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EXACT SOLUTION OF SPATIAL AXISYMMETRIC PROBLEMS OF THE FILTRATION THEORY WITH PARTIALLY UNKNOWN BOUNDARIES

Abstract. In the present work we suggest a general method of solution of spatial axisymmetric problems of steady liquid motion in a porous medium with partially unknown boundaries. The liquid motion of ground waters in a porous medium is subjected to the Darcy law. The porous medium is undeformable, isotropic and homogeneous. The velocity potential $\varphi(z,\rho)$ and the flow function $\psi(z,\rho)$ are mutually connected and separately they satisfy different equations of elliptic type, where z is the coordinate of the axis of symmetry, and ρ is the distance to that axis.

To the domain $S(\sigma)$ of the liquid motion on the plane of complex velocity there corresponds a circular polygon. The mapping $\omega=\varphi+i\psi$ belongs to the class of quasi-conformal mappings. Using the functions $\omega_0(\zeta)=\varphi_0(\xi,\eta)+i\psi_0(\xi,\eta)$, $\sigma(\zeta)=z(\xi,\eta)+i\rho(\xi,\eta)$ we map conformally the half-plane $\mathrm{Im}\,(\zeta)>0$ onto the domains $S(\sigma)$, $S(\omega_0)$ and $S(\omega_0'(\zeta)/\sigma'(\zeta))$. These functions satisfy all the boundary conditions, and the functions $\varphi_1(\xi,\eta)=\varphi(\xi,\eta)-\varphi_0(\xi,\eta)$, $\psi_1(\xi,\eta)=\psi(\xi,\eta)-\psi_0(\xi,\eta)$ satisfy the system of differential equations and also zero boundary conditions. The solution of these equations is reduced to a system of Fredholm integral equations of second kind which are solved uniquely by rapidly converging series.

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1. Liquid Motion with Axial Symmetry

In this paper we suggest an effective algorithm allowing one to construct solutions of spatial axisymmetric problems of filtration with partially unknown boundaries.

Let us consider some spatial axisymmetric problems (with partially unknown boundaries) of the theory of steady motion of incompressible liquid in a porous medium obeying the Darcy law. The porous medium is assumed to be non-deformable, isotropic and homogeneous ([1]–[39]).

The liquid motion is said to be axisymmetric if all velocity vectors lie in half-planes passing through some line which is called the symmetry axis. The picture of the liquid flow is the same in all such planes. The field of velocities of an axisymmetric liquid motion is completely described by the plane field taken from any of such half-planes. The symmetry axis is assumed to be the z-axis which is directed vertically downwards. The distance to the oz-axis is denoted by $\rho = \sqrt{x^2+y^2}$, v_z and v_ρ denote the coordinates of the vector of velocity $\vec{v}(v_z, v_\rho)$ which is connected with the velocity potential as follows: $\vec{v}(v_z, v_\rho) = \operatorname{grad}\varphi(z,\rho)$ ([1]–[39]).

Of an infinite set of half-planes we select arbitrarily the one passing through the symmetry axis on which the moving liquid occupies a certain simply connected domain $S(\sigma)$, where $\sigma = z+i\rho$. Some part of its boundary is unknown and should be defined.

The lines of intersection of the surface and the planes passing through the oz-axis of rotation are called meridians, and the lines of intersection with the planes perpendicular to the oz-axis are called parallels.

1.1. **The Notion of a Stream Function for an Axisymmetric Flow.**

Let us cite once again the definition of axisymmetric flow, analogous to that we presented above. The flow is called axisymmetric if the stream planes passing through the given axis, and every such plane has the same picture of distribution of flow lines ([1]–[6]). oz is assumed to be the symmetry axis of the cylindrical system of coordinates ρ, θ, z. Then it follows from the definition that the component of velocity, when the liquid flow is potential, has the form $v_\theta = 0$. Then the equation of continuity takes the form

$$\frac{\partial(\rho v_z)}{\partial z} + \frac{\partial(\rho v_\rho)}{\partial \rho} = 0. \tag{1.1}$$

Differential equation of any stream line for axisymmetric flow, $v_\rho\, dz - v_z\, d\rho = 0$, multiplied by ρ, is a full differential of some stream function $\psi(\rho,z)$, $d\psi = \rho v_\rho\, dz - \rho v_z\, d\rho$. Thus $v_z = \frac{1}{\rho}\frac{\partial\psi}{\partial\rho}$, $v_\rho = -\frac{1}{\rho}\frac{\partial\psi}{\partial z}$. On the other hand, $v_z = \frac{\partial\varphi}{\partial z}$, $v_\rho = \frac{\partial\varphi}{\partial\rho}$, and hence

$$v_z = \frac{\partial\varphi}{\partial z} = +\frac{1}{\rho}\frac{\partial\psi}{\partial\rho}\,, \quad v_\rho = \frac{\partial\varphi}{\partial\rho} = -\frac{1}{\rho}\frac{\partial\psi}{\partial z}\,. \tag{1.2}$$

If the liquid flow is irrotational, i.e. potential, $\frac{\partial v_z}{\partial \rho} = \frac{\partial v_\rho}{\partial z}$, then the stream function should satisfy the equation

$$\frac{\partial}{\partial z}\Big(\frac{1}{\rho}\,\frac{\partial \psi}{\partial z}\Big) + \frac{\partial}{\partial \rho}\Big(\frac{1}{\rho}\,\frac{\partial \psi}{\partial \rho}\Big) = 0. \tag{1.3}$$

Recall that $\varphi(z,\rho)$ is a harmonic function of the cylindrical system of coordinates. Unlike the plane case, the stream function $\psi(z,\rho)$ is not harmonic. It follows from (1.2) that

$$\frac{\partial \varphi}{\partial z}\,\frac{\partial \psi}{\partial z} + \frac{\partial \varphi}{\partial \rho}\,\frac{\partial \psi}{\partial \rho} = 0. \tag{1.4}$$

The system (1.1), (1.3) can be rewritten as

$$\Delta\varphi(z,\rho) + \frac{1}{\rho}\,\frac{\partial \varphi}{\partial \rho} = 0, \tag{1.5}$$

$$\Delta\psi(z,\rho) - \frac{1}{\rho}\,\frac{\partial \psi}{\partial \rho} = 0, \tag{1.6}$$

where Δ is the Laplace operator. We rewrite the system (1.5), (1.6) as follows:

$$\frac{\partial^2 \varphi}{\partial z^2} + 4\alpha\,\frac{\partial^2 \varphi}{\partial \alpha^2} + 4\,\frac{\partial \varphi}{\partial \alpha} = 0, \tag{1.7}$$

$$\frac{\partial^2 \psi}{\partial z^2} + 4\alpha\,\frac{\partial^2 \psi}{\partial \alpha^2} = 0, \tag{1.8}$$

where $\alpha = \rho^2$.

It can be seen from (1.7) and (1.8) that for $\alpha = \rho^2 \neq 0$ the system is elliptic. Hence $\varphi(z,\rho) = \text{const}$ and $\psi(z,\rho) = \text{const}$ are orthogonal. However, the mapping $f(z+i\rho) = \varphi(z,\rho) + i\psi(z,\rho)$ is not conformal. The mappings under consideration constitute a class of quasi-conformal mappings. The system (1.2) is elliptic only in the domains not adjoining the axis of rotation. The system degenerates on that axis and quasi-conformity is violated.

When the point $z+i\rho$ approaches the axis of rotation, the ratio of half-axes of these ellipses infinitely increases. Such violation of quasi-conformity is a geometric criterion of degeneration of a system on the axis of rotation. In the domains whose closure do not intersect the axis of rotation, the mappings $f = \varphi + i\psi$ satisfying the system (1.2) are quasi-conformal, possessing owing to the system (1.2) the principal properties of quasi-conformal mappings ([1]–[39]).

A linear elliptic equation is said to be degenerated if in some part of its domain of definition the quadratic form is defined nonpositively.

It can be seen from (1.7) and (1.8) that the given system for $\alpha = \rho^2 \neq 0$ is elliptic.

Along the oz-axis, as $\alpha \to 0$, we have

$$\frac{\partial^2 \varphi}{\partial z^2} + 4\,\frac{\partial \varphi}{\partial \alpha} = 0, \tag{1.9}$$

$$\frac{\partial^2\psi}{\partial z^2} = 0. \tag{1.10}$$

Along the oz-axis of symmetry we have

$$\begin{gathered}
\lim_{\rho\to 0}\frac{\partial\varphi}{\partial\rho} = 0, \quad \lim_{\rho\to 0}\frac{\partial\psi}{\partial\rho} = 0, \quad \lim_{\rho\to 0}\frac{\partial\psi}{\partial z} = 0, \\
\lim_{\rho\to 0}\frac{1}{\rho}\frac{\partial\varphi}{\partial\rho} = \frac{\partial^2\varphi}{\partial\rho^2}, \quad \lim_{\rho\to 0}\frac{1}{\rho}\frac{\partial\psi}{\partial\rho} = \frac{\partial^2\psi}{\partial\rho^2}.
\end{gathered} \tag{1.11}$$

We map the half-plane $\mathrm{Im}(\zeta) > 0$ (or $\mathrm{Im}(\zeta) < 0$) of the complex plane $\zeta = \xi + i\eta$ conformally onto the domains $S(\sigma)$,

$$\sigma(\zeta) = z(\xi,\eta) + i\rho(\xi,\eta). \tag{1.12}$$

The system (1.2) takes on the plane $\xi + i\eta$ the form

$$\frac{\partial\varphi}{\partial\xi} = \frac{1}{\rho(\xi,\eta)}\frac{\partial\psi}{\partial\eta}, \tag{1.13}$$

$$\frac{\partial\varphi}{\partial\eta} = -\frac{1}{\rho(\xi,\eta)}\frac{\partial\psi}{\partial\xi}, \tag{1.14}$$

that is,

$$\frac{\partial}{\partial\xi}\left[\rho(\xi,\eta)\frac{\partial\varphi(\xi,\eta)}{\partial\xi}\right] + \frac{\partial}{\partial\eta}\left[\rho(\xi,\eta)\frac{\partial\varphi(\xi,\eta)}{\partial\eta}\right] = 0, \tag{1.15}$$

$$\frac{\partial}{\partial\xi}\left[\frac{1}{\rho(\xi,\eta)}\frac{\partial\psi(\xi,\eta)}{\partial\xi}\right] + \frac{\partial}{\partial\eta}\left[\frac{1}{\rho(\xi,\eta)}\frac{\partial\psi(\xi,\eta)}{\partial\eta}\right] = 0. \tag{1.16}$$

From (1.13) and (1.14) follows the condition (1.4).

The boundary conditions have the following forms.

(1) On the free (depression) surface:

$$\varphi(z,\rho) - kz = \mathrm{const}, \tag{1.17}$$

$$\psi(z,\rho) = \mathrm{const}, \tag{1.18}$$

where $k = \mathrm{const}$ is the filtration coefficient;

(2) Along the boundary of water basins:

$$\varphi(z,\rho) = \mathrm{const}, \tag{1.19}$$

$$a_1 z + b_1\rho + c_1 = 0, \quad a_1, b_1, c_1 = \mathrm{const}; \tag{1.20}$$

(3) Along the leaking intervals:

$$\varphi(z,\rho) - kz = \mathrm{const}, \tag{1.21}$$

$$a_2 z + b_2\rho + c_2 = 0, \quad a_2, b_2, c_2 = \mathrm{const}; \tag{1.22}$$

(4) Along the symmetry axis, when a segment of the oz-axis of symmetry coincides with a segment of the boundary of $S(\sigma)$:

$$\rho = 0, \tag{1.23}$$

$$\psi(z,\rho) = 0, \tag{1.24}$$

but if the symmetry axis does not coincide with some part of the boundary of the flow domain $S(\sigma)$, then

$$\rho \neq 0, \quad \rho = \text{const}, \quad const \neq 0, \tag{1.25}$$

$$\psi(z,\rho) = \text{const}, \quad const \neq 0; \tag{1.26}$$

(5) Along impermeable boundaries:

$$\psi(z,\rho) = \text{const}, \tag{1.27}$$

$$a_3 z + b_3 \rho + c_3 = 0, \quad a_3, b_3, c_3 = \text{const}; \tag{1.28}$$

(6) Along the impermeable boundary, the velocity vector is directed along that boundary.

(7) The velocity vector is perpendicular to the boundary of water basins.

(8) Along the free surface (depression curve) we have

$$v_z^2 + v_\rho^2 - k v_z = 0. \tag{1.29}$$

It has been stated in our work [31] that on the plane of complex velocity we have circular polygons of particular types. But this class of problems is wide enough. There are axisymmetric spatial problems with partially unknown boundaries when the boundary of the domain does not involve the symmetry axis, but as is mentioned above, there are problems when the boundary of the domain involves the axis of symmetry or its parts.

For circular polygons, in particular, for linear polygons, we are able to solve plane problems of filtration with partially unknown boundaries. Statement and solution of the corresponding plane problems of filtration with partially unknown boundaries can be found in [26]–[39].

Suppose we have solved the plane problem, i.e. constructed analytic functions by which the half-plane $\mathrm{Im}(\zeta) > 0$ (or $\mathrm{Im}(\zeta) < 0$) of the plane $\zeta = \xi + i\eta$ is mapped conformally onto a circular polygon.

