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Mikheil Basheleishvili

INVESTIGATION OF THE BOUNDARY VALUE PROBLEMS OF STATICS OF AN ELASTIC MIXTURE

Abstract. Both the domains D^+ and D^- are considered, where the third and the fourth problems are formulated. Green's formulas are written and by means them uniqueness theorems are proved for the third and fourth problems.

For the third and fourth problems, in the domains D^+ and D^- Fredholm integral equations are derived and existence theorem are proved.

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რეზიუმე. განიხილება როგორც D^+ ასევე D^- არეები, სადაც ჩამოყალიბებულია მესამე და მეოთხე სასაზღვრო ამოცანები. დაწერილია გრინის ფორმულები და მათი გამოყენებით დამტკიცებულია მესამე და მეოთხე სასაზღვრო ამოცანების ამონახსნების ერთადერთობის თეორემები.

მესამე და მეოთხე ამოცანებისათვის D^+ და D^- არეებში მიღებულია ფრედჰოლმის ინტეგრალური განტოლებები და დამტკიცებულია ამონახსნის არსებობის თეორემები.

1. THE BASIC EQUATIONS. BOUNDARY VALUE PROBLEMS

The system of basic (homogeneous) equations of statics of an elastic mixture for two dimensions is of the form ([1])

$$\begin{aligned} &a_1\Delta u' + b_1 \operatorname{grad}\operatorname{div} u' + c\Delta u'' + d \operatorname{grad}\operatorname{div} u'' = 0,\\ &c\Delta u' + d \operatorname{grad}\operatorname{div} u' + a_2\Delta u'' + b_2 \operatorname{grad}\operatorname{div} u'' = 0, \end{aligned} \tag{1.1}$$

where

$$\begin{gathered} a_1 = \mu_1 - \lambda_5, \quad b_1 = \mu_1 + \lambda_1 - \lambda_5 - \rho^{-1}\alpha_2\rho_2,\\ a_2 = \mu_2 - \lambda_5, \quad c = \mu_3 + \lambda_5, b_2 = \mu_2 + \lambda_1 + \lambda_5 + \rho^{-1}\alpha_2\rho_2,\\ d = \mu_3 + \lambda_3 - \lambda_5 - \rho^{-1}\alpha_2\rho_1 \equiv \mu_3 + \lambda_4 - \lambda_5 + \rho^{-1}\alpha_2\rho_2,\\ \rho = \rho_1 + \rho_2, \quad \alpha_2 = \lambda_3 - \lambda_4. \end{gathered} \tag{1.2}$$

ρ_1 and ρ_2 appearing in (1.2) are the partial densities and μ_1, μ_2, μ_3, λ_1, λ_2, λ_3, λ_4, λ_5 are real constants characterizing physical properties of an elastic mixture and satisfying certain inequalities. $u' = (u_1, u_2)$ and $u'' = (u_3, u_4)$ are partial displacements.

Introducing the variables

$$z = x_1 + ix_2, \quad \overline{z} = x_1 - ix_2,$$

that is

$$x_1 = \frac{z + \overline{z}}{2}, \quad x_2 = \frac{z - \overline{z}}{2i},$$

the system (1.1) can be rewritten in the form ([2])

$$\frac{\partial^2 \mathcal{U}}{\partial z \partial \overline{z}} + \mathcal{K}\frac{\partial^2 \overline{\mathcal{U}}}{\partial \overline{z}^2} = 0, \tag{1.3}$$

where

$$\mathcal{U} = \begin{pmatrix} u_1 + iu_2\\ u_3 + iu_4 \end{pmatrix} = m\varphi(z) - Kmz\overline{\varphi'(z)} + \overline{\psi(z)}, \tag{1.4}$$

$$\mathcal{M} = \begin{bmatrix} m_1 & m_2\\ m_2 & m_3 \end{bmatrix}, \quad m_1 = l_1 + \frac{l_4}{2}, \quad m_2 = l_2 + \frac{l_5}{2}, \quad m_3 = l_3 + \frac{l_6}{2},$$

$$l_1 = \frac{a_2}{d_2}, \ \ l_2 = -\frac{c}{d_2}, \ \ l_3 = \frac{a_1}{d_2},$$

$$l_1 + l_4 = \frac{a_2 + b_2}{d_1}, \ \ l_2 + l_5 = -\frac{c + d}{d_1}, \ \ l_3 + l_6 = \frac{a_1 + b_1}{d_1},$$

$$\mathcal{K} = \begin{bmatrix} k_1 & k_3\\ k_2 & k_4 \end{bmatrix}, \quad km = -\frac{l}{2}, \tag{1.5}$$

$$l = \begin{bmatrix} l_4 & l_5\\ l_5 & l_6 \end{bmatrix}, \quad m^{-1} = \frac{1}{\Delta_0}\begin{bmatrix} m_3 & -m_2\\ -m_2 & m_1 \end{bmatrix}, \quad \Delta_0 = m_1 m_3 - m_2^2 > 0,$$

$$\delta_0 k_1 = 2(a_2 b_1 - cd) + b_1 b_2 - d^2, \quad \delta_0 k_2 = 2(da_1 - cb_1),$$

$$\delta_0 k_3 = 2(da_2 - cb_2), \quad \delta_0 k_4 = 2(a_1 b_2 - cd) + b_1 b_2 - d^2,$$

$$\delta_0 = (2a_1 + b_1)(2a_2 + b_2) - (2c + d)^2 \equiv 4d_1 d_2 \Delta_0,$$

$$d_1=(a_1+b_1)(a_2+b_2)-(c+d)^2>0,\ \ d_2=a_1a_2-c^2>0,$$

$\varphi(z)$ and $\psi(z)$ are analytic vectors.

