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PSEUDODIFFERENTIAL OPERATORS WITH OPERATOR VALUED SYMBOLS. FREDHOLM THEORY AND EXPONENTIAL ESTIMATES OF SOLUTIONS

Dedicated to 120 Birthday of Academician,
Professor N. Muskhelishvili

Abstract. We consider a class of pseudodifferential operators with operator-valued symbols $a = a(x, \xi)$ having power growth with respect to the variables x and ξ. Moreover we consider the symbols analytically extended with respect to ξ onto a tube domain in $\mathbb{C}^n$ with a base being a ball in $\mathbb{R}^n$ with a radius depending on the variable x.

The main results of the paper are the Fredholm theory of pseudodifferential operators with operator valued symbols and exponential estimates at infinity of solutions of pseudodifferential equations $Op(a)u = f$.

We apply these results to Schrödinger operators with operator-valued potentials and to the spectral properties of Schrödinger operators in quantum waveguides.

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რეზიუმე. ჩვენ განვიხილავთ ფსევდოდიფერენციალურ ოპერატორებს ოპერატორულ მნიშვნელობებიანი სიმბოლოებით $a = a(x, \xi)$, რომელთაც აქვთ ხარისხოვანი ზრდა x და ξ ცვლადების მიმართ. უფრო მეტიც, ჩვენ განვიხილავთ სიმბოლოებს, რომლებიც უშვებენ ანალიზურ გაგრძელებას ξ ცვლადის მიმართ მილისებრ არეზე $\mathbb{C}^n$-ში, რომლის ფუძე წარმოადგენს ბირთვს $\mathbb{R}^n$-ში და ამ ბირთვის რადიუსი დამოკიდებულია x ცვლადზე.

ნაშრომის ძირითადი შედეგია ოპერატორულ მნიშვნელობებიანი სიმბოლოების მქონე ფსევდოდიფერენციალური ოპერატორებისთვის ფრედჰოლმის თეორია და $Op(a)u = f$ ფსევდოდიფერენციალური განტოლებების ამონახსნების ექსპონენციალური შეფასებები უსასრულობაში.

მიღებულ შედეგებს ვიყენებთ ოპერატორულ მნიშვნელობებიაბი პოტენციალების მქონე შროდინგერის ოპერატორებისათვის და კვანტური ტალღების გამტარებში შროდინგერის ოპერატორების სპექტრალური თვისებებისათვის.

1. Introduction

We consider the class of pseudodifferential operators

$$(Op(a)u)(x) = (2\pi)^{-n} \int\limits_{\mathbb{R}^n} d\xi \int\limits_{\mathbb{R}^n} a(x,\xi)u(y)e^{i(x-y)\cdot\xi}\, dy, \quad u \in S(\mathbb{R}^n, \mathcal{H}_1), \quad (1)$$

with symbols a with values in the space of bounded linear operators acting from a Hilbert space $\mathcal{H}_1$ into a Hilbert space $\mathcal{H}_2$. In (1), $S(\mathbb{R}^n, \mathcal{H}_1)$ is the space of $\mathcal{H}_1$-valued infinitely differentiable functions rapidly decreasing with all their derivatives. We consider the symbols which can have a power growth at infinity with respect to the variables x and ξ. Moreover, we suppose that the symbol a can be analytically extended with respect to ξ onto a tube domain $\mathbb{R}^n + i\{\eta \in \mathbb{R}^n : \ |\eta| < b(x)\}$, where b is a continuous positive function.

The main results of the paper are the Fredholm theory of pseudodifferential operators and exponential estimates at infinity of solutions of pseudodifferential equations $Op(a)u = f$. We apply these results to the Schrödinger operators with operator-valued potentials and discuss applications to quantum waveguides.

Our approach is based on the construction of the local inverse operator at infinity and on estimates of commutators of pseudodifferential operators with exponential weights (First the idea of this approach for scalar pseudodifferential operators with bounded symbols appeared in the paper [20], and later also for scalar pseudodifferential operators with symbols admitting a power, exponential and super-exponential grows and local discontinuities in [31], [32], [34]. [35].)

Estimates of exponential decay are intensively studied in the literature. We would like to emphasize Agmon's monograph [1] where the exponential estimates of the behavior of solutions of second order elliptic operators have been obtained in terms of a special metric (now called the Agmon metric). See also [4], [18], [19], [16], [20], [24], [25], [28], [31], [32], [5], [6], [35]. In [36], [37] the authors established the relation between the essential spectrum of pseudodifferential operators and exponential decay of their solutions at infinity. The recent paper [33] is devoted to local exponential estimates of solutions of finite-dimensional h-pseudodifferential operators with applications to the tunnel effect for Schrödinger, Dirac and Klein–Gordon operators.

It turns out that many problems in mathematical physics are reduced to the study of associated pseudodifferential operators with operator-valued symbols. In particular, this happens for problems of wave propagation in acoustic, electromagnetic and quantum waveguides (see for instance [3] and references cited there).

This paper is organized as follows. In Section 2 we present some auxiliary facts on operator-valued pseudodifferential operators. Some standard references for the theory of pseudodifferential operators are [17], [39], [40], whereas operator-valued pseudodifferential operators have been studied in [21], [22]. The approach in the latter books follows ideas by Hörmander and employs a special partition of unity connected with a metric defining the class of pseudodifferential operators. We will follow here the approach of [30], which based on the notion of a formal symbol. A main point is the representation of the symbol of a product of pseudodifferential operators and of a double pseudodifferential operators in form of an operator-valued double oscillatory integral. This approach allows us to extend the theory of scalar pseudodifferential operators to pseudodifferential operators with operator-valued symbols, and it provides us with an pseudodifferential operator calculus which is convenient for applications.

In Section 3 we examine the local invertibility at infinity of operator-valued pseudodifferential operators in suitable spaces and discuss their Fredholm property. Section 4 is devoted to the exponential estimates at infinity of solutions of operator-valued pseudodifferential operators. In the concluding Section 5 we study the Fredholm property of Schrödinger operators and derive exponential estimates at infinity of solutions of Schrödinger equations with operator-valued increasing potentials. These general results are then applied to the Fredholm property of Schrödinger operators with increasing potentials for quantum waveguides, for which we obtain exponential estimates of eigenfunctions. Note that spectral problems for quantum waveguides have attracted many attention in the last time. See, for instance, [3], [10], [13], [9].

2. Pseudodifferential Operators with Operator Valued Symbols and its Fredholm Properties

2.1. **Notations.**

- Given Banach spaces X, Y, we denote the Banach space of all bounded linear operators acting from X in Y by $\mathcal{L}(X, Y)$. In case $X = Y$, we simply write $\mathcal{L}(X)$.
- Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Then we denote by $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ the points of the dual space with respect to the scalar product $\langle x, \xi \rangle = x \cdot \xi = x_1\xi_1 + \cdots + x_n\xi_n$.
- For $j = 1, \dots, n$, let $\partial_{x_j} := \frac{\partial}{\partial x_j}$ and $D_{x_j} := -i\frac{\partial}{\partial x_j}$. More generally, given a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, set $|\alpha| := \alpha_1 + \cdots + \alpha_n$ and

$$\partial_x^\alpha := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} \quad \text{and} \quad D_x^\alpha := D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}.$$

- Let $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$ for $\xi \in \mathbb{R}^n$.
- Let X be a Banach space. We denote by
  - (i) $C^\infty(\mathbb{R}^n, X)$ the set of all infinitely differentiable functions on $\mathbb{R}^n$ with values in X;
  - (ii) $C_0^\infty(\mathbb{R}^n, X)$ the set of all functions in $C^\infty(\mathbb{R}^n, X)$ with compact supports;

(iii) $C^\infty_{b,N}(\mathbb{R}^n, X)$ the set of all functions $a \in C^\infty(\mathbb{R}^n, X)$ such that for some $N \geq 0$

$$\sup_{x\in\Omega} \sum_{|\alpha|\leq k} \langle x\rangle^{-N} \left\|(\partial_x^\alpha a)(x)\right\|_X < \infty$$

for every $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We will write $C_b^\infty(\mathbb{R}^n, X)$ if $N = 0$.

(iv) $S(\mathbb{R}^n, X)$ the set of all functions $a \in C^\infty(\mathbb{R}^n, X)$ such that

$$\sup_{x\in\mathbb{R}^n} \langle x\rangle^k \sum_{|\alpha|\leq k} \left\|(\partial_x^\alpha a)(x)\right\|_X < \infty$$

for every $k \in \mathbb{N}_0$.

In each case, we omit X whenever $X = \mathbb{C}$.

- Let $\mathcal{H}$ be a Hilbert space and $u \in S(\mathbb{R}^n, \mathcal{H})$. Then we denote by

$$\widehat{u}(\xi) = (Fu)(\xi) := \int_{\mathbb{R}^n} u(x) e^{-ix\cdot\xi}\, dx$$

the Fourier transform of u. Note that $F : S(\mathbb{R}^n, \mathcal{H}) \to S(\mathbb{R}^n, \mathcal{H})$ is an isomorphism with inverse

$$(F^{-1}\widehat{u})(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{u}(\xi) e^{ix\cdot\xi}\, d\xi.$$

We write $S'(\mathbb{R}^n, \mathcal{H})$ for the space of distributions over $S(\mathbb{R}^n, \mathcal{H})$ and define the Fourier transform of distributions in $S'(\mathbb{R}^n, \mathcal{H})$ via duality. Note that $F : S'(\mathbb{R}^n, \mathcal{H}) \to S'(\mathbb{R}^n, \mathcal{H})$ is an isomorphism.

- In what follows we consider separable Hilbert spaces $\mathcal{H}$ only.

2.2. **Oscillatory vector-valued integrals.** 1^0. Let B be a Banach space, and let a be a function in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n, B)$ for which there exist m_1, $m_2 \in \mathbb{R}$ such that

$$|a|_{r,t} := \sum_{|\alpha|\leq r,\, |\beta|\leq t} \sup_{\mathbb{R}^n\times\mathbb{R}^n} \left\|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)\right\|_B \langle x\rangle^{-m_1} \langle\xi\rangle^{-m_2} < \infty \tag{2}$$

for all r, $t \in \mathbb{N}_0$. Further let $\chi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ be such that $\chi(x,\xi) = 1$ for all points (x,ξ) in a neighborhood of the origin. Let $R > 0$. In what follows we call $\chi_R(x,\xi) := \chi(x/R, \xi/R)$ a cut-off function.

