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THE DIRICHLET PROBLEM FOR VARIABLE EXPONENT SMIRNOV CLASS HARMONIC FUNCTIONS IN DOUBLY-CONNECTED DOMAINS

Dedicated to the memory of academician Niko Muskhelishvili

Abstract. In the present work we consider the Dirichlet problem in a doubly-connected domain D with an arbitrary piecewise smooth boundary Γ in a class of those harmonic functions which are real parts of analytic in D functions of Smirnov class $E^{p_1(t),p_2(t)}(D)$ with variable exponents $p_1(t)$ and $p_2(t)$. It is shown that depending on the geometry of Γ and the functions p_i, $i = 1, 2$, the problem may turn out to be uniquely and non-uniquely solvable or, generally speaking, unsolvable at all. In the latter case we have found additional (necessary and sufficient) conditions for the given on the boundary functions ensuring the existence of a solution. In all cases, where solutions do exist, they are constructed in quadratures.

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რეზიუმე. ნაშრომში უბან-უბან გლუვსაზღვრიან ორადბმულ D არეში ამოხსნილია დირიხლეს ამოცანა იმ ჰარმონიულ ფუნქციათა კლასში, რომლებიც წარმოადგენენ სმირნოვის ცვლადმაჩვენებლიანი $E^{p_1(t),p_2(t)}(D)$ კლასის ანალიზური ფუნქციების ნამდვილ ნაწილებს. გამოვლენილია საზღვრის რთული გეომეტრიული ბუნებისა და p_i, $i = 1, 2$, ფუნქციათა გავლენა ამოხსნადობის სურათზე. მაშინ, როცა სასაზღვრო პირობის ნებისმიერი მარჯვენა მხარისათვის ამოცანა არ არის ამოხსნადი, სასაზღვრო ფუნქციისათვის დადგენილია ის აუცილებელი და საკმარისი პირობა, რომელიც განაპირობებს ამოხსნადობას. ამოხსნადობის ყველა შემთხვევაში ამონახსნები აგებულია ეფექტურად.

1. Introduction

In boundary value problems of the theory of analytic functions natural sets for unknown functions are generalized Hardy classes, Smirnov classes $E^p(D)$ of analytic in the domain D functions Φ whose integral p-means along certain curves, converging to the boundary Γ of D, are uniformly bounded. This may be accounted for the fact that this class involves all bounded in D functions. Functions of this class possess angular boundary values on Γ and, what is of great importance, for $p \geq 1$ they are representable by the Cauchy type integral with density from the Lebesgue class $L^p(\Gamma)$. The above-said makes it possible to take boundary values from $L^p(\Gamma)$.

Recently, the theory of Lebesgue spaces with a variable exponent $p(t)$ and their applications is being elaborated very intensively [1]–[5]. Such spaces allow one to take much better into account local singularities of functions than for the constant p. Therefore it is advisable to solve boundary value problems under the assumption that the known boundary functions belong to more natural class $L^{p(\cdot)}(\Gamma)$. As following from the above, in [6]–[9] we studied boundary value problems of the theory of analytic functions in classes of functions representable by the Cauchy type integral with density from $L^{p(t)}(\Gamma)$. For the constant p and for Carleson curves Γ this class coincides with the Smirnov class $E^p(D)$ (see [10, p. 29]). In case of a variable exponent, the question on behavior of integral means of Cauchy type integrals near the boundary remained open. Thus there naturally arose a problem of introducing into consideration such generalized Smirnov classes with a variable exponent which, preserving important properties inherent in the functions from classes with a constant exponent, would possess boundary functions belonging to the Lebesgue class with a variable exponent.

This may probably be attained in different ways. In [11]–[12], we introduced natural, in our opinion, Hardy and Smirnov classes with a variable exponent in simply connected domains responding our purpose in view. In [13], we introduced the same, but this time the weighted classes among which are those for doubly-connected domains. In the latter case, Smirnov class depends on two functions prescribed on the curves composing the boundary of a domain. Therein, the Dirichlet problem is considered for harmonic functions which are real parts of analytic Smirnov class functions for a circular ring (such functions we call harmonic Smirnov class functions). The Dirichlet problem in the classes $E^{p(\cdot)}(D)$ for simply connected domains with arbitrary piecewise smooth boundaries is investigated in [14], where the problem is reduced to the Dirichlet problem for a ring in the weighted Hardy class with a weight not necessarily of power type, so conventional for such kind of problems. Generalizing the well-known Muskhelishvili's method of reducing the Riemann–Hilbert problem (which includes the Dirichlet problem in the statement under consideration) to the boundary value Riemann problem ([15, §§ 40–41]), we have succeeded in studying the appearing problem with that general weight and, as a result,

we have obtained a full picture of solvability of the Dirichlet problem in the class $\operatorname{Re} E^{p(\cdot)}(D)$ when the boundary D is an arbitrary piecewise smooth curve, and $p(t)$ is a function satisfying a certain Log-Hölder condition.

Certainly, it is desirable to solve the Dirichlet problem for doubly-connected domains, other than a circular ring. In the present work we solve the Dirichlet problem in doubly-connected domains with piecewise smooth boundaries in Smirnov classes $\operatorname{Re} E^{p_1(\cdot),p_2(\cdot)}(D)$. The problem is reduced to the problem for a circular ring in the weighted Hardy class. But in this case we will fail to reduce it to the Riemann problem (even in the case of a constant exponent we have to find another ways for its investigation (see, e.g., [16])). Having used the results obtained in [6], [12] and [14] and revealed new properties of conformal mapping of a circular ring onto a doubly-connected domain with piecewise smooth boundaries (see item 3^0), enable us to investigate the obtained problem and to get a full picture of the Dirichlet problem in view. Depending both on the geometry of the boundary and on the functions $p_1(t)$ and $p_2(t)$, the problem may turn out to be uniquely or non-uniquely solvable, and in the presence of cusps on the boundary it may, generally speaking, be unsolvable. When the problem fails to be solvable, necessary and sufficient conditions are found regarding the given boundary functions which guarantee the existence of a solution. Solutions are constructed for all cases in which they exist.

In our opinion, the novelty of the results of the present paper is a progress in the theory of plane boundary value problems in three directions: investigation of the problem in the framework of the variable exponent analysis, passage from simply- to doubly-connected domains and solution of the problem when boundaries of the domain have complicated geometrical structure.

It should be noted that the plane boundary value problems for analytic and harmonic functions were investigated in the framework of the classical function spaces in domains with nonsmooth boundaries by G. Iakovlev, I. I. Danilyuk, M. Dauge, V. Kondrat'ev, V. Maz'ya, V. Maz'ya and A. Solov'ev, V. Rabinovich and B. V. Schulze, N. Tarkhanov, R. Duduchava, A. Saginashvili and T. Latsabidze, R. Duduchava and B. S. Silberman, V. Kokilashvili and V. Paatashvili, G. Khuskivadze and V. Paatashvili, E. Gordadze, etc.

2^0. Notation, Definitions and Auxiliary Statements

2.1. **Lebesgue classes with a variable exponent.** Let $\Gamma = \{t \in \mathbb{C},\ t = t(s),\ 0 \le s \le l\}$ be a simple rectifiable curve whose equation is given with respect to the arc abscissa, and $p = p(t(s))$ be a positive measurable function given on Γ. If ω is measurable, almost everywhere different from zero function on Γ, then by $L^{p(\cdot)}(\Gamma, \omega)$ we denote the set of all measurable

on Γ functions f for which

$$\|f\|_{L^{p(\cdot)}(\Gamma;\omega)} = \inf\left\{\lambda > 0 : \int\limits_0^1 \Big|\frac{f(t(s))\omega(t(s))}{\lambda}\Big|\, dx \le 1\right\}$$

is finite.

2.2. **One Property of a Derivative of Conformal Mapping of a Circle Onto the Domain with a Piecewise Smooth Boundary.** If a simple closed piecewise smooth curve Γ is given with a finite number of angular points t_k, $k = \overline{1,n}$, at which angle values with respect to the domain D bounded by the curve Γ are equal to $\pi\nu_k$, $0 \le \nu_k \le 2$, then we write $\Gamma \in C_D^1(t_1,\dots,t_n;\nu_1,\dots,\nu_n)$. The set of the same curves, but piecewise Lyapounov ones, we denote by $C_D^{1,L}(t_1,\dots,t_n;\nu_1,\dots,\nu_n)$.

Let D be a doubly-connected domain bounded by simple closed piecewise smooth curves Γ_1 and Γ_2. Assume $\Gamma = \Gamma_1 \cup \Gamma_2$. If $t_1,\dots,t_n$ are all angular points on Γ with angle values $\pi\nu_k$ with respect to D, we likewise write $\Gamma \in C_D^1(t_1,\dots,t_n;\nu_1,\dots,\nu_n)$.

If D is the domain bounded by the curve $\Gamma \in C_D^1(t_1,\dots,t_n;\nu_1,\dots,\nu_n)$, $z = z(w)$ is a conformal mapping of the circle $U = \{w : |w| < 1\}$ onto the domain D, and $z(a_k) = t_k$, then

$$z'(w) \sim \prod_{k=1}^{n}(w-a_k)^{\nu_k-1}\exp\int\limits_\gamma \frac{\psi(\tau)}{\tau - w}\,d\tau, \quad w \in U, \tag{1}$$

where ψ is a real continuous function on Γ, and the writing $f \sim g$ denotes that $0 < m \le \inf|\frac{f}{g}| \le \sup|\frac{f}{g}| = M < \infty$ (see [10, p. 144]).

2.3. **The classes of functions $P(\Gamma)$, $\widetilde{P}(\Gamma)$ and $Q(\Gamma)$.** Let Γ be a simple closed rectifiable curve. By $P_{1+\varepsilon}(\Gamma)$, $\varepsilon \ge 0$, we denote the set of positive measurable on Γ functions p for which the following conditions are fulfilled:

(a) there exists a constant A depending on p and ε such that for any t_1, $t_2 \in \Gamma$ we have $|p(t_1) - p(t_2)| < A|\ln|t_1 - t_2||^{1+\varepsilon}$;
(b) $\inf p(t) = \underline{p} > 1$.

Assume $P(\Gamma) = P_1(\Gamma)$, $\widetilde{P}(\Gamma) = \bigcup\limits_{\varepsilon>0} P_{1+\varepsilon}(\Gamma)$.