For general discussion we assume that we have a circular polygon with number of vertices m. To find such an analytic function, we have to solve a nonlinear third order Schwarz differential equation whose solution is reduced to that of a differential Fuchs class equation. The Schwarz equation, and hence the corresponding Fuchs class equation involves $2(m-3)$ essential unknown parameters. After integration of the Schwarz equation there appear six additional parameters of integration. To find these parameters, we write a system of $2(m-3)$ higher transcendent equations and a system consisting of six equations. The boundary conditions for the problem of filtration contain additional unknown parameters. Further, using the solutions of the plane problems, we construct solutions $\varphi(\xi,\eta)$, $\psi(\xi,\eta)$ for the systems (1.13)–(1.16) of differential equations of spatial axisymmetric problems. They allow one to construct the functions which map quasiconformally the half-plane $\mathrm{Im}(\zeta) \geq 0$ onto the domain of the complex potential and onto the domains of the complex velocity, i.e. the onto $S(\omega_0)$ and $S(\omega_0'(\zeta)/\sigma'(\zeta))$.

For three analytic functions

$$\sigma(\zeta) = z(\xi,\eta) + i\rho(\xi,\eta), \quad \omega_0(\zeta) = \varphi_0(\xi,\eta) + i\psi_0(\xi,\eta),$$
$$w_0(\zeta) = \omega_0'(\zeta)/\sigma'(\zeta)$$

we introduce the notation

$$\begin{gathered}\Delta z(\zeta,\eta) = 0, \ \ \Delta\rho(\zeta,\eta) = 0, \ \ \Delta\varphi_0(\zeta,\eta) = 0, \\ \Delta\psi_0(\zeta,\eta) = 0, \ \ \mathrm{Im}(\zeta) \geq 0,\end{gathered} \tag{1.30}$$

which map conformally the half-plane $\mathrm{Im}(\zeta) \geq 0$ onto the domain $S(\sigma)$ of liquid motion, the domains of the complex potential $S(\omega_0)$ and the domains of the complex velocity $S(\omega_0'(\zeta)/\sigma'(\zeta))$.

Below for the half-plane we will need the Dirichlet problem. Suppose that on the real axis there is a function $u(\xi)$ bounded by a finite number of points of discontinuity. To find a value at the point $\zeta = \xi + i\eta$ of the harmonic in the upper half-plane function, we have to use the Poisson integral

$$u(\xi) = \frac{1}{\pi} \int\limits_{-\infty}^{+\infty} u(t)\, \frac{\eta}{(t-\xi)^2 + \eta^2}\, dt, \tag{1.31}$$

where $\zeta = \xi + i\eta$.

2. Solution of the System (1.13), (1.14)

We rewrite the system (1.13), (1.14) as follows:

$$\Delta\varphi(\xi,\eta) + a(\xi,\eta)\,\frac{\partial\varphi}{\partial\xi} + b(\xi,\eta)\,\frac{\partial\varphi}{\partial\eta} = 0, \tag{2.1}$$

$$\Delta\psi(\xi,\eta) - a(\xi,\eta)\,\frac{\partial\psi}{\partial\xi} - b(\xi,\eta)\,\frac{\partial\psi}{\partial\eta} = 0, \tag{2.2}$$

where

$$a(\xi,\eta) = \frac{1}{\rho(\xi,\eta)}\,\frac{\partial\rho}{\partial\xi}\,, \quad b(\xi,\eta) = \frac{1}{\rho(\xi,\eta)}\,\frac{\partial\rho}{\partial\eta}\,, \quad \Delta = \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\,. \tag{2.3}$$

Below we will pass to the consideration of the problem of solvability of the system of differential equations (2.1), (2.2) with respect to the functions $\varphi(\xi,\eta)$ and $\psi(\xi,\eta)$ which should satisfy both the compatibility conditions (1.13) and (1.14) and the mixed boundary conditions (1.17)–(1.28) on the known and unknown parts of the boundary. First of all, we replace $\varphi(\xi,\eta)$ and $\psi(\xi,\eta)$ by $\varphi_0(\xi,\eta) + \varphi_1(\xi,\eta)$, $\psi_0(\xi,\eta) + \psi_1(\xi,\eta)$, where $\varphi_0(\xi,\eta)$ and $\psi_0(\xi,\eta)$ are conjugate, harmonic in the domain $\mathrm{Im}(\zeta) > 0$ functions satisfying the boundary conditions. This transformation makes it possible for the unknown functions $\varphi_1(\xi,\eta)$ and $\psi_1(\xi,\eta)$ to satisfy the zero boundary conditions. Note that the system of equations (2.1) and (2.2) will alter hereat.

As is said above, a solution of the system (2.1) and (2.2) will be sought with regard for (1.13) and (1.14) in the form

$$\varphi(\xi,\eta)=\varphi_0(\xi,\eta)+\varphi_1(\xi,\eta), \tag{2.4}$$

$$\psi(\xi,\eta)=\psi_0(\xi,\eta)+\psi_1(\xi,\eta), \tag{2.5}$$

where $\varphi_0(\xi,\eta)$, $\psi_0(\xi,\eta)$ are conjugate harmonic functions,

$$\Delta\varphi_0(\xi,\eta)=0,\quad \Delta\psi_0(\xi,\eta)=0, \tag{2.6}$$

which satisfy the Cauchy–Riemann conditions

$$\frac{\partial\varphi_0}{\partial\xi}=\frac{\partial\psi_0}{\partial\eta},\quad \frac{\partial\varphi_0}{\partial\eta}=-\frac{\partial\psi_0}{\partial\zeta} \tag{2.7}$$

and also all the boundary conditions.

By means of the functions $\omega_0(\xi)=\varphi_0(\xi,\eta)+i\psi_0(\xi,\eta)$, $\sigma(\zeta)=z(\xi,\eta)+i\rho(\xi,\eta)$, the half-plane $\mathrm{Im}(\zeta)>0$ (or $\mathrm{Im}(\zeta)<0$) of the plane $\zeta=\xi+i\eta$ is, as is said above, mapped conformally onto the domains $S(\omega)$, $S(\sigma)$, $S(w)$, where $w(\zeta)=\omega'(\zeta)/\sigma'(\zeta)$. The functions $z(\xi,\eta)$ and $\rho(\xi,\eta)$ should satisfy the conditions

$$\Delta z(\xi,\eta)=0,\quad \Delta\rho(\xi,\eta)=0, \tag{2.8}$$

$$\frac{\partial z}{\partial\xi}=\frac{\partial\rho}{\partial\eta},\quad \frac{\partial z}{\partial\eta}=-\frac{\partial\rho}{\partial\xi}\,. \tag{2.9}$$

The system (2.1), (2.2) can be written with respect to $\varphi_1(\xi,\eta)$, $\psi_1(\xi,\eta)$ as follows:

$$\begin{aligned}\Delta\varphi_1(\xi,\eta)+a(\xi,\eta)\,\frac{\partial\varphi_1(\xi,\eta)}{\partial\xi}+b(\xi,\eta)\,\frac{\partial\varphi_1(\xi,\eta)}{\partial\eta}=\\ =-\Big[\Delta\varphi_0(\xi,\eta)+a(\xi,\eta)\,\frac{\partial\varphi_0}{\partial\xi}+b(\xi,\eta)\,\frac{\partial\varphi_0}{\partial\eta}\Big],\end{aligned} \tag{2.10}$$

$$\begin{aligned}\Delta\psi_1(\xi,\eta)-a(\xi,\eta)\,\frac{\partial\psi_1(\xi,\eta)}{\partial\xi}-b(\xi,\eta)\,\frac{\partial\psi_1(\xi,\eta)}{\partial\eta}=\\ =-\Big[\Delta\psi_0(\xi,\eta)-a(\xi,\eta)\,\frac{\partial\psi_0}{\partial\xi}-b(\xi,\eta)\,\frac{\partial\psi_0}{\partial\eta}\Big].\end{aligned} \tag{2.11}$$

To simplify our investigation and solution of the system (2.10), (2.11), we have deliberately left in the right-hand sides of (2.10) and (2.11) the terms $\Delta\varphi_0(\xi,\eta)$, $\Delta\psi_0(\xi,\eta)$ which are, according to (2.6), equal to zero.

Transforming the unknown functions $\varphi_1(\xi,\eta)$, $\psi_1(\xi,\eta)$, $\varphi_0(\xi,\eta)$, $\psi_0(\xi,\eta)$ as

$$\varphi_1(\xi,\eta)=\rho^{-1/2}(\xi,\eta)\varphi_2(\xi,\eta),\quad \psi_1(\xi,\eta)=\rho^{1/2}(\xi,\eta)\psi_2(\xi,\eta), \tag{2.12}$$

$$\varphi_0(\xi,\eta)=\rho^{-1/2}(\xi,\eta)\varphi_2^*(\xi,\eta),\quad \psi_0(\xi,\eta)=\rho^{1/2}(\xi,\eta)\psi_2^*(\xi,\eta), \tag{2.13}$$

we obtain

$$\begin{aligned}
\frac{\partial\varphi_1}{\partial\xi} &= -\frac{1}{2}\,\rho^{-3/2}\,\frac{\partial\rho}{\partial\xi}\,\varphi_2(\xi,\eta) + \rho^{-1/2}\,\frac{\partial\varphi_2}{\partial\xi}\,,\\
\frac{\partial\varphi_1}{\partial\eta} &= -\frac{1}{2}\,\rho^{-3/2}\,\frac{\partial\rho}{\partial\eta}\,\varphi_2(\xi,\eta) + \rho^{-1/2}\,\frac{\partial\varphi_2}{\partial\eta}\,,
\end{aligned} \tag{2.14}$$

$$\begin{aligned}
\frac{\partial^2\varphi_1}{\partial\xi^2} &= \frac{3}{4}\,\rho^{-5/2}\Big(\frac{\partial\rho}{\partial\xi}\Big)^2 - \frac{1}{2}\,\rho^{-3/2}\,\frac{\partial^2\rho}{\partial\xi^2}\,\varphi_2-\\
&\quad -\frac{1}{2}\,\rho^{-3/2}\,\frac{\partial\rho}{\partial\xi}\,\frac{\partial\varphi_2}{\partial\xi} - \frac{1}{2}\,\rho^{-3/2}\,\frac{\partial\rho}{\partial\xi}\,\frac{\partial\varphi_2}{\partial\xi} + \rho^{-1/2}\,\frac{\partial^{\varphi}_2}{\partial\xi^2}\,,
\end{aligned} \tag{2.15}$$

$$\begin{aligned}
\frac{\partial^2\varphi_1}{\partial\eta^2} &= \frac{3}{4}\,\rho^{-5/2}\Big(\frac{\partial\rho}{\partial\eta}\Big)^2\varphi_2 - \frac{1}{2}\,\rho^{-3/2}\,\frac{\partial^2\rho}{\partial\eta^2}\,\varphi_2-\\
&\quad -\frac{1}{2}\,\rho^{-3/2}\,\frac{\partial\rho}{\partial\eta}\,\frac{\partial\varphi_2}{\partial\eta} - \frac{1}{2}\,\rho^{-3/2}\,\frac{\partial\rho}{\partial\eta}\,\frac{\partial\varphi_2}{\partial\eta} + \rho^{-1/2}\,\frac{\partial^{\varphi}_2}{\partial\eta^2}\,,
\end{aligned} \tag{2.16}$$

$$\begin{aligned}
\Delta\varphi_1 &= \frac{3}{4}\,\rho^{-5/2}\Big[\Big(\frac{\partial\rho}{\partial\xi}\Big)^2 + \Big(\frac{\partial\rho}{\partial\eta}\Big)^2\Big]\varphi_2-\\
&\quad -\rho^{-3/2}\Big(\frac{\partial\rho}{\partial\xi}\,\frac{\partial\varphi_2}{\partial\xi} + \frac{\partial\rho}{\partial\eta}\,\frac{\partial\varphi_2}{\partial\eta}\Big) + \rho^{-1/2}\Delta\varphi_2,
\end{aligned} \tag{2.17}$$

$$\begin{aligned}
\Delta\varphi_1 + \frac{1}{\rho}\,\Big(\frac{\partial\rho}{\partial\xi}\,\frac{\partial\varphi_2}{\partial\xi} + \frac{\partial\rho}{\partial\eta}\,\frac{\partial\varphi_2}{\partial\eta}\Big) &= \frac{3}{4}\,\rho^{-5/2}\Big[\Big(\frac{\partial\rho}{\partial\xi}\Big)^2 + \Big(\frac{\partial\rho}{\partial\eta}\Big)^2\Big]\varphi_2-\\
&\quad -\rho^{-3/2}\Big(\frac{\partial\rho}{\partial\xi}\,\frac{\partial\varphi_2}{\partial\xi} + \frac{\partial\rho}{\partial\eta}\,\frac{\partial\varphi_2}{\partial\eta}\Big) + \rho^{-1/2}\Delta\varphi_2+\\
&\quad +\frac{1}{\rho}\,\frac{\partial\rho}{\partial\xi}\,\Big(-\frac{1}{2}\,\rho^{-3/2}\,\frac{\partial\rho}{\partial\xi}\,\varphi_2 + \rho^{-1/2}\,\frac{\partial\varphi_2}{\partial\xi}\Big)+\\
&\quad +\frac{1}{\rho}\,\frac{\partial\rho}{\partial\eta}\,\Big(-\frac{1}{2}\,\rho^{-3/2}\,\frac{\partial\rho}{\partial\eta}\,\varphi_2 + \rho^{-1/2}\,\frac{\partial\varphi_2}{\partial\eta}\Big),
\end{aligned} \tag{2.18}$$