Proposition 1. Let $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n, B)$ satisfy the estimates (2). Then the limit

$$\mathcal{I}(a) := \lim_{R\to\infty} (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} \chi_R(x,\xi) a(x,\xi) e^{-ix\cdot\xi}\, dx\, d\xi$$

exists in the norm topology of B and

$$\mathcal{I}(a) = (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} \langle\xi\rangle^{-2k_2} \langle D_x\rangle^{2k_2} \big\{\langle x\rangle^{-2k_1} \langle D_\xi\rangle^{2k_1} a(x,\xi)\big\} e^{-ix\cdot\xi}\, dx\, d\xi$$

for all

$$2k_1 > n + m_1, \;\; 2k_2 > n + m_2. \tag{3}$$

This limit is independent on k_1, k_2 satisfying (3) and the choice of χ. Moreover,

$$\begin{aligned}\|\mathcal{I}(a)\|_B \le C \sum_{|\alpha|\le 2k_1,\, |\beta|\le 2k_2} \sup_{\mathbb{R}^n\times\mathbb{R}^n} \left\|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)\right\|_B \langle x\rangle^{-m_1}\langle \xi\rangle^{-m_2} = \\ = C|a|_{2k_1,2k_2}.\end{aligned} \tag{4}$$

The element $\mathcal{I}(a) \in B$ is called the *oscillatory integral.*

In what follows the double integral

$$\iint\limits_{\mathbb{R}^{2n}} a(x,\xi) e^{-ix\cdot\xi}\, dx\, d\xi$$

is understood as oscillatory.

Proposition 2. Let $a \in C^\infty(\mathbb{R}^n, B)$ and for all β

$$\|\partial_x^\beta a(x)\|_B \le C_\beta \langle x\rangle^N, \;\; N > 0.$$

Then, for each $x \in \mathbb{R}^n$,

$$(2\pi)^{-n} \iint\limits_{\mathbb{R}^{2n}} a(x+y) e^{-iy\cdot\xi}\, dy\, d\xi = a(x). \tag{5}$$

Propositions 1 and 2 are proved as in the scalar case by integrating by parts (see for instance [30]).

2.3. **Pseudodifferential operators.** Let $\mathcal{H}$ and $\mathcal{H}'$ be Hilbert spaces. A function $p \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{L}(\mathcal{H}', \mathcal{H}))$ is said to be a *weight function* in the class $O(\mathcal{H}, \mathcal{H}')$ if the operator $p(x,\eta)$ is invertible for each $(x,\eta) \in \mathbb{R}^n \times \mathbb{R}^n$ and for all α, β there are constants $C_{\alpha\beta} > 0$ such that

$$\begin{aligned}\left\|p(y,\eta)^{-1} \partial_x^\beta \partial_\xi^\alpha p(x+y, \xi+\eta)\right\|_{\mathcal{L}(\mathcal{H}')} \le C_{\alpha\beta}\big(1+|y|+|\eta|\big)^N, \\ \left\|\big(\partial_x^\beta \partial_\xi^\alpha p(x+y,\xi+\eta)\big) p^{-1}(y,\eta)\right\|_{\mathcal{L}(\mathcal{H})} \le C_{\alpha\beta}\big(1+|y|+|\eta|\big)^N\end{aligned} \tag{6}$$

for some $N > 0$ and arbitrary pairs (x,ξ), $(y,\eta) \in \mathbb{R}^n \times \mathbb{R}^n$.

Example 3. We give an important example of a weight function. Let L be an unbounded self-adjoint positive operator in a Hilbert space $\mathcal{H}$ with a dense in $\mathcal{H}$ domain D_L and $L \ge \delta I$, E_μ, $\mu \in [\delta, \infty)$ be the family of the spectral projectors of the self-adjoint operator L. Then the operator $L^m, m \ge 0$ is defined by means of the spectral decomposition as

$$L^m u = \int\limits_{\delta}^{+\infty} \mu^m\, dE_\mu u$$

with domain

$$D_{L^m} = \left\{ u \in \mathcal{H} : \int\limits_{\delta}^{+\infty} \mu^{2m} \, \|dE_\mu u\|^2_{\mathcal{H}} < \infty \right\}.$$

One can introduce in D_{L^m} the structure of the Hilbert space $\mathcal{H}_{L^m}$ by the scalar product

$$\langle u, v\rangle_{\mathcal{H}_{L^m}} = \int\limits_{\delta}^{+\infty} \mu^{2m} \, \langle dE_\mu u, v\rangle_{\mathcal{H}}.$$

We denote by $\mathcal{H}_{L^{-m}}$ the dual space to $\mathcal{H}_{L^m}, m > 0$ with respect to the scalar product $\langle u, v\rangle_{\mathcal{H}}$. Note that the operator $L^m : \mathcal{H}_{L^m} \to \mathcal{H}$ is an isomorphism of the Hilbert spaces with inverse $L^{-m} : \mathcal{H} \to \mathcal{H}_{L^m}$. Let

$$p^m(x, \xi) = \Big(\big(\langle \xi \rangle + q(x)\big) I + L \Big)^m, m \in \mathbb{R},$$

where $q(x) \geq 1$ for all $x \in \mathbb{R}^n$ and

$$\left|\partial_x^\beta q(x+y) q^{-1}(x)\right| \leq C_\beta q(x) \langle y\rangle^r, \;\; r > 0, \;\; C > 0. \tag{7}$$

Inequality (7) implies that for every $\mu \geq 0$

$$\mu + q(x+y) \leq \mu + Cq(x)\langle y\rangle^r \leq C\langle y\rangle^r (\mu + q(x)), \tag{8}$$

and

$$\langle \xi + \eta\rangle + \mu \leq \sqrt{2}\, \langle \xi\rangle\langle\eta\rangle + \mu \leq \sqrt{2}\, \langle \eta\rangle \big(\langle\xi\rangle + \mu\big). \tag{9}$$

Applying (8) and (9) we obtain that for every $\mu \geq 0$

$$\langle \xi + \eta\rangle + q(x+y) + \mu \leq C\langle\eta\rangle\langle y\rangle^r \big(\langle\xi\rangle + q(x) + \mu\big). \tag{10}$$

It follows from (10) that for every $m \in \mathbb{R}$

$$\big(\langle \xi + \eta\rangle + q(x+y) + \mu\big)^m \leq C\langle\eta\rangle^{|m|}\langle y\rangle^{|m|r} \big(\langle\xi\rangle + q(x) + \mu\big)^m. \tag{11}$$

The spectral representation for $p^m(x, \xi)$, $m \in \mathbb{R}$

$$p^m(x, \xi) = \int\limits_{\mathbb{R}_+} \big(\langle\xi\rangle + q(x) + \mu\big)^m \, dE_\mu$$

yields the estimates

$$\begin{aligned} &\left\|p^m(x,\xi)^{-1} p^m(x+y, \xi+\eta)\right\|^2_{\mathcal{L}(\mathcal{H}_m)} = \\ &= \left\|L^m p^m(x,\xi) p^{-m}(x+y,\xi+\eta) L^{-m}\right\|^2_{\mathcal{L}(\mathcal{H})} \leq \\ &\leq \sup_{\mu \in [\delta, \infty)} \left| \frac{\big(\langle \xi + \eta\rangle + q(x+y) + \mu\big)^m}{\big(\langle\xi\rangle + q(x) + \mu\big)^m} \right| \leq C\langle\eta\rangle^{|m|}\langle y\rangle^{|m|r}. \end{aligned} \tag{12}$$

In the same way we obtain that

$$\left\|p^{-m}(x,\xi) p^m(x+y,\xi+\eta)\right\|^2_{\mathcal{L}(\mathcal{H})} \leq C\langle\eta\rangle^{|m|}\langle y\rangle^{|m|r}$$

and corresponding estimates (6) for derivatives. Hence $p^m \in O(\mathcal{H}_{L^m}, \mathcal{H})$ for every $m \in \mathbb{R}$.

Let now $\mathcal{H}_1$, $\mathcal{H}_1'$, $\mathcal{H}_2$ and $\mathcal{H}_2'$ be Hilbert spaces and $p_1 \in O(\mathcal{H}_1, \mathcal{H}_1')$ and $p_2 \in O(\mathcal{H}_2, \mathcal{H}_2')$. We say that a function $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ belongs to $S(p_1, p_2)$ if

$$|a|_{l_1,l_2} :=$$
$$:= \sum_{|\alpha|\le l_1,\, |\beta|\le l_2} \sup_{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n} \left\| p_2^{-1}(x,\xi)\partial_x^\beta \partial_\xi^\alpha a(x,\xi) p_1(x,\xi)\right\|_{\mathcal{L}(\mathcal{H}_1',\mathcal{H}_2')} < \infty \quad (13)$$

for every l_1, $l_2 \in \mathbb{N}_0$. The semi-norms $|a|_{l_1,l_2}$ define a Frechet topology on $S(p_1, p_2)$. The (operator-valued) functions in $S(p_1, p_2)$ are called symbols.

With each symbol $a \in S(p_1, p_2)$, we associate the pseudodifferential operator $Op(a)$ which acts at $u \in S(\mathbb{R}^n, \mathcal{H}_1)$ by

$$Op(a)u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} a(x,\xi)\widehat{u}(\xi) e^{ix\cdot\xi}\, d\xi =$$
$$= (2\pi)^{-n} \int_{\mathbb{R}^n} d\xi \int_{\mathbb{R}^n} a(x,\xi) u(y) e^{i(x-y)\cdot\xi}\, dy. \quad (14)$$

We denote the set of all pseudodifferential operators with symbols in $S(p_1, p_2)$ by $OPS(p_1, p_2)$.

We will also need double symbols and their associated double pseudodifferential operators. Let again $p_1 \in O(\mathcal{H}_1, \mathcal{H}_1')$ and $p_2 \in O(\mathcal{H}_2, \mathcal{H}_2')$. A function $a : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is said to belong to the class $S_d(p_1, p_2)$ of *double symbols* if there exist $N > 0$ such that

$$|a|_{l_1,l_2,l_3} = \sum_{|\alpha|\le l_1,\, |\beta|\le l_2,\, |\gamma|\le l_3} \sup_{(x,y,\xi)\in\mathbb{R}^{3n}} \langle y\rangle^{-N} \times$$
$$\times \left\| p_2(x,\xi)^{-1} \partial_x^\beta \partial_y^\gamma \partial_\xi^\alpha a(x, x+y, \xi) p_1(x,\xi)\right\|_{\mathcal{L}(\mathcal{H}_1',\mathcal{H}_2')} < \infty \quad (15)$$

for each l_1, l_2, $l_3 \in \mathbb{N}_0$. We correspond to each double symbol $a \in S_d(p_1, p_2)$ the *double pseudodifferential operator*

$$Op_d(a)u(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} d\xi \int_{\mathbb{R}^n} a(x,y,\xi) u(y) e^{i(x-y)\cdot\xi}\, dy, \quad (16)$$

$u \in S(\mathbb{R}^n, \mathcal{H}_1)$ and denote the class of all double pseudodifferential operators by $OPS_d(p_1, p_2)$. Note that the estimates (6) and (13) imply that if $a \in S(p_1, p_2)$ or $S_d(p_1, p_2)$ there exist $M > 0$, $N > 0$ and constants $C_{\alpha\beta}$ and $C_{\alpha\beta\gamma}$ such that

$$\left\| \partial_x^\beta \partial_\xi^\alpha a(x,\xi) \right\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)} \le C_{\alpha\beta} \big(1 + |x| + |\xi|\big)^N \quad (17)$$

and

$$\left\| \partial_x^\beta \partial_y^\gamma \partial_\xi^\alpha a(x,y,\xi) \right\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)} \le C_{\alpha\beta\gamma} \big(1 + |x| + |\xi|\big)^N \langle y\rangle^M \quad (18)$$

for all multiindeces α, β, γ.

