in (a,b], \end{cases}$$

is the unique sequential solution of (1.2). □

Remark 4.13. Notice that if $a < c < b$ and the functions x_1^* and x_2^* are, respectively, the sequential solutions to

$$x(t) = \widetilde{x}_1 + \int_a^t \mathrm{d}[A]x + f(t) - f(a), \quad t \in [a,c],$$

and

$$x(t) = \widetilde{x}_2 + \int_c^t \mathrm{d}[A]x + f(t) - f(c), \quad t \in [c,b],$$

where $\widetilde{x}_2 = x_1^*(c)$, then the function

$$x^*(t) = \begin{cases} x_1^*(t) & \text{if } \ t \in [a,c], \\ x_2^*(t) & \text{if } \ t \in (c,b] \end{cases}$$

is a sequential solution to (1.2).

Theorem 4.14. *Assume that* $A \in BV([a,b], L(X))$, $f \in G([a,b], X)$ *and*

$$\mathcal{U}(A) \cup \mathcal{U}(f) = \{s_1, s_2, \dots, s_m\} \subset [a,b].$$

Then for each $\widetilde{x} \in X$, *there is exactly one sequential solution* x^* *of equation* (1.2) *on* $[a,b]$.

Moreover,

$$x^*(t) = w_\ell(1) + \int_{s_\ell}^{t} \mathrm{d}[\widetilde{A}_\ell]x^* + \widetilde{f}_\ell(t) - \widetilde{f}_\ell(s_\ell) \ \ \textit{for} \ \ t \in [s_\ell, s_{\ell+1}), \ \ \ell \in \mathbb{N} \cap [0, m],$$

$$x^*(t) = v_\ell(1) \ \ \textit{for} \ \ t = s_\ell, \ \ \ell \in \mathbb{N} \cap [1, m+1],$$

where $s_0 = a, s_{m+1} = b, w_0(1) = \widetilde{x}$ *and, for* $\ell \in \mathbb{N} \cap [0, m]$,

$$\widetilde{A}_\ell(t) = \begin{cases} A(s_\ell+) & \textit{if} \ \ t = s_\ell, \\ A(t) & \textit{if} \ \ t \in (s_\ell, s_{\ell+1}], \end{cases} \qquad \widetilde{f}_\ell(t) = \begin{cases} f(s_\ell+) & \textit{if} \ \ t = s_\ell, \\ f(t) & \textit{if} \ \ t \in (s_\ell, s_{\ell+1}] \end{cases}$$

and v_ℓ *and* w_ℓ *denote, respectively, the solutions on* $[0, 1]$ *of the initial value problems*

$$v'_\ell = [\Delta^- A(s_\ell)]v_\ell + [\Delta^- f(s_\ell)], \quad v_\ell(0) = x^*(s_\ell-)$$

and

$$w'_\ell = [\Delta^+ A(s_\ell)]w_\ell + [\Delta^+ f(s_\ell)], \quad w_\ell(0) = x^*(s_\ell).$$

Proof. Having in mind Remark 4.13, we deduce the assertion of Theorem 4.14 by a successive use of Lemmas 4.11 and 4.12. Towards this end, it suffices to choose a division $D = \{\alpha_0, \alpha_1, \ldots, \alpha_r\}$ of $[a, b]$ such that for each subinterval $[\alpha_{k-1}, \alpha_k], k = 1, 2, \ldots, r$, either the assumptions of Lemma 4.11 or those of Lemma 4.12 are satisfied with α_{k-1} in place of a and α_k in place of b. □

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