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THE INVESTIGATION OF THE FOURTH BOUNDARY VALUE PROBLEM FOR EQUATIONS OF STATICS OF ELASTIC MIXTURE FOR A CIRCLE AND AN INFINITE PLANE WITH A CIRCULAR HOLE

Abstract. The fourth boundary value problem for a circle and for an infinite domain with a circular hole is formulated. The theorem on the uniqueness of a solution is proved. Singular integral equations with Hilbert kernel are obtained for solving the problems. The formula of permutation of singular integrals with Hilbert kernel is used. The solutions of the above-mentioned problems are represented in terms of the Poisson formula.

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[illegible]

1. Statement of the Fourth Boundary Value Problem and the Uniqueness Theorems

The system of the basic (homogeneous) equations of statics for an elastic mixture in two dimensions has the form ([1])

$$\begin{aligned} &a_1\Delta u' + b_1 \operatorname{grad}\operatorname{div} u' + c\Delta u'' + d \operatorname{grad}\operatorname{div} u'' = 0,\\ &c\Delta u' + d \operatorname{grad}\operatorname{div} u' + a_2\Delta u'' + b_2 \operatorname{grad}\operatorname{div} u'' = 0, \end{aligned} \tag{1.1}$$

where

$$\begin{gathered} a_1 = \mu_1 - \lambda_5, \quad b_1 = \mu_1 + \lambda_1 - \lambda_5 - \rho^{-1}\alpha_2\rho_2,\\ a_2 = \mu_2 - \lambda_5, \quad b_2 = \mu_2 + \lambda_1 + \lambda_5 + \rho^{-1}\alpha_2\rho_2,\\ d = \mu_3 + \lambda_3 - \lambda_5 - \rho^{-1}\alpha_2\rho_1 \equiv \mu_3 + \lambda_4 - \lambda_5 + \rho^{-1}\alpha_2\rho_2,\\ \rho = \rho_1 + \rho_2, \quad \alpha_2 = \lambda_3 - \lambda_4. \end{gathered} \tag{1.2}$$

In (1.2), ρ_1 and ρ_2 are the partial densities, and μ_1, μ_2, μ_3, λ_1, λ_2, λ_3, λ_4, λ_5 are real constants characterizing physical properties of an elastic mixture and satisfying certain inequalities. $u' = (u_1, u_2)$ and $u'' = (u_3, u_4)$ are the partial displacements.

If we introduce the variables

$$z = x_1 + ix_2, \quad \overline{z} = x_1 - ix_2,$$

i.e.,

$$x_1 = \frac{z+\overline{z}}{2}, \quad x_2 = \frac{z-\overline{z}}{2i},$$

then the system (1.1) can be rewritten in the form ([2])

$$\frac{\partial^2 \mathcal{U}}{\partial z \partial \overline{z}} + \mathcal{K}\frac{\partial^2 \overline{\mathcal{U}}}{\partial \overline{z}^2} = 0, \tag{1.3}$$

where

$$\mathcal{U} = \begin{pmatrix} u_1 + iu_2\\ u_3 + iu_4 \end{pmatrix} = m\varphi(z) - Kmz\overline{\varphi'(z)} + \overline{\psi(z)}, \tag{1.4}$$

$$\begin{gathered} \mathcal{M} = \begin{bmatrix} m_1 & m_2\\ m_2 & m_3 \end{bmatrix}, \quad m_1 = l_1 + \frac{l_4}{2}, \quad m_3 = l_3 + \frac{l_6}{2}, \quad m_2 = l_2 + \frac{l_5}{2},\\ l_1 = \frac{a_2}{d_2}, \ \ l_2 = -\frac{c}{d_2}, \ \ l_3 = \frac{a_1}{d_2},\\ l_1 + l_4 = \frac{a_2+b_2}{d_1}, \ \ l_2 + l_5 = -\frac{c+d}{d_1}, \ \ l_3 + l_6 = \frac{a_1+b_1}{d_1},\\ \mathcal{K} = \begin{bmatrix} k_1 & k_3\\ k_2 & k_4 \end{bmatrix}, \quad km = -\frac{l}{2}, \\ l = \begin{bmatrix} l_4 & l_5\\ l_5 & l_6 \end{bmatrix}, \quad m^{-1} = \frac{1}{\Delta_0}\begin{bmatrix} m_3 & -m_2\\ -m_2 & m_1 \end{bmatrix}, \ \ \Delta_0 = m_1m_3 - m_2^2 > 0,\\ \delta_0 k_1 = 2(a_2b_1 - cd) + b_1b_2 - d^2, \quad \delta_0 k_2 = 2(da_1 - cb_1),\\ \delta_0 k_3 = 2(da_2 - cb_2), \quad \delta_0 k_4 = 2(a_1b_2 - cd) + b_1b_2 - d^2, \end{gathered} \tag{1.5}$$