The following statements are valid.

(i) Let D be the domain bounded by a simple closed rectifiable curve Γ, $z = z(w)$ be a conformal mapping of the circle U onto D, and $p \in P(\Gamma)$. Then if $z' \in \bigcup\limits_{\delta>1} H^\delta$, where H^δ is the Hardy class of analytic in U functions, then the function $l(\tau) = p(z(\tau))$, $\tau \in \gamma$, $\gamma = \{\tau : |\tau| = 1\}$ belongs to $P(\gamma)$. In particular, if Γ is a piecewise smooth curve free from zero angles, and $p \in P(\Gamma)$, then $l \in P(\gamma)$ ([13, Theorem 7.2]).

(ii) If $\Gamma \in C^1_D(t_1,\dots,t_n;\nu_1,\dots,\nu_n)$, $0<\nu_k\le 2$ and $p\in \widetilde{P}(\Gamma)$, then $l\in \widetilde{P}(\Gamma)$ ([8, Lemma 2]).

If D is the interior domain bounded by a simple closed rectifiable curve Γ and $z=z(w)$ is a conformal mapping of the circle U with the boundary γ onto the domain D, then we say that the given on Γ function p belongs to the class $Q(\Gamma)$ if $p\in \widetilde{P}(\Gamma)$ and $l\in \widetilde{P}(\Gamma)$, $l(\tau)=p(z(\tau))$, $\tau\in\gamma$.

2.4. **The set of weighted functions $W^{p(\cdot)}(\Gamma)$.** Let p be a given on Γ measurable function, and $\underline{p}>1$. We say that a measurable, almost everywhere different from zero on Γ function ω belongs to the class $W^{p(\cdot)}(\Gamma)$ if the Cauchy singular operator

$$S_\Gamma : f\to S_\Gamma f,\quad (S_\Gamma f)(t)=\frac{\omega(t)}{\pi i}\int\limits_\Gamma \frac{f(\tau)}{\omega(\tau)}\,\frac{d\tau}{\tau-t}\,,\ \ t\in\Gamma,$$

is continuous in $L^{p(\cdot)}(\Gamma)$.

Examples.

(1) Let $p\in P(\gamma)$, $a_k\in\gamma$, $k=\overline{1,n}$. We fix an arbitrary branch of the functions $(w-a_k)^{\alpha_k}$, $\alpha_k\in\mathbb{R}$, analytic in the plane cut along the curve lying outside of U and connecting the point a_k with $z=\infty$. The function

$$\rho(w)=\prod_{k=1}^{n}(w-a_k)^{\alpha_k}, \tag{2}$$

$$-\frac{1}{p(a_k)}<\alpha_k<\frac{1}{p'(a_k)}\,,\quad p'(t)=\frac{p(t)}{p(t)-1} \tag{3}$$

belongs to $W^{p(\cdot)}(\gamma)$ [17].

(2) If $z=z(w)$ is a conformal mapping of U onto the domain D bounded by the curve from $C^1_D(t_1,\dots,t_n;\nu_1,\dots,\nu_n)$, $0\le\nu_k\le 2$, and $z(a_k)=t_k$, then the function

$$\omega(\tau)=\prod_{k=1}^{n}(\tau-a_k)^{\alpha_k}\exp\left(\frac{1}{p(\tau)}\int\limits_\gamma\frac{\psi(\zeta)}{\zeta-\tau}\,d\zeta\right),\ \ \tau\in\gamma, \tag{4}$$

where ψ is the function appearing in the relation (1) for $p\in P(\gamma)$ and $0<\nu_k<p(t_k)$, belongs to $W^{p(\cdot)}(\gamma)$ [17].

2.5. **The classes of functions $E^{p(\cdot)}(D;\omega)$, $H^{p(\cdot)}(\omega)$, $e^{p(\cdot)}(D;\omega)$, $h^{p(\cdot)}(\omega)$ and $h^{p(\cdot)}$.** Let D be the interior domain bounded by a simple closed rectifiable curve Γ, p be a given on Γ measurable function, ω be measurable, almost everywhere different from zero function in D, and $z=z(w)$ be a conformal mapping of the circle U onto D. By $E^{p(\cdot)}(D;\omega)$ we denote the set of those analytic in D functions Φ for which

$$\sup_{0<r<1}\int_0^{2\pi}\left|\Phi(z(re^{i\vartheta}))\omega(z(re^{i\vartheta}))\right|^{p(z(e^{i\vartheta}))}\left|z'(re^{i\vartheta})\right|d\vartheta<\infty.$$

The class $E^{p(\cdot)}(U;\omega)$ we denote by $H^{p(\cdot)}(\omega)$. These classes for the constant p and $\omega = 1$ coincide with Smirnov $E^p(D)$ and Hardy H^p classes, respectively (for these classes see, e.g., [18], [19, Chs. IX–X]).

When $\omega(z)$ is a function which almost everywhere on Γ has angular boundary values $\omega^+(\tau)$, then the functions Φ of the class $E^{p(\cdot)}(D;\omega)$ have likewise angular boundary values almost everywhere on Γ, and $\Phi^+ \in L^{p(\cdot)}(\gamma;\omega^+)$ [14].

Assume

$$e^{p(\cdot)}(D;\omega) = \Big\{u:\ u = \operatorname{Re}\Phi,\ \Phi \in E^{p(\cdot)}(D;\omega)\Big\},$$
$$h^{p(\cdot)}(\omega) = e^{p(\cdot)}(U;\omega), \quad h^{p(\cdot)} = h^{p(\cdot)}(1).$$

2.6. **The classes of functions $E^{p_1(\cdot),p_2(\cdot)}(D)$ and $e^{p_1(\cdot),p_2(\cdot)}(D)$ in doubly-connected domains D.** Let Γ_i, $i = 1, 2$, be simple closed rectifiable curves, where Γ_2 lies in the bounded domain with boundary Γ_1. By D we denote a doubly-connected domain with boundary $\Gamma = \Gamma_1 \cup \Gamma_2$. This domain can be conformally mapped onto the circular ring $K = \{w:\ \rho < |w| < 1\}$ (see, for e.g., [19, p. 207]). Let $w = w(z)$ be such a mapping and $z = z(w)$ be the inverse mapping.

Let p_i be a positive measurable on Γ_i, $i = 1, 2$, function. Assume $p_1(\vartheta) = p_1(z(e^{i\vartheta}))$, $p_2(\vartheta) = p_2(z(\rho e^{i\vartheta}))$. Further, let ω be the different from zero almost everywhere in D measurable function.

We say that the analytic in D function Φ belongs to the class $E^{p_1(\cdot),p_2(\cdot)}(D)$, if for any $r_0 \in (\rho;1)$ we have

$$\Bigg(\sup_{\rho<r<r_0}\int_0^{2\pi}\big|(\Phi\omega)(z(re^{i\vartheta}))\big|^{p_2(\vartheta)}\big|z'(re^{i\vartheta})\big|\,d\vartheta + \sup_{r_0<r<1}\int_0^{2\pi}\big|(\Phi\omega)(z(re^{i\vartheta}))\big|^{p_1(\vartheta)}|z'(re^{i\vartheta})|\,d\vartheta\Bigg) < \infty.$$

For $D = K$ and $p_1 = p_2 = p = \text{const} > 0$, $\omega = 1$, the class $E^{p,p}(K;1)$ coincides with the class $E^p(K)$ (for these classes, see [20]). If D is a doubly-connected domain bounded by simple rectifiable curves and $z = z(w)$ is a conformal mapping of the ring K onto D, then $z' \in E^1(K)$ (see, e.g., [16]).

Assume $E^{p_1(\cdot),p_2(\cdot)}(D) = E^{p_1(\cdot),p_2(\cdot)}(D;1)$. If $\Phi \in E^{p_1(\cdot),p_2(\cdot)}(D)$ and $p_i > 0$. Then $E^{p_1(\cdot),p_2(\cdot)}(D) \subset E^{p_0}(D)$, $p_0 = \min p_i$ and $\Phi^+(t) \in L^{p_i}(\Gamma_i)$, $t \in \Gamma_i$. Next, if $p_i \in P(\Gamma_i)$, then

$$\Phi(z) = \Phi_1(z) + \Phi_2(z), \quad \Phi_i \in E^{p_i(\cdot)}(D_i), \tag{5}$$

where D_i is the domain bounded by the curve Γ_i which contains D, and vice versa, if Φ is representable in the form (5), then $\Phi \in E^{p_1(\cdot),p_2(\cdot)}(D)$ ([13, Lemma 6.2]).

Let

$$e^{p_1(\cdot),p_2(\cdot)}(D)=\big\{u:\ u=\operatorname{Re}\Phi(z),\ \Phi\in E^{p_1(\cdot),p_2(\cdot)}(D)\big\}.$$

It follows from the above-said that if $u\in e^{p_1(\cdot),p_2(\cdot)}(D)$, then $u=u_1+u_2$, where $u_i\in e^{p_i}(D_i)$, $i=1,2$, and vice versa if $u=u_1+u_2$, $u_i\in e^{p_i(\cdot)}(D_i)$, then $u\in e^{p_1(\cdot),p_2(\cdot)}(D)$.

2.7. **Some auxiliary statements.**

Theorem A ([13, Theorem 6.3]). *Let D be a doubly-connected domain with the boundary $\Gamma=\Gamma_1\cup\Gamma_2$, $\Gamma\in C_D^{1,L}(t_1,\dots,t_n;\nu_1,\dots,\nu_n)$, $0<\nu_k\le 2$, $k=\overline{1,n}$. Then if $f_i\in L^{p_i(\cdot)}(\Gamma)$, then the Cauchy type integral*

$$\Phi(z)=(K_{\Gamma_1}f_1)(z)+(K_{\Gamma_2}f_2)(z)=\frac{1}{2\pi i}\int\limits_{\Gamma_1}\frac{f_1(t)}{t-z}\,dt+$$
$$+\frac{1}{2\pi i}\int\limits_{\Gamma_2}\frac{f_2(t)}{t-z}\,dt\,,\quad z\in D_1,$$

belongs to $E^{p_1(\cdot),p_2(\cdot)}(D)$.