$$\begin{aligned}
\Delta\varphi_1 + \frac{1}{\rho}\,\Big(\frac{\partial\rho}{\partial\xi}\,\frac{\partial\varphi_1}{\partial\xi} + \frac{\partial\rho}{\partial\eta}\,\frac{\partial\varphi_1}{\partial\eta}\Big) &=\\
&= \rho^{-1/2}\Big\{\Delta\varphi_2 + \frac{1}{4}\,\Big[\Big(\frac{\partial\rho}{\partial\xi}\Big)^2 + \Big(\frac{\partial\rho}{\partial\eta}\Big)^2\Big]\varphi_2\Big\},
\end{aligned} \tag{2.19}$$

$$\Delta\psi_1 - \frac{1}{\rho}\,\Big(\frac{\partial\rho}{\partial\xi}\,\frac{\partial\psi_1}{\partial\xi}+\frac{\partial\rho}{\partial\eta}\,\frac{\partial\psi_1}{\partial\eta}\Big) = -\Big[\Delta\psi_0 - \frac{1}{\rho}\Big(\frac{\partial\rho}{\partial\xi}\,\frac{\partial\psi_0}{\partial\eta}+\frac{\partial\rho}{\partial\eta}\,\frac{\partial\psi_0}{\partial\eta}\Big)\Big], \tag{2.20}$$

$$\begin{aligned}
\frac{\partial\psi_1}{\partial\xi} &= \frac{1}{2}\,\rho^{-1/2}\,\frac{\partial\rho}{\partial\xi}\,\psi_2 + \rho^{1/2}\,\frac{\partial\psi_2}{\partial\xi}\,, \quad \frac{\partial\psi_1}{\partial\eta} =\\
&= \frac{1}{2}\,\rho^{-1/2}\,\frac{\partial\rho}{\partial\eta}\,\psi_2 + \rho^{1/2}\,\frac{\partial\psi_2}{\partial\eta}\,,
\end{aligned} \tag{2.21}$$

$$\begin{aligned}
\frac{\partial^2\psi_1}{\partial\xi^2} &= -\frac{1}{4}\,\rho^{-3/2}\Big(\frac{\partial\rho}{\partial\xi}\Big)^2\psi_2 + \frac{1}{2}\rho^{-1/2}\,\frac{\partial^2\psi_2}{\partial\xi^2} + \frac{1}{2}\,\rho^{-1/2}\frac{\partial\rho}{\partial\xi}\,\frac{\partial\psi_2}{\partial\xi}+\\
&\quad +\frac{1}{2}\,\rho^{-1/2}\frac{\partial\rho}{\partial\xi}\,\frac{\partial\psi_2}{\partial\xi} + \rho^{1/2}\frac{\partial^2\psi_2}{\partial\xi^2}\,,
\end{aligned} \tag{2.22}$$

$$\frac{\partial^2\psi_1}{\partial\eta^2} = -\frac{1}{4}\,\rho^{-3/2}\Big(\frac{\partial\rho}{\partial\eta}\Big)^2\psi_2 + \frac{1}{2}\rho^{-1/2}\,\frac{\partial^2\psi_2}{\partial\eta^2}\,\psi_2 + \frac{1}{2}\,\rho^{-1/2}\frac{\partial\rho}{\partial\eta}\,\frac{\partial\psi_2}{\partial\eta} + $$
$$+\frac{1}{2}\,\rho^{-1/2}\frac{\partial\rho}{\partial\eta}\,\frac{\partial\psi_2}{\partial\eta} + \rho^{1/2}\frac{\partial^2\psi_2}{\partial\eta^2}\,, \tag{2.23}$$
$$\Delta\psi_1 = -\frac{1}{4}\,\rho^{-3/2}\Big[\Big(\frac{\partial\rho}{\partial\xi}\Big)^2 + \Big(\frac{\partial\rho}{\partial\eta}\Big)^2\Big]\psi_2 +$$
$$+\rho^{-1/2}\Big(\frac{\partial\rho}{\partial\xi}\,\frac{\partial\psi_2}{\partial\xi} + \frac{\partial\rho}{\partial\eta}\,\frac{\partial\psi_2}{\partial\eta}\Big) + \rho^{-1/2}\Delta\psi_2, \tag{2.24}$$
$$\Delta\psi_1 - \frac{1}{\rho}\,\Big(\frac{\partial\rho}{\partial\xi}\,\frac{\partial\psi_1}{\partial\xi} + \frac{\partial\rho}{\partial\eta}\,\frac{\partial\psi_1}{\partial\eta}\Big) = -\frac{1}{4}\,\rho^{-3/2}\Big[\Big(\frac{\partial\rho}{\partial\xi}\Big)^2 + \Big(\frac{\partial\rho}{\partial\eta}\Big)^2\Big]\psi_2 +$$
$$+\rho^{-1/2}\Big(\frac{\partial\rho}{\partial\xi}\,\frac{\partial\psi_2}{\partial\xi} + \frac{\partial\rho}{\partial\eta}\,\frac{\partial\psi_2}{\partial\eta}\Big) + \rho^{1/2}\Delta\psi_2, \tag{2.25}$$
$$\Delta\psi_1 - \frac{1}{\rho}\,\frac{\partial\rho}{\partial\xi}\,\Big(\frac{1}{2}\,\rho^{-1/2}\,\frac{\partial\rho}{\partial\xi}\,\psi_2 + \rho^{1/2}\,\frac{\partial\psi_2}{\partial\xi}\Big) -$$
$$-\frac{1}{\rho}\,\frac{\partial\rho}{\partial\eta}\,\Big(\frac{1}{2}\,\rho^{-1/2}\,\frac{\partial\rho}{\partial\eta}\,\psi_2 + \rho^{1/2}\,\frac{\partial\psi_2}{\partial\eta}\Big) =$$
$$= -\frac{1}{4}\,\rho^{-3/2}\Big[\Big(\frac{\partial\rho}{\partial\xi}\Big)^2 + \Big(\frac{\partial\rho}{\partial\eta}\Big)^2\Big]\psi_2 +$$
$$+\rho^{-1/2}\Big(\frac{\partial\rho}{\partial\xi}\,\frac{\partial\psi_2}{\partial\xi} + \frac{\partial\rho}{\partial\eta}\,\frac{\partial\psi_2}{\partial\eta}\Big) + \rho^{1/2}\Delta\psi_2, \tag{2.25$_1$}$$
$$\Delta\psi_1 = \rho^{1/2}\Big\{\Delta\psi_2 - \frac{3}{4}\,\frac{1}{\rho^2}\,\Big[\Big(\frac{\partial\rho}{\partial\xi}\Big)^2 + \Big(\frac{\partial\rho}{\partial\eta}\Big)^2\Big]\psi_2\Big\}. \tag{2.26}$$

Taking into account (2.13), we represent the functions $\varphi_0(\xi,\eta)$ and $\psi_0(\xi,\eta)$ analogously to (2.19) and (2.26) with respect to $\varphi_2^*(\xi,\eta)$, $\psi_2^*(\xi,\eta)$ and obtain

$$\Delta\varphi_0 + \frac{1}{\rho}\,\Big(\frac{\partial\rho}{\partial\xi}\,\frac{\partial\varphi_0}{\partial\xi} + \frac{\partial\rho}{\partial\eta}\,\frac{\partial\varphi_0}{\partial\eta}\Big) =$$
$$= \rho^{-1/2}\Big\{\Delta\varphi_2^* + \frac{1}{4}\,\rho^{-2}\Big[\Big(\frac{\partial\rho}{\partial\xi}\Big)^2 + \Big(\frac{\partial\rho}{\partial\eta}\Big)^2\Big]\varphi_2^*\Big\}, \tag{2.27}$$
$$\Delta\psi_0 - \frac{1}{\rho}\,\Big(\frac{\partial\rho}{\partial\psi}\,\frac{\partial\psi_0}{\partial\xi} + \frac{\partial\rho}{\partial\eta}\,\frac{\partial\psi_0}{\partial\eta}\Big) =$$
$$= \rho^{1/2}\Big\{\Delta\psi_2^* - \frac{3}{4}\,\rho^{-2}\Big[\Big(\frac{\partial\rho}{\partial\xi}\Big)^2 + \Big(\frac{\partial\rho}{\partial\eta}\Big)^2\Big]\psi_2^*\Big\}, \tag{2.28}$$

where $\Delta\varphi_0 = 0$, $\Delta\psi_0 = 0$.

Bearing in mind (2.19), (2.27), (2.26) and (2.28), we represent the system (2.10) and (2.11) as follows:

$$\rho^{-1/2}\big[\Delta\varphi_2 + \lambda\rho_1\varphi_2\big] = -\rho^{-1/2}\Big[\Delta\varphi_2^* + \frac{1}{4}\,\rho_1\varphi_2^*\Big], \tag{2.29}$$
$$\rho^{1/2}\big[\Delta\psi_2 - \mu\rho_1\psi_2\big] = -\rho^{1/2}\Big[\Delta\psi_2^* - \frac{3}{4}\,\rho_1\psi_2^*\Big], \tag{2.30}$$

where $\rho_1 = \frac{1}{\rho^2}\left[\left(\frac{\partial\rho}{\partial\xi}\right)^2 + \left(\frac{\partial\rho}{\partial\eta}\right)^2\right]$, $\lambda = \frac{1}{4}$, $\mu = \frac{3}{4}$.

The equalities (2.29) and (2.30) can be rewritten in the form

$$\Delta\varphi_2 + \frac{1}{4}\rho_1\varphi_2 = -\Big[\Delta\varphi_2^* + \frac{1}{4}\,\rho_1\varphi_2^*\Big], \tag{2.31}$$

$$\Delta\psi_2 - \frac{3}{4}\rho_1\psi_2 = -\rho^{-1/2}\Big[\Delta\psi_2^* - \frac{3}{4}\,\rho_1\psi_2^*\Big]. \tag{2.32}$$

Assuming that $\varphi_2^*(\xi,\eta)$ and $\psi_2^*(\xi,\eta)$ are known functions, we rewrite the equations (2.31), (2.32) as

$$\Delta(\varphi_2 + \varphi_2^*) = -\frac{1}{4}\,\rho_1(\varphi_2 + \varphi_2^*) \equiv f_1^*(\xi,\eta), \tag{2.33}$$

$$\Delta(\psi_2 + \psi_2^*) = \frac{3}{4}\,\rho_1(\psi_2 + \psi_2^*) \equiv f_2^*(\xi,\eta). \tag{2.34}$$

Consider the Poisson equation

$$\Delta u(\xi,\eta) = f_1^*(\xi,\eta), \quad (\xi_1,\eta_1) \in \mathrm{Im}(\zeta) > 0. \tag{2.35}$$

Define the function $u(\xi,\eta)$ by the formula

$$u(\xi,\eta) = -\frac{1}{2\pi}\iint\limits_{\mathrm{Im}(\zeta)>0} G(\xi,\eta;x,y) f_1(x,y)\,dx\,dy, \tag{2.36}$$

where

$$G(\xi,\eta;x,y) = \frac{1}{4\pi}\,\ln\frac{(\xi-x)^2 + (\eta+y)^2}{(\xi-x)^2 + (\eta-y)^2} \tag{2.37}$$

is Green's function of the Dirichlet problem for the harmonic in $\mathrm{Im}(\zeta) > 0$ function, while the function $f_1^*(\xi,\eta)$ is bounded and has continuous first derivatives bounded in $\mathrm{Im}(\zeta) > 0$, $U(\xi,\eta)$ is a regular solution of the Poisson equation (2.35). It is proved that (2.36) satisfies the boundary condition [4]

$$\lim_{(\xi,\eta)\to(\xi_0,\eta_0)} u(\xi,\eta) = 0, \quad (\xi,\eta) \in \mathrm{Im}(\zeta) > 0, \quad (\xi_0,\eta_0) \in \mathrm{Im}(\zeta_0). \tag{2.38}$$

Using (2.35) and (2.36) with respect to (2.33) and (2.34), we obtain

$$\varphi_2(\xi,\eta) = -\varphi_2^*(\xi,\eta) + \frac{1}{2\pi}\iint\limits_{\mathrm{Im}(\zeta)>0} G(\xi,\eta;x,y) f_1^*(x,y)\,dx\,dy, \tag{2.39}$$

$$\psi_2(\xi,\eta) = -\psi_2^*(\xi,\eta) + \frac{1}{2\pi}\iint\limits_{\mathrm{Im}(\zeta)>0} G(\xi,\eta;x,y) f_2^*(x,y)\,dx\,dy. \tag{2.40}$$

The equalities (2.39) and (2.40) can be written as follows:

$$\varphi_2(\xi,\eta) = -\rho^{1/2}\varphi_0(\xi,\eta) +$$

$$+\frac{1}{8\pi}\iint\limits_{\mathrm{Im}(\zeta)>0} G(\xi,\eta;x,y)\rho_1(x,y)\big[\varphi_2(x,y) + \rho^{1/2}\varphi_0(x,y)\big]\,dx\,dy, \tag{2.41}$$

$$\psi_2(\xi,\eta) = -\psi^*(\xi,\eta) + \\ + \frac{3}{8\pi} \iint\limits_{\mathrm{Im}(\zeta)>0} G(\xi,\eta;x,y)\rho_1(x,y)\big[\psi_2(x,y)+\psi_2^*(x,y)\big]\,dx\,dy. \qquad (2.42)$$