$$\delta_0 = (2a_1 + b_1)(2a_2 + b_2) - (2c + d)^2 \equiv 4d_1 d_2 \Delta_0,$$
$$\delta_1 = (a_1 + b_1)(a_2 + b_2) - (c + d)^2 > 0, \;\; d_2 = a_1 a_2 - c^2 > 0,$$

$\varphi(z)$ and $\psi(z)$ are analytic vectors. The stress vector is of the form

$$\mathcal{T}\mathcal{U} = \begin{pmatrix} (TU)_2 - i(TU)_1 \\ (TU)_4 - i(TU)_3 \end{pmatrix} = \frac{\partial}{\partial s}\big(-2\varphi(z) + 2\mu\mathcal{U}\big), \tag{1.6}$$

where

$$\frac{\partial}{\partial s(x)} = n_1 \frac{\partial}{\partial x_2} - n_2 \frac{\partial}{\partial x_1}\,, \tag{1.7}$$

n_1 and n_2 are the projections onto the axes x_1 and x_2 of the basis normal vector. It is evident that the basis tangent vector is $s(x) = (-n_2, n_1)$, and $(TU)_k$ is the projection of the stress vector onto the axes x_k $(k = \overline{1,4})$,

$$\mu = \begin{bmatrix} \mu_1 & \mu_3 \\ \mu_3 & \mu_2 \end{bmatrix}, \quad \det \mu = \mu_1 \mu_2 - \mu_3^2 > 0. \tag{1.8}$$

Here we present the definition of a regular solution ([3]).

The vector $\mathcal{U}$ is a regular solution if it and its partial derivatives of the first order are continuous up to the boundary, while the second derivatives satisfy the system (1.3).

We now can formulate the fourth boundary value problem: Find a regular solution in a circular domain D^+ which on the boundary (i.e. on the circumference of radius R) satisfies the boundary conditions

$$(sU)^+ = f(t), \quad (nTU)^+ = F(t), \tag{1.9}$$

where f and F are given continuous functions on the circumference. Here the sign "+" denotes limiting values from the inside. If instead of D^+ we have D^- (i.e. an infinite plane with a hole), then the boundary conditions are

$$(SU)^- = f(t), \quad (nTU)^- = F(t), \tag{1.10}$$

where the sign "−" denotes limiting values from the outside. For the domain D^-, to the regularity conditions we add the conditions at infinity:

$$\mathcal{U} = O(1), \quad \frac{\partial U}{\partial u_k} = O(\rho^{-2}), \;\; k = 1, 2, \quad \rho = \sqrt{x_1^2 + x_2^2}\,. \tag{1.11}$$

Thus we have the following formulas ([4]):

$$\int_{D^+} E(u, u)\, dy_1\, dy_2 = \int_S uTu\, ds \equiv \operatorname{Im} \int_S \mathcal{U}T\overline{\mathcal{U}}\, ds, \tag{1.12}$$

$$\int_{D^-} E(u, u)\, dy_1\, dy_2 = -\int_S uTu\, ds \equiv -\operatorname{Im} \int_S \mathcal{U}T\overline{\mathcal{U}}\, ds, \tag{1.13}$$

where s is the circumference of radius R,

$$\operatorname{Im} \mathcal{U}T\overline{\mathcal{U}} = \sum_{k=1}^{4} u_k (T\overline{\mathcal{U}})k \equiv u_n (T\overline{\mathcal{U}})_n + u_s (T\overline{\mathcal{U}})_s,$$

u_n, u_s, $(T\overline{u})_n$, $(T\overline{u})_s$ are the normal and tangential components of the displacement and stress vectors, $E(u,u)$ is the doubled potential energy of the form