Theorem B ([6, Theorem 6.1]). *Let $\rho\in W^{p(\cdot)}(\gamma)$, $\frac{1}{\rho}\in L^{p'(\cdot)+\varepsilon}(\gamma)$, $\varepsilon>0$, and φ be a real continuous function. Then the function $\rho(t)\exp\int\limits_\gamma\frac{\varphi(\tau)}{\tau-t}\,d\tau$, $t\in\gamma$, belongs to $W^{p(\cdot)}(\gamma)$.*

Corollary. *Let $p\in P(\gamma)$, $\chi_\varphi(t)=\exp\int\limits_\gamma\frac{\varphi(\tau)}{\tau-t}\,d\tau$, $t\in\gamma$, where φ is a real continuous function on γ, ρ be given by the equality (2) and the conditions (3) be fulfilled. Then the function*

$$\rho(t)\chi_\varphi(t)=\prod_{k=1}^{n}(t-a_k)^{\alpha_k}\chi_\varphi(t),\quad t\in\gamma,\quad -\frac{1}{p(\alpha_k)}<\alpha_k<\frac{1}{p'(\alpha_k)}\,,$$

belongs to $W^{p(\cdot)}(\gamma)$.

Theorem C ([12, Theorem 3]). *If $\Phi\in H^{p(\cdot)}$, $\underline{p}>0$, and $\Phi^+\in L^{p_1(\cdot)}(\gamma)$, $p_1\in P(\gamma)$, then $\Phi\in H^{\widetilde{p}(\cdot)}$, $\widetilde{p}(t)=\max(p(t),p_1(t))$.*

2.8. **The Dirichlet problem in the class $h^{p(\cdot)}(\omega)$.**

Theorem D ([14, Theorem 3]). *Let $p\in P(\gamma)$ and ω be given by the equality (4), where*

$$-\frac{1}{p(a_k)}<\alpha_k<\frac{1}{p'(a_k)}\,,\quad k=\overline{1,m},$$
$$\frac{1}{p'(a_k)}<\alpha_k<\frac{1}{p'(a_k)}+1,\quad k=\overline{m+1,m+j},$$
$$\alpha_k=\frac{1}{p'(a_k)}\,,\quad k=\overline{m+j+1,m+j+s},$$

$$\alpha_k = -\frac{1}{p(a_k)}, \quad k = \overline{m+j+s+1, n}.$$

Then for the Dirichlet problem

$$\begin{cases} \Delta u = 0, & u \in h^{p(\cdot)}(\omega), \\ u^+(t) = f(t), & t \in \gamma, \quad f \in L^{p(\cdot)}(\gamma;\omega), \end{cases}$$

to be solvable, it is necessary and sufficient that the conditions

$$\frac{\omega_1(s)}{\pi i} \int\limits_\gamma \frac{g(\tau)}{\omega_1(\tau)} \frac{d\tau}{\tau - \zeta} \in L^{p(\cdot)}(\gamma), \quad g = f\omega, \tag{6}$$

be fulfilled; here

$$\omega_1(\zeta) = \rho(\zeta)\omega(\zeta), \quad \rho(w) = \prod_{k=m+1}^{m+j+s} (w - a_k)^{-1} \ \text{ if } \ j + s \geq 1$$

and

$$\rho(w) = 1 \ \text{ if } \ j + s = 0.$$

If (6) *holds, then a general solution is given by the equality*

$$u(w) = u_0(w) + u_f(w), \tag{7}$$

where

$$u_0(w) = \sum_{k=m+1}^{m+j+s} M_k(p) \operatorname{Re} \frac{a_k + w}{a_k - w}, \tag{8}$$

$$M_k(p) = \begin{cases} 0 \ \text{ when } \ m+j<k\leq m+j+s, \ \text{ if } \ X_k \overline{\in} H^{p(\cdot)}, \\ X_k(w) = (w - a_k)^{-\frac{1}{p(a_k)}} \chi_\varphi(w), \\ M_k \ \text{ is an arbitrary constant} \\ \text{for } \ m + 1 \leq k \leq k + j \ \text{ and} \\ m + j < k \leq m + j + s, \ \text{ but } \ X_k \in H^{\rho(\cdot)}, \end{cases} \tag{9}$$

$$u_f(w) = \operatorname{Re}\left[\frac{1}{\rho(w)} \left(\frac{1}{2\pi i} \int\limits_\gamma \frac{f(\zeta)\rho(\zeta)}{\zeta - w}\, d\zeta - \right.\right.$$

$$\left.\left. - \frac{(-1)^{j+1} w^{j+1}}{2\pi i} \prod_{k=m+1}^{m+j} a_k \int\limits_\gamma \frac{f(\zeta)}{\zeta} \frac{\overline{\rho(\zeta)}}{\zeta - w}\, d\zeta \right)\right], \tag{10}$$

$$\prod_{k=m+1}^{m+j} a_k = 1 \ \text{ if } \ j = 0.$$

2.9. **The classes of functions $H^{p(\cdot)}(C\overline{U}_\rho;\widetilde{\omega})$ and the Dirichlet Problem in $H^{p(\cdot)}(C\overline{U}_\rho;\widetilde{\omega})$.** Let $U_\rho=\{w:\ |w|<\rho\}$, $\rho\in(0,1)$ and $C\overline{U}_\rho=\mathbb{C}\setminus U_\rho$. Assume

$$\widetilde{\omega}(w)=\prod_{k=1}^{n}\Big(\frac{w-a_k}{w-w_0}\Big)^{\alpha_k}\chi_\varphi(w),\ \ a_k\in\gamma_\rho,$$

$$\gamma_\rho=\{w:|;|w|=\rho\},\ \ w_0\in U,\ \ w\in C\overline{U}_\rho,\ \ \operatorname{Im}\varphi=0,\ \ \varphi\in C(\gamma_\rho).$$

We say that an analytic in $C\overline{U}_\rho$ function Φ belongs to the class $H^{p(\cdot)}(C\overline{U}_\rho;\widetilde{\omega})$ if

$$\sup_{r>\rho}\int_0^{2\pi}\big|\Phi(re^{i\vartheta})\widetilde{\omega}(re^{i\vartheta})\big|^{p(\rho e^{i\vartheta})}=\sup_{r>\rho}\int_{|\zeta|=r}\big|\Phi(\zeta)\widetilde{\omega}(\zeta)\big|^{p(\zeta)}\,d\zeta<\infty.$$

For the Dirichlet problem

$$\begin{cases}\Delta u=0, & u\in h^{p(\cdot)}(C\overline{U}_\rho;\widetilde{\omega})=\operatorname{Re}H^{p(\cdot)}(C\overline{U}_\rho;\widetilde{\omega}),\\ u^+(t)=f(t), & t\in\gamma_\rho,\ \ f\in L^{p(\cdot)}(\gamma_\rho;\widetilde{\omega}^+),\end{cases}$$

the analogue of Theorem D is valid. However, due to the fact that the functions of the class $H^{p(\cdot)}(C\overline{U}_\rho;\widetilde{\omega})$ must vanish at infinity, the condition

$$\int_0^{2\pi}f(\rho e^{i\vartheta})\,d\vartheta=0 \tag{11}$$

is to be fulfilled, and hence a general solution of the homogeneous problem is, this time, given by the equality

$$u_0(w)=\sum_{k=m+1}^{m+j+s}M_k(p)\operatorname{Re}\frac{w+a_k}{w-a_k},$$

where $M_k(p)$ are defined according to (9), but with the additional condition

$$\sum_{k=m+1}^{m+j+s}M_k(p)=0. \tag{12}$$

3^0. On The Conformal Mapping of a Circular Ring onto a Doubly-Connected Domain with a Piecewise Smooth Boundary

In [16], we proved the following

Statement 1. *If D is a doubly-connected domain with the boundary $\Gamma=\Gamma_1\cup\Gamma_2$, $\Gamma\in C^1_D(t_1,\dots,t_n;\nu_1,\dots,\nu_n)$, $0\le\nu_k\le 2$, $k=\overline{1,n}$ and $z=z(w)$ is a conformal mapping of the ring $K=\{w:\ \rho<|w|<1\}$ onto D such that $z(a_k)=t_k$, $z(\gamma_i)=\Gamma_i$, $\gamma_1=\{t:\ |t\|=1\}$, $\gamma_2=\{t:\ |t|=\rho\}$, then*

$$z'(w)\sim\prod_{k=1}^{n}(w-a_k)^{\nu_k-1}z_0(w),$$

where

$$[z_0(w)]^{\pm 1} \in \bigcap_{\delta>1} E^{\delta}(K),$$

$$[z_0(e^{i\vartheta})]^{\pm 1} \in \bigcap_{\delta>1} W^{\delta}(\gamma_1), \quad [z_0(\rho e^{i\vartheta})]^{\pm 1} \in \bigcap_{\delta>1} W^{\delta}(\gamma_2).$$

In addition to the above statement we first prove that the statement below is valid.

Statement 2. *Under the assumptions of Statement 1, for every point t_k, $k = \overline{1,n}$, there exists a real continuous on γ function φ_k such that in some subdomain $G_k \subset K$ containing the point $a_k = w(t_k)$ and with the boundary having with the boundary $\gamma_1 \cup \gamma_2$ of the ring K a common arc l_k, $a_k \in l_k$, we have*

$$z'(w) \sim (w - a_k)^{\nu_k - 1} \chi_{\varphi_k}(\zeta_k(w)),$$

where $\zeta = \zeta_k(w)$ is a conformal mapping of the circle U onto G_k.