The vector of forces has the form

$$\mathcal{T}\mathcal{U}=\begin{pmatrix}(TU)_2-i(TU)_1\\(TU)_4-i(TU)_3\end{pmatrix}=\frac{\partial}{\partial s}\big(-2\varphi(z)+2\mu\mathcal{U}\big),\tag{1.6}$$

where

$$\frac{\partial}{\partial s(x)}=n_1\,\frac{\partial}{\partial x_2}-n_2\,\frac{\partial}{\partial x_1}\,,\tag{1.7}$$

n_1 and n_2 are the projections of the unit vector on the axes x_1 and x_2. Obviously, the unit vector of the tangent is $s(x)=(-n_2,n_1)$; $(TU)_k$ is the projection of the force vector on the axes x_k $(k=\overline{1,4})$,

$$\mu=\begin{bmatrix}\mu_1 & \mu_3\\ \mu_3 & \mu_2\end{bmatrix},\quad \det\mu=\mu_1\mu_2-\mu_3^2>0.\tag{1.8}$$

Here we give the definition of a regular solution in the domain D^+.

The vector $\mathcal{U}$ is a regular solution in the domain D^+ for the equation (1.3) if this vector and its first order derivatives are continuous up to the boundary, while the second order derivatives lie in the domain D^+ and satisfy the equation (1.3).

We can now formulate the third boundary value problem.

Find a regular solution in the finite domain D^+ which on the boundary (i.e. on S) satisfies the boundary conditions

$$(nU)^+=f(t),\quad (sTU)^+=F(t),\tag{1.9}$$

where f and F are given continuous functions on S. The sign "+" refers to interior limiting values. If instead of D^+ we take $D^-=E_2\setminus\overline{D}^+$, where $\overline{D}^+=D^+\cup S$ and E_2 is the two-dimensional infinite plane, then the boundary conditions take the form

$$(nU)^-=f(t),\quad (sTU)^-=F(t),\tag{1.10}$$

where the sign "$-$" refers to exterior limiting values. For the domain D^-, to the conditions of regularity we add the following conditions at infinity:

$$\mathcal{U}=O(1),\quad \frac{\partial U}{\partial u_k}=O(\rho^{-2}),\ \ k=1,2,\quad \rho=\sqrt{x_1^2+x_2^2}\,.\tag{1.11}$$

If the point is on the boundary, then t is the affix of the point z.

The fourth boundary value problem in the domains D^+ and D^- is defined analogously. The boundary conditions now are the following:

$$(sU)^+=f(t),\quad (nTU)^+=F(t),\tag{1.12}$$

or

$$(sU)^-=f(t),\quad (nTU)^-=F(t),\tag{1.13}$$

where f and F are given continuous functions.

Below we will need the following Green's formulas ([2]):

$$\int\limits_{D^+} E(u,u)\,dy_1\,dy_2=\int\limits_S uTu\,ds\equiv \operatorname{Im}\int\limits_S \mathcal{U}T\overline{\mathcal{U}}\,ds, \tag{1.14}$$

$$\int\limits_{D^-} E(u,u)\,dy_1\,dy_2=-\int\limits_S uTu\,ds\equiv -\operatorname{Im}\int\limits_S \mathcal{U}T\overline{\mathcal{U}}\,ds, \tag{1.15}$$

where

$$\operatorname{Im}\mathcal{U}T\overline{\mathcal{U}}=n(T\overline{\mathcal{U}})_n+s(T\overline{\mathcal{U}})_s, \tag{1.16}$$

$(T\overline{\mathcal{U}})_n$ and $(T\overline{\mathcal{U}})_s$ are, respectively, the normal and the tangential components of the force vector, and $E(u,u)$ is the doubled potential energy of the form

$$\begin{gathered}
E(u,u)=\\
=(b_1-\lambda_5)\Big(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\Big)^2+2(d+\lambda_5)\Big(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\Big)\Big(\frac{\partial u_3}{\partial x_1}+\frac{\partial u_4}{\partial x_2}\Big)^2+\\
+(b_2-\lambda_5)\Big(\frac{\partial u_3}{\partial x_1}+\frac{\partial u_4}{\partial x_2}\Big)^2+\\
+\mu_1\Big[\Big(\frac{\partial u_1}{\partial x_1}-\frac{\partial u_2}{\partial x_2}\Big)^2+\Big(\frac{\partial u_2}{\partial x_1}+\frac{\partial u_1}{\partial x_2}\Big)^2\Big]+\\
+2\mu_3\Big[\Big(\frac{\partial u_1}{\partial x_1}-\frac{\partial u_2}{\partial x_2}\Big)\Big(\frac{\partial u_3}{\partial x_1}-\frac{\partial u_4}{\partial x_2}\Big)+\Big(\frac{\partial u_2}{\partial x_1}+\frac{\partial u_1}{\partial x_2}\Big)\Big(\frac{\partial u_4}{\partial x_1}+\frac{\partial u_3}{\partial x_2}\Big)\Big]+\\
+\mu_2\Big[\Big(\frac{\partial u_3}{\partial x_1}-\frac{\partial u_4}{\partial x_2}\Big)^2+\Big(\frac{\partial u_4}{\partial x_1}+\frac{\partial u_3}{\partial x_2}\Big)^2\Big]-\\
-\lambda_5\Big[\Big(\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\Big)^2-\Big(\frac{\partial u_4}{\partial x_1}-\frac{\partial u_3}{\partial x_2}\Big)^2\Big]. \qquad (1.17)
\end{gathered}$$

Let us prove the theorem allowing us to solve the third boundary value problem: a regular solution in the domain D^+ satisfying the homogeneous conditions of the third boundary value problem is identical zero, if S is not a parabolic type line without center.