$$
\begin{aligned}
E(u,u) &= \\
&= (b_1-\lambda_5)\Big(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\Big)^2 + 2(d+\lambda_5)\Big(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\Big)\Big(\frac{\partial u_3}{\partial x_1}+\frac{\partial u_4}{\partial x_2}\Big)^2 + \\
&\quad +(b_2-\lambda_5)\Big(\frac{\partial u_3}{\partial x_1}+\frac{\partial u_4}{\partial x_2}\Big)^2 + \\
&\quad +\mu_1\Big[\Big(\frac{\partial u_1}{\partial x_1}-\frac{\partial u_2}{\partial x_2}\Big)^2+\Big(\frac{\partial u_2}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\Big)^2\Big]+ \\
&\quad +2\mu_3\Big[\Big(\frac{\partial u_1}{\partial x_1}-\frac{\partial u_2}{\partial x_2}\Big)\Big(\frac{\partial u_3}{\partial x_1}-\frac{\partial u_4}{\partial x_2}\Big)+\Big(\frac{\partial u_2}{\partial x_1}+\frac{\partial u_1}{\partial x_2}\Big)\Big(\frac{\partial u_4}{\partial x_1}+\frac{\partial u_3}{\partial x_2}\Big)\Big]+ \\
&\quad +\mu_2\Big[\Big(\frac{\partial u_3}{\partial x_1}-\frac{\partial u_4}{\partial x_2}\Big)^2+\Big(\frac{\partial u_4}{\partial x_1}+\frac{\partial u_3}{\partial x_2}\Big)^2\Big]- \\
&\quad -\lambda_5\Big[\Big(\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\Big)^2-\Big(\frac{\partial u_4}{\partial x_1}-\frac{\partial u_3}{\partial x_2}\Big)^2\Big].
\end{aligned}
\tag{1.14}
$$

To solve the fourth boundary value problem, we prove the following

Theorem. *a regular solution in the domain D^+ satisfying the homogeneous conditions of the fourth boundary value problem is the identical zero, if s is not a straight line.*

Proof. The use is made here of the formula (1.12). If in (1.12) we have $f=F=0$, then it follows from (1.14) that

$$u_1=c_1-\varepsilon x_2,\quad u_2=c_2+\varepsilon x_1,\quad u_3=c_3-\varepsilon x_2,\quad u_4=c_4+\varepsilon x_1,\tag{1.15}$$

where c_k $(k=1,4)$ and ε are arbitrary constants.

Using now Green's formula

$$0=\int\limits_S (SU)^+\,ds=\int\limits_{D^+}\Big(\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\Big)\,dx_1\,dx_2=2\varepsilon\int\limits_{D^+}dx_1\,dx_2=2\varepsilon D,$$

where D is the space of the domain D^+, we obtain $\varepsilon=0$, and from (1.15) we get

$$n_1u_2-n_2u_1=c_1\,\frac{dx_2}{ds}+c_2\,\frac{dx_1}{ds}=0\ \text{ or finally,}\ \ c_2x_1+c_1x_2=c,\tag{1.16}$$

where c is a constant. (1.16) is the equation of a straight line.

Thus the theorem for the domain D^+ is complete.

Let us now prove the uniqueness for the domain D^-. Since (1.11) holds, therefore $\varepsilon=0$, and from the condition $(SU)^-=0$ we obtain

$$c_2x_1+c_1x_2=c.$$

This is the equation of a straight line. Thus if the boundary S of the domain D^- is not a straight line, then the uniqueness of a solution of the fourth boundary value problem holds. □

2. Solving of the Fourth Boundary Value Problem in the Domain D^+

The analytic vector $\varphi(z)$ from (1.4) has in a circular domain D^+ the form

$$\varphi(z) = \frac{m^{-1}}{2\pi i} \int\limits_S \frac{\partial \ln \sigma}{\partial s(y)} (ng + sh)\, dS, \tag{2.1}$$

where g and h are unknown scalar values, $\sigma = z - \zeta$, the point z corresponds to the point $x = (x_1, x_2)$, while ζ corresponds to the point $y = (y_1, y_2) \in S$, n and s are respectively the basis normal and tangent vectors, oriented in the same way as the axes x_1 and x_2, m^{-1} is the inverse to m matrix. Since $\det m > 0$, m^{-1} does exist.