Proof. Let G_k be such a subdomain of the ring K which is bounded by a closed Lyapounov curve Λ_k having one arc l_k, common with the boundary of K, where $a_k \in l_k$, and l_k does not contain points from the set $\{a_1, \dots, a_n\}$, except the point a_k. Consider the restriction of the function $z(w)$ on G_k and denote it by $z_k(w)$. Then z_k is a conformal mapping of G_k onto some simply connected domain D_k, the subdomain of D containing the point t_k but not containing another angular points of Γ. Let $w = w_k(\zeta)$ be a conformal mapping of U onto G_k, and $w_k(b_k) = a_k$; then $z_k(w_k(\zeta))$ is a conformal mapping of U onto D_k. Since the boundary of the domain D_k is a piecewise smooth curve with one angular point t_k, therefore $z_k'(w_k(\zeta)) \sim (\zeta - b_k)^{\nu_k - 1} \widetilde{z}_0(\zeta)$, where

$$\widetilde{z}_0(\zeta) \sim \chi_{\varphi_k}(\zeta), \ \ \operatorname{Im} \varphi_k = 0, \ \ \varphi_k \in C(\gamma)$$

(see item 2.4). Thus we find that

$$z_k'(w) \sim \big(\zeta_k(w) - \zeta_k(a_k)\big) \chi_{\varphi_k}(\zeta_k(w)).$$

Since $w_k(\zeta)$ is a conformal mapping of the circle U onto the domain G_k with Lyapounov boundary, therefore $\zeta_k(w) - \zeta_k(a_n) \sim (w - a_k)\zeta_{k_0}(w)$, where ζ_{k_0} belongs to the Hölder class, and $\zeta_{k_0}(a_k) \neq 0$ (see, e.g., [10, p. 146]). Consequently,

$$z_k'(w) \sim (w - a_k)^{\nu_k - 1} \chi_{\varphi_k}(\zeta_k(w)). \qquad \square$$

Theorem 1. *Let D be a doubly-connected domain with the boundary $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma \in C_D^1(t_1, \dots, t_n; \nu_1, \dots, \nu_n)$, $z = z(w)$ conformally map the ring $K = \{w : \ \rho < |w| < 1\}$ with the boundary $\gamma = \gamma_1 \cup \gamma_2$, $\gamma_1 = \{\tau : \ |\tau| = 1\}$, $\gamma_2 = \{\tau : \ |\tau| = \rho\}$ onto the domain D, and $z(a_k) = t_k$. Then*

$$z'(w) \sim \prod_{k=1}^{n} (w - a_k)^{\nu_k - 1} \exp \int\limits_{\gamma_1} \frac{\psi_1(\tau)}{\tau - w}\, d\tau \exp \int\limits_{\gamma_2} \frac{\psi_2(\tau)}{\tau - w}\, d\tau\,, \qquad (13)$$

where ψ_i is a continuous real function on γ_i, $i = 1, 2$.

Proof. Consider first only the points t_k lying on Γ_1. Let G_k be subdomains of the ring K with Lyapounov boundaries constructed when proving Statement 2. If $\zeta_k = \zeta_k(w)$ maps conformally the circle U onto G_k, $\varphi_k(t) = \varphi_k(\zeta_k(\tau))$, then

$$\int_\gamma \frac{\varphi_k(\tau)}{\tau - \zeta_k(w)}\, d\tau = \int_{\Lambda_k} \frac{\varphi_k(t)\zeta_k'(t)}{\zeta_k(t) - \zeta_k(w)}\, dt =$$
$$= \int_{\Lambda_k} \varphi_k(t)\Big[\frac{\zeta_k'(t)}{\zeta_k(t)-\zeta_k(w)} - \frac{1}{t-w}\Big]\, dt + \int_{\Lambda_k} \frac{\varphi_k(t)}{t-w}\, dt =$$
$$= I_{k1}(w) + I_{k2}(w).$$

Since ζ_k' is Hölder continuous in U and $|\zeta(t) - \zeta(w)| \geq m|t - w|$, it is not difficult to show that

$$\Big|\frac{\zeta_k'(t)}{\zeta_k(t) - \zeta_k(w)} - \frac{1}{t-w}\Big| \leq \frac{M}{|t-w|^\lambda}\,, \quad \lambda < 1,$$

(analogously to the reasoning in [21, p. 18], where the difference $\frac{\zeta'(s)}{\zeta(s)-\zeta(s_0)} -$ ctg $\frac{s-s_0}{2}$ is considered); thus we have that $(\exp I_{k_1})^{\pm 1}$ are bounded functions. Consequently,

$$z'(w) \sim (w-a)^{\nu_k - 1}\chi_{\varphi_k}(w), \quad w \in G_k.$$

The curve Λ_k contains the arc $l_k \in \gamma_1$, hence

$$\int_{\Lambda_k} \frac{\varphi_k(t)}{t-w}\, dt = \int_{l_k} \frac{\varphi_k(t)}{t-w}\, dt + \int_{\Lambda_k - l_k} \frac{\varphi_k(t)}{t-w}\, dt.$$

The second summand here is a bounded in the subdomain $G_k' \subset G_k$ function adjoining to γ_1 along the curve $l_k' \subset l_k$, such that $\overline{l_k'} \subset l_k$. Therefore in G_k' we have

$$z_k'(w) \sim (w - a_k)^{\nu_k - 1}\chi_{\varphi_k}(w),$$

where we recall that

$$\chi_{\varphi_k}(w) = \exp \int_\gamma \frac{\varphi_k(t)}{t-w}\, dt.$$

The domains G_k and G_k' can be chosen in such a way that the following conditions are fulfilled:

(i) $\cup G_k'$ covers a one-sided neighborhood $G^{(1)}$ of the curve γ_1, i.e., a set of those $w \in K$ for which $1 - \delta < |w| < 1$;

(ii) the curves l_k intersect only with the curves l_{k-1} and l_{k+1} ($l_0 = l_{n_1}$, $l_{n_1+1} = l_1$), where n_1 is the number of points a_k lying on γ_1. Then

$$z'(w) \sim \prod_{a_k \in \gamma_1} (w - a_k)^{\nu_k - 1} \exp \int_{l_k} \frac{\varphi_k(t)}{t-w}\, dt, \quad w \in G^{(1)}.$$

Assume

$$\psi_1(t) =$$
$$= \begin{cases} \varphi_k(t), & t\in l_k\setminus\widetilde{l}_k,\ \widetilde{l}_k=l_k\setminus\big[(l_k\cap l_{k-1}\cup(l_k\cap l_{k+1})\big], \\ \varphi_k(t)+\varphi_{k-1}(t), & t_k\in l_k\setminus(l_k\cap l_{k-1}), \\ \varphi_k(t)+\varphi_{k+1}(t), & t_k\in l_k\setminus(l_k\cap l_{k+1}), \end{cases} \qquad t\in\gamma_1.$$

Then for $n_1>0$ we have

$$\prod_{k=1}^{k_1}\exp\int\limits_{l_k}\frac{\varphi_k(t)}{t-w}\,dt\sim\exp\int\limits_{\gamma_1}\frac{\psi_1(t)}{t-w}\,dt,\ \ w\in G^{(1)}.$$

Obviously, ψ_1 is continuous on γ_1. Thus

$$z'(w)\sim\prod_{a_k\in\gamma_1}(w-a_k)^{\nu_k-1}\exp\int\limits_{\gamma_1}\frac{\psi_1(t)}{t-w}\,dt,\ \ w\in G^{(1)}. \tag{14_1}$$

Analogously, we prove the existence of the neighborhood $G^{(2)}$ of the curve γ_2 and of the function $\psi_2\in C(\gamma_2)$ such that

$$z'(w)\sim\prod_{a_k\in\gamma_2}(w-a_k)^{\nu_k-1}\exp\int\limits_{\gamma_2}\frac{\psi_2(t)}{t-w}\,dt,\ \ w\in G^{(2)}. \tag{14_2}$$

The validity of the theorem follows from the above-proven relations (14_1)–(14_2). □

4^0. The Dirichlet Problem in the Class $e^{p_1(\cdot),p_2(\cdot)}(D)$; its Reduction to the Problem in the Class $h^{l_1(\cdot),l(\cdot)}(K;\omega)$

4.1. **Statement of the problem.** Let D be a doubly-connected domain bounded by simple closed curves Γ_1 and Γ_2, where Γ_2 lies in the bounded domain with boundary Γ_1.

We assume that $\Gamma\in C^1_D(t_1,\dots,t_n;\nu_1,\dots,\nu_n)$ and $p_i\in Q(\Gamma_i)$, $i=1,2$.

Consider the Dirichlet problem formulated as follows: find a harmonic function $\mathcal{U}(z)$ satisfying the conditions

$$\begin{cases}\mathcal{U}(z)\in e^{p_1(\cdot),p_2(\cdot)}(D),\\ \mathcal{U}\big|_{\Gamma_i}=f_i,\ \ f_i\in L^{p_i(\cdot)}(\Gamma_i),\ \ i=1,2.\end{cases} \tag{15}$$

4.2. **Reduction of the problem** (15) **to the problem for a ring.**

Lemma 1. *If $\mathcal{U}(z)\in e^{p_1(\cdot),p_2(\cdot)}(D)$ and $p_i\in Q(\Gamma_i)$, then the function $u(w)=\mathcal{U}(z(w))$, where $z=z(w)$ is a conformal mapping of the ring K onto D, belongs to the class $h^{l_1(\cdot),l_2(\cdot)}(K;\omega)$, where $l_i(\zeta)=p_i(z(\zeta))$, $\zeta\in\gamma_i$,*

$$\omega(w)=\prod_{k=1}^{n}(w-a_k)^{\frac{\nu_k-1}{p(t_k)}}\exp\int\limits_{\gamma_1}\frac{\psi_1(\zeta)}{l_1(\zeta)}\,\frac{d\zeta}{\zeta-w}\,\exp\int\limits_{\gamma_2}\frac{\psi_2(\zeta)}{l_2(\zeta)}\,\frac{d\zeta}{\zeta-w}\,, \tag{16}$$

ψ_i, $i = 1, 2$, are the functions defined by means of z' appearing in (13), *$a_k = w(t_k)$, $p(t_k) = p_i(t_k)$, $t_k \in \Gamma_i$.*

Proof. Let $\mathcal{U}(z) = \operatorname{Re}\Phi(z)$, $\Phi \in E^{p_1(\cdot),p_2(\cdot)}(D)$. This implies that

$$I = \Bigg(\sup_{r_0<r<1} \int_0^{2\pi} \left|\phi(z(re^{i\vartheta}))\right|^{p_1(\vartheta)} \left|z'(re^{i\vartheta})\right| d\vartheta + \\ + \sup_{\rho<r\le r_0} \int_0^{2\pi} \left|\phi(z(re^{i\vartheta}))\right|^{p_2(\vartheta)} \left|z'(re^{i\vartheta})\right| d\vartheta \Bigg) < \infty,$$

where

$$p_1(\vartheta) = p_1(z(e^{i\vartheta})) \ \ \big(= (l_1(\zeta)), \ \ \zeta \in \gamma_1 \big), \\ p_2(\vartheta) = p_2(z(\rho e^{i\vartheta})) \ \ \big(= (l_2(\zeta)), \ \ \zeta \in \gamma_2 \big).$$