We rewrite the equations (2.40) and (2.42) in the form

$$\varphi_2(\xi,\eta) = f_3(\xi,\eta) + \frac{1}{8\pi} \iint\limits_{\mathrm{Im}(\zeta)>0} G(\xi,\eta;x,y)\rho_1(x,y)\varphi_2(x,y)\,dx\,dy, \qquad (2.43)$$

$$\psi_2(\xi,\eta) = f_4(\xi,\eta) + \frac{3}{8\pi} \iint\limits_{\mathrm{Im}(\zeta)>0} G(\xi,\eta;x,y)\rho_1(x,y)\psi_2(x,y)\,dx\,dy, \qquad (2.44)$$

where

$$f_3(\xi,\eta) = -\,\rho^{1/2}\varphi_0(\xi,\eta) + \\ + \frac{1}{8\pi} \iint\limits_{\mathrm{Im}(\zeta)>0} G(\xi,\eta;x,y)\rho_1(x,y)\rho^{1/2}\varphi_0(x,y)\,dx\,dy, \qquad (2.45)$$

$$f_4(\xi,\eta) = -\,\psi^*(\xi,\eta) + \\ + \frac{3}{8\pi} \iint\limits_{\mathrm{Im}(\zeta)>0} G(\xi,\eta;x,y)\rho_1(x,y)\psi_2^*(x,y)\,dx\,dy. \qquad (2.46)$$

Thus we have obtained the second kind Fredholm's integral equations (2.43) and (2.44) with respect to $\varphi_2(\xi,\eta)$ and $\psi_2(\xi,\eta)$. The problems (2.33) and (2.34) are, respectively, equivalent to the integral equations (2.43) and (2.44) which will be solved by using exact methods.

Solutions of the integral equations (2.43) and (2.44) will be sought by the method of successive approximations in the form of the series

$$\varphi_2(\xi,\eta) = \sum_{n=0}^{\infty} \lambda^n \varphi_{2(n)}(\xi,\eta), \qquad (2.47)$$

$$\psi_2(\xi,\eta) = \sum_{n=0}^{\infty} \mu^n \psi_{2(n)}(\xi,\eta), \qquad (2.48)$$

where $\lambda = \frac{1}{8\pi}$, $\mu = \frac{3}{8\pi}$.

Substituting the series (2.47) and (2.48) respectively into the integral equations and equating the coefficients with the same powers of the parameters λ and μ, we obtain

$$\varphi_{2(0)}(\xi,\eta) = f_3(\xi,\eta), \qquad (2.49)$$

. . . .

$$\varphi_{2(n)}(\xi,\eta) = \iint\limits_{\mathrm{Im}(\zeta)>0} G(\xi,\eta;x,y)\rho_1(x,y)\varphi_{2(n-1)}(x,y)\,dx\,dy, \qquad (2.50)$$

. . . .

$$\psi_{2(0)}(\xi,\eta) = f_4(\xi,\eta), \tag{2.51}$$

. . . .

$$\psi_{2(n)}(\xi,\eta) = \iint\limits_{\operatorname{Im}(\zeta)>0} G(\xi,\eta;x,y)\rho_1(x,y)\psi_{2(n-1)}(x,y)\,dx\,dy, \tag{2.52}$$

. . . .

$$n = 1,2,3,\dots .$$

The parameters $\lambda = \frac{1}{8\pi}$ and $\mu = \frac{3}{8\pi}$ of the integral equations (2.43) and (2.44) are small enough; this ensures the convergence of the series (2.47) and (2.48). Recall here that as initial approximations, as usual, have been taken the free terms $f_3(\xi,\eta)$ and $f_4(\xi,\eta)$.

Basing on (2.47)–(2.52), we can construct general formulas which allow one to express any approximations through the free terms by means of iterated kernels.

Assuming that the series (2.47) and (2.48) are constructed, we can multiply them respectively by $\rho^{-1/2}$ and $\rho^{1/2}$. We obtain

$$\varphi(\xi,\eta) = \varphi_0(\xi,\eta) + \varphi_1(\xi,\eta), \tag{2.53}$$

$$\psi(\xi,\eta) = \psi_0(\xi,\eta) + \psi_1(\xi,\eta). \tag{2.54}$$

Recall that the boundary conditions along the oz-axis of symmetry, when some parts of oz coincide with the boundary $S(\sigma)$, have in the coordinates (z,ρ) the form

$$\begin{gathered} \rho \to 0, \quad \alpha = \rho^2, \\ \frac{\partial\varphi}{\partial\rho} = \frac{\partial\varphi}{\partial\alpha}\frac{\partial\alpha}{\partial\rho} = \frac{\partial\varphi}{\partial\alpha}\cdot 2\rho \to 0; \quad \frac{\partial\psi}{\partial\rho} = \frac{\partial\psi}{\partial\alpha}\frac{\partial\alpha}{\partial\rho} = \frac{\partial\psi}{\partial\alpha}\cdot 2\rho \to 0. \end{gathered} \tag{2.55}$$

In the coordinates (ξ,η), the boundary conditions along the oz-axis have the form

$$\begin{gathered} \rho(\xi,\eta) = 0, \quad \frac{\partial\rho}{\partial\xi} = 0, \quad \frac{\partial\rho}{\partial\eta} = 0; \\ \Big(\frac{\partial\varphi}{\partial\xi}\Big)_{\rho\to 0} = \Big(\frac{\partial\varphi}{\partial z}\frac{\partial z}{\partial\zeta} + \frac{\partial\varphi}{\partial\alpha}\frac{\partial\alpha}{\partial\rho}\frac{\partial\rho}{\partial\xi}\Big)_{\rho\to 0} = \Big(\frac{\partial\varphi}{\partial z}\frac{\partial z}{\partial\xi}\Big)_{\rho\to 0}, \\ \Big|\frac{\partial\varphi}{\partial\alpha}\frac{\partial\alpha}{\partial\rho}\frac{\partial\rho}{\partial\xi}\Big|_{\rho\to 0} \to 0, \end{gathered} \tag{2.56}$$

$$\begin{gathered} \Big(\frac{\partial\varphi}{\partial\rho}\Big)_{\rho\to 0} = \Big(\frac{\partial\varphi}{\partial\alpha}\frac{\partial\alpha}{\partial\rho}\Big)_{\rho\to 0} \to 0, \\ \Big(\frac{\partial\varphi}{\partial\eta}\Big) = \Big(\frac{\partial\varphi}{\partial z}\frac{\partial z}{\partial\eta} + \frac{\partial\varphi}{\partial\alpha}\frac{\partial\alpha}{\partial\rho}\frac{\partial\rho}{\partial\eta}\Big)_{\rho\to 0} = \Big(\frac{\partial\varphi}{\partial z}\frac{\partial z}{\partial\eta}\Big)_{\rho\to\infty} \quad \frac{\partial\rho}{\partial\eta} \to 0. \end{gathered} \tag{2.57}$$

Suppose that the oz-axis of symmetry (or its parts) does not coincide with the boundary of the filtration domain $S(\sigma)$,

$$\rho = \text{const} \neq 0, \quad \alpha = \rho^2,$$

$$\frac{\partial\varphi}{\partial\rho} = \frac{\partial\varphi}{\partial\alpha}\,\frac{\partial\alpha}{\partial\rho} = \frac{\partial\varphi}{\partial\alpha}\cdot 2\rho = \frac{\partial\varphi}{\partial\alpha}\cdot 2\,\text{const}, \quad \rho \to \text{const}\,. \tag{2.58}$$

$$\Big(\frac{\partial\psi}{\partial\alpha}\,\frac{\partial\alpha}{\partial\rho}\Big)_{\rho\to\text{const}} = \Big(\frac{\partial\psi}{\partial\alpha}\Big)\cdot 2\,\text{const}, \quad (\psi)_{\rho=\text{const}} = \text{const},$$

$$\Big(\frac{\partial\psi}{\partial\xi}\Big)_{\rho\to\text{const}} \to 0, \quad \Big(\frac{\partial\psi}{\partial\eta}\Big)_{\rho\to\text{const}} \to 0,$$

$$\Big(\frac{\partial\varphi}{\partial\eta}\Big)_{\rho=\text{const}} = \Big(\frac{\partial\varphi}{\partial z}\,\frac{\partial z}{\partial\eta} + \frac{\partial\varphi}{\partial\alpha}\,\frac{\partial\alpha}{\partial\rho}\,\frac{\partial\rho}{\partial\eta}\Big)_{\rho=\text{const}} = \Big(\frac{\partial\varphi}{\partial z}\,\frac{\partial z}{\partial\eta}\Big)_{\rho=\text{const}}, \tag{2.59}$$

$$\frac{\partial\rho}{\partial\eta} = 0.$$

Below we present another way of solution of the system (2.10) and (2.11).

1^0. Green's function belongs to the class of fundamental solutions of the Laplace equation. It is determined as a harmonic function of a pair of points $(P;Q)$, is symmetric with respect to P and Q, equals to zero on the boundary and is analytic at all points P of the domain D_i, except of the points $P=Q$ at which it has logarithmic singularity, i.e. at the point P of the neighborhood of Q the relation

$$G(P;Q) = \frac{1}{2\pi}\,\ln r(P;Q) + g(P;Q) \tag{2.60}$$

is fulfilled, where $r = \sqrt{(x-\xi)^2 + (y-\eta)^2}$ is the distance between the points P and Q. Moreover, Green's function, as a function of P, should have everywhere inside of D_i, except the point of Q, continuous derivatives up to the second order and satisfy the Laplace equation, while on the boundary it should satisfy the limiting condition. Next, $G(P;Q)$, as function of P, should have singularity at the point Q corresponding to the initial charge (or to the mass) concentrated at the point Q. Green's function of the Laplace operator for the plane simply connected domain under the limiting condition $U_\ell = 0$ is tightly connected with the function which transforms conformally the above-mentioned domain onto the circle $|W| \leq 1$.

$G(P;Q)$ is a harmonic in the domain D_i function of the coordinates x and y ([4], [17], [33]–[36]).

If d is the diameter of the domain D_i, then the inequality

$$0 \leq G(P;Q) \leq \ln\Big(\frac{d}{r}\Big) \tag{2.61}$$

is valid. Green's function for the circle of radius $R=1$ has the form

$$G(P;Q) = \frac{1}{2\pi}\,\ln\Big(\frac{\rho r_1}{r}\Big), \tag{2.62}$$

where $\rho = \sqrt{\xi^2 + \eta^2}$ is the distance of the point $Q(\xi, \eta)$ from the center of the circle. r_1 is defined as follows: $r_1 = \sqrt{(x - \xi/\rho^2)^2 + (y - \eta/\rho^2)^2}$.

2^0. Consider the inhomogeneous equation

$$\Delta U(x, y) = -\varphi(x, y). \tag{2.63}$$

We seek for a solution of (2.63) continuous up to the contour of the domain and satisfying the limiting equation $U\big|_\ell = 0$. There may be only one such solution ([35]).

The unknown solution has the form

$$U(x, y) = \iint\limits_{D_i} G(x, y; \xi, \eta)\varphi(\xi, \eta)\, d\xi\, d\eta, \tag{2.64}$$

that is,

$$U(x, y) = \frac{1}{2\pi} \iint\limits_{D_i} \varphi(\xi, \eta) \ln \frac{1}{r}\, d\xi\, d\eta + \\ + \iint\limits_{D_i} g(x, y; \xi, \eta)\varphi(\xi, \eta) \ln \frac{1}{r}\, d\xi\, d\eta, \tag{2.65}$$

otherwise.

The first summand of (2.65) has inside of D_i continuous derivatives up to the second order, and its Laplace operator is equal to $[-\varphi(\xi, \eta)]$. It is proved that the second summand of (2.65) can be differentiated with respect to the coordinates (x, y) of the point $P(x, y)$ as many times as desired under the integral sign. This implies that this summand is a function harmonic inside of D_i, because $g(P; Q)$ is a harmonic function of the point $P(x, y)$. $g(P; Q)$ is a harmonic function of the point Q with limiting values $\left(\frac{1}{2\pi} \ln r\right)$, where $r = \sqrt{(x - \xi)^2 + (y - \eta)^2}$. It is assumed that $P(x, y)$ is inside of D_i. The formula (2.65) provides us with the solution of the equation (2.63) satisfying the condition $U\big|_\ell = 0$. Recall that there exists a generalized solution of (2.63).

3^0. The linear-fractional conformal mapping of the half-plane $\mathrm{Im}(\zeta) > 0$ onto the circle $|W| < 1$ has the form

$$W = \frac{1 + i\zeta}{i + zt}, \quad \zeta = \xi + i\eta, \quad w = u + iv. \tag{2.66}$$

It follows from (2.66) that

$$u = \frac{2\xi}{\xi^2 + (1 + \eta)^2}, \quad v = \frac{\xi^2 + \eta^2 - 1}{\xi^2 + (1 + \eta)^2}. \tag{2.67}$$

On the other hand, from (2.66) we have

$$\zeta = \frac{i + w}{1 + iw}, \quad \xi = \frac{2u}{u^2 + (1 - v)^2}, \quad \eta = \frac{1 - v^2 - u^2}{u^2 + (1 - v)^2}. \tag{2.68}$$

4^0. Harmonic and analytic functions of a complex variable.