From (2.1) we have

$$\overline{\varphi'(z)} = -\frac{m^{-1}}{2\pi i} \int\limits_S \frac{\partial}{\partial s(y)} \frac{1}{\overline{\sigma}} (n\overline{g} + s\overline{h})\, dS. \tag{2.2}$$

Taking into account (2.1) and (2.2), in (1.4) we have

$$\begin{aligned} \mathcal{U}(x) = \frac{1}{2\pi i} \int\limits_S \frac{\partial \ln \sigma}{\partial s(y)} (ng + sh)\, dS + \\ + \frac{K}{2\pi i} \int\limits_S \frac{\partial}{\partial s(y)} \frac{z}{\overline{\sigma}} (n\overline{g} + s\overline{h})\, dS + \overline{\Psi(z)}. \end{aligned} \tag{2.3}$$

Let $\overline{\psi(z)}$ be chosen in terms of

$$\overline{\psi(z)} = -\frac{1}{2\pi i} \int\limits_S \frac{\partial \ln \overline{\sigma}}{\partial s(y)} (ng + sh)\, dS - \frac{K}{2\pi i} \int\limits_S \zeta \frac{\partial}{\partial s(y)} \frac{1}{\overline{\sigma}} (n\overline{g} + s\overline{h})\, dS.$$

Then $\mathcal{U}(x)$ takes the form

$$\mathcal{U}(x) = \frac{1}{\pi} \int\limits_S \frac{\partial \theta}{\partial s} (ng + sh)\, dS + \frac{K}{2\pi i} \int\limits_S \frac{\partial}{\partial s(y)} \frac{\sigma}{\overline{\sigma}} (n\overline{g} + s\overline{h})\, dS, \tag{2.4}$$

where

$$\theta = \operatorname{arctg} \frac{y_2 - x_2}{y_1 - x_1}\,.$$

Instead of $\mathcal{U}(x)$ we consider the expression

$$\begin{aligned} \mathcal{U}(x) = \frac{1}{\pi} \int\limits_S \Big(\frac{\partial \theta}{\partial s(y)} - \frac{1}{2R} \Big) (ng + sh)\, dS + \\ + \frac{K}{2\pi i} \int\limits_S \Big(\frac{\partial}{\partial s(y)} e^{2i\theta} + \frac{iz}{R^2} \Big) (n\overline{g} + s\overline{h})\, dS. \end{aligned} \tag{2.5}$$

Obviously, if (2.1) is a solution of (1.3), then (2.5) is likewise a solution of the equation (1.3) because the difference between them is a first degree function with respect to the coordinates of the point x.

When the point x lies on the boundary s, then $z = Re^{it}$, and $\zeta = Re^{i\tau}$, $2\theta = \pi + t + \tau$ and $e^{2i\theta} = -e^{-i(t+\tau)}$.

Let us now pass to the limit in (2.5) as x tends to a point of the boundary S. We have

$$U^{+}(t) = ng + sh, \tag{2.6}$$

whence

$$(SU)^{+} = h = f(t), \tag{2.7}$$

where $f(t)$ is known. Thus the function g remains still unknown. It will be defined below.

Our aim now is to establish the connection between the limiting values of the displacement and stress vectors. From (2.1) we have

$$\varphi^{+}(t) = \frac{m^{-1}}{2}\,(ng + sh) + \frac{m^{-1}}{4\pi}\int\limits_{0}^{2\pi}(ng + sh)\,dt + \\ + \frac{m^{-1}}{2\pi}\int\limits_{0}^{2\pi}\operatorname{ctg}\frac{\tau - t}{2}\,(ng + sh)\,d\tau. \tag{2.8}$$

It should be noted that on the boundary S

$$\frac{\partial}{S(x)} = \frac{1}{R}\,\frac{d}{dt}\,,$$

hence

$$\frac{\partial}{\partial t}\,\operatorname{ctg}\frac{\tau - t}{2}\,(ng + sh)\,d\tau = -\frac{\partial}{\partial \tau}\,\operatorname{ctg}\frac{\tau - t}{2}\,(ng + sh)\,\frac{1}{2}\,,$$

and from (2.8) we obtain

$$2\frac{d\varphi^{+}}{dt} = m^{-1}\,\frac{d}{dt}\,(ng + sh) + \frac{im^{-1}}{2\pi}\int\limits_{0}^{2\pi}\operatorname{ctg}\frac{\tau - t}{2}\,\frac{d}{d\tau}\,(ng + sh)\,d\tau. \tag{2.9}$$

Taking into account (2.9) and (1.6), we have

$$(2\mu - m^{-1})\,\frac{dU^{+}}{dt} + \frac{im^{-1}}{2\pi}\int\limits_{0}^{2\pi}\operatorname{ctg}\frac{\tau - t}{2}\,\frac{dU^{+}}{d\tau}\,dt = RF(t), \tag{2.10}$$

where $F = (TU)^{+}$.