From Lemmas 4 and 5 of [14], it directly follows that the functions

$$\big(|z'(w)|^{\frac{1}{p_1(\vartheta)}} (\omega(w))^{-1}\big)^{\pm 1}, \ \ w = re^{i\vartheta},$$

are bounded in K. We now obtain

$$I = \Bigg(\sup_{r_0<r<1} \int_0^{2\pi} \left|\phi(z(re^{i\vartheta}))\omega(z(re^{i\vartheta}))\right|^{p_1(\vartheta)} d\vartheta + \\ + \sup_{\rho<r\le r_0} \int_0^{2\pi} \left|\phi(z(re^{i\vartheta}))\omega(z(re^{i\vartheta}))\right|^{p_2(\vartheta)} d\vartheta \Bigg) < \infty.$$

This means that $\Phi(z(re^{i\vartheta})) = \Phi(z(w)) \in H^{l_1(\cdot),l_2(\cdot)}(K;\omega)$ and since $\mathcal{U}(z(w)) = \operatorname{Re}\Phi(z(w))$, we have $u(w) \in h^{l_1(\cdot),l_2(\cdot)}(K;\omega)$. □

Corollary. *It follows from Lemma* 1 *that every solution $U(z)$ of the problem* (15) *generates a solution $u(w) = \mathcal{U}(z(w))$ of the problem*

$$\begin{cases} \Delta u = 0, & u \in h^{l_1(\cdot),l_2(\cdot)}(K;\omega), \\ u\big|_{\gamma_i} = g_i, & g_i(\tau) = f(z(\tau)), \ \ f_i \in L^{p_i(\cdot)}(\Gamma_i), \end{cases} \tag{17}$$

where $l_i(\tau) = p_i(z(\tau))$, $\tau \in \gamma_i$, and ω is the function given by the equality (16). *Conversely, if $u(w)$ is a solution of the problem* (17), *then $\mathcal{U}(z) = u(w(z))$ is a solution of the problem* (15).

Remark. The condition $p_i \in Q(\Gamma_i)$ allows us to conclude that $l_i(\zeta) = p_i(z(\zeta))$, $\zeta \in \gamma_i$, belongs to $P(\gamma_i)$, $i = 1, 2$.

5^0. Solution of the Problem (17)

5.1. **On some properties of solution of the problem** (17)**.** Under the assumptions adopted in item 4.1 regarding Γ and the function p_i, every function $u \in h^{l_1(\cdot),l_2(\cdot)}(K;\omega)$ is representable in the form $u = u_1 + u_2$, where $u_i \in h^{l_i(\cdot)}(K_i;\omega_i)$, $u_2(\infty) = 0$, and

$$\omega_1(w) = \prod_{a_k\in\gamma_1} (w-a_k)^{\frac{\nu_k-1}{l_1(a_k)}} \exp\int\limits_{\gamma_1} \frac{\psi_1(\zeta)}{l_1(\zeta)}\,\frac{d\zeta}{\zeta-w}\,, \tag{18_1}$$

$$w \in K_1,\ \ l_1(a_k) = p_1(t_k),\ \ a_k = w(t_k),$$

$$\omega_2(w) = \prod_{a_k\in\gamma_2} \Big(\frac{w-a_k}{w-w_0}\Big)^{\frac{\nu_k-1}{l_2(a_k)}} \exp\int\limits_{\gamma_2} \frac{\psi_2(\zeta)}{l_2(\zeta)}\,\frac{d\zeta}{\zeta-w}\,, \tag{18_2}$$

$$w \in K_2,\ \ |w_0| < \rho,\ \ l_2(a_k) = p_2(t_k).$$

According to Lemma 3 of [14], we have $\omega_i \in H^{l_i(\cdot)}(K_i)$. Moreover, if u_2 is representable by the Poisson integral, then

$$\int\limits_0^{2\pi} u_2(\rho e^{i\vartheta})\,d\vartheta = 0.$$

If now $u = u_1+u_2$ is a solution of the problem (17), then we can conclude that

$$\begin{cases} \Delta u_1 = 0, & u_1 \in h^{l_1(\cdot)}(K_1;\omega_1), \\ u_1^+(e^{i\vartheta}) = g_1(e^{i\vartheta}) - u_2(e^{i\vartheta}), & g_1 \in L^{l_1(\cdot)}(\Gamma_1;\omega_1^+), \end{cases} \tag{19_1}$$

and

$$\begin{cases} \Delta u_2 = 0, & u_2 \in h^{l_2(\cdot)}(K_2;\omega_2), \\ u_2^+(e^{i\vartheta}) = g_2(\rho e^{i\vartheta}) - u_1(\rho e^{i\vartheta}), & g_2 \in L^{l_2(\cdot)}(\Gamma_2;\omega_2^+). \end{cases} \tag{19_2}$$

5.2. **The problem** (17) **in case** γ_i **has points at which** $l_i(a_k) = \nu_k(z(a_k))$ **or** $\nu_k(z(a_k)) = 0$**.**

Lemma 2. *If* Γ *has points at which* $\nu_k = 0$*, then the problem* (17) *is, generally speaking, unsolvable.*

Proof. Let, for example, $t_0 \in \Gamma_1$, $\nu(t_0) = 0$ and $z(t_0) = a$. Then $\frac{\nu-1}{l(a)} = -\frac{1}{l(a)}$, and

$$\omega_1(w) = (w-a)^{-\frac{1}{l(a)}}\chi_{\varphi_1}(w),\ \ \varphi_1 = \psi_1/l_1.$$

Assume now that the statement of the lemma is invalid. Then for any fixed $g_2 \in L^{l_2(\cdot)}(\gamma_2;\omega_2^+)$ and any $g_1 \in L^{l_1(\cdot)}(\gamma_1;\omega_1^+)$ there exists a solution of the problem (17), $u = u_1 + u_2$, where $u_1 = \operatorname{Re}\Phi_1$, $\Phi_1 \in H^{l_1(\cdot)}(K_1;\omega_1) = H^{l_1(\cdot)}(\omega_1)$, and for u_1 the condition (19_1) is fulfilled. Thus $\Phi_1 = \omega_1^{-1}F_1$, $F_1 \in H^{l_1(\cdot)} \subset H^1$. But every function Φ of H^1 is representable by the

Schwarz integral with density $\operatorname{Re}\Phi$ (see, e.g., [22, p. 84]). This easily results in

$$\Phi(w)=\frac{1}{\pi i}\int\limits_{\gamma_i}\frac{\operatorname{Re}\Phi}{t-w}\,dt-\overline{\Phi(0)}. \tag{20}$$

Therefore

$$\Phi_1(w)=\frac{1}{\pi i}\int\limits_{\gamma_1}\frac{\operatorname{Re}\Phi_1(t)}{t-w}\,dt+\text{const}=\frac{1}{\pi i}\int\limits_{\gamma_1}\frac{g_1(t)-u_2(t)}{t-w}\,dt+\text{const}\,.$$

Since u_2 is differentiable on γ_1, the Cauchy type integral $K_{\gamma_1}u_2$ is a bounded function, and hence, it belongs to $H^{l_1(\cdot)}(\omega_1)$ because $\omega_1\in H^{l_1(\cdot)}$. The last statement follows directly from Theorem C. Thus since $\Phi_1=K_{\gamma_1}g_1-K_{\gamma_1}u_2+\text{const}$ and $\Phi_1\in H^{l_1(\cdot)}(\omega_1)$, it follows that $K_{\gamma_1}g_1\in H^{l_1(\cdot)}(\omega_1)$ for any $g_1\in L^{l_1(\cdot)}(\gamma_1;\omega_1^+)$. By virtue of Sokhotskiĭ–Plamelj formulas, we conclude that $S_{\gamma_1}g_1\in L^{l_1(\cdot)}(\gamma_1;\omega_1^+)$ for any $g_1\in L^{l_1(\cdot)}(\gamma_1;\omega_1^+)$. This implies that $\omega_1^+\in W^{l_1(\cdot)}(\gamma_1)$ [10, Theorem 3.3]. Moreover, $(\omega_1^+)^{-1}\sim(\tau-a)^{\frac{1}{l_1(a)}}(\chi_{\varphi_1}(\tau)^{-1}$, and hence $(\omega_1^+)^{-1}\in L^{l_1'(\cdot)+\varepsilon}(\gamma_1)$. Therefore according to Theorem B, we conclude that $\omega_1^+(w)\chi_\psi(w)$ belongs to $W^{l_1(\cdot)}(\gamma_1)$ for any real, continuous on γ_1 function ψ. Assuming $\psi=-\varphi_1$, we obtain $(\tau-a)^{-\frac{1}{l(a)}}\in W^{l_1(\cdot)}(\gamma_1)$, but this is impossible because $(\tau-a)^\alpha$ belongs to $W^{l_1(\cdot)}(\gamma_1)$ if and only if $\alpha\in\big(-\frac{1}{l(a)}\,,\frac{1}{l'(a)}\big)$ [17].

Similar contradiction is obtained under the assumption that $t_0\in\gamma_2$.

