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ON HARNACK'S INEQUALITY AND INTERIOR REGULARITY FOR A CLASS OF NONUNIFORMLY DEGENERATED ELLIPTIC EQUATIONS OF NONDIVERGENT TYPE

Abstract. Nonuniformly degenerating elliptic equations of nondivergent type are considered. Harnack type inequalities and an a priori estimate of the Hölder norm are proved for positive solutions of such equations.

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Let E_n be an n-dimensional space of points $x=(x_1,x_2,\dots,x_n)$, $n\geq 2$, D be a bounded domain in E_n with the boundary ∂D, $0\in\partial D$. We will consider in D the equation

$$L_x u=\sum_{i,j=1}^{n} a_{ij}(x)\,\frac{\partial^2 u}{\partial x_i\partial x_j}=0 \tag{1}$$

assuming that $\|a_{ij}(x)\|$ is a real matrix with smooth elements in $D\setminus\{0\}$. It is also assumed that for all $x\in D$, $\xi\in E_n$ the condition

$$\gamma\sum_{i=1}^{n}\lambda_i(x)\xi_i^2\leq\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j\leq\gamma^{-1}\sum_{i=1}^{n}\lambda_i(x)\xi_i^2 \tag{2}$$

is fulfilled, where $\gamma\in(0,1]$ is a constant, $\lambda_i=g_i(\rho(x))$, $\rho(x)=\sum\limits_{i=1}^{n}\omega_i(|x_i|)$, $g_i(t)=\big(\frac{\omega_i^{-1}(t)}{t}\big)^2$, $i=1,2,\dots,n$. Here $\omega_i(t)$ are strictly monotonic, upwards convex functions on $(0,\operatorname{diam}D]$, $\omega_i(0)=0$, $\omega_i^{-1}(t)$ are the functions inverse to $\omega_i(t)$. There exist constants $\eta,\alpha,\beta\in(1,\infty)$, $\sigma>2$, $A>0$ such that

$$\alpha\omega_i(R)\leq\omega_i(\eta R)\leq\beta\omega_i(R), \tag{3}$$

$$\left(\frac{\omega_i^{-1}(R)}{R}\right)^{\sigma-1}\int\limits_0^{\omega_i^{-1}(R)}\left(\frac{\omega_i(t)}{t}\right)^{\sigma}dt\leq AR, \tag{4}$$

$i=1,2,\dots,n$, for $R\in(0,2d]$, $d=\operatorname{diam}D$.

The aim of this paper is to prove Harnack's inequality for positive solutions and to obtain an estimate of the Hölder norm for the class of equations (1) that depends only on α, β, η, A, n, σ, γ, but does not depend on the smoothness of the coefficients $a_{ij}(x)$. The method employed in the paper is analogous to that described in [1] which is applicable to the investigation of problems for nondivergent uniformly elliptic equations of second order. The case of power functions $\omega_i(t)$ is considered in [2], while the case for divergent equations in [3], [4].

For a measurable function $u(x)$ in D we set that $\fint\limits_D u\,dx=\frac{1}{\operatorname{mes}D}\int\limits_D u\,dx$ and denote by p' the number conjugate to $1\leq p<\infty$, $\frac{1}{p}+\frac{1}{p'}=1$ ($\frac{1}{1}+\frac{1}{\infty}=1$). For $x_0\in E_n$ we denote $\Pi_R(x_0)=\{x\in E_n: |x_i-x_i^0|\leq\omega_i^{-1}(R);\ i=1,2,\dots,n\}$. In the following results with regard to the operator L_x we assume that the conditions (2)–(4) are fulfilled. Uniform estimates of the solution and Green's function $G_y(x)$ of the Dirichlet problem are proved, which do not depend on the smoothness of the coefficients of the equation.

We set $\Lambda(x)=\Big(\prod\limits_{j=1}^{n}\lambda_j(x)\Big)^{1/(n-1)}$, $\theta(x)=\Lambda(x)^{-(n-1)}$. The following weighted Sobolev space is introduced: the completion $W_\theta^{2,n}(D)$ ($\dot W_\theta^{2,n}(D)$) of the subspace of functions $u(x)\in C^2(D)\cap C(\overline{D})$ ($u(x)=0$, $x\in\partial D$) with respect to the norm

$$\|u\| = \|u\|_2 + \sum_{|\alpha|=2} \|D^\alpha u\|_{n,\theta}\,,$$

where $\|D^\alpha u\|_{n,\theta} = \|\theta^{1/n} D^\alpha u\|_n$, $\|\cdot\|_n$ denotes the Lebesgue norm in the space $L_n(D)$.