We use here the formula of permutation of singular integrals with the Hilbert kernel ([4], p. 144):

$$\frac{1}{(2\pi)^2}\int\limits_{0}^{2\pi}\operatorname{ctg}\frac{t - \varphi}{2}\,dt\int\limits_{0}^{2\pi}\operatorname{ctg}\frac{\tau - t}{2}\,\frac{dU^{+}}{d\tau}\,d\tau = -\frac{dU^{+}}{d\varphi}\,. \tag{2.11}$$

Then (2.10) yields

$$\frac{2\mu-m^{-1}}{2\pi}\int_0^{2\pi}\operatorname{ctg}\frac{\tau-\varphi}{2}\,\frac{dU^+}{d\tau}\,d\tau-im^{-1}\,\frac{dU^+}{d\varphi}=\frac{R}{2\pi}\int_0^{2\pi}\operatorname{ctg}\frac{t-\varphi}{2}\,F(t)\,dt.$$

Replacing φ by t and t by τ, we obtain

$$\frac{2\mu-m^{-1}}{2\pi}\int_0^{2\pi}\operatorname{ctg}\frac{\tau-t}{2}\,\frac{dU^+}{d\tau}\,d\tau-im^{-1}\,\frac{dU^+}{dt}=$$
$$=\frac{iR}{2\pi}\int_0^{2\pi}\operatorname{ctg}\frac{\tau-t}{2}\,F(\tau)\,d\tau. \tag{2.12}$$

We multiply (2.12) by i and add to (2.10). As a result, we obtain

$$A'(2A'-2E)m^{-1}\,\frac{dU^+}{dt}=RF(t)+\frac{R}{2\pi}\int_0^{2\pi}\operatorname{ctg}\frac{\tau-t}{2}\,F(\tau)\,d\tau, \tag{2.13}$$

where

$$\begin{aligned}
A'&=\begin{bmatrix}A_1 & A_3\\ A_2 & A_4\end{bmatrix},\\
A_1&=\frac{d_1+d_2+a_1b_2-cd}{d_1}+\lambda_5\Big(\frac{a_1+c}{d_2}+\frac{a_2+b_2+c+d}{d_1}\Big),\\
A_2&=\frac{cb_1-da_1}{d_1}-\lambda_5\Big(\frac{a_1+c}{d_2}+\frac{a_1+b_1+c+d}{d_1}\Big),\\
A_3&=\frac{cb_2-da_2}{d_1}-\lambda_5\Big(\frac{a_2+c}{d_2}+\frac{a_2+b_2+c+d}{d_1}\Big),\\
A_4&=\frac{d_1+d_2+a_2b_1-cd}{d_1}+\lambda_5\Big(\frac{a_1+c}{d_2}+\frac{a_1+b_1+c+d}{d_1}\Big).
\end{aligned}$$

The values appearing here are defined in (1.5).

On the basis of (2.13) we can write

$$\frac{dU^+}{dt}=m(A'-2E)^{-1}(A')^{-1}\cdot R\Big[F(t)+\frac{1}{2\pi}\int_0^{2\pi}\operatorname{ctg}\frac{\tau-t}{2}\,F(\tau)\,d\tau\Big]. \tag{2.14}$$

Multiplying (2.14) by $n(t)$, we find that

$$\frac{dU^+}{dt}=m(A'-2E)^{-1}(A')^{-1}R\times$$
$$\times\Big[F(t)n(t)+\frac{1}{2\pi}\int_0^{2\pi}\operatorname{ctg}\frac{\tau-t}{2}\,F(\tau)\big[n(t)-n(\tau)+n(\tau)\big]\,d\tau\Big].$$

Note that

$$\big[n(t)-n(\tau)\big]\operatorname{ctg}\frac{\tau-t}{2}=-s(t)-s(\tau).$$

Taking now into account that the principal vector and the principal moment of external stresses are equal to zero, the last formula results in

$$\frac{dU^+}{dt} = \\ = m(A'-2E)^{-1}(A')^{-1}R\Big[F(t)n(t)+\frac{1}{2\pi}\int\limits_0^{2\pi}\operatorname{ctg}\frac{\tau-t}{2}\,F(\tau)n(\tau)\,d\tau\Big]. \quad (2.15)$$