Let

$$w = f(z) = u(x,y) + iv(x,y), \quad z = x + iy \tag{2.69}$$

be some function of the complex variable $z = x + iy$; u and v are the real functions of the variables x and y. The Cauchy–Riemann conditions $u_x = v_y$, $u_y = -v_x$ are necessary and sufficient for the function to be analytic. It follows from these conditions that $\Delta U = 0$, $\Delta V = 0$, where Δ is the Laplace operator.

Consider the transformation

$$x = x(u,v), \quad y = y(u,v), \tag{2.70}$$

$$u = u(x,y), \quad v = v(x,y), \tag{2.71}$$

where $u(x,y)$, $v(x,y)$ are conjugate harmonic functions. Then the above transformation is equivalent to (2.69).

By virtue of the Cauchy–Riemann conditions, the relations [36]

$$u_x^2 + u_y^2 = u_x^2 + v_x^2 = v_y^2 + v_x^2 = |f'(z)|^2, \quad u_x v_x + u_y v_y = 0 \tag{2.72}$$

should be satisfied for the functions u and v.

Let us find out how the Laplace operator changes under that transformation. We obtain

$$U = U\big[x(u,v), y(u,v)\big] = \widetilde{U}(u,v), \tag{2.73}$$

$$U_{xx} + U_{yy} = (\widetilde{U}_{uu} + \widetilde{V}_{uu})|f'(z)|^2, \tag{2.74}$$

whence it follows that, as a result of the transformation $w = f(z) = u + iv$, the function $U(x,y)$, harmonic in the domain G, transforms into the function $\widetilde{U} = \widetilde{U}(u,v)$, harmonic in the domain G', if only $|f'(z)|^2 \neq 0$.

Consider the equations

$$\Delta\varphi + \frac{1}{4}\,\rho_1\varphi = 0, \tag{2.75}$$

$$\Delta\psi - \frac{3}{4}\,\rho_1\psi = 0, \tag{2.76}$$

where $\rho_1 = \frac{1}{\rho^2}\big[\big(\frac{\partial\rho}{\partial x}\big)^2 + \big(\frac{\partial\rho}{\partial y}\big)^2\big]$. The transformations

$$\varphi = \varphi\big[x(u,v), y(u,v)\big] = \widetilde{\varphi}(u,v), \tag{2.77}$$

$$\psi = \psi\big[x(u,v), y(u,v)\big] = \widetilde{\psi}(u,v) \tag{2.78}$$

result in the equalities

$$\varphi_{xx} + \varphi_{yy} + \frac{1}{4}\,\frac{1}{\rho^2(x,y)}\Big[\Big(\frac{\partial\rho}{\partial x}\Big)^2 + \Big(\frac{\partial\rho}{\partial y}\Big)^2\Big]\varphi(x,y) = \Big\{\widetilde{\varphi}_{uu} + \widetilde{\varphi}_{vv} +$$

$$+\frac{1}{4}\,\frac{1}{\widetilde{\rho}(u,v)}\Big[\Big(\frac{\partial\widetilde{\rho}}{\partial u}\Big)^2 + \Big(\frac{\partial\widetilde{\rho}}{\partial v}\Big)^2\Big]\widetilde{\varphi}(u,v)\Big\}|f'(z)|^2 = 0, \quad |f'(z)|^2 \neq 0, \tag{2.79}$$

$$\psi_{xx} + \psi_{yy} - \frac{3}{4}\,\frac{1}{\rho^2(x,y)}\Big\{\Big(\frac{\partial\rho}{\partial x}\Big)^2 + \Big(\frac{\partial\rho}{\partial y}\Big)^2\Big\}\psi(x,y) =$$

$$= \left\{ \widetilde{\psi}_{uu} + \widetilde{\psi}_{vv} - \frac{3}{4} \frac{1}{\widetilde{\rho}(u,v)} \left[\left(\frac{\partial \widetilde{\rho}}{\partial u} \right)^2 + \left(\frac{\partial \widetilde{\rho}}{\partial v} \right)^2 \right] \widetilde{\varphi}(u,v) \right\} |f'(z)|^2 = 0, \quad (2.80)$$

since $|f'(z)| \neq 0$, and hence from (2.79) and (2.80) we have

$$\widetilde{\varphi}_{uu} + \widetilde{\varphi}_{vv} + \frac{1}{4} \frac{1}{\widetilde{\rho}^2} \left[\left(\frac{\partial \widetilde{\rho}}{\partial u} \right)^2 + \left(\frac{\partial \widetilde{\rho}}{\partial v} \right)^2 \right] \widetilde{\varphi} = 0, \quad (2.81)$$

$$\widetilde{\psi}_{uu} + \widetilde{\psi}_{vv} - \frac{3}{4} \frac{1}{\widetilde{\rho}^2} \left[\left(\frac{\partial \widetilde{\rho}}{\partial u} \right)^2 + \left(\frac{\partial \widetilde{\rho}}{\partial v} \right)^2 \right] \widetilde{\psi} = 0. \quad (2.82)$$

It follows from the above-said that using the transformations (2.68), (2.71) and Green's function (2.62), we can reduce the problem (2.31) (or (2.32)) to the solution of Fredholm's integral equation of second kind, where the given functions, the kernel and the right-hand side are defined in the domain of the unit circle. In this case, for the convergence of Neumann's series we can indicate a simpler condition. In particular, if the kernel is bounded, then for the convergence of Neumann's series there exist more plausible condition.

In hydrodynamics, there exists the method of sources and channels. This method has been for the first time applied by Rankin to the spatial problem of body streamline. The method consists in the replacement of the body streamline by such a system of sources and channels that the body surface is one of the stream surfaces; note that the algebraic sum of abundance sources should be equal to zero. The choice of a system of sources and channels by means of a preassigned surface form of a body streamline is of great mathematical difficulty ([1-3], [7]).

Below we will present an algorithm of finding the functions $\varphi_0(\xi,\eta)$, $\psi_0(\xi,\eta)$, $z(\xi,\eta)$ and $\rho(\xi,\eta)$. Recall that the plane of liquid motion coincides with that of the complex variable $\sigma = z + i\rho$, $i = -\sqrt{-1}$.

In the domain $S(\sigma)$ with the boundary $\ell(\sigma)$ we seek for a complex potential (i.e., a potential divided by the filtration coefficient) $\omega(\sigma) = \varphi_0(z,\rho) + i\psi_0(z,\rho)$. The velocity potential $\varphi_0(z,\rho)$ and the stream function $\psi_0(z,\rho)$ satisfy the Cauchy–Riemann conditions and the boundary conditions

$$a_{kj}\varphi_0(z,\rho) + a_{k2}\psi_0(z,\rho) + ak_3 z + a_{k4}\rho = f_k, \quad k = 1,2, \quad (z,\rho) \in \ell(\sigma), \quad (2.83)$$

where a_{kj}, f_k, $k = 1,2$, $j = 1,\dots,n$, are given piecewise constant real functions; f_k, $k = 1,2$, depend on an unknown parameter Q of the filtrated liquid discharge.

Using (2.60), we can find a part of the boundary $\ell(\omega_0)$ of $S(\omega_0)$ and the boundary $\ell(w_0)$ of the domain of complex velocity $S(w_0)$, where $w_0(z) = d\omega_0/d\sigma = \frac{\omega_0'(\zeta)}{\sigma'(\zeta)}$, excluding some coordinates of those vertices of circular polygons which are connected with cut ends. By means of the functions $\omega_0(\sigma)$ and $w(\sigma)$, we map conformally the domain $\ell(\sigma)$ with the boundary $\ell(\sigma)$ onto the domains $S(\omega_0)$ and $S(w)$. The domain $S(w)$ is a circular polygon.

Angular points of the boundaries $\ell(\sigma)$, $\ell(\omega_0)$ and $\ell(w)$ which can be encountered at least at one of them when passing in the positive direction we denote by A_k, $k=1,\dots,n$.

The half-plane $\mathrm{Im}(\zeta)>0$ of the plane $\zeta=\xi+i\eta$ is mapped conformally onto the domains $S(\sigma)$, $S(\omega_0)$ and $S(w_0)$. We denote the corresponding mapping functions by $\sigma(\zeta)$, $\omega_0(\zeta)$, $w(\zeta_0)=\omega_0'(\zeta)/\sigma'(\zeta)$, $d\omega_0(\zeta)/d\zeta=\omega_0'(\zeta)$, $d\sigma(\zeta)/d\zeta=\sigma'(\zeta)$. To the angular points A_k, $k=1,\dots,n$, there correspond the points $\zeta=e_k$, $k=1,\dots,n$, along the axis t with $-\infty<e_1<e_2<\cdots<e_{n-1}<e_n<+\infty$, and $\xi=e_{n+1}=0$ is mapped into the nonangular point A_∞ of the boundary $\ell(\sigma)$ which is located between the points A_n and A_1.

3. Construction of the Functions $d\omega_0(\zeta)/d\sigma(\zeta)$, $\omega_0(\zeta)$, $\sigma(\zeta)$

We denote by $\sigma(\xi)=z(\xi)+i\rho(\xi)$, $\omega_0(\xi)=\varphi_0(\xi)+i\psi_0(\xi)$, $w_0(\xi)=u_0(\xi)+iv_0(\xi)$ the boundary values of the functions $\sigma(\zeta)$, $\omega_0(\zeta)$ and $w_0(\zeta)$ as $\zeta\to\xi$, $\zeta\in\mathrm{Im}(\zeta)>0$. By $\overline{\sigma}(\xi)$, $\overline{\omega}_0(\xi)$ and $\overline{w}_0(\xi)$ we denote the complex conjugate functions corresponding to the functions $\sigma(\xi)$, $\omega_0(\xi)$ and $w_0(\xi)$.

Introduce the vectors $\Phi_0(\xi)=[\omega_0(\xi),\sigma(\xi)]$, $\overline{\Phi}_0(\xi)=[\overline{\omega}_0(\xi),\overline{\sigma}(\xi)]$, $\Phi_0'(\xi)=[\omega_0'(\xi),\sigma'(\xi)]$, $\overline{\Phi}_0{}'(\xi)=[\overline{\omega}_0{}'(\xi),\overline{\sigma}{}'(\xi)]$, $f(\xi)=[f_1(\xi),f_2(\xi)]$. Then the boundary conditions ([26]–[31])

$$\Phi_0(\xi)=g(\xi)\overline{\Phi}_0(\xi)+i\cdot 2G^{-1}f(\xi),\ \ -\infty<\xi<+\infty, \tag{3.1}$$

where $G^{-1}(\xi)\overline{G}(\xi)=g(\xi)$ is a piecewise constant nonsingular second rank matrix with the points of discontinuity $\xi=e_k$, $k=1,\dots,n$; $G^{-1}(\xi)$ and $\overline{G}(\xi)$ are, respectively, the inverse and complex conjugate matrices to the matrix $G(\xi)$, and the vector $f(\xi)$ is defined by means of (2.83).

Differentiating (3.1) along the boundary ξ, we obtain

$$\Phi_0'(\xi)=g(\xi)\Phi_0'(\xi),\ \ -\infty<\xi<+\infty. \tag{3.2}$$

It can be verified that the equality $\overline{g}(\xi)=\overline{G}^{\,-1}(\xi)G(\xi)$ holds. For the points $\xi=e_j$, $j=1,\dots,n$, let us consider the characteristic equation ([1]–[31])

$$\det\left[g_{j+1}^{-1}(e_j+0)g_j(e_j-0)-\lambda E\right] \tag{3.3}$$

with respect to the parameter λ, where E is the unit matrix, $g_j(\xi)$, $e_j<\xi<e_{j+1}$, $g_{j+1}^{-1}(e_j+0)$, $g_j(e_j-0)$ are the limiting values of the matrices $g_{j+1}^{-1}(\xi)$, $g_i(\xi)$ at the point $\xi=e_j$ from the right and from the left, respectively.

By means of the roots λ_{kj} of the equation (3.3) we define uniquely the numbers $\alpha_{kj}=(2\pi i)^{-1}\ln\lambda_{kj}$, $k=1,2$; $j=1\dots,n$ ([1]–[30]).

Suppose that among the points A_k, $k=1\dots,n$, of the boundaries $\ell(\sigma)$ and $\ell(\omega_0)$ there exist removable points to which on the boundary $\ell(w_0)$ of $S(w_0)$ there correspond regular nonangular points ([26]–[30]).

For the sake of simplicity we assume that the removable singular point coincides with the point $\xi=e_j$ to which on the boundaries $\ell(\sigma)$ and $\ell(\omega_0)$ there correspond the angles $\pi/2$, while on the boundary $\ell(w_0)$ the angle π.

To remove this point from the homogeneous boundary conditions (3.2), we introduce a new unknown vector $\Phi_1(\xi)$ ([26]–[30])

$$\Phi_0'(\xi) = \chi_1(\xi)\Phi_1(\xi), \tag{3.4}$$

where

$$\chi_1(\xi = \sqrt{\frac{\xi - e_{j-1}}{\xi - e_j}}\,. \tag{3.5}$$

After the passage from the vector $\Phi'(\xi)$ to $\Phi_1(\xi)$, we multiply the matrix $g_i(\xi)$ by (-1).