Proof. The use is made of the formula (1.14). In (1.9), if $f=F=0$, then it follows from (1.14) that

$$u_1=c_1-\varepsilon x_2,\quad u_2=c_2+\varepsilon x_1,\quad u_3=c_3-\varepsilon x_2,\quad u_4=c_4+\varepsilon x_1, \tag{1.18}$$

where c_k $(k=1,4)$ and ε are arbitrary constants.

We write

$$n\mathcal{U}=(n_1(u_1+cu_2)+n_2(u_3+cu_4)),\quad n_1=\frac{dx_2}{ds}\,,\quad n_2=-\frac{dx_2}{ds}\,.$$

Then

$$0=(n\mathcal{U})^+=(u_1+cu_2)\,\frac{dx_2}{ds}-(u_3+iu_4)\,\frac{dx_1}{ds}\,.$$

Thus we easily get

$$\begin{aligned}(c_1-\varepsilon x_2)\,\frac{dx_2}{ds}-(c_3-\varepsilon x_2)\,\frac{dx_1}{ds}=0,\\ (c_2+\varepsilon x_1)\,\frac{dx_2}{ds}-(c_4+\varepsilon x_1)\,\frac{dx_1}{ds}=0,\end{aligned}$$

that is,

$$\frac{d}{ds}\Big[(c_1+c_2)x_2-(c_3+c_4)x_1-\frac{\varepsilon}{2}\,(x_1^2+x_2^2-2x_1x_2)\Big]=0. \tag{1.19}$$

(1.19) results in

$$\frac{\varepsilon}{2}\,(x_1-x_2)^2-(c_1+c_2)x_2+(c_3+c_4)x_1=c,$$

where c is a real constant. On the basis of (1.19) we can write the so-called discriminant D_1 and the higher terms discriminant D_2. In our case, using the well-known formulas from the analytic geometry, we obtain

$$D_1=-\varepsilon\begin{vmatrix}1&-1\\-1&1\end{vmatrix}=0,\quad D_2=\begin{vmatrix}1&-1&A\\-1&1&B\\A&B&-\frac{2}{\varepsilon}\,C\end{vmatrix},$$

where

$$A=\frac{c_3+c_4}{\varepsilon}\,,\quad B=\frac{c_1+c_2}{\varepsilon}\,.$$

Since $D_1=0$, this implies that the line S is without center, of parabolic type. The condition $D_2=A(-A-B)-B(A+B)=-(A+B)^2=0$ implies that $A+B=0$ or $c_1+c_2+c_3+c_4=0$, and in this case the line is represented by conjugate lines. Thus we have proved that the uniqueness of a solution of the third boundary value problem takes place if s is not a parabolic type line without center, or conjugate lines.

Just in the same way we can prove the uniqueness of a solution of the third boundary value problem in the domain D^-.

The fourth boundary value problem in the domains D^+ and D^- is considered analogously and it is proved that the uniqueness of a solution in the domain D^+ takes place if S is not a parabolic type line without center, and in case of the domain D^-, S is not a straight line.

2. Solution of the Third Boundary Value Problem of Statics of an Elastic Mixture in the Domain D^+

Consider the expression $-2\varphi(z)+2\mu\mathcal{U}(z)$.

Taking into account the formula (1.4), we obtain

$$-2\varphi(z)+2\mu\mathcal{U}(z)=(A-2E)\varphi(z)-2\mu\mathcal{K}mz\overline{\varphi'(z)}+2\mu\overline{\psi(z)}, \tag{2.1}$$

where $A=2\mu m$, and E is the unit matrix.

We seek for $\varphi(z)$ in the form

$$\varphi(z)=\frac{(A-2E)^{-1}}{2\pi i}\int\limits_S \ln\Big(1-\frac{z}{\zeta}\Big)g(y)\,dS, \tag{2.2}$$

where $\det(A-2E)^{-1}>0$, g is a vector, complex in general, which will be defined below.

Inserting (2.2) into (2.1), we find that

$$-2\varphi(z)+2\mu\mathcal{U}=$$
$$=\frac{1}{2\pi i}\int\limits_S \ln\Big(1-\frac{z}{\zeta}\Big)g\,dS+\frac{2\mu\mathcal{K}m(A-2E)^{-1}}{2\pi i}\int\limits_S \frac{zg}{\overline{\sigma}}\,dS+2\mu\overline{\psi(z)}. \tag{2.3}$$

We choose $\overline{\psi(z)}$ in the form

$$2\mu\overline{\psi(z)}=\frac{1}{2\pi i}\int\limits_S \Big[\ln\Big(1-\frac{z}{\zeta}\Big)-\ln\Big(1-\frac{\overline{z}}{\overline{\zeta}}\Big)\Big]g\,dS+$$
$$+\frac{2\mu\mathcal{K}m(A-2E)^{-1}}{2\pi i}\int\limits_S \frac{\zeta\overline{g}}{\overline{\sigma}}\,dS \tag{2.4}$$

and insert (2.4) into (2.3). Thus we obtain

$$-2\varphi(z)+2\mu\mathcal{U}=\frac{1}{2\pi i}\int\limits_S \ln\frac{1-\frac{z}{\zeta}}{1-\frac{\overline{z}}{\overline{\zeta}}}\,g\,dS+\frac{\mu\mathcal{K}m(A-2E)^{-1}}{\pi i}\int\limits_S \frac{\sigma}{\overline{\sigma}}\,\overline{g}\,dS, \tag{2.5}$$

where $\sigma=z-\zeta$, $\overline{\sigma}=\overline{z}-\overline{\zeta}$.