Here $(A')^{-1}$ and $(A'-2E)^{-1}$ always exist since

$$\det A' = 4\Delta_0\Delta_1 > 0,$$
$$\Delta_0 = \frac{4\Delta_1 + 2a + b - \lambda_5(2a_0+b_0)}{4d_1d_2} > 0, \ \ \Delta_1 = \mu_1\mu_2 - \mu_3^2 > 0,$$
$$a = \mu_1(b_2-\lambda_5) + \mu_2(b_1-\lambda_5) - 2\mu_3(d+\lambda_5) > 0,$$
$$b = (b_1-\lambda_5)(b_2-\lambda_5) - (d+\lambda_5)^2 > 0,$$
$$a_0 = \mu_1+\mu_2+2\mu_3 > 0, \quad b_0 = b_1+b_2+2d > 0,$$
$$\det(A'-2E) = \Delta_2,$$
$$\Delta_2 d_1 d_2 = \big[\Delta_1 - 2\lambda_5(a_1+a_2+2c)\big](b_1b_2-d^2) - 2\lambda_5 d_2(b_1+b_2+2d) \equiv \\ \equiv \big[\Delta_1 - 2\lambda_5(a_1+a_2+2c)\big]\,\big[(b_1-\lambda_5)(b_2-\lambda_5)-(d+\lambda_5)^2\big] - \\ -\lambda_5(b_1+b_2+2d)\Delta_1 > 0.$$

In view of (2.6), we have

$$\frac{dU^+}{dt} = n\Big(\frac{dg}{dt} - h\Big) + s\Big(\frac{dh}{dt} + g\Big),$$

whence

$$\frac{dU^+}{dt}\,n = \frac{dg}{dt} - f,$$

where f is the given function. Then (2.15) can be rewritten in the form

$$\frac{dg}{dt} = f + m(A'-2E)^{-1}(A')^{-1}R\Big[F\cdot n + \frac{1}{2\pi}\int\limits_0^{2\pi}\operatorname{ctg}\frac{\tau-t}{2}\,F(\tau)n(\tau)\,d\tau\Big] \quad (2.16)$$

from which g is written in quadratures.

Substituting g and h into (2.6), we obtain U^+ in quadratures.

Thus we have obtained U^+ by means of the Poisson formula. Consequently, the solution of the fourth boundary value problem in the circular domain D^+ is complete.

3. Solving of the Fourth Boundary Value Problem in an Infinite Domain with a Circular Hole

A solution is sought as follows:

$$\varphi(z)=\frac{m^{-1}}{2\pi i}\int\limits_0^{2\pi}\frac{\partial\ln\sigma}{\partial s(y)}\,(ng+sh)\,dS, \tag{3.1}$$

where the values appearing here are defined by us in Section 2. The functions g and h will be defined below. From (3.1) we have

$$\overline{\varphi'(z)}=\frac{m^{-1}}{2\pi i}\int\limits_0^{2\pi}\frac{\partial}{\partial s(y)}\,\frac{1}{\overline{\sigma}}\,(n\overline{g}+s\overline{h})\,dS. \tag{3.2}$$

Substituting (3.1) and (3.2) into (1.4), we get

$$\begin{aligned}\mathcal{U}(x)=\frac{1}{2\pi i}\int\limits_S\frac{\partial\ln\sigma}{\partial s(y)}\,(ng+sh)\,dS+\\ +\frac{K}{2\pi i}\int\limits_S\frac{\partial}{\partial s(y)}\,\frac{z}{\overline{\sigma}}\,(n\overline{g}+s\overline{h})\,dS+\overline{\psi(z)}.\end{aligned} \tag{3.3}$$

We write $\overline{\psi(z)}$ in the form

$$\overline{\psi(z)}=-\frac{1}{2\pi i}\int\limits_S\frac{\partial\ln\overline{\sigma}}{\partial s(y)}\,(ng+sh)\,dS-\frac{K}{2\pi i}\int\limits_S\frac{\partial}{\partial s(y)}\,\frac{\zeta}{\overline{\sigma}}\,(n\overline{g}+s\overline{h})\,dS.$$