Integrating by parts one can prove as in the scalar case that the pseudodifferential operators (14) and (16) can be written of the form of double oscillatory integrals depending on the parameter $x \in \mathbb{R}^n$,

$$(Op(a)u)(x) = (2\pi)^{-n} \iint\limits_{\mathbb{R}^{2n}} a(x,\xi)u(x+y)e^{-iy\cdot\xi}\, d\xi\, dy, \tag{19}$$

$$(Op_d(a)u)(x) = (2\pi)^{-n} \iint\limits_{\mathbb{R}^{2n}} a(x,x+y,\xi)u(x+y)e^{-iy\cdot\xi}\, d\xi\, dy, \tag{20}$$

and that the operators $Op(a)$ and $Op_d(a)$ in (19) and (20) are defined on $C_b^\infty(\mathbb{R}^n, \mathcal{H}_1)$.

For $\xi \in \mathbb{R}^n$, define $e_\xi : \mathbb{R}^n \to \mathbb{C}$ by $e_\xi(x) := e^{ix\cdot\xi}$. Let now A be a continuous linear operator from $C_b^\infty(\mathbb{R}^n, \mathcal{H}_1)$ to $C_{b,N}^\infty(\mathbb{R}^n, \mathcal{H}_2)$, $N \geq 0$, and let $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$. Then there is a bounded linear operator $\sigma_A(x,\xi) : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$e_{-\xi}(x)\big[A(e_\xi \otimes \varphi)\big](x) = \sigma_A(x,\xi)\varphi \tag{21}$$

for every $\varphi \in \mathcal{H}_1$. The function $\sigma_A : \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is then called the *formal symbol* of A.

We will suppose that there exists $N \geq 0, C > 0$ such that

$$\|\sigma_A(x,\xi)\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)} \leq C\big(1 + |x| + |\xi|\big)^N. \tag{22}$$

Proposition 4. Let $A : C_b^\infty(\mathbb{R}^n, \mathcal{H}_1) \to S'(\mathbb{R}^n, \mathcal{H}_2)$ be a continuous linear operator with a formal symbol σ_A. Then A acts at functions $u \in S(\mathbb{R}^n, \mathcal{H}_1)$ via

$$(Au)(x) = (2\pi)^{-n} \int\limits_{\mathbb{R}^n} e^{ix\cdot\xi}\sigma_A(x,\xi)\widehat{u}(\xi)\, d\xi. \tag{23}$$

Proof. Let $u \in S(\mathbb{R}^n, \mathcal{H}_1)$. Then

$$u(x) = (2\pi)^{-n} \int\limits_{\mathbb{R}^n} \widehat{u}(\xi)e_\xi(x)\, d\xi.$$

Let $\{\phi_j\}$ be an orthonormal basis of $\mathcal{H}_1$ and write $\widehat{u}(\xi) = \sum\limits_{j=1}^{\infty} \widehat{u}_j(\xi)\phi_j$ with Fourier coefficients $\widehat{u}_j(\xi) = \langle \widehat{u}(\xi), \phi_j\rangle_{\mathcal{H}_1}$. Hence,

$$\begin{aligned}(Au)(x) &= (2\pi)^{-n} \int\limits_{\mathbb{R}^n} \sum_{j=1}^{\infty} \widehat{u}_j(\xi)(A(e_\xi \otimes \phi_j))(x)\, d\xi = \\ &= (2\pi)^{-n} \int\limits_{\mathbb{R}^n} \sum_{j=1}^{\infty} \widehat{u}_j(\xi)e^{ix\cdot\xi}\sigma_A(x,\xi)\phi_j\, d\xi = \\ &= (2\pi)^{-n} \int\limits_{\mathbb{R}^n} e^{ix\cdot\xi}\sigma_A(x,\xi)\widehat{u}(\xi)\, d\xi. \end{aligned} \tag{24}$$

The last integral exists according to estimate (22). □

Proposition 5. Let $A = Op(a) \in OPS(p_1, p_2)$. Then A has a formal symbol σ_A which coincides with a.

Proof. Let $\xi \in \mathbb{R}^n$ and $\varphi \in \mathcal{H}_1$. Then, by (19),

$$\big(A(e_\xi \otimes \varphi)\big)(x) = (2\pi)^{-n} \iint\limits_{\mathbb{R}^{2n}} a(x,\eta)\varphi e^{i(x+y)\cdot\xi} e^{-iy\cdot\eta}\, d\eta\, dy =$$

$$= e^{ix\cdot\xi}(2\pi)^{-n} \iint\limits_{\mathbb{R}^{2n}} a(x,\xi+\eta)\varphi e^{-iy\cdot\eta}\, d\eta\, dy. \tag{25}$$

Using equality (5) we obtain from (25)

$$\sigma_A(x,\xi)\varphi = e^{-ix\cdot\xi} A(e_\xi \otimes \varphi)(x) = a(x,\xi)\varphi$$

which gives the assertion. □

The next propositions describe the main properties of pseudodifferential operators with operator-valued symbols.

Proposition 6. Every operator in $OPS(p_1, p_2)$ is bounded from $S(\mathbb{R}^n, \mathcal{H}_1)$ to $S(\mathbb{R}^n, \mathcal{H}_2)$.

The proof makes use of estimates (17) and runs completely similar to the proof for scalar pseudodifferential operators (see, for instance, [30]).

Hence the composition of pseudodifferential operators is well defined. But below we will prove that the product of pseudodifferential operators is a pseudodifferential operator again.

Proposition 7.

(i) Let $A^1 = Op(a_1) \in OPS(p_1, p_2)$ and $A^2 = Op(a_2) \in OPS(p_2, p_3)$. Then $A^2A^1 \in OPS(p_1, p_3)$, and the symbol of A^2A^1 is given by

$$\sigma_{A^2A^1}(x,\xi) = (2\pi)^{-n} \iint\limits_{\mathbb{R}^{2n}} a_2(x,\xi+\eta)a_1(x+y,\xi)e^{-iy\cdot\eta}\, dy\, d\eta. \tag{26}$$

(ii) Let $A = Op_d(a) \in OPS_d(p_1, p_2)$. Then $A \in OPS(p_1, p_2)$, and the symbol of A is given by

$$\sigma_A(x,\xi) = (2\pi)^{-n} \iint\limits_{\mathbb{R}^{2n}} a(x, x+y, \xi+\eta)e^{-iy\cdot\eta}\, dy\, d\eta. \tag{27}$$

The double integrals in (26), (27) are understood as oscillatory integrals.

Proof. The proof mimics the proof for the scalar case (see [30]).

(i) Let $\varphi \in \mathbb{H}_1$. Then, applying formula (5) we obtain

$$\begin{aligned}\sigma_{A^2A^1}(x,\xi)\phi &= e^{-ix\cdot\xi}A_2\big[A_1(e_\xi\phi)\big](x) = \\ &= e^{-ix\cdot\xi}A_2(a_1(\cdot,\xi)e_\xi\phi)(x) = \\ &= (2\pi)^{-n}\iint\limits_{\mathbb{R}^{2n}} a_2(x,\eta)a_1(y,\xi)e^{-i(x-y)\cdot(\xi-\eta)}\phi\,dy\,d\eta = \\ &= (2\pi)^{-n}\iint\limits_{\mathbb{R}^{2n}} a_2(x,\xi+\eta)a_1(x+y,\xi)e^{-iy\cdot\eta}\phi\,dy\,d\eta.\end{aligned}$$

Hence, formula (26) holds. Further we have to show that

$$\sigma_{A^2A^1}(x,\xi) = (2\pi)^{-n}\times$$
$$\times\iint\limits_{\mathbb{R}^{2n}}\langle y\rangle^{-2k_1}\langle D_\eta\rangle^{2k_1}\Big\{\langle\eta\rangle^{-2k_2}\langle D_y\rangle^{2k_2}a_2(x,\xi+\eta)a_1(x+y,\xi)\Big\}e^{-iy\cdot\eta}dyd\eta. \quad (28)$$

Application of the Leibnitz formula leads to the estimates

$$p_3^{-1}(x,\xi)\mathcal{I}_{\gamma,\delta}(x,\xi)p_1(x,\xi) = (2\pi)^{-n}\iint\limits_{\mathbb{R}^{2n}}\langle y\rangle^{-2k_1}\langle\eta\rangle^{-2k_2}p_3^{-1}(x,\xi)\times$$
$$\times\,\partial_\eta^\gamma a_2(x,\xi+\eta)\partial_y^\delta a_1(x+y,\xi)p_1(x,\xi)e^{-iy\cdot\eta}\,dy\,d\eta. \quad (29)$$

Applying the next estimates following from (6)

$$\begin{aligned}\big\|p_3^{-1}(x,\xi)p_3(x,\xi+\eta)\big\| &\le C\langle\eta\rangle^{M_3},\\ \big\|p_2^{-1}(x,\xi+\eta)p_2(x,\xi)\big\| &\le C\langle\eta\rangle^{M_2},\\ \big\|p_2^{-1}(x+y,\xi)p_2(x,\xi)\big\| &\le C\langle y\rangle^{M_2},\\ \big\|p_1^{-1}(x+y,\xi)p_1(x,\xi)\big\| &\le C\langle y\rangle^{M_1},\end{aligned} \quad (30)$$

and choosing $2k_1 > n+M_1+M_2, 2k_2 > n+M_2+M_3$, we obtain the estimate

$$\big\|p_3^{-1}(x,\xi)\mathcal{I}_{\gamma,\delta}(x,\xi)p_1(x,\xi)\big\|_{\mathcal{B}(\mathcal{H}_1',\mathcal{H}_3')} \le C|a_2|_{l_1,l_2}|a_1|_{l_1,l_2},$$

for some $l_1, l_2 \in \mathbb{N}$. In the same way one can show that

$$\big\|p_3^{-1}(x,\xi)\partial_x^\beta\partial_\xi^\alpha\sigma_{A^2A^1}(x,\xi)p_1(x,\xi)\big\|_{\mathcal{B}(\mathcal{H}_1',\mathcal{H}_3')} \le C|a_2|_{l_1,l_2}|a_1|_{l_1,l_2},$$

for some l_1, $l_2 \in \mathbb{N}$.