Inserting (2.5) into (2.1) and then into (1.6), we get

$$\mathcal{T}\mathcal{U}=\frac{\partial}{\partial s(x)}\bigg\{\int\limits_S \ln\frac{\sigma}{\overline{\sigma}}\,\frac{\overline{\zeta}}{\zeta}\,g\,dS+\frac{\mu\mathcal{K}m(A-2E)^{-1}}{\pi i}\int\limits_S \frac{\sigma}{\overline{\sigma}}\,\overline{g}\,dS\bigg\}. \tag{2.6}$$

Taking into account (2.2) and (2.4), the expression (1.4) takes the form

$$\mathcal{U}=\frac{m(A-2E)^{-1}}{2\pi i}\int\limits_S \ln\Big(1-\frac{z}{\zeta}\Big)g\,dS+\frac{\mathcal{K}m(A-2E)^{-1}}{2\pi i}\int\limits_S \frac{z}{\overline{\sigma}}\,\overline{g}\,dS+\overline{\varphi(z)}.$$

Substitutibg here the value $\overline{\psi(z)}$ from (2.4), we obtain

$$\mathcal{U}=\frac{m(A-2E)^{-1}}{2\pi i}\int\limits_S \ln\Big(1-\frac{z}{\zeta}\Big)g\,dS-$$
$$-\frac{(2\mu)^{-1}}{2\pi i}\int\limits_S \ln\Big(1-\frac{\overline{z}}{\overline{\zeta}}\Big)g\,dS+\frac{\mathcal{K}m(A-2E)^{-1}}{2\pi i}\int\limits_S \frac{\sigma}{\overline{\sigma}}\,\overline{g}\,dS. \tag{2.7}$$

The vector $\mathcal{U}$ is continuous up to the boundary.

Taking into account (2.6) and (2.7), in case of the third boundary value problem to find g we obtain the integral equation of the form

$$(nU)^{+}=f(z),\quad (sTU)^{+}=F(z). \tag{2.8}$$

The use is now made of the system (2.8) which we rewrite as follows:

$$
\begin{gathered}
(nU)^+ = f,\\
sg+\frac{s}{2\pi i}\left\{\int\limits_S\frac{\partial\theta}{\partial s}\,g\,dS+\frac{s\mu\mathcal{K}m(A-2E)^{-1}}{\pi i}\int\limits_S e^{2i\theta}\,\frac{\partial\theta}{\partial s}\,\overline{g}\,dS\right\}=F(t),\ t\in S,
\end{gathered}
\tag{2.9}
$$

where

$$
\theta = \operatorname{arctg}\frac{y_2-x_2}{y_1-x_1}\,,\quad x=(x_1,x_2)\in S. \tag{2.10}
$$

To investigate the equation (2.9), besides the vector U we will need the vector V ([2]):

$$
V = i\big[-m\varphi(z)+\frac{l}{2}\,z\overline{\varphi'(z)}+\overline{\psi(z)}\big].
$$

Relying on [2], we have $U+iV=2m\varphi(z)$ and

$$
\begin{aligned}
TU &= \frac{\partial}{\partial s(x)}\,\big[(A-2E)\varphi(z)+Bz\overline{\varphi'(z)}+2\mu\overline{\psi(z)}\big],\\
TV &= i\,\frac{\partial}{\partial s(x)}\,\big[-(A-2E)\varphi(z)+Bz\overline{\varphi'(z)}+2\mu\overline{\psi(z)}\big],
\end{aligned}
\tag{2.11}
$$

where $B=\mu l$.

Using the vectors U and V, we can show that

$$
NU=-im^{-1}\frac{\partial V}{\partial s(x)}\,,\quad NV=im^{-1}\frac{\partial U}{\partial s(x)}\,, \tag{2.12}
$$

where N is the pseudostress operator ([2]).

The operator N is of great importance for investigation of boundary value problems of statics of elastic mixtures.

For the solvability of the problem we have to investigate the system (2.9). Towards this end, we consider the homogeneous equation obtained from (2.9), when $f=F=0$. Let it have a nontrivial solution which we denote by g_0. Introduce the notation:

$$
U(x,g_0)=U^{(0)}(x),\quad V(x,g_0)=V^{(0)}(x). \tag{2.13}
$$

From the uniqueness theorem we find that $U^{(0)}(x)=0$, $x\in D^+$. Then $LU^{(0)}=0$ ([2], p. 434) and $TV^{(0)}(x)=0$. But as is known, $(TV^{(0)})^+=(TV^{(0)})^-$. Using in this case Green's formula in the domain D^-, we have $V^{(0)}(x)=0$, $x\in D^-$. Thus we obtain $TU^{(0)}(x)=0$, $x\in D^-$. Obviously, for the vector g_0 we have $TU^{(0)}(x)=0$, $x\in D^+$, and $TU^{(0)}(x)=0$, $x\in D^-$. Consequently, $(TU^{(0)}(t))^+=0$, $t\in S$, and $(TU^{(0)}(t))^-=0$, $t\in S$. But since there takes place the formula $2g_0=(TU^{(0)}(t))^+-(TU^{(0)}(t))^-=0$, we find that the homogeneous equation corresponding to (2.9) has a trivial solution. In this case, the inhomogeneous equation (2.9) has always a unique solution for an arbitrary right-hand side f and F.

Thus we have proved that the third boundary value problem of statics of an elastic mixture has always a unique solution if s is not a parabolic type line without center.