The boundary conditions with respect to $\Phi_1(\xi)$ take the form

$$\Phi_1(\xi) = g^*(\xi)\overline{\Phi}_1(\xi), \tag{3.6}$$

where

$$g^*(\xi) = [\chi_1(\xi)]^{-1} g(\xi)[\overline{\chi}_1(\xi)]. \tag{3.7}$$

On the contour $\ell(w_0)$ we renumerate singular points and denote them by a_j, $j = 1, \dots, m$. We denote the characteristic points defined uniquely and corresponding to the points $t = a_j$ again by α_{kj}, $k = 1, 2$, $j = 1, \dots, m$. They satisfy the Fuchs condition.

Now we write the Fuchs class equation ([1]–[39])

$$u''(\xi) + p(\xi)u'(\zeta) + g(\xi)u(\xi) = 0, \tag{3.8}$$

where

$$p(\xi) = \sum_{j=1}^{m} (1 - \alpha_{1j} - \alpha_{2j})(\xi - a_j)^{-1}, \tag{3.9}$$

$$q(\xi) = \sum_{j=1}^{m} \left[\alpha_{1j}\alpha_{2j}(\xi - a_j)^{-2} + c_j(\xi - a_j)^{-1}\right], \tag{3.10}$$

where c_j are unknown real accessory parameters satisfying the conditions

$$N_1 = \sum_{j=1}^{m} c_j = 0. \tag{3.11}$$

By means of matrices, we write the equation (3.8) in the form of the system ([26]–[39])

$$\chi'(\xi) = \chi(\xi)\Phi(\xi), \tag{3.12}$$

$$\chi(\xi) = \begin{pmatrix} u_1(\xi) & u_1'(\xi) \\ u_2(\xi) & u_2'(\xi) \end{pmatrix}, \quad \Phi(\xi) = \begin{pmatrix} 0 & -g(\xi) \\ 1 & -p(\xi) \end{pmatrix}. \tag{3.13}$$

Further, using linearly independent solutions $u_1(\xi)$ and $u_2(\xi)$ of the equation (3.8), we construct the general solution

$$w(\xi) = \frac{pu_1(\xi) + qu_2(\xi)}{ru_1(\xi) + su_2(\xi)}\,, \tag{3.14}$$

of the Schwarz equation ([26]–[30])

$$\{w;\xi\} = \frac{w'''(\xi)}{w'(\xi)} - 1,5\left(\frac{w'(\xi)}{w'(\xi)}\right)^2 = R(\xi), \tag{3.15}$$

where

$$R(\xi) = 2q(\xi) - p'(\xi) - 0,5[p(\xi)]^2 =$$
$$= \sum_{j=1}^{m}\Big\{0,5\big[1-(\alpha_{1j}-\alpha_{2j})^2\big](\zeta-a_j)^{-2} + c_j^*(\xi-a_j)^{-1}\Big\}, \tag{3.16}$$

$$\alpha_{1j} - \alpha_{2j} = \nu_j, \quad j = 1,\dots,m,$$
$$c_j^* = 2c_i - \beta_j \sum_{k=1,\,k\neq j}^{m} \beta_k(a_j-a_k)^{-1}, \tag{3.17}$$
$$\beta_k = 1 - a_{1k} - a_{2k}, \quad k = 1,\dots,m,$$

p, q, r and s are constants of integration of (3.14) which satisfy the condition $ps-rq\neq 0$, $\pi\nu_j$ is the interior angle at the vertex B_j of the circular polygon.

Using (3.14), we map conformally the half-plane $\mathrm{Im}(\zeta)>0$ (or $\mathrm{Im}(\zeta)<0$) onto the domain $S(w)$ with the boundary $\ell(w)$. Expanding the functions $R(\zeta)$ into the serie of powers of $1/\zeta$, we obtain

$$R(\zeta) = \sum_{k=1}^{\infty} M_k\zeta^{-k}. \tag{3.18}$$

Since the point $\zeta=\infty$ is the image of the nonsingular point of the boundary $\ell(\sigma)$, the conditions ([1]–[31])

$$M_1 = \sum_{k=1}^{m} c_k^* = 0, \tag{3.19}$$

$$M_2 = \sum_{k=1}^{m}\big[a_kc_k^* + 0,5(1-\nu_k^2)\big] = 0, \tag{3.20}$$

$$M_3 = \sum_{k=1}^{m}\big[a_k^2c_k^* + a_k(1-\nu_k^2)\big] = 0 \tag{3.21}$$

should be fulfilled. From the condition (3.19) follows (3.11), and vice versa. We can obtain the conditions (3.20) and (3.21) in somewhat different way and in another form. Taking into account (3.12), the conditions (3.19)–(3.21) allow one to define three parameters c_j, $j=1,2,3$, of the parameters c_j, $j=1,\dots,m$. Moreover, we choose arbitrarily three of the parameters $t=a_j$, $j=1,\dots,m$ and fix them according to the Riemann theorem. Therefore $R(\zeta)$ defined by the formula (3.16) will depend on $2(m-3)$ unknown parameters a_j, c_j, $j=1,\dots,m-3$. The equation (3.18) near the point $\xi=a_j$ can be rewritten as [26–31]

$$(\zeta-a_j)^2u''(\xi) + (\xi-a_j)p_j(\xi)u'(\xi) + q_j(\xi)u(\xi) = 0, \tag{3.22}$$

where

$$p_j(\xi) = p_{0j} + \sum_{n=1}^{\infty} p_{nj}(\xi - a_j)^n, \tag{3.23}$$

$$p_{nj} = (-1)^{n-1} \sum_{k=1,\, k\neq j}^{m} \beta_k (a_j - a_k)^{-n}, \quad \beta_k = 1 - \alpha_{1k} - al_{2k}, \tag{3.24}$$

$$q_j = \alpha_{1j}\alpha_{2j} + c_j(\xi - a_j) + \sum_{n=2}^{\infty} q_{nj}(\xi - a_j)^n, \tag{3.25}$$

$$q_{nj} = (-1)^{n-2} \sum_{k=2,\, k\neq j}^{m} \big[\alpha_{1k}\alpha_{2k}(n-1) + \\ + c_k(a_j - a_k)\big](a_j - a_k)^{-n}, \quad n = 2,3,\dots, \tag{3.26}$$

$$q_{0j} = \alpha_{1j}\alpha_{2j}, \quad q_{1j} = c_j, \ \ j = 1,\dots,m. \tag{3.27}$$

4. Local Solutions

Local solutions of the equation (3.32) for the points $\xi = a_j$, $j = 1,\dots,m$, are sought in the form

$$u_j(\xi) = (\xi - a_j)^{\alpha_j}\widetilde{u}_j(\xi), \quad \widetilde{u}_j(\xi) = 1 + \sum_{n=1}^{\infty} \gamma_{nj}(\xi - a_j)^n, \tag{4.1}$$

where γ_{0j}, $n = 1,\dots,\infty$, $j = 1,\dots,m$, are defined by the recurrence formulas ([26]–[31])

$$f_{0j}(\alpha_j) = \alpha_j(\alpha_{j-1}) + p_{nj}\alpha_j + q_{0j} = 0, \tag{4.2}$$

$$\gamma_{1j} f_{0j}(\alpha_j + 1) + f_{1j}(\alpha_j) = 0, \tag{4.3}$$

$$\gamma_{2j} f_{0j}(\alpha_j + 2) + \gamma_{1j} f_{1j}(\alpha_j + 1) + f_{2j}(\alpha_j) = 0, \tag{4.4}$$

. . . .

where

$$f_n(\alpha_j) = \alpha_{1j} p_{nj} + q_{nj}. \tag{4.5}$$

If the difference $\alpha_{1j} - \alpha_{2j}$, $j = 1,\dots,m$, is noninteger, then using the formulas (4.3)–(4.5), we construct the linearly independent solutions (3.32),

$$u_{ki}(\xi) = (\xi - a_j)^{\alpha_{kj}}\widetilde{u}_{kj}(\xi), \\ \widetilde{u}_{kj}(\xi) = 1 + \sum_{n=1}^{\infty} \gamma_{nj}^{k}(\xi - a_j)^n, \ \ k = 1,2, \ \ j = 1,\dots,m. \tag{4.6}$$

However, if $\alpha_{1j} - \alpha_{2j} = n$, $n = 0,1,2$, then $u_{1j}(\xi)$ is constructed by the formulas (4.3)–(4.5), while $u_{2j}(\xi)$ by the Frobenius method ([24], [26]–[31]). Note that for $\alpha_{1j} - \alpha_{2j} = 0$, the function $u_{2j}(\zeta)$ has the form

$$u_{2j}(\xi) = u_{1j}(\xi)\ln(\xi - a_j) + (\xi - a_j)^{\alpha_{1j}} \sum_{n=1}^{\infty} \gamma_{nj}^{2}(\xi - a_j)^n, \tag{4.7}$$

where

$$\gamma_{nj}^2 = \Big\{\frac{d\gamma_{1j}(\alpha_j)}{d\alpha_j}\Big\}_{\alpha_j=\alpha_{2j}}.$$

If $\alpha_{1j} - \alpha_{2j} = n$, $n = 1, 2$, then for the construction of $u_{2j}(\xi)$ we have to differentiate the equality

$$u_{2j}(\xi) = (\xi - a_j)^{\alpha_j}\Big[\alpha_j - \alpha_{2j} + \sum_{n=1}^{\infty}\gamma_{nj}^2(\xi - a_j)^n\Big] \tag{4.8}$$

with respect to α_j, then take $\alpha_j \to \alpha_{2j}$, and obtain

$$u_{2j}(\xi) = (\xi - a_j)^{\alpha_{2j}}\Big[\sum_{n=1}^{\infty}\lim_{\alpha_j\to\alpha_{2j}}\gamma_{nj}(\alpha_i)(t - a_j)^n\Big]\ln(t - a_j) +$$

$$+ (t - a_j)^{\alpha_{2j}}\Big\{1 + \sum_{n=1}^{\infty}\Big[\frac{d\gamma_{1j}(\alpha_j)}{d\alpha_j}\Big]_{\alpha_j=\alpha_{2j}}(t - a_j)^n\Big\}. \tag{4.9}$$

P. Ya. Polubarinova–Kochina has proved that a solution for the cut end $u_{2j}(\xi)$, where $\alpha_{1j} - \alpha_{2j} = 2$, does not involve a logarithmic term. Moreover, for such points she also obtained an algebraic equation connecting the parameters a_j, c_j, $j = 1, \dots, m$. To construct $u_{2j}(\xi)$ uniquely, we suggested in our works the following method. For the point $t = a_j$, the equality (4.4) fails to be fulfilled since

$$f_{0j}(\alpha_j + 2) = 0, \ \ \alpha_j \to \alpha_{2j}. \tag{4.10}$$

For the equality (4.4) to take place as $\alpha_j \to \alpha_{2j}$, it will be necessary and sufficient to require the condition

$$\gamma_{1j}f_1(\alpha_j + 1) + f_2(\alpha_j) = 0, \ \ \alpha_{1j} \to \alpha_{2j} + 2. \tag{4.11}$$

After simplification, (4.11) takes the form ([26]–[31])

$$q_{2j} + q_{1j}^2 + q_{1j}p_{1j} = 0. \tag{4.12}$$

To construct $u_{2j}(\xi)$ uniquely, it suffices to construct $\gamma_{2j}^2(\alpha_{2j})$ and then make use of the formulas (4.3)–(4.5) ([26]–[39]). Indeed, suppose $\alpha_{1j} \neq \alpha_{2j}$. Then using (4.4), we find $\gamma_{2j}(\alpha_j)$ and obtain

$$\gamma_{2j}(\alpha_j) = -\frac{\gamma_{1j}(\alpha_j)f_{1j}(\alpha_j + 1) + f_{2j}(\alpha_j)}{f_0(\alpha_j + 2)}\,. \tag{4.13}$$

In the formula (4.13) we remove uncertainty and then pass to the limit $\alpha_j \to \alpha_{2j}$. We have

$$\gamma_{2j}^2 = -0,5\big[p_{1j}(p_{1j} + 2q_{1j}) + p_{2j}\big]. \tag{4.14}$$

Then we define local solutions near the point $t = \infty$. The functions $p(\xi)$ and $q(\xi)$ near the point $t = \infty$ can be represented in the form

$$p(\xi) = \xi^{-1}\sum_{n=0}^{\infty}p_{n\infty}\xi^{-4}, \quad q(\xi) = \xi^{-2}\sum_{n=0}^{\infty}q_{n\infty}\xi^{-n}, \tag{4.15}$$

where

$$p_{n\infty} = \sum_{k=1}^{m} \beta_k a_k^n, \quad p_{0\infty} = 6, \tag{4.16}$$

$$q_{n\infty} = \sum_{k=1}^{m} \big[\alpha_{1k}\alpha_{2k}(n+1) + c_k a_k\big] a_k^n. \tag{4.17}$$