(ii) Following the proof of (i) we have to estimate the integrals

$$p_2^{-1}(x,\xi)\mathcal{I}_{\gamma,\delta}(x,\xi)p_1(x,\xi) = (2\pi)^{-n}\times$$
$$\times\iint\limits_{\mathbb{R}^{2n}}\langle y\rangle^{-2k_1}\langle\eta\rangle^{-2k_2}p_2^{-1}(x,\xi)\partial_\eta^\gamma\partial_y^\delta a(x,x+y,\xi+\eta)p_1(x,\xi)e^{-iy\cdot\eta}\,dy\,d\eta. \quad (31)$$

Applying (30) and the estimate

$$\big\|p_2^{-1}(x,\xi+\eta)\partial_\eta^\gamma\partial_x^\delta a(x,x+y,\xi+\eta)p_1(x,\xi+\eta)\big\|_{\mathcal{L}(\mathcal{H}_1',\mathcal{H}_2')}\le C|a|_{l_1,0,l_3}\langle y\rangle^N,$$

and choosing $2k_1 > n + N$, $2k_2 > n + M_1 + M_2$ we obtain

$$\left\| p_2^{-1}(x,\xi)\mathcal{I}_{\gamma,\delta}(x,\xi)p_1(x,\xi) \right\|_{\mathcal{L}(\mathcal{H}_1',\mathcal{H}_2')} \le C|a|_{l_1,0,l_3}.$$

In the same way we obtain the estimate

$$\left\| p_2^{-1}(x,\xi)\partial_x^\beta\partial_\xi^\alpha\sigma_A(x,\xi)p_1(x,\xi) \right\|_{\mathcal{L}(\mathcal{H}_1',\mathcal{H}_3')} \le C|a|_{l_1,l_2,l_3}. \qquad \square$$

An operator A^* is called the *formal adjoint* to the operator $A \in OPS(p_1,p_2)$ if, for arbitrary functions $u \in S(\mathbb{R}^n,\mathcal{H}_1)$ and $v \in S(\mathbb{R}^n,\mathcal{H}_2)$,

$$\langle Au, v\rangle_{L^2(\mathbb{R}^n,\mathcal{H}_2)} = \langle u, A^*v\rangle_{L^2(\mathbb{R}^n,\mathcal{H}_1)}. \tag{32}$$

Proposition 8. Let $A = Op(a) \in OPS(p_1,p_2)$. Then $A^* \in OPS(p_2^*,p_1^*)$, and the symbol of A^* is given by

$$\sigma_{A^*}(x,\xi) = (2\pi)^{-n} \iint\limits_{\mathbb{R}^{2n}} a^*(x+y,\xi+\eta)e^{i(x-y)\cdot\xi}\, dy\, d\xi, \tag{33}$$

where

$$\langle a(x,\xi)u, v\rangle_{\mathcal{H}_2} = \langle u, a^*(x,\xi)v\rangle_{\mathcal{H}_1}$$

for all $u \in \mathcal{H}_1$ and $v \in \mathcal{H}_2$. The double integrals in (33) are understood as oscillatory integrals.

The assertion of Proposition 8 follows from Proposition 7 (ii).

By Proposition 8 and formula (32), one can think of operators in $OPS(p_1,p_2)$ as acting from $S'(\mathbb{R}^n,\mathcal{H}_1)$ to $S'(\mathbb{R}^n,\mathcal{H}_2)$.

Theorem 9 (Calderon–Vaillancourt)**.** *If* $A = Op(a) \in OPS(I_{\mathcal{H}_1}, I_{\mathcal{H}_2}) := OPS(\mathcal{H}_1,\mathcal{H}_2)$, *then* A *is bounded as operator from* $L^2(\mathbb{R}^n,\mathcal{H}_1)$ *to* $L^2(\mathbb{R}^n,\mathcal{H}_2)$*, and there exists constants* $C > 0$ *and* $2k_1, 2k_2 > n$ *such that*

$$\|A\|_{\mathcal{L}(L^2(\mathbb{R}^n,\mathcal{H}_1),L^2(\mathbb{R}^n,\mathcal{H}_2))} \le C \sum_{|\alpha|\le 2k_1,|\beta|\le 2k_2} \sup_{(x,\xi)\in\mathbb{R}^{2n}} \left\| a_{(\alpha)}^{(\beta)}(x,\xi) \right\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)}.$$

Proposition 10 (Beals)**.** Let $A = Op(a) \in OPS(\mathcal{H}_1,\mathcal{H}_2)$ be invertible as operator from $L^2(\mathbb{R}^n,\mathcal{H}_1)$ to $L^2(\mathbb{R}^n,\mathcal{H}_2)$. Then $A^{-1} \in OPS(\mathcal{H}_2,\mathcal{H}_1)$.

2.4. **Sobolev spaces $H(\mathbb{R}^n,p_h)$.** Let $p \in O(\mathcal{H}',\mathcal{H})$. We denote by $p_h, h > 0$ the symbol $p_h(x,\xi) = p(x,h\xi)$.

Proposition 11. . Let $p \in O(\mathcal{H}_1,\mathcal{H}_2)$. Then for every $h > 0$

$$\begin{aligned} Op(p_h)Op(p_h^{-1}) &= I_{\mathcal{H}_2} + hOp(r_h^2), \\ Op(p_h^{-1})Op(p_h) &= I_{\mathcal{H}_1} + hOp(r_h^1), \end{aligned} \tag{34}$$

where $Op(r_h^j) \in OPS(\mathcal{H}_j,\mathcal{H}_j)$, $j = 1,2$, and

$$\sup_{h>0} \|Op(r_h^j)\|_{\mathcal{L}(\mathcal{H}_j)} < \infty, \quad j = 1,2.$$

For the proof see [33], Proposition 7 and Corollary 14.

Corollary 12. *For $h > 0$ small enough*

$$Op(p_h)Op(p_h)^{-1} = I_{\mathcal{H}_2}, Op(p_h)^{-1}Op(p_h) = I_{\mathcal{H}_1}, \tag{35}$$

where

$$Op(p_h)^{-1} = Op(p_h^{-1})(I_{\mathcal{H}_2} + hOp(r_h^2))^{-1} = (I_{\mathcal{H}_2} + hOp(r_h^1))^{-1}Op(p_h^{-1}).$$

In what follows for $p \in O(\mathcal{H}_1, \mathcal{H}_2)$ we fix $h > 0$, such that there exists $Op(p_h)^{-1}$.

We denote by $H(\mathbb{R}^n, p_h)$ the Banach space which is the closure of $S(\mathbb{R}^n, \mathcal{H})$ with respect to the norm

$$\|u\|_{H(\mathbb{R}^n, p_h)} := \|Op(p_h)u\|_{L^2(\mathbb{R}^n, \mathcal{H}')}.$$

It turns out that then $Op(p_h) : H(\mathbb{R}^n, p_h) \to L^2(\mathbb{R}^n, \mathcal{H}_1)$ is an isomorphism. Using these facts one easily gets the following versions of Proposition 9 and 10, respectively.

Proposition 13. Let $Op(a) \in OPS(p_1, p_2)$. Then $Op(a)$ is bounded as operator from $H(\mathbb{R}^n, p_{1,h})$ to $H(\mathbb{R}^n, p_{2,h})$, and

$$\|A\|_{\mathcal{L}(H(\mathbb{R}^n, p_{1,h}), H(\mathbb{R}^n, p_{2,h}))} \le C|a|_{l_1, l_2},$$

where $C > 0$ and $l_1, l_2 \in \mathbb{N}$ are independent of A.

Proposition 14. Let $A = Op(a) \in OPS(p_1, p_2)$ be invertible as operator from $H(\mathbb{R}^n, p_{1,h})$ to $H(\mathbb{R}^n, p_{2,h})$. Then $A^{-1} \in OPS(p_2, p_1)$.

Let $a \in C_b^\infty(\mathbb{R}^n)$ and $\mathcal{H}$ be a Hilbert space. In what follows we write $aI_{\mathcal{H}}$ for the operator of multiplication by a acting on $S'(\mathbb{R}^N, \mathcal{H})$. Note that this operator is bounded on $H(\mathbb{R}^n, p_h)$ for every weight function $p \in O(\mathcal{H}, \mathcal{H}')$.

We note one more import property of operators in $OPS(p_1, p_2)$ which follows easily from Propositions 7 (i) and 13.

Proposition 15. Let $A = Op(a) \in OPS(p_1, p_2)$. Further let $\varphi \in C_b^\infty(\mathbb{R}^n)$ and set $\varphi_R(x) := \varphi(x/R)$. Then, with $[A, \varphi_R] := A\varphi_R I_{\mathcal{H}_1} - \varphi_R I_{\mathcal{H}_2} A$

$$\lim_{R\to\infty} \big\|[A, \varphi_R]\big\|_{\mathcal{L}(H(\mathbb{R}^n, p_{1,h}), H(\mathbb{R}^n, p_{2,h}))} = 0. \tag{36}$$

2.5. **Pseudodifferential operators with slowly oscillating symbols.** We say that a symbol $a \in S(p_1, p_2)$ is *slowly oscillating at infinity* if, for all multi-indices α, β,

$$\big\|p_2^{-1}(x,\xi)\partial_x^\beta \partial_\xi^\alpha a(x,\xi) p_1(x,\xi)\big\|_{\mathcal{L}(\mathcal{H}_1', \mathcal{H}_2')} \le C_{\alpha\beta}^a(x), \tag{37}$$

where

$$\lim_{x\to\infty} C_{\alpha\beta}^a(x) = 0 \tag{38}$$

for all multi-indices α, β with $\beta \ne 0$. We denote this class of symbols by $S_{sl}(p_1, p_2)$ and write $OPS_{sl}(p_1, p_2)$ for the corresponding class of pseudodifferential operators. Furthermore, let $S^0(p_1, p_2)$ refer to the subset of $S_{sl}(p_1, p_2)$ of all symbols such that (38) holds for all multi-indices α, β.

Similarly, a double symbol $a \in S_d(p_1, p_2)$ is called *slowly oscillating* at infinity if, for all mutli-indices α, β and some $N > 0$

$$\left\| p_2^{-1}(x,\xi)\partial_x^{\beta}\partial_{\xi}^{\alpha} a(x, x+y, \xi) p_1(x,\xi)\right\|_{\mathcal{L}(\mathcal{H}_1', \mathcal{H}_2')} \leq C_{\alpha\beta}^a(x) \langle y \rangle^N ,$$

where

$$\lim_{x\to\infty} C_{\alpha\beta}^a(x) = 0$$

for all multi-indices α, β with $\beta \neq 0$. We denote the set of all slowly oscillating double symbols by $S_{d,sl}(p_1, p_2)$ and write $OPS_{d,sl}(p_1, p_2)$ for the corresponding class of double pseudodifferential operators.

The next proposition describes some properties of pseudodifferential operators with operator-valued slowly oscillating at infinity symbols which will be needed in what follows.

Proposition 16.

(i) Let $A^1 = Op(a_1) \in OPS_{sl}(p_1, p_2)$ and $A^2 = Op(a_2) \in OPS_{sl}(p_2, p_3)$. Then $A^2A^1 \in OPS_{sl}(p_1, p_3)$, and

$$\sigma_{A^2A^1}(x,\xi) = a_2(x,\xi)a_1(x,\xi) + r(x,\xi),$$

where $r \in S^0(p_1, p_3)$.