3. SOLUTION OF THE THIRD BOUNDARY VALUE PROBLEM OF STATICS OF AN ELASTIC MIXTURE IN THE DOMAIN D^-

The third boundary value problem has the following boundary conditions of the form:

$$(nU)^- = f, \quad (sTU)^- = F, \tag{3.1}$$

where the sign "$-$" refers to the exterior boundary values of the domain $D^- = E_2 \setminus \overline{D}^+$, and f and F are known continuous functions.

From [2] we write out the well-known formulas

$$U = m\varphi(z) + \frac{l}{2}\, z\overline{\varphi'(z)} + \overline{\psi(z)}, \quad V = i\Big[-m\varphi(z) + \frac{l}{2}\, z\overline{\varphi'(z)} + \overline{\psi(z)}\Big] \tag{3.2}$$

and

$$\begin{aligned} TU &= \frac{\partial}{\partial s(x)}\,\big[(A-2E)\varphi(z) + Bz\overline{\varphi'(z)} + 2\mu\overline{\psi(z)}\big],\\ TV &= i\,\frac{\partial}{\partial s(x)}\,\big[-(A-2E)\varphi(z) + Bz\overline{\varphi'(z)} + 2\mu\overline{\psi(z)}\big], \end{aligned} \tag{3.3}$$

where $\varphi(z)$ and $\psi(z)$ are analytic vectors and $A-2E$ is a nonsingular matrix, i.e., $\det(A-2E) > 0$, $B = \mu l$.

In the domain D^- we seek for $\varphi(z)$ in the form

$$\varphi(z) = \frac{(A-2E)^{-1}}{2\pi i} \int\limits_S (\ln\sigma - \ln z) g\, dS, \tag{3.4}$$

where g is an unknown vector, $\sigma = z - \zeta$. Hence we have

$$\overline{\varphi'(z)} = -\frac{(A-2E)^{-1}}{2\pi i} \int\limits_S \Big(\frac{1}{\sigma} - \frac{1}{\overline{z}}\Big)\overline{g}\, dS.$$

Substituting $\varphi(z)$ and $\overline{\varphi'(z)}$ into (3.3), we obtain

$$\begin{aligned} TU &= \frac{\partial}{\partial s(x)}\,\bigg[\frac{1}{2\pi i}\int\limits_S (\ln\sigma - \ln z) g\, dS - \\ &\qquad - \frac{B(A-2E)^{-1}}{2\pi i}\int\limits_S \Big(\frac{z}{\sigma} - \frac{z}{\overline{z}}\Big)\overline{g}\, dS + 2\mu\overline{\psi(z)}\bigg],\\ TV &= \frac{\partial}{\partial s(x)}\,\bigg[-\frac{1}{2\pi i}\int\limits_S (\ln\sigma - \ln z) g\, dS - \\ &\qquad - \frac{B(A-2E)^{-1}}{2\pi i}\int\limits_S \Big(\frac{z}{\overline{\sigma}} - \frac{z}{\overline{z}}\Big)\overline{g}\, dS + 2\mu\overline{\psi(z)}\bigg]. \end{aligned} \tag{3.5}$$

Choosing $\overline{\psi(z)}$ in the form

$$2\mu\overline{\psi(z)} = -\frac{1}{2\pi i}\int\limits_S (\ln\overline{\sigma} - \ln\overline{z}) g\, dS - \frac{B(A-2E)^{-1}}{2\pi i}\int\limits_S \Big(\frac{\zeta}{\overline{\sigma}} - \frac{\zeta}{\overline{z}}\Big)\overline{g}\, dS, \tag{3.6}$$

we get

$$
\begin{aligned}
TU &= \frac{\partial}{\partial s(x)}\left[\frac{1}{\pi i}\int\limits_S(\theta-\vartheta)g\,dS-\frac{B(A-2E)^{-1}}{\pi i}\int\limits_S\frac{\sigma}{\overline{\sigma}}\,\overline{g}\,dS\right],\\
TV &= \frac{\partial}{\partial s(x)}\left[\frac{1}{\pi i}\int\limits_S\ln r g\,dS-\frac{B(A-2E)^{-1}}{\pi i}\int\limits_S\frac{\sigma}{\overline{\sigma}}\,\overline{g}\,dS\right],
\end{aligned}
\tag{3.7}
$$

where

$$
\theta=\operatorname{arctg}\frac{y_2-x_2}{y_1-x_1}\,,\quad \vartheta=\operatorname{arctg}\frac{x_2}{x_1}\,. \tag{3.8}
$$

It is obvious from (3.7) that TV is defined in both domains D^+ and D^-. Moreover, the equality

$$
(TV)^+=(TV)^- \tag{3.9}
$$

holds.

We consider that $\varphi()$ and $\psi(z)$ appearing in (3.2) are defined by means of (3.4) and (3.6). Then U and V are single-valued vectors, continuous up to the boundary S.

Taking into account the boundary conditions of the third boundary value problem, we can write

$$
\begin{gathered}
(nU)^-=f(t),\\
-sg+\frac{1}{\pi}\int\limits_S(\theta-\vartheta)g\,dS-\frac{B(A-2E)^{-1}}{\pi i}\int\limits_S e^{2i\theta}\overline{g}\,dS=F(t),\quad t\in S,
\end{gathered}
\tag{3.10}
$$

where

$$
\begin{aligned}
U={}&\frac{m(A-2E)^{-1}}{2\pi i}\int\limits_S(\ln\sigma-\ln z)g\,dS-\\
&-\frac{e(A-2E)^{-1}}{2\pi i}\int\limits_S\Big(\frac{z}{\overline{\sigma}}-\frac{z}{\overline{z}}\Big)g\,dS-\frac{(2\mu)^{-1}}{2\pi i}\int\limits_S(\ln\overline{\sigma}-\ln\overline{z})g\,dS+\\
&+\frac{(2\mu)^{-1}B(A-2E)^{-1}}{2\pi i}\int\limits_S\Big(\frac{\zeta}{\overline{\sigma}}-\frac{z}{\overline{z}}\Big)\overline{g}\,dS.
\end{aligned}
\tag{3.11}
$$

Obviously U is a single-valued vector, continuous up to the boundary S. In this case, (3.10) is a system of Fredholm integral equations of second kind.