Then the displacement vector $U(x)$ in (3.3) takes the form

$$\mathcal{U}(x)=\frac{1}{\pi}\int\limits_S\frac{\partial\theta}{\partial s(y)}\,(ng+sh)\,dS+\frac{K}{2\pi i}\int\limits_S\frac{\partial}{\partial s(y)}\,\frac{\sigma}{\overline{\sigma}}\,(n\overline{g}+s\overline{h})\,dS, \tag{3.4}$$

where

$$\theta=\operatorname{arctg}\frac{y_2-x_2}{y_1-x_1}\,.$$

Instead of (3.4), we consider $\mathcal{U}(x)$ in terms of

$$\begin{aligned}\mathcal{U}(x)=\frac{1}{\pi}\int\limits_S\Big(\frac{\partial\theta}{\partial s(y)}-\frac{1}{2R}\Big)\,(ng+sh)\,dS+\\ +\frac{K}{2\pi i}\int\limits_S\Big(\frac{\partial}{\partial s(y)}\,e^{2i\theta}+\frac{i\zeta}{\overline{z}}\Big)(n\overline{g}+s\overline{h})\,dS.\end{aligned} \tag{3.5}$$

It is evident that if $\mathcal{U}(x)$ in (3.4) is a solution of the equation (1.3), then (3.5) is a solution of (1.3) as well.

Let now $z=Re^{it}$ and $\zeta=Re^{i\tau}$. Then taking into account the fact that $2\theta=\pi+t+\tau$, we pass to the limit as the point x tends to the point z. We obtain

$$(\mathcal{U})^-=-(ng+sh), \tag{3.6}$$

whence

$$S(\mathcal{U})^- = -h = f(t),$$

where f is a given complex function of period 2π. The function g will be defined below.

Using the formula (1.6) for the normal component of the stress vector, we get

$$T\mathcal{U}(x)n(x) = \frac{\partial}{\partial s(x)}\big[-2\varphi - 2\mu(ng+sh)\big]n(x) \tag{3.7}$$

or

$$\big(T\mathcal{U}(x)n(x)\big)^- = \frac{\partial}{\partial s(x)}\big[-2\varphi^+ - 2\mu\mathcal{U}^-\big]n(x) = RF(t). \tag{3.8}$$

Since

$$\frac{\partial}{\partial s(t)} = \frac{1}{R}\,\frac{d}{dt}\,,$$

we can rewrite (3.8) as follows:

$$\Big[-2\,\frac{d\varphi^+}{dt} - 2\mu\,\frac{d}{dt}\mathcal{U}^-\Big]n(t) = RF(t). \tag{3.9}$$

Calculations carried out in Section 2 allow us to find

$$\varphi^- = \frac{m^{-1}}{2}U^- - \frac{m^{-1}}{4}\int\limits_0^{2\pi} u^-\,d\varphi - \frac{im^{-1}}{2\pi}\int\limits_0^{2\pi}\operatorname{ctg}\frac{t-\varphi}{2}\,u^-\,dt,$$

which implies that

$$-2\,\frac{d\varphi^-}{dt} = -m^{-1}\,\frac{du^-}{dt} + \frac{im^{-1}}{2\pi}\int\limits_0^{2\pi}\operatorname{ctg}\frac{\tau-t}{2}\,\frac{du^-}{d\tau}\,d\tau.$$

Taking the last formulas into consideration, we can rewrite (3.9) as

$$\Bigg[(2\mu-m^{-1})\,\frac{du^-}{dt} + \frac{im^{-1}}{2\pi}\int\limits_0^{2\pi}\operatorname{ctg}\frac{\tau-t}{2}\,\frac{du^-}{d\tau}\,d\tau\Bigg]n(t) = RF(t), \tag{3.10}$$

where R is the radius of the circumference, and $F(t)$ is the normal component of the stress vector. From (3.10) we have

$$\begin{aligned}
&\frac{2\mu-m^{-1}}{2\pi}\int\limits_0^{2\pi}\operatorname{ctg}\frac{t-\varphi}{2}\,\frac{du^-}{dt}\,n(t)\,dt+\\
&+\frac{im^{-1}}{4\pi^2}\int\limits_0^{2\pi}\operatorname{ctg}\frac{t-\varphi}{2}\,\operatorname{ctg}\frac{\tau-t}{2}\,\frac{du^-}{d\tau}\,d\tau\cdot n(t) =\\
&= \frac{R}{2\pi}\int\limits_0^{2\pi}\operatorname{ctg}\frac{t-\varphi}{2}\,F(t)\,dt.
\end{aligned} \tag{3.11}$$