(ii) Let $A = Op_d(a) \in OPS_{d,sl}(p_1, p_2)$. Then $A \in OPS_{sl}(p_1, p_2)$, and

$$\sigma_A(x,\xi) = a(x,x,\xi) + r(x,\xi),$$

where $r \in S^0(p_1, p_2)$.

(iii) Let $A = Op(a) \in OPS(p_1, p_2)$. Then $A^* \in OPS(p_2^*, p_1^*)$, and

$$\sigma_{A^*}(x,\xi) = a^*(x,x,\xi) + r(x,\xi),$$

where $r \in S^0(p_2^*, p_1^*)$.

Proof. We prove (i). Statements (ii), (iii) are proved in the similar way. We use the representation (26) for $\sigma_{A^2A^1}$

$$\sigma_{A^2A^1}(x,\xi) = (2\pi)^{-n} \iint\limits_{\mathbb{R}^{2n}} a_2(x,\xi+\eta)a_1(x+y,\xi)e^{-iy\cdot\eta}\, dy\, d\eta. \tag{39}$$

For obtain estimate (37) for $\sigma_{A^2A^1}$ we have to estimate the integrals

$$\mathcal{I}_{\alpha,\beta,\gamma,\delta}(x,\xi) =$$

$$= (2\pi)^{-n}\iint\limits_{\mathbb{R}^{2n}} \langle y\rangle^{-2k_1}\langle \eta\rangle^{-2k_2}\partial_x^{\beta}\partial_{\xi}^{\alpha}a_2(x,\xi+\eta)\partial_x^{\gamma}\partial_{\xi}^{\delta}a_1(x+y,\xi)e^{-iy\cdot\eta}\, dy\, d\eta,$$

for $|\beta| \geq 1$ or $|\gamma| \geq 1$. Let $2k_1 > n+1+M_1+M_2$, $2k_2 > n+1+M_2+M_3$, Then similar to the proof of Proposition 7 we obtain

$$\left\|p_3^{-1}(x,\xi)\mathcal{I}_{\alpha,\beta,\gamma,\delta}(x,\xi)p_1(x,\xi)\right\|_{\mathcal{L}(\mathcal{H}_1',\mathcal{H}_2')} \leq \\ \leq C \iint\limits_{\mathbb{R}^{2n}} \langle y\rangle^{-2k_1-M_1-M_2}\langle\eta\rangle^{-2k_2-M_3-M_2} \times \\ \times \left\|p_3^{-1}(x,\xi+\eta)\partial_x^\beta\partial_\xi^\alpha a_2(x,\xi+\eta)p_2(x,\xi+\eta)\right\| \times \\ \times \left\|p_2^{-1}(x+y,\xi)\partial_x^\gamma\partial_\xi^\delta a_2(x+y,\xi)p_1(x+y,\xi)\right\| \, dy\, d\eta \leq \\ \leq C C_{\alpha\beta}^{a_2}(x) \sup_{y\in\mathbb{R}^n} \frac{C_{\gamma\delta}^{a_1}(x+y)}{\langle y\rangle}. \tag{40}$$

Estimate (40) shows that

$$\lim_{x\to\infty}\sup_{\xi\in\mathbb{R}^n}\left\|p_3^{-1}(x,\xi)\mathcal{I}_{\alpha,\beta,\gamma,\delta}(x,\xi)p_1(x,\xi)\right\|_{\mathcal{L}(\mathcal{H}_1',\mathcal{H}_2')} = 0.$$

Hence $\sigma_{A^2A^1} \in OPS_{sl}(p_1,p_3)$. Further, by the Lagrange formula

$$a_2(x,\xi+\eta) = a_2(x,\xi) + \sum_{j=1}^n \eta_j \int_0^1 \partial_{\xi_j} a_2(x,\xi+\theta\eta)\, d\theta. \tag{41}$$

Substituting (41) in (39) and applying formula (5) we obtain

$$\sigma_{A^2A^1}(x,\xi) = a_2(x,\xi)a_1(x,\xi) + r(x,\xi),$$

where

$$r(x,\xi) = (2\pi)^{-n} \times \\ \times \sum_{j=1}^n \int_0^1 d\theta \iint\limits_{\mathbb{R}^{2n}} \partial_{\xi_j} a_2(x,\xi+\theta\eta) D_{x_j} a_1(x+y,\xi) e^{-iy\cdot\eta}\, dy\, d\eta. \tag{42}$$

Because the integral (42) contains the derivative of $a_1 (\in S_{sl}(p_1,p_2))$ with respect to x one can prove that $r \in S^0(p_1,p_3)$ following to the proof that $\sigma_{A^2A^1} \in OPS_{sl}(p_1,p_3)$. □

3. Invertibility at Infinity and Fredholm Property of Pseudodifferential Operators

Let $\chi \in C_0^\infty(\mathbb{R}^n)$ be a function such that $\chi(x) = 1$ if $|x| \leq 1$ and $\chi(x) = 0$ if $|x| \geq 2$. Set $\phi := 1 - \chi$ and, for $R > 0$, $\chi_R(x) := \chi(x/R)$ and $\phi_R(x) := \phi(x/R)$. Further let

$$B_R := \{x \in \mathbb{R}^n : |x| < R\} \text{ and } B_R' := \{x \in \mathbb{R}^n : |x| > R\}.$$

We say that an operator $A : H(\mathbb{R}^n, p_1) \to H(\mathbb{R}^n, p_2)$ is *locally invertible at infinity* if there is an $R_0 > 0$ such that, for every $R > R_0$, there are operators $\mathcal{L}_R$ and $\mathcal{R}_R$ such that

$$\mathcal{L}_R A \phi_R I_{\mathcal{H}_1} = \phi_R I_{\mathcal{H}_1} \text{ and } \phi_R A \mathcal{R}_R = \phi_R I_{\mathcal{H}_2}. \tag{43}$$

Operators $\mathcal{L}_R$ and R_R with these properties are called *locally left and right inverses of* A, respectively.

Theorem 17. *Let* $A = Op(a) \in OPS_{sl}(p_1, p_2)$. *Assume there is a constant* $R_0 > 0$ *such that the operator* $a(x, \xi) : \mathcal{H}_1 \to \mathcal{H}_2$ *is invertible for every* $(x, \xi) \in B'_{R_0} \times \mathbb{R}^n$ *and that*

$$\sup_{(x,\xi)\in B'_{R_0}\times\mathbb{R}^n} \left\| p_1^{-1}(x,\xi) a(x,\xi)^{-1} p_2(x,\xi) \right\|_{\mathcal{L}(\mathcal{H}'_2,\mathcal{H}'_1)} < \infty.$$

Then the operator $A : H(\mathbb{R}^n, p_{1,h}) \to H(\mathbb{R}^n, p_{2,h})$ *is locally invertible at infinity.*

Proof. Given ϕ as above, choose $\varphi \in C_b^\infty(\mathbb{R}^n)$ such that $\varphi\phi = \phi$, and set $\varphi_R(x) := \varphi(x/R)$ for $R > R_0$. Condition (43) implies that the function $b_R(x, \xi) := \varphi_R(x) a(x, \xi)^{-1}$ belongs to $S(p_2, p_1)$. Hence, and by Proposition 16 (i),

$$Op(b_R) Op(a) \phi_R I_{\mathcal{H}_1} = (I_{\mathcal{H}_1} + Op(q_R)\psi_R I_{\mathcal{H}_1})\phi_R I_{\mathcal{H}_1},$$

where $q_R \in S^0(p_1, p_2)$. Moreover, one can prove that for all multi-indices α, β,

$$\lim_{x\to\infty} \sup_{\xi\in\mathbb{R}^n} \left\| p_1^{-1}(x,\xi) \partial_x^\beta \partial_\xi^\alpha q_R(x,\xi) p_1(x,\xi) \right\|_{\mathcal{L}(\mathcal{H}'_1)} = 0$$

uniformly with respect to $R > R_0$. It follows from Proposition 13 that there exists an $R' > R_0$ such that

$$\|Op(q_R)\psi_R I_{\mathcal{H}_1}\|_{\mathcal{L}(H(\mathbb{R}^n, p_1))} < 1$$

for every $R > R'$. Hence,

$$\left(I_{\mathcal{H}_1} + Op(q_R)\psi_R I_{\mathcal{H}_1}\right)^{-1} Op(b_R) Op(a) \phi_R I_{\mathcal{H}_1} = \phi_R I_{\mathcal{H}_1}, \tag{44}$$

and $Op(a)$ is locally invertible from the left at infinity, with a local left inverse operator given by

$$\mathcal{L}_R := \left(I_{\mathcal{H}_1} + Op(q_R)\psi_R I_{\mathcal{H}_1}\right)^{-1} Op(b_R) \in OPS(p_2, p_1).$$

In the same way, a local right inverse operator $\mathcal{R}_R \in OPS(p_2, p_1)$ can be constructed. It follows from the definition of the operators $\mathcal{L}_R$ and $\mathcal{R}_R$ that

$$\begin{aligned} &\sup_{R>R_0} \|\mathcal{L}_R\|_{\mathcal{L}(H(\mathbb{R}^n,p_{2,h}),H(\mathbb{R}^n,p_{1,h}))} < \infty, \\ &\sup_{R>R_0} \|\mathcal{R}_R\|_{\mathcal{L}(H(\mathbb{R}^n,p_{2,h}),H(\mathbb{R}^n,p_{1,}))} < \infty \end{aligned} \tag{45}$$

which finishes the proof. □

We say that a linear operator $A : H(\mathbb{R}^n, p_{1,h}) \to H(\mathbb{R}^n, p_{2,h})$ is *locally Fredholm* if, for every $R > 0$, there exist bounded linear operators $\mathcal{L}_R, \mathcal{D}_R : H(\mathbb{R}^n, p_{2,h}) \to H(\mathbb{R}^n, p_{1,h})$ and compact operators $T'_R : H(\mathbb{R}^n, p_{1,h}) \to H(\mathbb{R}^n, p_{1,h})$ and $T''_R : H(\mathbb{R}^n, p_{2,h}) \to H(\mathbb{R}^n, p_{2,h})$ such that

$$\mathcal{L}_R A \phi_R I_{\mathcal{H}_1} = \phi_R I_{\mathcal{H}_1} + T'_R \quad\text{and}\quad \phi_R A \mathcal{D}_R = \phi_R I_{\mathcal{H}_2} + T''_R. \tag{46}$$

Theorem 18. *Let $A = Op(a) \in OPS_{sl}(p_1, p_2)$ an operator which satisfies the conditions of Theorem 17. If A is a locally Fredholm operator, then A has the Fredholm property as operator from $H(\mathbb{R}^n, p_{1,h})$ to $H(\mathbb{R}^n, p_{2,h})$.*