Let us now investigate (3.10). To this end, let us consider the homogeneous equation obtained from (3.10), when $f=F=0$. Assume that it has a nontrivial solution which we denote by g_0. Introduce the notation

$$
U(x,g_0)\equiv U^{(0)}(x),\quad V(x,g_0)\equiv V^{(0)}(x). \tag{3.12}
$$

From the uniqueness theorem we obtain $U^{(0)}(x)=0$, $x\in D^-$. Then $LU^{(\cdot)}(x)=0$, $x\in D^-$, and $TV^{(0)}(x)=0$. Taking into account the property $(TV^{(0)})^-=(TV^{(0)})^+$ and using Green's formula in the domain D^-, we will have $V^{(0)}(x)=0$, $x\in D^-$. Then $LV^{(0)}(x)=0$ and $TU^{(0)}(x)=0$. Finally, using the formula $0 = (TU)^+ - (TU)^- = 2g_0$, we find that $g_0 = 0$. Hence our assumption that the homogeneous equation obtained by means of (3.10) for $f = F = 0$ has a nontrivial solution is invalid.

Thus we have proved that the system (3.10) has always a unique solution, when f and F are continuous functions and S is a parabolic type line without center.

4. SOLUTION OF THE FOURTH BOUNDARY VALUE PROBLEM IN THE DOMAIN D^+

The method of solution of the third boundary value problem in the domains D^+ and D^- described above fits for the solution of the fourth boundary value problem in the domains D^+ and D^-.

The boundary conditions for the fourth boundary value problem in the domain D^+ are

$$(sU)^+ = f(t), \quad (nTU)^+ = F(t), \quad t \in S, \tag{4.1}$$

where sU and nTU are the tangential components of the displacement vector and the normal components of the stress vector, respectively.

The conjugate vectors U and V have the form of (3.2). Moreover, the formulas (3.3) hold. In (3.3) we take

$$\varphi(z) = \frac{(A-2E)^{-1}}{2\pi i} \int\limits_S \ln\frac{\zeta - t}{\zeta}\, g(y)\, dS, \tag{4.2}$$

where $\zeta = (y_1, y_2) \in S$, and g is an unknown vector. (4.2) yields

$$\overline{\varphi'(z)} = -\frac{(A-2E)^{-1}}{2\pi i} \int\limits_S \frac{1}{\overline{\sigma}}\, \overline{g}\, dS. \tag{4.3}$$

Substituting (4.2) and (4.3) into (2.12), we obtain

$$\begin{aligned} TU &= \frac{\partial}{\partial s(x)}\left[\frac{1}{2\pi i}\int\limits_S \ln\frac{\zeta - z}{\zeta}\, g\, dS - \frac{B(A-2E)^{-1}}{2\pi i}\int\limits_S \frac{z}{\overline{\sigma}}\,\overline{g}\, dS + 2\mu\overline{\psi(z)}\right],\\ TV &= \frac{\partial}{\partial s(x)}\left[-\frac{1}{2\pi i}\int\limits_S \ln\frac{\zeta - z}{\zeta}\, g\, dS - \right.\\ &\qquad \left. - \frac{B(A-2E)^{-1}}{2\pi i}\int\limits_S \frac{z}{\overline{\sigma}}\,\overline{g}\, dS + 2\mu\overline{\psi(z)}\right]. \end{aligned} \tag{4.4}$$

In (4.4) we take $\overline{\psi(z)}$ as follows:

$$\begin{aligned} 2\mu\overline{\psi(z)} = &-\frac{1}{2\pi i}\int\limits_S \ln\frac{\overline{\zeta} - \overline{z}}{\zeta}\, g\, dS + \\ &+ \frac{B(A-2E)^{-1}}{2\pi i}\int\limits_S \frac{z}{\overline{\sigma}}\,\overline{g}\, dS - \frac{1}{2\pi i}\int\limits_S \ln\overline{\zeta} g\, dS. \end{aligned} \tag{4.5}$$

Then (4.4) takes the form

$$\begin{aligned} TU &= \frac{\partial}{\partial s(x)}\biggl[\frac{1}{\pi}\int\limits_S \theta g\,dS - \frac{B(A-2E)^{-1}}{\pi i}\int\limits_S \frac{\sigma}{\overline{\sigma}}\,\overline{g}\,dS\biggr], \\ TV &= \frac{\partial}{\partial s(x)}\biggl[\frac{1}{\pi}\int\limits_S \ln rg\,dS - \frac{B(A-2E)^{-1}}{\pi i}\int\limits_S \frac{\sigma}{\overline{\sigma}}\,\overline{g}\,dS\biggr]. \end{aligned} \tag{4.6}$$

It follows from (4.6) that

$$\begin{aligned} (TU)^+ &= -g(t) + \frac{\partial}{\partial s(t)}\biggl[\frac{1}{\pi}\int\limits_S \theta g\,dS - \frac{B(A-2E)^{-1}}{\pi i}\int\limits_S e^{2i\theta}\overline{g}\,dS\biggr], \\ (TV)^+ &= \frac{\partial}{\partial s(t)}\biggl[\frac{1}{\pi}\int\limits_S \ln rg\,dS - \frac{B(A-2E)^{-1}}{2\pi i}\int\limits_S e^{2i\theta}\overline{g}\,dS\biggr], \end{aligned} \tag{4.7}$$

where θ is defined by (3.8), and $r=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$.