Thus the lemma is proved. □

Lemma 3. *If Γ has points t_k at which $\nu_k=p(t_k)$, then the problem* (17) *is, generally speaking, unsolvable.*

Proof. For the sake of simplicity, let there is only one point $t_{k_0}\in\Gamma_i$, such that $\nu(t_{k_0})=p_i(t_{k_0})$ and $w(t_{k_0})=a$, $a\in\gamma_i$. Then

$$\omega_i(w)=(w-a)^{\frac{1}{l_i'(a)}}\chi_{\varphi_i}(w),\quad\varphi_i=\psi_i/l_i.$$

Assume that the problem is solvable for any given $g_i\in L^{l_i(\cdot)}(\gamma_i;\omega_i^+)$, $i=1,2$. For its solution $u=u_1+u_2$ we have $u_i=\operatorname{Re}\Phi_i$, $\Phi_i\in H^{l_i(\cdot)}(K_i;\omega_i)$. Consider the functions $\Psi_i(w)=\Phi_i(w)(w-a)^{-1}$. In this case $\Psi_i\in H^{l_i(\cdot)}(\widetilde{\omega}_i)$,

$$\widetilde{\omega}_i(w)=\omega_i(w)(w-a)^{-1}\sim(w-a)^{-\frac{1}{l_i(a)}}\chi_{\varphi_i}(\omega). \tag{21}$$

Let $\widetilde{g}_i(\tau)=g_i(\tau)(\tau-a)^{-1}$, then $\widetilde{g}_i\in L^{l_i(\cdot)}(\gamma_i;\widetilde{\omega}_i^+)$, and when g_i runs through $L^{l_i(\cdot)}(\gamma_i;\omega_i^+)$, then $\widetilde{g}_i$ runs through $L^{l_i(\cdot)}(\gamma_i;\widetilde{\omega}_i^+)$. The functions $\widetilde{u}_i=\operatorname{Re}\Psi_i$ belong to $h^{l_1(\cdot)}(\widetilde{\omega}_i)$, and $\widetilde{u}=\widetilde{u}_1+\widetilde{u}_2$ is a solution of the problem

$$\begin{cases}\Delta\widetilde{u}=0, & u\in h^{l_1(\cdot),l_2(\cdot)}(\omega),\\ \widetilde{u}_i\big|_{\gamma_i}=\mu_i, & \mu_i\in L^{l_i(\cdot)}(\gamma_i;\omega_i^+),\end{cases}$$

where $\mu_i\sim\widetilde{g}_i$.

According to our assumption, this problem turns out to be solvable for any μ_i from $L^{l_i(\cdot)}(\gamma_i;\widetilde{\omega}_i)$. But this is impossible on account of Lemma 2. □

From Lemmas 2 and 3 it follows

Statement 3. *If* Γ *has angular points* t_k *for which* $\nu_k = p(t_k)$*, or* $\nu_k = 0$ $(p(t_k) = p_i(t_k),\ t_k \in \Gamma_i)$*, then the problem* (17) (*and hence Problem* (15)) *is, generally speaking, unsolvable.*

5.3. **A necessary condition for the solvability of the problem** (17)**.** From the results set out in items 2.8 and 2.9 it follows that for the conditions (19_1) and (19_2) to be fulfilled, it is necessary that the functions $g_i(\tau) = f_i(z(\tau))$ satisfy the conditions

$$\frac{\rho_i(\tau)\omega_i^+(\tau)}{\pi i}\int\limits_{\gamma_i}\frac{g_i(\zeta)}{\omega_i^+(\zeta)}\,\frac{d\zeta}{\zeta-\tau}\in L^{l_i(\cdot)}(\gamma_i),\quad i=1,2, \tag{22}$$

where

$$\omega_i(w)=\prod_{a_k\in\gamma_i}(w-a_k)^{\frac{\nu_k-1}{l_i(a_k)}}\chi_{\varphi_i}(w),\quad \varphi_i=\psi_i/l_i,\quad \rho_i(w)=\prod_{k\in T_i}(w-a_k)^{-1}$$

for $T_i = \{k:\ \nu_k \geq p_i(t_k),\ t_k \in \Gamma_i\} \neq \varnothing$, and $\rho_i(w) = 1$ for $T_i = \varnothing$.

Lemma 4. *If the conditions* (22) *are fulfilled, then* $g_i \in L^{l_i(\cdot)-\varepsilon}(\gamma_i)$*,* $\varepsilon > 0$.

Proof. Since $g_i(\tau) = f_i(z(\tau))$ and $f_i \in L^{l_i(\cdot)}(\Gamma_i)$, in view of the fact that $l_i \in \widetilde{P}(\gamma_i)$, we obtain $g_i \in L^{l_i(\cdot)}(\gamma_i;\omega_i^+)$, i.e., $g_i = q_i(\omega_i^+)^{-1}$, where $q_i \in L^{l_i(\cdot)}(\gamma)$. Since in the absence of the points from the set $\{t_k:\ \nu_k = p(t_k)$ or $\nu_k = 0,\ t_k \in \Gamma_i\}$ the functions $\omega_i^+ q_i$ belong to $W^{l_i(\cdot)}(\gamma_i)$, the conditions (22) are equivalent to the conditions

$$\lambda_i(\tau)=\frac{\omega_i^+(\tau)}{\pi i}\int\limits_{i}\frac{q_i(\zeta)}{\omega_i^+(\zeta)}\,\frac{d\zeta}{\zeta-\tau}\in L^{l_i(\cdot)}(\gamma_i),$$

where this time

$$\omega_i(w)=\prod_{k\in T_i'}(w-a_k)^{-\frac{1}{l_i(a_k)}}\exp\int\limits_{\gamma_i}\frac{\psi_i(\zeta)}{l_i(\zeta)}\,\frac{d\zeta}{\zeta-w}\,,$$
$$T_i'=\{k:\ \nu_k=p(t_k),\ \text{or}\ \nu_k=0,\ t_k\in\Gamma_i\},$$

whence

$$\frac{1}{\pi i}\int\limits_{\gamma_i}\frac{q_i(\zeta)}{\omega_i^+(\zeta)}\,\frac{d\zeta}{\zeta-\tau}=\frac{\lambda_i(\tau)}{\omega_i^+(\tau)}\,,\ \text{ i.e.,}\ \ S_{\gamma_i}\frac{q_i}{\omega_i^+}=\frac{\lambda_i}{\omega_i^+}\,.$$

The above equality yields $S_{\gamma_i}\left(S_{\gamma_i}\frac{q_i}{\omega_i^+}\right)=S_{\gamma_i}\frac{\lambda_i}{\omega_i^+}$. Since $\frac{q_i}{\omega_i^+}\in L^{l_i(\cdot)}(\gamma_i)\subset L^{\underline{l_i}(\gamma_i)}$ and $\underline{l_i}>1$, we have $S_{\gamma_i}\left(S_{\gamma_i}\frac{q_i}{\omega_i^+}\right)=\frac{\lambda_i}{\omega_i^+}$ (see, e.g., [21, p. 35–36]). Finally, we obtain $q_i=\omega_i^+ S_{\gamma_i}\frac{\lambda_i}{\omega_i^+}$. But then $g_i=q_i(\omega_i^+)^{-1}=S_{\gamma_i}\frac{\lambda_i}{\omega_i^+}\in L^{\underline{l_i}(\cdot)-\varepsilon}(\gamma_i)$ (because $\lambda_i\in L^{l_i(\cdot)}(\gamma_i)$, and $(\omega_i^+)^{-1}\in\bigcap\limits_{\delta>1}L^{\delta}(\gamma_i)$). □

5.4. **On the existence of a particular solution $u = u_1 + u_2$ of the problem** (17) **whose summands u_i are representable by the Poisson integral in the domains K_i.** Relying on Lemma 4, we will show that under the conditions (22) there exists a particular solution of the problem (17) admitting the representation $u = u_1 + u_2$, where u_i, $i = 1, 2$, are representable by the Poisson integrals in the domains K_i with density from $L^{\underline{l}_i - \varepsilon}(\gamma_i)$, if and only if

$$\int_0^{2\pi i} g_1(e^{i\vartheta})\, d\vartheta = \int_0^{2\pi i} g_2(e^{i\vartheta})\, d\vartheta. \tag{23}$$

Lemma 5. *If $u = u_1 + u_2$ is a solution of the problem* (17)*, where g_i, $i = 1, 2$, satisfy the condition* (22)*, then the functions u_i are representable by the Poisson integrals in domains K_i with density from $L^{\underline{l}_i - \varepsilon}(\gamma_i)$, $\varepsilon > 0$.*

Proof. Let us prove that $u_1 \in h^{\underline{l}_i - \varepsilon}(K_1)$. u_1 satisfies the condition (19_1). Without restriction of generality, we may assume that $\{t_k : \nu_k > p(t_k), t_k \in \gamma_1\} = \varnothing$. (Indeed, in the presence of such points we would be able to reduce the problem (17) to an analogous problem of the class $h^{l_1(\cdot), l_2(\cdot)}(\omega_*)$, where $\omega_* = \omega \prod\limits_{\{k:\, \nu_k > \mu(t_k)\}} (t - t_k)^{-1}$.) Now, according to Theorem D, u_1 is given by the equality

$$u_1(w) = \operatorname{Re}\left\{ \frac{\omega_1(w)}{2\pi i} \int_{\gamma_1} \frac{g_1(e^{i\vartheta}) - u_2(e^{i\vartheta})}{\omega_1^+(e^{i\vartheta})} \frac{\zeta + w}{\zeta - w}\, d\zeta \right\}, \quad \zeta = e^{i\vartheta}.$$

Since $u_2(e^{i\vartheta}) = u_2(\zeta)$ is differentiable on γ_1 and $\omega_1 \in H^{l_1(\cdot) - \varepsilon}$, we can easily verify that

$$\frac{\omega_1(w)}{2\pi i} \int_{\gamma_1} \frac{u_2(\zeta)}{\omega_1^+(\zeta)} \frac{\zeta + w}{\zeta - w}\, d\zeta \in H^{l_1(\cdot) - \varepsilon}.$$

Therefore, our lemma will be completed if we show that

$$(G_1(g_1))(w) = \omega_1(w) \int_{\gamma_1} \frac{g_1(\zeta)}{\omega_1^+(\zeta)} \frac{d\zeta}{\zeta - w} \in H^{l_1(\cdot) - \varepsilon}.$$

As $\omega_1 \in H^\eta$, $\eta > 0$, and the Cauchy type integral with density $\frac{g_1}{\omega_1^+}$ belongs to $\bigcap\limits_{\delta < 1} H^\delta$ (see, e.g., [18, p. 33]), we have $G_1(g_1) \in H^{\eta_0}$, $\eta_0 > 0$. Show that $G_1(g_1) \in H^{\underline{l}_i - \varepsilon}$. Towards this end, owing to the well-known Smirnov's theorem (see also Theorem C of item 2.7), it suffices to prove that $(G_1 g_1)^+ \in L^{\underline{l}_i - \varepsilon}(\gamma_1)$, and to this end, by virtue of the Sokhotskiĭ–Plemelj formulas and Lemma 4 it suffices to establish that $\omega_1^+ \int\limits_{\gamma_1} \frac{g_1}{\omega_1^+} \in L^{\underline{l}_1 - \varepsilon}$. But this function in view of (22) belongs to $L^{l_1(\cdot)}(\gamma_1) \subset L^{\underline{l}_1 - \varepsilon}(\gamma_1)$. Analogously, we can prove that $u_2 \in h^{\underline{l}_2 - \varepsilon}(K_2)$. □

5.5. **Integral equation with respect to the functions $u_2(\rho e^{i\vartheta})$.** To solve the problem (17), we refer to the condition (19_2). This problem will be considered as the Dirichlet problem in the class $h^{l_2(\cdot)}(K_2;\omega_2^+)$. To make use of the results of item 2.9, we have to fulfil the condition (11).