Proof. Let R_0 be such that for every $R > R_0$ there exist local inverse operators $\mathcal{L}_R, \mathcal{R}_R \in OPS(p_2, p_1)$ of A. Set $\Lambda_R := \mathcal{B}_R \phi_R I_{\mathcal{H}_2} + \mathcal{L}_R \chi_R I_{\mathcal{H}_2}$. Then $\Lambda_R A = I_{\mathcal{H}_1} + T'_R + Q_R$ where $Q_R := \mathcal{B}_R[\phi_R, A] + \mathcal{B}_R[\chi_R, A]$ and where $T'_R : H(\mathbb{R}^n, p_{1,h}) \to H(\mathbb{R}^n, p_{1,h})$ is compact. Proposition 7 implies that

$$\lim_{R\to 0} \left\| [\phi_R, A] \right\|_{\mathcal{L}(H(\mathbb{R}^n, p_{1,h}), H(\mathbb{R}^n, p_{2,h}))} = \\ = \lim_{R\to 0} \left\| [\chi_R, A] \right\|_{\mathcal{L}(H(\mathbb{R}^n, p_{1,h}), H(\mathbb{R}^n, p_{2,h}))} = 0. \quad (47)$$

From (47) and (45) we conclude that $\|Q_R\|_{\mathcal{L}(H(\mathbb{R}^n, p_1))} < 1$ for large enough $R > 0$. Hence, $\Lambda'_R := (I_{\mathcal{H}_1} + Q_R)^{-1} \Lambda_R$ is a left regularizator of A whenever R_0 is large enough. In the same way, a regularizator from the right-hand side can be found. □

4. Pseudodifferential Operators with Analytical Symbols and Exponential Estimates

4.1. Operators and weight spaces.

In this section we consider the weight functions of the form

$$p_T(x, \xi) = \big(\langle \xi \rangle + q(x)\big) I + T, \quad (48)$$

where T is a self-adjoint operator in a Hilbert space $\mathcal{H}$ with a dense domain D_T. We suppose that T is positively defined. Let $\mathcal{H}_{T^m}, m \in \mathbb{R}$ be the Hilbert spaces introduced in Example 7, $q(x) \geq 1$ for every $x \in \mathbb{R}^n$. Moreover, $q \in C^\infty(\mathbb{R}^n)$ and

$$\left| \partial_x^\alpha q(x + y) q^{-1}(x) \right| \leq C_\alpha \langle y \rangle^r, r \geq 0. \quad (49)$$

The estimate (49) implies the estimate

$$\left| \partial_x^\alpha q(x) \right| \leq C_\alpha q(x). \quad (50)$$

In what follows we consider the weight functions of the form $p(x, \xi) = p_T^m(x, \xi)$. We say that the such weight function $p \in O(T^m, q)$.

Let $a \in S(p_1, p_2)$ where $p_j \in O(T_j^{m_j}, q)$, $j = 1, 2$. We denote by $S(p_1, p_2, B_{dq(x)})$ the class of symbols such that:

(1) for every $x \in \mathbb{R}^n$ the operator-valued function $\xi \mapsto a(x, \xi)$ can be extended analytically with respect to ξ into the tube domain $\mathbb{R}^n + iB_{dq(x)}$, where $B_{dq(x)} = \{\eta \in \mathbb{R}^n : |\eta| < dq(x)\}$, $d > 0$.

(2) for arbitrary multi-indices α, β there exists a constant $C_{\alpha\beta}$ such that

$$\left\| p_2^{-1}(x, \xi + i\eta) \partial_x^\beta \partial_\xi^\alpha a(x, \xi + i\eta) p_1(x, \xi + i\eta) \right\|_{\mathcal{L}(\mathcal{H}'_1, \mathcal{H}'_2)} \leq \\ \leq C_{\alpha\beta} \langle \xi + i\eta \rangle^{-|\alpha|} \quad (51)$$

for all $(x, \xi + i\eta) \in \mathbb{R}^n \times (\mathbb{R}^n + iB_{dq(x)})$, where

$$p_j(x, \xi + i\eta) = \Big(\big(1 + |\xi|^2 + |\eta|^2\big)^{1/2} + q_j(x) + T_j\Big)^m.$$

We denote by $OPS(p_1, p_2, B_{dq(x)})$ the class of pseudodifferential operators with symbols in $S(p_1, p_2, B_{dq(x)})$.

(3) If in estimates (51) $C_{\alpha\beta} = C_{\alpha\beta}(x)$ and $\lim\limits_{x\to\infty} C_{\alpha\beta}(x) = 0$ for $\beta \neq 0$ then we denote the corresponding classes of symbols and operators by $S_{sl}(p_1, p_2, B_{dq(x)})$.

(4) We say that a positive C^∞-function $w(x) = e^{v(x)}$ is *a weight in the class* $\mathcal{R}(dq)$ if $v \in C^\infty(\mathbb{R}^n)$ and

$$\big|\partial_x^\alpha(\nabla v(x))\big| < C_\alpha dq(x), \ \ C_0 = 1 \tag{52}$$

for every α and every point $x \in \mathbb{R}^n$. We say that a weight w is slowly oscillating if there exists $\delta \in (0, 1]$ such that

$$\big|\partial_x^\alpha(\nabla v(x))\big| \leq C_\alpha dq^{1-\delta|\alpha|}(x). \tag{53}$$

We denote by $\mathcal{R}_{sl}(dq)$ the class of slowly oscillating weights.

Theorem 19.

(i) *Let* $a \in S(p_1, p_2, B_{dq(x)})$ *where* $p_j \in O(T_j^{m_j}, q), j = 1, 2$ *and* $w = \exp v \in \mathcal{R}(dq)$. *Then* $w^{-1}Op(a)wI = Op_d(a_w) \in OPS_d(p_1, p_2)$, *where*

$$a_w(x, y, \xi) = a(x, \xi + i\theta_w(x, y)),$$

and

$$\theta_w(x, y) = \int_0^1 (\nabla v)((1 - t)x + ty)\, dt.$$

(ii) *Let* $a \in S_{sl}(p_1, p_2, B_{dq(x)})$ *where* $p_j \in O^{m_j}(T_j, q_j), j = 1, 2$ *and* $w = \exp v \in \mathcal{R}_{sl}(dq)$. *Then* $w^{-1}Op(a)wI = Op(\tilde{a}_w) \in OPS_{sl}(p_1, p_2)$ *where*

$$\widetilde{a}_w(x, \xi) = a\big(x, \xi + i\nabla v(x)\big) + r(x, \xi), \tag{54}$$

and $r \in S^0(p_1, p_2)$.

Proof. (i) Let $w = \exp v \in \mathcal{R}(\mu)$. By the theorem of the mean value there exists $t_0 \in [0, 1]$ such that

$$\theta_w(x, y) = (\nabla v)((1 - t_0)x + t_0 y).$$

Hence $\theta_w(x, y) \in B_{\mu(x)}$ for every pair (x, y). As in the scalar case (see [32]) we prove that

$$(w^{-1}Op(a)w)\varphi(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} d\xi \int_{\mathbb{R}^n} a\big(x, \xi + i\theta_w(x, y)\big)u(y)e^{i(x-y)\cdot\xi}\, dy$$

for $u \in S(\mathbb{R}^n, \mathcal{H}_1)$. The next step is to prove that the function $(x, y, \xi) \to a_w(x, y, \xi)$ satisfies estimates (15). Applying formulas

$$\partial_{x_k}\big(a_w(x, \xi + i\nabla v(x + t_0 y))\big) = \partial_{x_k} a_w(x, \xi + i\theta_w(x, y)) + \\ + i\sum_{k=1}^{n} \partial_{\xi_k} a_w\big(x, \xi + i\nabla v(x + t_0 y)\big)\, \frac{\partial \nabla v(x + t_0 y)}{\partial x_k}\,, \tag{55}$$

$$\partial_{y_k}\big(a_w(x, \xi + i\nabla v(x + t_0 y))\big) = \\ = i\sum_{k=1}^{n} \partial_{\xi_k} a_w\big(x, \xi + i\nabla v(x + t_0 y)\big)\, \frac{\partial \nabla v(x + t_0 y)}{\partial y_k}\,. \tag{56}$$

Taking into account that $\theta_w(x, x + y) = \nabla v(x + t_0 y)$, estimates (51), and the Leibnitz formula we obtain

$$\Big\| p_2^{-1}\big(x, \xi + i\nabla v(x + t_0 y)\big)\partial_x^{\beta}\partial_\xi^{\alpha} a\big(x, \xi + i\nabla v(x + t_0 y)\big)\times \\ \times p_1\big(x, \xi + i\nabla v(x + t_0 y)\big)\Big\|_{\mathcal{L}(\mathcal{H}_1', \mathcal{H}_2')} \le \\ \le C'_{\alpha\beta}\big\langle \xi + i\nabla v(x + t_0 y)\big\rangle^{-|\beta|} |\nabla v(x + t_0 y)|^{\beta} \le C'_{\alpha\beta} \tag{57}$$

for all α, β with some constants $C'_{\alpha\beta}$. Estimate (49) and spectral decomposition for the operator T yield the estimate

$$\big\| p\big(x, \xi + i\nabla v(x + t_0 y)\big) p^{-1}(x, \xi)\big\|_{\mathcal{L}(\mathcal{H})} \le C\langle y\rangle^N, \tag{58}$$

for some $C > 0$ and $N > 0$. Then estimates (57), (58) imply that

$$\big\| p_2^{-1}(x, \xi)\partial_x^{\beta}\partial_\xi^{\alpha} a_w(x, x + y, \xi) p_1(x, \xi)\big\|_{\mathcal{L}(\mathcal{H})} \le C_{\alpha\beta}\langle y\rangle^M$$

for some $C_{\alpha\beta} > 0$ and $M > 0$. Hence $a_w \in S_d(p_1, p_2)$.