It is obvious from (4.6) that the vector TV is defined on the whole plane and is continuous, i.e., we have

$$(TV)^+ = (TV)^-. \tag{4.8}$$

Calculating from (3.2) the generalized stress vector, we find that

$$\overset{\varkappa}{T}U = \begin{pmatrix} (\overset{\varkappa}{T}U)_2 - i(\overset{\varkappa}{T}U)_1 \\ (\overset{\varkappa}{T}U)_4 - i(\overset{\varkappa}{T}U)_3 \end{pmatrix}, \quad \overset{\varkappa}{T}V = \begin{pmatrix} (\overset{\varkappa}{T}V)_2 - i(\overset{\varkappa}{T}V)_1 \\ (\overset{\varkappa}{T}V)_4 - i(\overset{\varkappa}{T}V)_3 \end{pmatrix}, \tag{4.9}$$

where $\varkappa$ is a constant and

$$\begin{aligned} \overset{\varkappa}{T}U &= \frac{\partial}{\partial s(x)}\bigl[-2\varphi(z) + (2\mu-\varkappa)U\bigr], \\ \overset{\varkappa}{T}V &= \frac{\partial}{\partial s(x)}\bigl[-2\varphi(z) + (2\mu-\varkappa)V\bigr]. \end{aligned} \tag{4.10}$$

In (4.9), let $\varkappa = 2\mu - 2(A-E)^{-1}\mu$. Then

$$\begin{aligned} LU &= \frac{\partial}{\partial s(x)}\bigl[-2\varphi(z) + 2(A-E)\mu U\bigr], \\ LV &= \frac{\partial}{\partial s(x)}\bigl[-2\varphi(z) + 2(A-E)^{-1}\mu V\bigr]. \end{aligned}$$

Bearing in mind the arguments given in [2], we have

$$TU = -i(A-E)LV, \quad TV = i(A-E)LU, \tag{4.11}$$

where TU and TV are obtained from (4.9), when $\varkappa = 0$.

We can now rewrite (3.2) and (3.5) in the form

$$
\begin{aligned}
&(sU)^+ = f,\\
&(nTU)^+ = -ng(t) + n\,\frac{\partial}{\partial s(t)}\Bigg[\frac{1}{\pi}\int\limits_S \theta g\,dS - \\
&\qquad\qquad - \frac{B(A-2E)^{-1}}{\pi i}\int\limits_S e^{2i\theta}\overline{g}\,dS\Bigg] = F(t),
\end{aligned}
\tag{4.12}
$$

where under U we mean that φ, $\overline{\varphi'(z)}$ and $\overline{\psi(z)}$ are defined from (4.2) and (4.3).

Thus for finding an unknown vector g we have obtained a system of Fredholm integral equations of second kind. Assume that (4.11) has a nontrivial solution when $f = F = 0$, which we denote by g_0. Let

$$
U(x, g_0) = U^{(0)}(x), \quad V(x, g_0) = V^{(0)}(x). \tag{4.13}
$$

By the uniqueness theorem, when S is not a parabolic type line without center, we obtain

$$
U^{(0)}(x) = 0, \ \ x \in D^+.
$$

Then (4.11) yields $LU^{(0)}(x) = 0$ and

$$
TV^{(0)}(x) = 0, \ \ x \in D^+.
$$

But the vector $TV^{(0)}(x)$ crosses continuously the boundary S. In this case we have

$$
(TV^{(0)}(t))^+ = (TV^{(0)}(t))^- = 0.
$$

Using now the uniqueness theorem, in the domain D^- for the vector $V^{(0)}$ we find that

$$
V^{(0)}(x) = c, \ \ x \in D^-,
$$

where c is a constant vector.

Thus we have obtained that

$$
LV^{(0)}(x) = 0, \ \ x \in D^-,
$$

and using (4.11), we get

$$
TU^{(0)}(x) = 0, \ \ x \in D^-.
$$

Since

$$
(TU^{(0)}(t))^- - (TU^{(0)}(t))^+ = 2g_0(t)
$$

and

$$
(TU^{(0)}(t))^- = (TU^{(0)}(t))^+ = 0,
$$

we obtain $g_0(t) = 0$. Thus the homogeneous equation obtained from (4.12) for $f = F = 0$ has only the trivial solution. Hence the equation (4.12) has a unique solution, when f and F are arbitrary continuous functions.

Thus our investigation of the fourth boundary value problem in the domain D^+ is complete.

5. SOLUTION OF THE FOURTH BOUNDARY VALUE PROBLEM IN THE DOMAIN D^-

The fourth boundary value problem in the domain D^- is written as follows:

$$(sU)^- = f(t), \quad (nTU)^- = F(t), \quad t \in S, \tag{5.1}$$

where f and F are the known functions.