Lemma 6. *If the conditions* (22) *and* (23) *are fulfilled, and $u = u_1 + u_2$ is a solution of the problem* (17)*, then*

$$I = \int_0^{2\pi} \left[g_2(\rho e^{i\vartheta}) - u_1(\rho e^{i\vartheta})\right] d\vartheta = 0. \tag{24}$$

Proof. Since

$$u_1(\rho e^{i\vartheta}) = \frac{1}{2\pi} \int_0^{2\pi} u_1(e^{i\alpha}) \, \frac{1-\rho^2}{1+\rho^2-2\rho\cos(\alpha-\vartheta)} \, d\alpha,$$

we have

$$\begin{aligned} I &= \int_0^{2\pi} \left[g_2(\rho e^{i\vartheta}) - \frac{1}{2\pi} \int_0^{2\pi} u_1(e^{i\alpha}) \, \frac{1-\rho^2}{1+\rho^2-2\rho\cos(\alpha-\vartheta)} \, d\alpha \right] d\vartheta = \\ &= \int_0^{2\pi} g_2(\rho e^{i\vartheta}) \, d\vartheta - \int_0^{2\pi} u_1(e^{i\alpha}) \, \frac{1}{2\pi} \int_0^{2\pi} \frac{1-\rho^2}{1+\rho^2-2\rho\cos(\alpha-\vartheta)} \, d\vartheta \, d\alpha = \\ &= \int_0^{2\pi} g_2(\rho e^{i\vartheta}) \, d\vartheta - \int_0^{2\pi} u_1(e^{i\alpha}) \, d\alpha. \end{aligned}$$

Moreover, since $u_2 \in h^1(K_2)$, therefore $\int\limits_0^{2\pi} u_2(e^{i\alpha}) \, d\alpha = 0$. As far as $u = u_1 + u_2$ is a solution of the problem (17), we have $u_1(e^{i\alpha}) + u_2(e^{i\alpha}) = g_1(e^{i\alpha})$. Consequently, $\int\limits_0^{2\pi} u_1(e^{i\alpha}) \, d\alpha = \int\limits_0^{2\pi} g_1(e^{i\alpha}) \, d\alpha$, and by virtue of the assumption (23), we conclude that $I = 0$.

Thus assuming that the conditions (22) and (23) are fulfilled, we have the equality (24), and hence we are able to apply the results of item 2.9 according to which the problem (19_2) is solvable, and its solution u_2 is given by the formula $u_2 = u_f(\cdot)$, where u_f is given by the equality (10) in which f is replaced by $[g_2(\tau) - u_1(\tau)]$. Thus the restriction of the function $u_2(re^{i\vartheta})$ on γ_1 is contained in the set of functions

$$u_2(e^{i\vartheta}) + \sum_{k\in T_2} M_k(p) \operatorname{Re} \frac{e^{i\vartheta}+a_k}{e^{i\vartheta}-a_k} = u_2(e^{i\vartheta}) + u_{2,0}(\vartheta),$$
$$T_2 = \left\{k : \ \nu_k \geq p_2(t_k), \ t_k \in \Gamma_2\right\},$$

where

$$u_{2,0}(e^{i\vartheta}) = \sum_{k\in T_2} M_k(p)\,\mathrm{Re}\left[(e^{i\vartheta}+a_k)(e^{i\vartheta}-a_k)^{-1}\right],\quad \sum_{k\in T_2} M_k(p) = 0,$$

and the real constants $M_k(p)$ are defined by virtue of (9).

u_1 now is contained in the set of functions satisfying the condition

$$\begin{cases} u_1 \in h^{l_1(\cdot)}(K_1;\omega_1), \quad K_1 = U, \\ u_1(\tau) = g_1(\tau) - u_2(\tau) - \sum\limits_{k\in T_2} M_k(p)\,\mathrm{Re}\,\dfrac{\tau+a_k}{\tau-a_k}, \quad \tau\in\gamma_1. \end{cases} \tag{25}$$

In view of Theorem D, if for g_1 the condition (22) is fulfilled, then the problem (25) is solvable, and

$$u_1(w) = \sum_{k\in T_1} M_k(p)\,\mathrm{Re}\,\frac{w+a_k}{w-a_k} + u_{1\widetilde{g}}(w), \quad w\in U, \tag{26}$$

where $u_{1\widetilde{g}}$ is a particular solution of the problem (25) representable by the Poisson integral. Hence

$$\begin{aligned} u_{1\widetilde{g}}(re^{i\vartheta}) &= \frac{1}{2\pi}\int_0^{2\pi} g_1(e^{i\alpha})\,\frac{1-r^2}{1+r^2-2r\cos(\alpha-\vartheta)}\,d\alpha - \\ &\quad - \frac{1}{2\pi}\int_0^{2\pi} u_2(e^{i\alpha})\,\frac{1-r^2}{1+r^2-2r\cos(\alpha-\vartheta)}\,d\alpha - \\ &\quad - \sum_{k\in T_2}\frac{1}{2\pi}\int_0^{2\pi} M_k(p)\,\mathrm{Re}\,\frac{e^{i\alpha}+a_k}{e^{i\alpha}-a_k}\,\frac{1-r^2}{1+r^2-2r\cos(\alpha-\vartheta)}\,d\alpha = \\ &= (Pg_1)(r,\vartheta) - (Pu_2)(r,\vartheta) - (Pu_{2,0})(r,\vartheta), \end{aligned} \tag{27}$$

where

$$(Pu_{2,0})(r,\vartheta) =$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\sum_{k\in T_2} M_k(p)\,\mathrm{Re}\,\frac{e^{i\alpha}+a_k}{e^{i\alpha}-a_k}\,\frac{1-r^2}{1+r^2-2r\cos(\alpha-\vartheta)}\,d\alpha, \quad re^{i\vartheta}\in K_1.$$

This implies that in the ring K, we have

$$u(w) =$$

$$= \sum_{k\in T_1} M_k(p)\,\mathrm{Re}\,\frac{w+a_k}{w-a_k} - (Pu_2)(w) + (Pg_1)(w) - (Pu_{2,0})(w) + u_2(w).$$

Since u is a solution of the problem (17), therefore $u(\rho e^{i\vartheta}) = g_2(\rho e^{i\vartheta})$, and the last equality results in

$$u_2(\rho e^{i\vartheta}) - \frac{1}{2\pi}\int_0^{2\pi} u_2(e^{i\alpha})\,\frac{1-\rho^2}{1+\rho^2-2\rho\cos(\alpha-\vartheta)}\,d\alpha + (Pg_1)(\rho,\vartheta) -$$

$$- (Pu_{2,0})(\rho,\vartheta) + \sum_{k\in T_1} M_k(p)\,\mathrm{Re}\,\frac{\rho e^{i\vartheta}+a_k}{\rho e^{i\vartheta}-a_k} = g_2(\rho e^{i\vartheta}). \quad (28)$$

As far as $u_2 \in h^1(K_2)$, we have

$$u_2(re^{i\vartheta}) = \frac{1}{2\pi}\int_0^{2\pi} u_2(\rho e^{i\beta})\,\frac{r^2-\rho^2}{r^2+\rho^2-2r\rho\cos(\beta-\vartheta)}\,d\beta.$$

Hence

$$u_2(e^{i\alpha}) = \frac{1}{2\pi}\int_0^{2\pi} u_2(\rho e^{i\beta})\,\frac{1-\rho^2}{1+\rho^2-2\rho\cos(\beta-\vartheta)}\,d\beta.$$

Substituting this value into (28), we obtain

$$u_2(\rho e^{i\vartheta}) - \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{1}{2\pi}\int_0^{2\pi} u_2(\rho e^{i\beta})\,\frac{1-\rho^2}{1+\rho^2-2\rho\cos(\alpha-\beta)}\,d\beta\right]\times$$

$$\times\,\frac{1-\rho^2}{1+\rho^2-2\rho\cos(\alpha-\vartheta)}\,d\alpha + (Pg_1)(\rho,\vartheta) - (Pu_{2,0})(\rho,\vartheta) +$$

$$+ \sum_{k\in T_1} M_k(p)\,\mathrm{Re}\,\frac{\rho e^{i\vartheta}+a_k}{\rho e^{i\vartheta}-a_k} = g_2(\rho e^{i\vartheta}), \quad (29)$$

that is,

$$u_2(\rho e^{i\vartheta}) + (Nu_2)(\rho,\vartheta) = \widetilde{g}_2(\rho,\vartheta),$$

where

$$(Nu_2)(\rho,\vartheta) = \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{1}{2\pi}\int_0^{2\pi} u_2(\rho e^{i\beta})\,\frac{1-\rho^2}{1+\rho^2-2\rho\cos(\alpha-\beta)}\,d\beta\right]\times$$

$$\times\,\frac{1-\rho^2}{1+\rho^2-2\rho\cos(\alpha-\vartheta)}\,d\alpha, \quad (30)$$

$$\widetilde{g}_2(\rho,\vartheta) = g_2(\rho e^{i\vartheta}) - (Pg_1)(\rho,\vartheta) + (Pu_{2,0})(\rho,\vartheta) -$$

$$- \sum_{k\in T_1} M_k(p)\,\mathrm{Re}\,\frac{\rho e^{i\vartheta}+a_k}{\rho e^{i\vartheta}-a_k}\,.$$

The lemma is proved. □

5.6. **Solution of the equation** (29) **and construction of a solution of the problem** (17)**.** Since it suffices to find only one solution of the equation (29), we put $M_k(p) = 0$, $k \in T_1$. Then $\widetilde{g}_2$ satisfies the condition (22), and hence by Lemma 4, $g_2 \in L^{l_2-\varepsilon}$.

We will now proceed to investigating the equation (29).