(ii) Let now $a \in S_{sl}(p_1, p_2, \mu)$ and $w \in \mathcal{R}_{sl}(dq)$. Again applying the definition of $S_{sl}(p_1, p_2, \mu)$ and estimate (53) we obtain as in (57)

$$\Big\| p_2^{-1}\big(x, \xi + i\nabla v(x + t_0 y)\big)\partial_x^{\beta}\partial_\xi^{\alpha} a\big(x, \xi + i\nabla v(x + t_0 y)\big)\times \\ \times p_1\big(x, \xi + i\nabla v(x + t_0 y)\big)\Big\|_{\mathcal{L}(\mathcal{H}_1', \mathcal{H}_2')} \le \\ \le C'_{\alpha\beta}(x)\big\langle \xi + i\nabla v(x + t_0 y)\big\rangle^{-|\beta|} |\nabla v(x + t_0 y)|^{\beta} \le C'_{\alpha\beta}(x), \tag{59}$$

where

$$\lim_{x\to\infty} C'_{\alpha\beta}(x) = 0,$$

if $\beta \neq 0$. Estimates (58), (59) imply that

$$\big\| p_2^{-1}(x, \xi)\partial_x^{\beta}\partial_\xi^{\alpha} a_w(x, x + y, \xi) p_1(x, \xi)\big\|_{\mathcal{L}(\mathcal{H})} \le C_{\alpha\beta}(x)\langle y\rangle^M,$$

where $\lim_{x\to\infty} C_{\alpha\beta}(x) = 0$ if $\beta \neq 0$. Formula (54) now follows from Proposition 16 (ii). □

4.2. **Exponential estimates.** For a C^∞-weight w, let $H(\mathbb{R}^n, p_h, w)$ denote the space of distributions with norm

$$\|u\|_{H(\mathbb{R}^n,p_h,w)} := \|wu\|_{H(\mathbb{R}^n,p_h)} < \infty. \tag{60}$$

Theorem 20. *Let $a \in S(p_{1,h}, p_{2,h}, B_{dq(x)})$ where $p_j \in O(T_j^{m_j}, q), j = 1, 2$ and $w = \exp v \in \mathcal{R}(dq)$. Then the operator $Op(a) : H(\mathbb{R}^n, p_{1,h}, w) \to H(\mathbb{R}^n, p_{2,h}, w)$ is bounded.*

Theorem 21. *Let $a \in S_{sl}(p_1, p_2, B_{dq(x)})$ where $p_j \in O(T_j^{m_j}, q)$, $j = 1, 2$ and $w = \exp v \in \mathcal{R}_{sl}(\mu)$ be a weight with $\lim_{x\to\infty} v(x) = +\infty$. Assume that the operators $a(x, x, \xi + it\nabla v(x))$ are invertible for all enough large x, all $\xi \in \mathbb{R}^n$, $t \in [-1, 1]$, and*

$$\lim_{x\to\infty} \sup_{(\xi,t)\in\times\mathbb{R}^n\times[-1,1]} \left\|p_1^{-1}(x, \xi)a^{-1}(x, \xi + it\nabla v(x))p_2(x, \xi)\right\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{61}$$

Finally, let $A = Op(a)$ be locally Fredholm as operator from $H(\mathbb{R}^n, p_{1,h})$ to $H(\mathbb{R}^n, p_{2,h})$.

If $f \in H(\mathbb{R}^n, p_{2,h}, w)$ then every solution of the equation $Au = f$, which a priori belongs to $H(\mathbb{R}^n, p_{1,h}, w^{-1})$, a posteriori belongs to $H(\mathbb{R}^n, p_{1,h}, w)$.

Proof. Condition (61) implies that the operators A_{w^t} are locally invertible at infinity, and the local Fredholm property of A moreover implies that these operators are locally Fredholm for each $t \in [-1, 1]$. Hence, by Theorem 18, each operator $A_{w^t} : H(\mathbb{R}^n, p_{1,h}) \to L^2(\mathbb{R}^n, p_{2,h})$ has the Fredholm property. Note that the symbol of A_{w^t} is given by

$$\sigma_{A_{w^t}}(x, \xi) = (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} a(x, y, \xi + it\theta_w(x, y))e^{-iy\cdot\xi} \, dy \, d\xi. \tag{62}$$

This formula shows that the mapping $[-1, 1] \to S(p_1, p_2)$, $t \mapsto \sigma_{A_{w^t}}$ is continuous. Thus, and by Proposition 13, the mapping

$$[-1, 1] \to \mathcal{L}(H(\mathbb{R}^n, p_{1,h}), H(\mathbb{R}^n, p_{2,h})), \quad t \mapsto A_{w^t}$$

is continuous. This shows that the Fredholm index of the operator $A_{w^t} : H(\mathbb{R}^n, p_1) \to H(\mathbb{R}^n, p_2)$ does not depend on $t \in [-1, 1]$. Hence, the operator A, considered as operator from $H(\mathbb{R}^n, p_{1,h}, w)$ to $H(\mathbb{R}^n, p_{2,h}, w)$, and the same operator A, but now considered as operator from $H(\mathbb{R}^n, p_{1,h}, w^{-1})$ to $H(\mathbb{R}^n, p_{2,h}, w^{-1})$, are Fredholm with the same Fredholm indices. Further, since $H(\mathbb{R}^n, p_h, w)$ is a dense subset of $H(\mathbb{R}^n, p_h, w^{-1})$ for $j = 1, 2$, we conclude that the kernel of A, considered as operator from $H(\mathbb{R}^n, p_{1,h}, w)$ to $H(\mathbb{R}^n, p_{2,h}, w)$, coincides with the kernel of A, now considered as operator from $H(\mathbb{R}^n, p_{1,h}, w^{-1})$ to $H(\mathbb{R}^n, p_{2,h}, w^{-1})$. Finally, if $u \in H(\mathbb{R}^n, p_{1,h}, w^{-1})$ is a solution of the equation $Au = f$ with $f \in H(\mathbb{R}^n, p_{2,h}, w)$, then $u \in H(\mathbb{R}^n, p_{1,h}, w^{-1})$ (see, for instance, [23, p. 308]). □

5. Schrödinger Operators with Operator-Valued Potentials

5.1. **Fredholm property.** Let T be a positive self-adjoint operator on a separable Hilbert space $\mathcal{H}$ with a dense domain D_T. Suppose that, for each $x \in \mathbb{R}^n$, we are given a bounded linear operator $L(x) : D_{T^{1/2}} \to D_{T^{-1/2}}$ which is symmetric on $D_{T^{1/2}}$, i.e.,

$$\big\langle L(x)\varphi, \psi\big\rangle_{\mathcal{H}} = \big\langle \varphi, L(x)\psi\big\rangle_{\mathcal{H}} \text{ for all } \varphi,\, \psi \in D_{T^{1/2}}.$$

We assume that the function $x \mapsto L(x)$ is strongly differentiable and that

$$\sup_{x\in\mathbb{R}^n} \left\| \big(T + \langle x\rangle^m I\big)^{-1/2} \partial_x^\beta L(x) \big(T + \langle x\rangle^m I\big)^{-1/2} \right\|_{\mathcal{L}(\mathcal{H})} < \infty, \ \ m \geq 0 \quad (63)$$

for every multiindex β. Moreover, we suppose that

$$\lim_{x\to\infty} \left\| \big(T + \langle x\rangle^m I\big)^{-1/2} \partial_x^\beta L(x) \big(T + \langle x\rangle^m I\big)^{-1/2} \right\|_{\mathcal{L}(\mathcal{H})} = 0 \quad (64)$$

if $\beta \neq 0$.

We consider the Schrödinger operator

$$(\mathbf{H}u)(x) := -\partial_{x_j} \rho^{jk}(x) \partial_{x_k} u(x) + L(x)u(x), \ \ x \in \mathbb{R}^n, \quad (65)$$

on the Hilbert space $L^2(\mathbb{R}^n, \mathcal{H})$ of vector-functions with values in $\mathcal{H}$. In (65) and in what follows, we make use of the Einstein summation convention. We will assume that $\rho^{jk} \in C_b^\infty(\mathbb{R}^n, \mathcal{L}(\mathcal{H}))$ and

$$\lim_{x\to\infty} \partial_{x_l} \rho^{jk}(x) = 0 \ \text{ for } \ l = 1, \dots, n; \quad (66)$$

$\rho^{kj}(x) = (\rho^{jk}(x))^*$, and there is a $C > 0$ such that, for every $\varphi \in \mathcal{H}$,

$$\langle \rho^{jk}(x)\xi_j\xi_k\varphi, \varphi\rangle_{\mathcal{H}} \geq C|\xi|^2 \|\varphi\|^2_{\mathcal{H}}. \quad (67)$$

Let

$$p(x,\xi) := \Big(\big(|\xi|^2 + \langle x\rangle^m\big)I + T\Big)^{1/2},$$

and write $H(\mathbb{R}^n, p)$ for the Hilbert space with norm

$$\|u\|_{H(\mathbb{R}^n,p_h)} := \|Op(p_h)u\|_{L^2(\mathbb{R}^n,\mathcal{H})},$$

for fixed $h > 0$ enough small. The estimates (63), (64) and (66) imply that $\mathbf{H}$ is a pseudodifferential operator in the class $OPS_{sl}(p^{-1}, p)$ with symbol

$$\sigma_{\mathbb{H}}(x,\xi) = \rho^{jk}(x)\xi_j\xi_k + i\frac{\partial \rho^{jk}(x)}{\partial x_j}\xi_k + L(x).$$

The following theorem states conditions of the Fredholmness of the operator $\mathbf{H} : H(\mathbb{R}^n, p_h) \to H(\mathbb{R}^n, p_h^{-1})$.

Theorem 22. *Let conditions* (63)–(67) *hold, and assume there are constants* $R > 0$ *and* $C > 0$ *such that*

$$\Re\langle L(x)\varphi, \varphi\rangle_{\mathcal{H}} \geq \gamma\big\langle\big(T + \langle x\rangle^m I\big)\varphi, \varphi\big\rangle_{\mathcal{H}}, \ \ \gamma > 0 \quad (68)$$

for every $x \in B'_R$ *and every vector* $\varphi \in D_{T^{1/2}}$*. If the operator* $\mathbf{H} : H(\mathbb{R}^n, p_h) \to H(\mathbb{R}^n, p_h^{-1})$ *is locally Fredholm, then it is already a Fredholm operator.*

Proof. Conditions (67) and (68) imply that there exist $C > 0$ and $R > 0$ such that, for every $x \in B'_R$ and every $\varphi \in D_{T^{1/2}}$,

$$\Re\langle\sigma_{\mathbf{H}}(x,\xi)\varphi,\varphi\rangle_{\mathcal{H}} \geq C\Big\langle\big((|\xi|^2+\langle x\rangle^m)I+T\big)\varphi,\varphi\Big\rangle_{\mathcal{H}}. \tag{69}$$

It follows from estimate (69) that, for every $x \in B'_R$ and every $\psi \in \mathcal{H}$,

$$\Re\Big\langle\big((|\xi|^2+\langle x\rangle^m)I+T\big)^{-1/2}\sigma_{\mathbf{H}}(x,\xi)\big((|\xi|^2+\langle x\rangle^m)I+T\big)^{-1/2}\psi,\psi\Big\rangle_{\mathcal{H}} \geq \\ \geq C\|\psi\|^2_{\mathcal{H}}. \tag{70}$$

This estimate yields that the operator

$$\Big(\big(|\xi|^2+\langle x\rangle^m I\big)+T\Big)^{-1/2}\sigma_{\mathbf{H}}(x,\xi)\Big(\big(|\xi|^2+\langle x\rangle^m I\big)I+T\Big)^{-1/2}$$

is invertible on $\mathcal{H}$ for every $x \in B'_R$ and every $\xi \in \mathbb{R}^n$ and that

$$\sup_{(x,\xi)\in B'_R\times\mathbb{R}^n}\Big\|\big((|\xi|^2+\langle x\rangle^m)I+T\big)^{1/2}\sigma^{-1}_{\mathbf{H}}(x,\xi)\big((|\xi|^2+\langle x\rangle^m)I+T\big)^{1/2}\Big\|_{\mathcal{L}(\mathcal{H})} < \\ < C^{-1}. \tag{71}$$

Hence, the conditions of Theorem 18 are satisfied, and $\mathbf{H}$ has the Fredholm property as operator from $H(\mathbb{R}^n, p_h)$ to $H(\mathbb{R}^n, p_h^{-1})$. □

5.2. **Exponential estimates.**

Theorem 23. *Let*

$$\mathbf{H}u(x) = -\Delta u(x) + L(x)u(x) = f(x), \tag{72}$$

be the Schrödinger equation with potential $x \to L(x)$ satisfies conditions (63), (64) *and* (68). *Let $w(x) = \exp d\langle x\rangle^{\frac{m+2}{2}}$ be the weight, where*

$$d = \frac{\sqrt{\gamma}}{\frac{m}{2}+1} - \varepsilon, \quad \varepsilon > 0$$

and $f \in H(\mathbb{R}^n, p_h, w)$. Then every solution of the equation (72) a priory in the space $H(\mathbb{R}^n, p_h, w^{-1})$ a posteriori belongs to the space $H(\mathbb{R}^n, p_h, w)$.