The vector $\varphi(z)$ is sought in the form

$$\varphi(z) = \frac{(A-2E)^{-1}}{2\pi i} \int\limits_S \ln \frac{\zeta - z}{\zeta}\, g(y)\, dS, \tag{5.2}$$

where $\zeta = y_1 + iy_2 \in S$, and g is an unknown vector. It follows from (5.2) that

$$\overline{\varphi'(z)} = -\frac{(A-2E)^{-1}}{2\pi i} \int\limits_S \frac{\overline{g}}{\overline{\sigma}}\, dS. \tag{5.3}$$

Substituting (5.2) and (5.3) into (3.3), we obtain

$$\begin{aligned} TU &= \frac{\partial}{\partial s(x)}\bigg[\frac{1}{2\pi i}\int\limits_S \ln\frac{\zeta-z}{\zeta}\, g\, dS - \frac{B(A-2E)^{-1}}{2\pi i}\int\limits_S \frac{z}{\overline{\sigma}}\,\overline{g}\, dS + 2\mu\overline{\psi(z)}\bigg], \\ TV &= \frac{\partial}{\partial s(x)}\bigg[-\frac{1}{2\pi i}\int\limits_S \ln\frac{\zeta-z}{\zeta}\, g\, dS - \\ &\qquad - \frac{B(A-2E)^{-1}}{2\pi i}\int\limits_S \frac{z}{\overline{\sigma}}\,\overline{g}\, dS + 2\mu\overline{\psi(z)}\bigg]. \end{aligned} \tag{5.4}$$

In (5.4) we take $\overline{\psi(z)}$ such that

$$\begin{aligned} 2\mu\overline{\psi(z)} = &-\frac{1}{2\pi i}\int\limits_S \ln\frac{\overline{\zeta}-\overline{z}}{\zeta}\, g\, dS + \\ &+ \frac{B(A-2E)^{-1}}{2\pi i}\int\limits_S \frac{\zeta}{\overline{\sigma}}\,\overline{g}\, dS - \frac{1}{2\pi i}\int\limits_S \ln\overline{\zeta} g\, dS. \end{aligned} \tag{5.5}$$

Then (5.4) takes the form

$$\begin{aligned} TU &= \frac{\partial}{\partial s(x)}\bigg[\frac{1}{\pi}\int\limits_S \theta g\, dS - \frac{B(A-2E)^{-1}}{\pi i}\int\limits_S \frac{\sigma}{\overline{\sigma}}\,\overline{g}\, dS\bigg], \\ TV &= \frac{\partial}{\partial s(x)}\bigg[\frac{1}{\pi}\int\limits_S \ln r g\, dS - \frac{B(A-2E)^{-1}}{\pi i}\int\limits_S \frac{\sigma}{\overline{\sigma}}\,\overline{g}\, dS\bigg]. \end{aligned} \tag{5.6}$$

where

$$\theta = \operatorname{arctg}\frac{y_2 - x_2}{y_1 - x_1}\,. \tag{5.7}$$

From (5.6) it follows

$$
\begin{aligned}
(TU)^- &= g(t) + \frac{\partial}{\partial s(t)}\left[\frac{1}{\pi}\int\limits_S \theta g\,dS - \frac{B(A-2E)^{-1}}{\pi i}\int\limits_S e^{2i\theta}\overline{g}\,dS\right],\\
(TV)^- &= \frac{\partial}{\partial s(t)}\left[\frac{1}{\pi}\int\limits_S \ln r g\,dS - \frac{B(A-2E)^{-1}}{2\pi i}\int\limits_S e^{2i\theta}\overline{g}\,dS\right].
\end{aligned}
\tag{5.8}
$$

It is evident from (5.6) that $(TV)^-$ is defined on the whole plane and is continuous, i.e., we have

$$
(TV)^- = (TV)^+. \tag{5.9}
$$

We now write U and (5.6) in the form

$$
\begin{aligned}
&(sU)^- = f,\\
&(nTU)^- = ng(t) + n\,\frac{\partial}{\partial s(t)}\left[\frac{1}{\pi}\int\limits_S \theta g\,dS - \right.\\
&\qquad\left. - \frac{B(A-2E)^{-1}}{\pi i}\int\limits_S e^{2i\theta}\overline{g}\,dS\right] = F(t),
\end{aligned}
\tag{5.10}
$$

where under U we mean that φ, $\overline{\varphi'(z)}$ and $\overline{\psi(z)}$ are defined from (5.2), (5.3) and (5.5).

(5.10) is a system of Fredholm integral equations of second kind. Let us investigate the system (5.10). Towards this end, we assume that (5.10) has a nontrivial solution, when $f = fF = 0$, which we denote by g_0. Let

$$
U(x, g_0) = U^{(0)}(x), \quad V(x, g_0) = V^{(0)}(x). \tag{5.11}
$$

By the uniqueness theorem , when S is not a straight line, we obtain

$$
U^{(0)}(x) = c, \;\; x \in D^-,
$$

where c is a constant. Then we find from (4.11) that $LU^{(0)}(x) = 0$ and

$$
TV^{(0)}(x) = 0, \;\; x \in D^-.
$$

The vector $TV^{(0)}(x)$ crosses continuously the boundary S, and we have

$$
(TV^{(0)}(t))^- = (TV^{(0)}(t))^+ = 0.
$$

Using now the uniqueness theorem in the domain D^- and assuming that S is not a parabolic type line without center, we have

$$
V^{(0)}(x) = 0, \;\; x \in D^+.
$$

Hence we obtain $LV^{(0)}(x) = 0$, $x \in D^+$, and from (4.11) it follows that

$$
TU^{(0)}(x) = 0, \;\; x \in D^+.
$$

Taking into account the formula

$$
(TU^{(0)}(t))^- - (TU^{(0)}(t))^+ = 2g_0
$$

and the fact that $(TU^{(0)}(t))^- = (TU^{(0)}(t))^+ = 0$, we find that

$$g_0 = 0.$$

Thus we have proved that a solution of the fourth boundary value problem in the domain D^- always exists if f and F are continuous functions.

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Author's address:

I. Vekua Institute of Applied Mathematics
Iv. Javakhishvili Tbilisi State University
2, University St., Tbilisi 0186
Georgia