Local solutions near the point $\xi = \infty$ have the form

$$u_\infty(\xi) = \xi^{-\infty} \sum_{n=1}^{\infty} \gamma_{n\infty} \xi^{-(\alpha_\infty + n)}, \tag{4.18}$$

where $\gamma_{n\infty}$, $n = 1, \dots, \infty$, are defined by the formulas

$$f_{0\infty}(\alpha_\infty) = \alpha_\infty(\alpha_\infty + 1) - p_{0\infty}\alpha_\infty + q_{0\infty} = 0, \tag{4.19}$$

$$\alpha_{1\infty} f_{0\infty}(\alpha_\infty + 1) - p_{1\infty}\alpha_\infty + q_{1\infty} = 0, \tag{4.20}$$

$$\alpha_{2\infty} f_{0\infty}(\alpha_\infty + 2) + \gamma_{1\infty}(\alpha_\infty + 1) - p_{2\infty}\alpha_\infty + q_{2\infty} = 0, \tag{4.21}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

where

$$f_{k\infty} = q_{k\infty} - (\alpha_\infty + k) p_{k\infty}. \tag{4.22}$$

Since $t = \infty$ is the image of the nonangular point, the equation (4.19) should have the roots $\alpha_{1\infty} = 3$, $\alpha_{2\infty} = 2$. Consequently,

$$q_{0\infty} = \sum_{k=1}^{m} \big[\alpha_{1k}\alpha_{2k} + a_k c_k\big] = 6. \tag{4.23}$$

As far as $\alpha_{1\infty} - \alpha_{2\infty} = 1$, the equations (4.20)–(4.22) allow one to define only one solution $u_{1\infty}(\xi)$. To find $u_{2\infty}(\xi)$ as $\alpha_\infty \to \alpha_{2\infty}$, it is necessary and sufficient that the condition

$$q_{1\infty} - p_{1\infty}\alpha_{2\infty} = 0 \tag{4.24}$$

takes place. To determine $\gamma_{1\infty}^2$, we act as follows. By virtue of (4.20), for $\alpha_{1\infty} \neq \alpha_{2\infty}$, we find $\gamma_{1\infty}$ and obtain

$$\gamma_{1\infty} = \frac{p_{1\infty}\alpha_\infty - q_{1\infty}}{f_{0\infty}(\alpha_\infty + 1)}\,. \tag{4.25}$$

Since the numerator and the denominator in (4.25) vanish as $\alpha_\infty \to \alpha_{2\infty}$, we have to remove uncertainty. We obtain ([26]–[31]) uniquely

$$\gamma_{1\infty}^2 = p_{1\infty}. \tag{4.26}$$

Having defined $\gamma_{1\infty}^2$, we can find the remaining $\gamma_{n\infty}^2$, $n = 2, \dots, \infty$, by using the formulas (4.20)–(4.22). Consequently, $u_{2\infty}(\xi)$ is defined uniquely. Finally, we obtain

$$u_{k\infty}(\xi) = \xi^{-\alpha_{k\infty}} + \sum_{n=1}^{\infty} \gamma_{n\infty}^k \xi^{-\alpha_{k\infty} - n}, \quad k = 1, 2. \tag{4.27}$$

The system (3.19), (3.20), (3.21) coincides respectively with the system (3.11), (4.23), (4.24), and vice versa.

Local solutions $u_{kj}(\xi)$, $k=1,2$, $j=1,\dots,m$, contain multi-valued functions of which we choose one-valued branches

$$\exp\big[\beta_{kj}\ln(\xi-a_j)\big]>0,\quad t>a_j,$$
$$\Big\{\exp\big[\alpha_{kj}\ln(\xi-a_j)\big]\Big\}^{-1}=\exp\big[i\pi\alpha_{kj}\big]\Big\{\exp\ln(a_j-\xi)\Big\},\ \ a_j-\xi>0,$$
$$\Big\{\exp\big[\alpha_{kj}\ln(\xi-a_j)\big]\Big\}^{-1}=\exp\big[-i\pi\alpha_{kj}\big]\Big\{\exp\big[\alpha_{kj}\ln(a_j-\xi)\big]\Big\},\quad a_j-t>0.$$

For the equation (3.8), in the neighborhood of every singular point $\xi=a_j$, $j=1,\dots,m+1$, and in the neighborhood of the points $t=a_1^*=(a_j+a_{j+1})/2$, $j=1,\dots,m-1$, we construct respectively $u_{kj}(\xi)$, $k=1,2$, $j=1,\dots,m+1$ and $\gamma_{kj}(\xi)$, $k=1,2$, $j=1,\dots,m-1$.

A solution of (3.6) is sought by means of the matrix $TX(\xi)$, where $\chi(\xi)$ is a solution of (3.12). If $\chi(\xi)$ is a solution of (3.12), then $TX(\xi)$ is likewise a solution of (3.12), where the constant matrix is defined as

$$T=\begin{pmatrix}p & q\\ r & s\end{pmatrix},\quad \det T\neq 0, \tag{4.28}$$

p, q, r and s are constants of integration of the equation (3.14).

5. Fundamental Matrices

The local fundamental matrices $\Theta_j(\xi)$, $\sigma_j(\xi)$, $\Theta_j^*(\xi)$, $\overline{\Theta}_j(\xi)$, where $\overline{\Theta}_j(\xi)$ is the matrix complex-conjugate to the matrix $\Theta_j(\xi)$, are defined as follows:

$$\Theta_j(\xi)=\begin{pmatrix}u_{1j}(\xi) & u'_{1j}(\xi)\\ u_{2j}(\xi) & u'_{2j}(\xi)\end{pmatrix},\quad a_j<\xi<a_{j+1},\ \ j=1,\dots,j-1, \tag{5.1}$$

$$\Theta_j^*(\xi)=\begin{pmatrix}u_{1j}^*(\xi) & u'^*_{1j}(\xi)\\ u_{2j}^*(\xi) & u'^*_{2j}(\xi)\end{pmatrix},\quad a_{j-1}<\xi<a_j, \tag{5.2}$$

$$\sigma_j(\xi)=\begin{pmatrix}\sigma_{1j}(\xi) & \sigma'_{1j}(\xi)\\ \sigma_{2j}(\xi) & \sigma'_{2j}(\xi)\end{pmatrix},\quad \xi=\frac{a_j+a_{j+1}}{2}=a_j^*,\ j=1,\dots,m-1, \tag{5.3}$$

$$\Theta_j^*(\xi)=\vartheta_j^{\pm}\Theta_j^*(\xi),\quad a_{j-1}<\xi<a_j, \tag{5.4}$$

$$\Theta_\infty(\xi)=\begin{pmatrix}u_{1\infty}(\xi) & u'_{1\infty}(\xi)\\ u_{2\infty}(\xi) & u'_{2\infty}(\xi)\end{pmatrix}, \tag{5.5}$$

$$\begin{gathered}\vartheta_j^{\pm}=\begin{pmatrix}\exp(\pm i\pi\alpha_{1j}) & 0\\ 0 & \exp(\pm i\pi\alpha_{2j})\end{pmatrix},\\ \overline{\vartheta}_j^{+}=\vartheta_j^{-},\ \ \alpha_{1j}-\alpha_{2j}\neq n,\ \ n=0,1,2,\end{gathered} \tag{5.6}$$

while if $\alpha_{1j}-\alpha_{2j}=n$, $n=0,1,2$, we have

$$\vartheta_j^{\pm}=\exp\big[\pm i\pi\alpha_{2j}\big]\begin{pmatrix}1 & 0\\ \pm i\pi & 1\end{pmatrix},\quad n=0,2, \tag{5.7}$$

$$\vartheta_j^{\pm} = \exp\big[\pm i\pi\alpha_{2j}\big]\begin{pmatrix} -1 & 0 \\ \mp\pi i & 1 \end{pmatrix}, \ \ n = 1. \tag{5.8}$$

One Essential Remark. The fact that the series $u_{kj}(\xi)$, $k = 1,2$, $j = 1,\dots,m$, converge weakly, makes the process of calculations difficult. To remove this drawback, we act as follows ([26]–[31]). We replace the series $u_{kj}(\xi)$, $k = 1,2$, $j = 1,\dots,m+1$, by strongly and uniformly convergent functional series. Towards this end, it suffices to write the series $u_{kj}(\xi)$, $k = 1,2$, $j = 1,\dots,m+1$, in a somewhat different form:

$$\begin{gathered} u_{kj}(\xi) = (\xi - a_j)^{\alpha_{kj}}\widetilde{u}_{kj}(\xi - a_j), \\ \widetilde{u}_{kj}(\xi - a_j) = 1 + \sum_{n=1}^{\infty}\gamma_{nj}^k(\xi - a_j), \ \ k = 1,2; \ \ j = 1,\dots,m, \end{gathered} \tag{5.9}$$

$$u_{k\infty}(\xi) = \xi^{-\alpha_{k\infty}}\Big(1 + \sum_{n=1}^{\infty}\gamma_{n\infty}^k(\xi)\Big), \tag{5.10}$$

where γ_{nj}^k, $\gamma_{n\infty}^k$ are defined through $f_{nj}(\alpha_j)$ and $f_{k\infty}(\alpha_j)$ as follows:

$$f_{nj}\big[(\xi - a_j),\beta_k\big] = \alpha_{kj}p_{nj}(\xi - a_j) + q_{mj}(\xi - a_j), \tag{5.11}$$

$$p_{nj}(\xi - a_j) = -\sum_{k=0,\,k\neq j}^{m}\beta_j\Big(\frac{\xi - a_j}{a_j - a_k}\Big)^n, \ \ n = 1,2,\dots, \tag{5.12}$$

$$q_{1j}(\xi - a_j) = c_j(\xi - a_j), \tag{5.13}$$

$$\begin{gathered} q_{nj}(\xi - a_j) = (-1)^{n-2}\sum_{k=1,\,k\neq j}^{m}\big[\alpha_{1k}\alpha_{2k}(n-1) + c_k(a_j - a_k)\big]\times \\ \times\Big(\frac{\xi - a_j}{a_j - a_k}\Big)^n, \ \ n = 1,2,\dots, \end{gathered} \tag{5.14}$$

$$\Big|\frac{\xi - a_j}{a_j - a_k}\Big| < 1, \ \ k \neq j,$$

$$\begin{gathered} p_{n\infty}(\xi) = \sum_{k=1}^{\infty}\beta_k\Big(\frac{a_k}{\xi}\Big)^n, \\ q_{n\infty}(\xi) = \sum_{k=1}^{\infty}\big[\alpha_{1j}\alpha_{2j}(n+1) + c_k a_k\big]\Big(\frac{a_k}{\xi}\Big)^n, \ \ n = 01,2,\dots. \end{gathered} \tag{5.15}$$

The local matrix $\Theta_j^-(\xi)$ is complex conjugate to the matrix $\Theta_j^+(\xi)$. The real matrices $\Theta_j(\xi)$, $\Theta_j^*(\xi)$ are local solutions of the system of equations (3.22). Suppose that the elements of these matrices converge on some part of the interval $a_{j-1} < \xi < a_j$, on which the matrices $\Theta_j^*(\xi)$ and $\Theta_{j-1}(\xi)$ are connected by the following matrix identity ([26]–[31]):

$$\Theta_j^*(\xi) = T_{j-1}\Theta_{j-1}(\xi), \ \ j = m, m-1,\dots,2, \tag{5.16}$$

from which the matrices T_{j-1} are defined uniquely. Assume also that the domains of convergence of the matrices $\Theta_j^*(\xi)$ and $\Theta_{j-1}(\xi)$ do not intersect. In this case, we construct at the point $\xi = a_j^* = (a_{j-1} + a_j)/2$ the matrix $\sigma_j(\xi)$ which converges in the interval $a_{j-1} < \xi < a_j$. It is seen that one can always pass from the matrix $\Theta_j^*(\xi)$ to the matrix $\Theta_{j-1}(\xi)$ with the following sequence:

$$\Theta_j^*(\xi) = T_{a_j}\sigma_j(\xi), \quad \sigma_j(\xi) = T_{j-1}^*(\xi)\Theta_{j-1}(\xi). \tag{5.17}$$

It follows from the above-said that $\Theta_m(\xi)$ can be analytically continued along the whole axis ξ.

To define the functions $\omega_0'(\xi)$ and $z'(\xi)$ in the interval $(-\infty, +\infty)$, we consider the matrices ([26]–[31])

$$\chi^{\pm}(\xi) = T\Theta_m^*(\xi), \ \ \xi > a_m; \quad \Theta_m^+(\xi) = \Theta_m^-(\zeta), \ \ a_m < \xi < +\infty. \tag{5.18}$$

From (5.18) we have $T = \overline{T}$.