Proof. We have

$$\Re\langle\sigma_{\mathbb{H}}(x,\xi+it\nabla v(x))\varphi,\varphi\rangle \geq \Big\langle\Big(|\xi|^2 - t^2d^2\Big(\frac{m}{2}\Big)^2\langle x\rangle^m\Big)I + L(x)\varphi,\varphi\Big\rangle \geq \\ \geq \Big\langle\Big(|\xi|^2 + \Big(\gamma - t^2d^2\Big(\frac{m}{2}\Big)^2\Big)\langle x\rangle^m\Big)I + T\varphi,\varphi\Big\rangle \geq \\ \geq C\Big\langle\Big(|\xi|^2+\langle x\rangle^m\Big)I + T\varphi,\varphi\Big\rangle, \tag{73}$$

for some $C > 0$ and for every $\varphi \in D_{T^{1/2}}$. As in the proof of Theorem 22, we conclude from (73) that

$$\sup_{(x,\xi,t)\in B'_R\times\mathbb{R}^n\times[-1,1]}\Big\|\big((|\xi|^2+\langle x\rangle^m)I+T\big)^{1/2}\sigma^{-1}_{\mathbb{H}}\big(x,\xi+it\nabla v(x)\big)\times \\ \times\big((|\xi|^2+\langle x\rangle^m)I+T\big)^{1/2}\Big\|_{\mathcal{L}(\mathcal{H})} < \infty.$$

Thus, all conditions of Theorem 21 are satisfied. □

5.3. **Quantum waveguides.** Let $\mathcal{D}$ be a bounded domain in $\mathbb{R}^m_y$ with a sufficiently regular boundary, and let Φ be a real valued function in the space $C^\infty(\Pi)$, where $\Pi = \mathbb{R}^n \times \mathcal{D}$. We suppose that for all β, γ there exist $C_{\beta\gamma} > 0$ such that

$$\left|\partial_x^\beta \partial_y^\gamma \Phi(x, y)\right| \leq C_{\beta\gamma} \langle x \rangle^{m-\delta|\beta|}, \quad \delta \in (0, 1]. \tag{74}$$

We consider the spectral problem for the Schrödinger equation in the quantum waveguide, i.e. the problem

$$\big((\mathbf{H} - \lambda I)u\big)(x, y) = \big(-\Delta_x - \Delta_y + \Phi(x, y) - \lambda\big)u(x, y) = 0, \tag{75}$$
$$(x, y) \in \mathbb{R}^n \times \mathcal{D} =: \Pi, \quad u\big|_{\partial\mathcal{D}} = 0, \quad k \in \mathbb{N}.$$

This problem describes the bound states of a quantum system with the electric potential Φ on the configuration space Π. We suppose that

$$\liminf_{x\to\infty} \inf_{y\in\mathcal{D}} \Phi(x, y)\langle x \rangle^{-m} \geq \gamma > 0. \tag{76}$$

The operator $\mathbf{H} - \lambda I$ can be realized as a pseudodifferential operator with operator-valued symbol $\sigma_{\mathbf{H}-\lambda I}(x, \xi) = |\xi|^2 I + L_\lambda(x)$, where

$$\big(L_\lambda(x)\varphi\big)(y) = \big(-\Delta_y + \tilde{\Phi}(x) - \lambda I\big)\varphi(y) \ \text{ for } \ y \in \mathcal{D}, \ \ \varphi\big|_{\partial\mathcal{D}} = 0$$

is the operator of the Dirichlet problem in $\mathcal{D}$ depending on the parameter $x \in \mathbb{R}^n$, where $(\tilde{\Phi}(x)\varphi)(x) := \Phi(x, y)\varphi(y)$ for $y \in \mathcal{D}$.

Let T be the operator of the Dirichlet problem for the Laplacian $-\Delta_y$ in the domain $\mathcal{D}$, considered as an unbounded operator on $\mathcal{H} = L^2(\mathcal{D})$ with domain $\mathring{H}^2(\mathcal{D}) = \{\varphi \in H^2(\mathcal{D}) : \ \varphi|_{\partial\mathcal{D}} = 0\}$ where $H^2(\mathcal{D})$ is the standard Sobolev space on $\mathcal{D}$. It is well-known that T is a positive define operator.

We set $p(x, \xi) = ((\xi^2 + \langle x \rangle^m)I + T)^{1/2}$. Then

$$\left\| p^{-1}(x, \xi)\partial_x^\beta \partial_\xi^\alpha \sigma_{\mathbf{H}-\lambda I}(x, \xi) p^{-1}(x, \xi) \right\|_{\mathcal{L}(L^2(\mathcal{D}))} \leq C_{\alpha\beta}$$

for all α, β. Hence $\sigma_{\mathbf{H}-\lambda I} \in S(p^{-1}, p)$. Moreover one can prove that condition (76) provides that $\sigma_{\mathbf{H}-\lambda I} \in S_{sl}(p^{-1}, p)$.

Let $H_h(\mathbb{R}, p)$ is the set of the distributions $u \in S'(\mathbb{R}^n, \mathcal{H})$ such that

$$\|u\|_{H_h(\mathbb{R}^n,p)} := \left\| (-h^2\Delta_x + \langle x \rangle^m + T)^{1/2} u \right\|_{L^2(\mathbb{R}^n,\mathcal{H})} < \infty,$$

where $h > 0$ is small enough such that $Op(h^2|\xi|^2 + \langle x \rangle^m + T)^{1/2}$ is invertible operator. One can prove that the $H_h(\mathbb{R}^n, p)$ within equivalent norms coincides with the closure of $C_0^\infty(\Pi)$ in the norm

$$\|u\|_{H(\mathbb{R}^n,p)} = \Big(\|u\|^2_{\mathring{H}^1(\Pi)} + \|\langle x \rangle^m u\|_{L^2(\Pi)} \Big)^{1/2}.$$

Consider now the problem of Fredholmness of the operator

$$\mathbf{H} - \lambda I : H_h(\mathbb{R}^n, p) \to H_h(\mathbb{R}^n, p^{-1}).$$

Theorem 24. *The operator* $\mathbf{H} - \lambda I : H_h(\mathbb{R}^n, p) \to H_h(\mathbb{R}^n, p^{-1})$ *is a Fredholm operator for every* $\lambda \in \mathbb{C}$.

Proof. It follows from standard local elliptic estimates for the Dirichlet problem in bounded domains that the operator $\mathbf{H} - \lambda I : H(\mathbb{R}^n, p_h) \to H(\mathbb{R}^n, p_h^{-1})$ is locally Fredholm. Conditions (76) implies condition (68) of Theorem 22. Hence $\mathbf{H} - \lambda I$ is locally invertible at infinity for every $\lambda \in \mathbb{C}$. It implies by Theorem 18 that $\mathbf{H} - \lambda I : H(\mathbb{R}^n, p_h) \to H(\mathbb{R}^n, p_h^{-1})$ is the Fredhom operator for every $\lambda \in \mathbb{C}$. □

Note that the operator $\mathbf{H}$ can be considered as an unbounded closed operator in $L^2(\Pi)$ with the domain $H(\mathbb{R}^n, p_h)$. Theorem 24 has the following corollary.

Corollary 25. *The operator* $\mathbf{H}$ *as unbounded has a discrete spectrum.*

Proof. Let $\lambda < \mu = \inf_\Pi \Phi(x, y)$. Then $\mathbf{H} - \lambda I : H(\mathbb{R}^n, p_h) \to H(\mathbb{R}^n, p_h^{-1})$ is inverible. Hence by the Theorem on the Analytic Fredholmness $\mathbf{H} - \lambda I : H(\mathbb{R}^n, p_h) \to H(\mathbb{R}^n, p_h^{-1})$ is invertible for all $\lambda \in \mathbb{R}$ except of a discrete set Λ of points λ for which $\ker(\mathbf{H} - \lambda I)$ has a finite dimension. Taking into account that the spectrum of $\mathbf{H}$ as unbounded operator coincides with the spectrum of $\mathbf{H}$ as a bounded operator acting from $H(\mathbb{R}^n, p_h)$ in $H(\mathbb{R}^n, p_h^{-1})$, and that $\mathbf{H} - \lambda I$ is a Fredholm operator as unbounded if and only if $\mathbf{H} - \lambda I : H(\mathbb{R}^n, p_h) \to H(\mathbb{R}^n, p_h^{-1})$ is a Fredholm operator we obtain the assertion of the corollary. □

Theorem 23 implies the exponential estimates of eigenfunctions of $\mathbf{H}$.

Theorem 26. *Every eigenfunction* u_λ *of the operator* $\mathbf{H}$ *belongs to* $H(\mathbb{R}^n, p_h, w)$, *where* $w(x) = \exp d\langle x\rangle^{\frac{m+2}{2}}$ *with*

$$d = \frac{\sqrt{\gamma}}{\frac{m}{2} + 1} - \varepsilon, \quad \varepsilon > 0.$$

In particular

$$\int_\Pi \left|u_\lambda(x, y)\right|^2 e^{2d\langle x\rangle^{\frac{m+2}{2}}} \, dx\, dy < \infty.$$

Example 27. Let the potential Φ be of the form

$$\Phi(x, y) = \Psi(x, y) + |x|^2,$$

where $\Psi \in C_b^\infty(\Pi)$. Hence (75) is a spectral problem for a perturbed Harmonic oscillator in the wavegide Π. In this case $p(x, \xi) = (1 + |\xi|^2 + |x|^2 + T)^{1/2}$. The unbounded operator $\mathbf{H}$ with domain $H(\mathbb{R}^n, p_h)$ has a discrete spectrum and the eigenfunctions u_λ satisfies the estimates

$$\int_\Pi |u_\lambda(x, y)|^2 e^{(1-\varepsilon)|x|^2} \, dx\, dy < \infty.$$

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