Note here that

$$[n(t)-n(\tau)]\operatorname{ctg}\frac{\tau-t}{2}=-s(t)-s(\tau).$$

(3.11) can now be rewritten as follows:

$$\begin{aligned}\frac{(A-E)m^{-1}}{2\pi}\int_0^{2\pi}\operatorname{ctg}\frac{t-\varphi}{2}\,\frac{du^-}{dt}\,n(t)\,dt+\\+\frac{im^{-1}}{4\pi^2}\int_0^{2\pi}\operatorname{ctg}\frac{t-\varphi}{2}\,dt\int_0^{2\pi}\operatorname{ctg}\frac{\tau-t}{2}\,\frac{du^-}{d\tau}\,n(\tau)\,d\tau=\\=\frac{R}{2\pi}\int_0^{2\pi}\operatorname{ctg}\frac{t-\varphi}{2}\,F(t)\,dt,\end{aligned}\tag{3.12}$$

where $A=2\mu m$.

By the formula of permutation of singular integrals with the Hilbert kernel, (3.12) yields

$$\begin{aligned}\frac{(A-E)m^{-1}}{2\pi}\int_0^{2\pi}\operatorname{ctg}\frac{t-\varphi}{2}\,\frac{du^-}{dt}\,n(t)\,dt-im^{-1}\,\frac{du^-}{dt}\cdot n(t)+\\+\frac{im^{-1}}{2\pi}\int_0^{2\pi}\frac{du^-}{d\tau}\,n(\tau)\,d\tau=\frac{R}{2\pi}\int_0^{2\pi}\operatorname{ctg}\frac{\tau-t}{2}\,F(\tau)\,d\tau.\end{aligned}\tag{3.13}$$

(3.10) and (3.13) result in

$$\begin{aligned}[(A-E)^2-E]m^{-1}\,\frac{du^-}{dt}\,n(t)+\frac{im^{-1}}{2\pi}\int_0^{2\pi}u^-s(\tau)\,d\tau=\\=\frac{R}{2\pi}\int_0^{2\pi}\operatorname{ctg}\frac{\tau-t}{2}\,F(\tau)\,d\tau,\end{aligned}$$

that is

$$\begin{aligned}A(A-2E)m^{-1}\,\frac{du^-}{dt}\,n(t)+\frac{im^{-1}}{2\pi}\int_0^{2\pi}u^-s(\tau)\,d\tau=\\=\frac{R}{2\pi}\int_0^{2\pi}\operatorname{ctg}\frac{\tau-t}{2}\,F(\tau)\,d\tau.\end{aligned}\tag{3.14}$$

In view of the fact that $u^-=-ng-sf$, we have

$$\frac{du^-}{dt}=-n\Big(\frac{dg}{dt}-f\Big)-s\Big(\frac{df}{dt}+g\Big),$$

and (3.14) takes the form

$$-A(A-2E)m^{-1}\Big(\frac{dg}{dt}-f\Big)-\frac{im^{-1}}{2\pi}\int\limits_0^{2\pi} f\,d\tau=\frac{R}{2\pi}\int\limits_0^{2\pi}\operatorname{ctg}\frac{\tau-t}{2}\,F(\tau)\,d\tau,$$

whence we obtain

$$\frac{dg}{dt}=f-m(A-2E)^{-1}A^{-1}\bigg[\frac{R}{2\pi}\int\limits_0^{2\pi}\operatorname{ctg}\frac{\tau-t}{2}\,F(\tau)\,d\tau+im^{-1}f_0\bigg], \quad (3.15)$$

where

$$f_0=\frac{1}{2\pi}\int\limits_0^{2\pi} f\,d\tau.$$

By virtue of (3.15), we define the function g_0 to within a constant which is, in its turn, defined in such a way that g to be periodic of period 2π.

Thus we have obtained g in quadratures. Substituting g and h into (3.6), we obtain the boundary value of the displacement vector explicitly. Finally, we have obtained the expression for the displacement vector in an integral form, i.e., in the form of the Poisson formula.

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