We continue the matrix (5.18) along the real axis ξ and use the notation

$$\chi^*(\xi) = \chi(\xi), \quad \vartheta_j^+ = \vartheta_j.$$

We obtain

$$\begin{aligned}
&\chi(\xi) = T\vartheta_m\Theta_m^*(\xi), \ \ a_{m-1} < \xi < a_m,\\
&\chi(\xi) = T\vartheta_m T_{m-1}\Theta_{m-1}(\xi), \ \ a_{m-1} < \xi < a_m,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\chi(\xi) = T\vartheta_m T_{m-1}\vartheta_{m-1}\Theta_{m-1}^*(\xi)\cdots T_1\vartheta_1\Theta_1^*(\xi), \ \ \xi < a_1,\\
&\chi(\xi) = T\vartheta_m T_{m-1}\vartheta_{m-1}\cdots T_1\vartheta_1 T_{-\infty}\Theta_\infty(\xi), \ \ -\infty < \xi < \infty,
\end{aligned} \tag{5.19}$$

where

$$\begin{aligned}
\Theta_m^*(\xi) &= T_{m-1}\Theta_{m-1}(\xi), \ \ a_{m-1} < \xi < a_m,\\
\Theta_{m-1}^*(\xi) &= T_{m-2}\Theta_{m-2}(\xi), \ \ a_{m-2} < \xi < a_{m-1},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
\Theta_2^*(\xi) &= T_1\Theta_1(\xi), \ \ a_1 < \xi < a_2,\\
\Theta_1^*(\xi) &= T_{-\infty}\Theta_\infty(\xi), \ \ -\infty < \xi < a_1,\\
\Theta_m(\xi) &= T_m\Theta_\infty(\xi), \ \ a_m < \xi < +\infty.
\end{aligned} \tag{5.20}$$

(5.20) allows one to determine the matrices $T_1, T_2, \dots, T_{m-1}, T_{-\infty}, T_{+\infty}$. Substituting the matrices (5.19) into the boundary conditions (3.6) and then multiplying successively every matrix equality from the left by $[\Theta_j^*(\xi)]^{-1}$, $j = m, m-1, \dots, 1$, we obtain the system of matrix equations ([26]–[31])

$$\begin{aligned}
T\vartheta_m &= g_{m-1}T\vartheta_m^-, \ \ \xi = a_m,\\
T\vartheta_m T_{m-1}\vartheta_{m-1} &= g_{m-2}T\overline{\vartheta}_m T_{m-1}\overline{\vartheta}_{m-1}, \ \ t = a_{m-1},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
T\vartheta_m T_{m-1}\vartheta_{m-1}\cdots T_1\vartheta_1 &= T\overline{\vartheta}_m T_{m-1}\overline{\vartheta}_{m-1}\cdots T_1\overline{\vartheta}_1, \ \ \xi = a_1.
\end{aligned} \tag{5.21}$$

The number of matrix equations is m. Every matrix equation gives two real equations. Consequently, we obtain the system consisting of $2m$ equations with respect to the parameters p, q, r, s, a_j, c_j, $j = 1, \dots, m$. From the system (5.20) we define the elements of the matrices T_j, $j = 1, \dots, m-1$, and substitute them in (5.21).

According to Riemann's theorem, we can choose arbitrarily three of the parameters $\xi = a_j$, $j = 1, \dots, m$, and fix them. Thus we obtain the system of equations (3.11), (4.23, (4.25).

Suppose that one of the vertices of the circular polygon has a cut with the angle 2π at the cut end. If to that point on the contour $\ell(\sigma)$ there corresponds a regular nonangular point, then instead of two equations we have only one, (4.12). Under such an assumption we will have a system of $2(m+1)$ equations with respect to $2m+1$ parameters (a_j, $j = 1, \dots, m-3$, c_j, $j = 1, \dots, j$, p, q, r, s).

From the system (3.19), (4.23), (4.24), (4.12) we can define four accessory parameters and then substitute them in the remaining equations.

For the sake of simplicity, we assume that on the plane of complex velocity there is a circular pentagon whose one vertex has a cut with the angle 2π at the cut end. In this case, the homogeneous problem (3.6) is reduced to a system of three higher transcendent equations. It is assumed that such a system of equations has a solution.

If we denote $v_1(\xi)$ and $v_2(\xi)$, where

$$v_1(\xi) = pu_1(\xi) + qu_2(\xi), \tag{5.22}$$

$$v_2(\xi) = ru_1(\xi) + su_2(\xi) \tag{5.23}$$

are the components of the vector $\Phi_(\xi)$, or what comes to the same thing, the components of the first row of the matrix $\chi(\xi)$, then by the formula

$$w(\xi) = \frac{v_1(\xi)}{v_2(\xi)} \tag{5.24}$$

we obtain the general solution (3.14). The components $\omega'(z)$ and $z'(\xi)$ of the vector $\Phi'(\xi)$ are defined by the equalities

$$d\omega_0(\xi) = v_1(\xi)\chi_1(\xi)d\xi, \quad -\infty < \xi < +\infty, \tag{5.25}$$

$$d\sigma(\xi) = v_2(\xi)\chi_1(\xi)d\xi, \quad -\infty < \xi < +\infty, \tag{5.26}$$

where $v_1(\xi)\chi_1(\xi)$, $v_2(\xi)\chi_1(\xi)$ satisfy the boundary conditions (3.1) and those at the singular points $\xi = e_j$, $j = 1, \dots, n$, $\xi = \infty$. The integration of (5.25) and (5.26) in the intervals $(-\infty < \xi)$, (e_j, ξ), $j = 1, \dots, n$, provides us with

$$\omega_0(\xi) = \int_{-\infty}^{\xi} v_1(\xi)\chi_1(\xi)\, d\xi + \omega(-\infty), \tag{5.27}$$

$$\sigma(\xi) = \int_{-\infty}^{\xi} v_2(\xi)\chi_1(\xi)\, d\xi + \sigma(-\infty), \tag{5.28}$$

$$\omega(\xi) = \int_{e_j}^{\xi} v_1(\xi)\chi_1(\xi)\,d\xi + \omega(e_j+), \tag{5.29}$$

$$\sigma(\xi) = \int_{e_j}^{\xi} v_2(\xi)\chi_1(\xi)\,d\xi + \sigma(e_j,+0). \tag{5.30}$$

Considering (5.29) and (5.30) for $\xi = e_{j+1}$, we obtain a system of equations with respect to the removable singular points $\xi = e_{j+1}$ and another unknown parameters. The equations (5.29) and (5.30) allow one to determine the parametric equation of the depression curve.

6. One Essential Remark

Consider one simplest integral Fredholm equation of the second kind ([33]–[39])

$$u(x) - \lambda \int_a^b k(x,t)u(t)\,dt = f(x), \tag{6.1}$$

where the unknown function $u(x)$ depends on the real variable x which changes in the same interval $[a,b]$ as the integration variable t; this requirement refers to all classes of integral equations we deal with in the present work. The interval may be finite or infinite. The functions $k(x,t)$ and $f(x)$ are assumed to be known and defined almost everywhere respectively in the square $a \le x \le b$, $a \le t \le b$ and in the interval $[a,b]$. The function $k(x,t)$ is called the kernel of the integral equation. It is assumed that the kernel $k(x,t)$ of Fredholm's equation satisfies the inequality

$$\int_a^b \int_a^b |k(x,t)|^2\,dx\,dt < \infty, \tag{6.2}$$

while the free term of Fredholm's equation satisfies the inequality

$$\int_a^b |f(x)|^2\,dx < \infty. \tag{6.3}$$

It is necessary to consider Fredholm's equations of more general type.

Let Ω^- be a measurable set in the space of any number of variables, x and t be points of that set, and μ be a nonnegative measure defined in Ω. The equation

$$u(x) - \lambda \int_\Omega k(x,t)u(t)\,d\mu(t) = f(x) \tag{6.4}$$

is likewise called the Fredholm equation whose kernel $k(x,t)$ and free term $f(x)$ satisfy respectively the inequalities

$$\int_{\Omega}\int_{\Omega} |k(x,t)|^2 \, d\mu(x)\, d\mu(t) < 0, \quad \int_{\Omega} |f(x)|^2 \, d\mu(x) < \infty. \tag{6.5}$$

The kernel $k(x,t)$ satisfying (6.5) is called the Fredholm one.

The unknown function $u(x)$ is quadratically summable in (a,b), and hence belongs to the functional space $L_2(a,b)$. A solution of the equation (6.4) belongs to the space $L_2(\mu,\Omega)$ of functions which are quadratically summable with respect to μ. The inequalities (6.3) and (6.5) imply that the free term of the equation belongs to the same space. The parameter λ may take both real and complex values.

Denote the volume element by dx, and the integral (6.5) by B_k^2:

$$\int_{\Omega}\int_{\Omega} |k(x,t)|^2 \, dx\, dt = B_k^2. \tag{6.6}$$

As is known, Fredholm's equation has either finite, or countable set of characteristic numbers; if there is a countable set of numbers, then they tend to infinity. But there are kernels which have no characteristic numbers at all, for example, Volterra kernels. A complete characteristic of such kernels is given in the following Lalesko's theorem. Let $k(x,t)$ be a Fredholm kernel and $k_n(x,t)$ be its iteration. For the kernel $k(x,t)$ to have no characteristic numbers, it is necessary and sufficient that

$$A_n = \int_{\Omega} k_n(x,x)\, dx = 0, \quad n = 3,4,5,\dots . \tag{6.7}$$

Note that the numbers A_n are called the traces of the kernel $k(x,t)$. Lalesko has proved his theorem for bounded kernels, and a general proof has been given by S. N. Krachkovskiĭ ([33]–[39]).

The determinant and Fredholm's minors are represented as a quotient of two entire functions of λ. Note that the poles of the resolvent, the characteristic numbers of the kernel $k(x,t)$, do not depend on x and t. Thus the resolvent should be of the form

$$\Gamma(x,t;\lambda) = \frac{D(x,t;\lambda)}{D(\lambda)}\,, \tag{6.8}$$

where $D(x,t;\lambda)$ and $D(\lambda)$ are entire functions of λ. If we succeed in constructing these functions, then we will be able to find the resolvent, and a solution of the integral equation will be constructed by the well-known formula. For the numerator and the denominator of the fraction in (6.8) we give representations in the form of the so-called Fredholm series

$$\begin{aligned} D(x,t;\lambda) &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} B_n(x,t)\lambda^n, \\ D(\lambda) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} C_n\lambda^n, \end{aligned} \tag{6.9}$$

where the formulas

$$\begin{aligned} C_0 = 1, \quad B_0(x,t) = k(x,t), \quad C_n &= \int\limits_{\Omega} B_{n-1}(x,x)\,dx, \quad n>0, \\ B_n = C_n k(x,t) - n &\int\limits_{\Omega} k(x,\tau)B_{n-1}(\tau,t)\,d\tau \end{aligned} \tag{6.10}$$

allow one to calculate recursively the coefficients $B_n(x,t)$ and C_n.

Below, we will need the well-known formula ([33]–[39])

$$\frac{D'(\lambda)}{D(\lambda)} = -\sum_{n=1}^{\infty} A_n\lambda^{n-1}, \tag{6.11}$$

where

$$A_n = \int\limits_{\Omega} k_n(x,x)\,dx, \quad n = 1,2,3,\dots, \tag{6.12}$$

are the traces of the kernel $k(x,t)$ mentioned above.

If the kernel is not continuous having second order discontinuities, then the integrals (6.12) defining the coefficients $c_1, c_2, c_3, \dots$ from the formulas (6.10), make no sense. For example, when the kernel $k(x,t)$ contains as a multiplier Green's function $G[P,Q]$ of the Dirichlet problem for harmonic functions which is symmetric with respect to P and Q, equals to zero on the boundary C and is analytic at all points P of the domain D except the points $P = Q$ where it has logarithmic singularity, the kernel $k(x,t)$ will have logarithmic singularity as well. Then the integral $\int\limits_{\Omega} k(x,x)\,dx$ defining the coefficient c_1 makes no sense. This difficulty can be disregarded by putting, for example, the density $c_1 = 0$.

The iterated kernel $k_2(s,t)$ has the form

$$k_2(s,t) = \int\limits_a^b k(s,t_1)k(t_1,t)\,dt_1. \tag{6.13}$$

The integral $k_2(s,t)$ has sense for any s and t from $[a,b]$ since in the unfavorable case, when s and t coincide, we have the following estimate of the integrand:

$$\big|k(s,t_1k(t_1,s)\big| \leq \frac{M_1}{|s-t_1|^{\varepsilon_1}}\,, \quad \varepsilon_1 > 0. \tag{6.14}$$

It is proved that the function $k_2(s,t)$ is continuous in the square $a \le x \le b$, $a \le t_1 \le b$. The functions

$$k_n(s,t) = \int\limits_a^b k(s,t_1)k_{n-1}(t_1,t)\,dt_1, \quad n = 1,2,3,\dots, \tag{6.15}$$

$$\big|k(s,t_1k_{n-1}(t_1,t)\big| < \frac{M_{n-1}}{|s-t_1|^{\varepsilon_1}}, \quad \varepsilon_{n-1} > 0, \tag{6.16}$$

are estimated analogously. The integral $k_n(s,t)$, $n = 1,2,3\dots$, makes sense for any positions s and t from $[a,b]$, and the estimates of the integrand have the form (6.16). Thus we have to put

$$A_n = \int\limits_\Omega k_n(s,s)\,ds = 0, \quad n = 1,2,3,\dots,n, \tag{6.17}$$

and then $k_n(x,x) = 0$, $n = 1,2,3,\dots$, $c_n = 0$, $n = 1,2,3,\dots,n$. Taking into account (6.17), we obtain from (6.11) that

$$D'(\lambda) = 0, \tag{6.18}$$

and from (6.18) we have

$$D(\lambda) = 1. \tag{6.19}$$

Consequently, the kernel of the integral equation (2.43) has no characteristic numbers. Analogously, one can prove that the considered in our work [39] kernel of the integral equation (3.35) has no characteristic numbers.

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