We fix $\varepsilon > 0$ such that $\widetilde{l}_2 = \underline{l}_2 - \varepsilon > 1$ and consider the equation (29) in the space $L_2^{\widetilde{l}_2}(\gamma_2)$.

The kernel of the operator N is a continuous function, hence it is completely continuous in $L_2^{\widetilde{l}_2}(\gamma)$. The equations $u + Nu = 0$ and $v + N^*v = 0$ in the capacity of solutions have only constant functions [16]. Taking this fact into account, the equation (29) is solvable if and only if

$$\int_0^{2\pi} \widetilde{g}_2(\rho, \vartheta)\, d\vartheta = 0,$$

i.e.,

$$\int_0^{2\pi} (Pg_1)(\rho, \vartheta)\, d\vartheta + \sum_{k \in T_1} M_k(p) \int_0^{2\pi} \operatorname{Re} \frac{\rho e^{i\vartheta} + a_k}{\rho e^{i\vartheta} - a_k}\, d\vartheta - \\ - \int_0^{2\pi} (Pu_{2,0})(\rho, \vartheta)\, d\vartheta = \int_0^{2\pi} g_2(\rho e^{i\vartheta})\, d\vartheta. \quad (31)$$

It can be easily verified that

$$\int_0^{2\pi} (Pg_1)(\rho, \vartheta)\, d\vartheta = \int_0^{2\pi} g_1(e^{i\vartheta})\, d\vartheta,$$

and

$$\int_0^{2\pi} \frac{\rho e^{i\vartheta} + a_k}{\rho e^{i\vartheta} - a_k} \frac{de^{i\vartheta}}{i\rho e^{i\vartheta}} = \int_{\gamma_2} \frac{t + a_k}{t - a_k} \frac{dt}{it} = 1, \quad |a_k| = 1.$$

Moreover,

$$\int_0^{2\pi} (Pu_{2,0})\, d\vartheta = \\ = \int_0^{2\pi} \frac{1}{2\pi} \int_0^{2\pi} \sum_{k \in T_2} M_k(p) \operatorname{Re} \frac{e^{i\vartheta} + a_k}{e^{i\vartheta} - a_k} \frac{1 - \rho^2}{1 + \rho^2 - 2\rho \cos(\alpha - \vartheta)}\, d\alpha\, d\vartheta = \\ = \sum_{k \in T_2} M_k(p) \int_0^{2\pi} \operatorname{Re} \frac{e^{i\alpha} + a_k}{e^{i\alpha} - a_k}\, d\alpha = 0.$$

With regard for the above equations, the condition (31) takes the form

$$\int_0^{2\pi} g_1(e^{i\vartheta})\, d\vartheta - \sum_{k \in T_1} M_k(p) = \int_0^{2\pi} g_2(\rho e^{i\vartheta})\, d\vartheta.$$

It now suffices to take $M_k(p) = 0$, $k \in T_1$, and the last equality coincides with the condition (23).

Thus if the conditions (22) and (23) are fulfilled, and $M_k(p) = 0$, $k \in T_1$, then the condition (31) is likewise fulfilled, and hence the equation (29) is solvable. Since the homogeneous equation has in the capacity of solutions only constant functions, it follows that a solution satisfying the condition $\int_0^{2\pi} u_2(\rho e^{i\vartheta})\, d\vartheta = 0$ is unique. Further, let $u_2 = \operatorname{Re}\Phi_2$, $\Phi_2 \in H^{\tilde{l}_2}$. If we apply the formula (20) to the function Φ_2 and take into account that $u_2(\rho e^{i\vartheta}) = g_2(\rho e^{i\vartheta}) - u_1(\rho e^{i\vartheta})$, then making use of the condition (22) and the fact that $\omega_2 \in h^{l_2(\cdot)}$, we can conclude that $u_2 \in h^{l_2(\cdot)}(C\overline{U}_\rho; \omega_2)$.

Having the function $u_2(\rho e^{i\vartheta})$ at hand, by means of the Poisson integral we find the function $u_2(w)$, $|w| > \rho$, and hence the function $u_2(e^{i\alpha})$, as well. Next, the equality (27) allows us to find the function $u_{1\tilde{g}}$. In view of the condition (22) and taking into account the fact that $\omega_1 \in h^{l_1(\cdot)}$ (see proof of Lemma 2), from the equality (27) we conclude that $u_{1\tilde{g}} \in h^{l_1(\cdot)}(\omega_1)$. Having $u_{1\tilde{g}}$, by means of the equality (26) we find the function $u_1(w)$. It is not difficult to verify that if $u(w) = u_1(w) + u_2(w)$, where $u_i \in h^{l_i(\cdot)}(K_i; \omega_i)$, and ω_i are given by the equalities (18_i), then $u \in h^{l(\cdot)}(K; \omega_1\omega_2) = h^{l(\cdot)}(K; \omega)$, in which ω is the function defined by the equality (16). Thus we have proved that u is a solution of the problem (17) of the class $h^{l(\cdot)}(K; \omega)$.

6^0. The Basic Result Referring to the Dirichlet Problem of the Class $e^{p_1(\cdot),p_2(\cdot)}(D)$

Having a picture of solvability of the Dirichlet problem (17) in the ring K, we can, relying on corollary of Lemma 1, get a picture of solvability of the Dirichlet problem in the class $e^{p_1(\cdot),p_2(\cdot)}(D)$. From the results of item 5^0 it follows

Theorem 2. *Let*

(1) *the doubly-connected domain D be bounded by simple closed curves Γ_1 and Γ_2, where Γ_2 lies inside of Γ_1, while $\Gamma = \Gamma_1 \cup \Gamma_2$ belongs to $C^1_D(t_1, \dots, t_n; \nu_1, \dots, \nu_n)$, $0 \le \nu_k \le 2$;*

(2) *$w = w(z)$ be a conformal mapping of the domain D onto the ring $K = \{w : \rho < |w| < 1\}$; $z = z(w)$ be the inverse mapping, $z(t_k) = a_k$, $z(\gamma_i) = \Gamma_i$, where $\gamma_1 = \{\tau : |\tau| = 1\}$, $\gamma_2 = \{\tau : |\tau| = \rho\}$;*

(3) *$p_i \in Q(\Gamma_i)$.*

Then for the Dirichlet problem

$$\begin{cases} \Delta \mathcal{U} = 0, & \mathcal{U} \in e^{p_1(\cdot),p_2(\cdot)}(D), \\ \mathcal{U}\big|_{\Gamma_i} = f_i, & f_i \in L^{p_i}(\Gamma_i), \end{cases}$$

the following statements are valid.

I. *The homogeneous problem is solvable and its general solution is given by the equality*

$$\begin{gathered}\mathcal{U}_0(z)=\sum_{k\in T_1\cup T_2} M_k(p)\,\mathrm{Re}\,\frac{w(z)+a_k}{w(z)-a_k}\,,\\ T_i=\big\{k:\ \nu_k\geq p(t_k),\ t_k\in\Gamma_i\big\},\quad \sum_{k\in T_2} M_k(p)=0,\end{gathered}\tag{32}$$

$$M_k(p)=\begin{cases}0, & \text{if } 0\leq\nu_k<p(t_k), \quad \text{or } \nu=p(t_k)\\ & \quad \text{for } a_k\in\gamma_i,\ \ X_{k,i}\,\overline{\in}\,H^{l_i(\cdot)},\\ & \quad X_{k,i}(w)=(w-a_k)^{-\frac{1}{l_i(a_k)}}\exp\displaystyle\int_{\gamma_i}\frac{\psi_i(\tau)}{l_i(\tau)}\,\frac{d\tau}{\tau-w}\,,\\ M_k & \text{is an arbitrary constant for } p(t_k)<\nu_k\leq 2 \ \text{ or}\\ & \quad \nu_k=p(t_k),\ \ X_{k,i}\in H^{l_i(\cdot)},\end{cases}$$

where ψ_i are functions defined by means of z' appearing in (13).

II. *If among the points t_k there are such that $\nu_k=p(t_k)$ or $\nu_k=0$, then the Dirichlet problem is, generally speaking, unsolvable.*

III. *The problem is solvable if and only if for the functions $g_i(\tau)=f_i(z(\tau))$, $\tau\in\gamma_i$, the following conditions are fulfilled:*

(a)

$$\omega_i^+(\zeta)\int_{\gamma_i}\frac{g_i(\tau)}{\omega_i^+(\tau)}\,\frac{d\tau}{\tau-\zeta}\in L^{l_i(\cdot)}(\gamma_i),\tag{33}$$

where

$$\omega_i(w)=\prod_{k\in T_i'}(w-a_k)^{-\frac{1}{l_i(a_k)}}\exp\int_{\gamma_i}\frac{\psi_i(\tau)}{l_i(\tau)}\,\frac{d\tau}{\tau-w}\,,$$

$$T_i'=\big\{k:\ \nu_k=p(t_k)\ \text{or}\ \nu_k=0,\ t_k\in\Gamma_i\big\};$$

(b)

$$\int_0^{2\pi} g_1(e^{i\vartheta})\,d\vartheta=\int_0^{2\pi} g_2(\rho e^{i\vartheta})\,d\vartheta.\tag{34}$$

IV. *If the conditions* (33) *and* (34) *are fulfilled, then the Dirichlet problem is solvable, and its general solution is given by the equality*

$$\mathcal{U}(z)=\mathcal{U}_0(z)+u^*(w(z)),$$

where $\mathcal{U}_0(z)$ is given by the equality (32)*, and $u^*(w)=u_1(w)+u_2(w)$, where $u_1(w)$ is the function given by the equalities* (27) *and* (26)*, and*

$$u_2(w)=u_2(re^{i\vartheta})=\int_0^{2\pi} u_2(\rho e^{i\alpha})\,\frac{r^2-\rho^2}{r^2+\rho^2-2r\rho\cos(\alpha-\vartheta)}\,d\alpha,$$

where $u_2(\rho e^{i\vartheta})$ is a solution of the equation (29).

Remark. For $p(t) = \text{const}$, the work [10] presents an easily verifiable condition referring to the function $f(t)$ which guarantees the existence of a solution. It consists in that the function

$$f(t) \prod_{\{\nu(t_k)=0,\ \nu(t_k)=p(t_k)\}} \ln|w(\tau) - w(t_k)|$$

is to belong to $L^p(\Gamma)$.

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