Denote by $G_y(x)$ Green's function of the Dirichlet problem

$$L_x u = -f(x) \quad \text{in} \quad D, \quad u|_{\partial D} = 0,$$

i.e., $L_x G_y(x) = -\delta_y(x)$ in D, $G_y(x)|_{\partial D} = 0$, where $\delta_y(x)$ is Dirac's delta function with singularity at the point $y \in D$. The representation

$$u(x) = \int_D G_y(x)\, L_y u\, dy \tag{5}$$

is valid and the function $G_y(x)$ is the solution of the problem

$$L_y^* v = \sum_{i,j=1}^n \frac{\partial^2}{\partial y_i \partial y_j}\, (a_{ij}(y) v(y)) = -\delta_x(y), \quad v|_{\partial D} = 0, \quad y \in D,$$

with respect to the variable y (see, e.g., [5]).

We will make a frequent use of the representation (5). As is known, in studying the question of the existence of Green's function of the Dirichlet problem in the domain D for elliptic equations, we are faced with serious difficulties depending on the degeneration character and geometric structure of the domain D. We consider sufficiently smooth domains and infinitely differentiable coefficients in $D \setminus \{0\}$ (Hölder coefficients in $D \setminus \{0\}$ can also be considered).

In Lemma 3 below it will be shown to which class the function $u(x)$ should belong so that the representation (5) be fulfilled for it.

Lemma 1. *Let the condition* (2) *be fulfilled for the elements of the matrix* $\|a_{ij}(x)\|$ $(i, j = 1, 2, \ldots, n)$. *Then the estimate*

$$\gamma^n \prod_{j=1}^n \lambda_j(x) \le \det \|a_{ij}(x)\| \le \gamma^{-n} \prod_{j=1}^n \lambda_j(x) \tag{6}$$

is valid.

Proof. Let us use the following formula for the determinant of the matrix $A = \|a_{ij}(x)\|$:

$$\frac{\pi^{n/2}}{(\det \|A\|)^{1/2}} = \int_{E_n} e^{-(Ay,y)}\, dy \tag{7}$$

(see [6, p. 125]). By virtue of (2), we have

$$\int_{E_n} e^{-(A(x)y,y)}\, dy \le \int_{E_n} e^{-\gamma \sum\limits_{i=1}^n \lambda_i(x) y_i^2}\, dy. \tag{8}$$

By the transformation $y_i = \frac{1}{\sqrt{\gamma\lambda_i(x)}}\,\xi_i$, $i = 1,\dots,n$, we obtain

$$\int\limits_{E_n} e^{-(A(x)y,y)}\,dy \le \Bigl(\prod_{j=1}^{n}\sqrt{\lambda_j(x)\gamma}\Bigr)^{-1}\int\limits_{E_n} e^{-|\xi|^2}\,d\xi = \frac{\pi^{n/2}}{\Bigl(\prod\limits_{j=1}^{n}\lambda_j(x)\Bigr)^{1/2}\gamma^{n/2}}.$$

Using this inequality, from (7) we obtain the left estimate (6). To obtain the right estimate (6), we use the right inequality (2). Then

$$\int\limits_{E_n} e^{-(A(x)y,y)}\,dy \ge \int\limits_{E_n} e^{-\gamma^{-1}\sum\limits_{j=1}^{n}\lambda_j(x)y_j^2}\,dy = \frac{\pi^{n/2}\gamma^{n/2}}{\Bigl(\prod\limits_{j=1}^{n}\lambda_j(x)\Bigr)^{1/2}}.$$

This inequality and (7) yield the right estimate (6). □

Lemma 2. *Let $D \subset \Pi_R(x_0)$, $u(x) \in \dot W^{2,n}_\theta(D)$ and the conditions* (2)–(4) *be fulfilled. Then there exists a constant $C > 0$ depending on n, γ such that we have the estimate*

$$\sup_{x\in D}|u(x)| \le C\Bigl(\prod_{j=1}^{n}\omega_j^{-1}(R)\Bigr)^{1/n}\Biggl(\int\limits_{D}\frac{|L_x u|^n\,dx}{\prod\limits_{j=1}^{n}\lambda_j(x)}\Biggr)^{1/n}. \tag{9}$$

Proof. First we will show the validity of the estimate (9) for a function $u \in C^2(D)\cap C(\overline{D})$, $u|_{\partial D} = 0$. Let $\frac12 \sup\limits_{x\in D}|u(x)| \le |u(x_0)| \le \sup\limits_{x\in D}|u(x)|$. We make the transformation of the variables $x \to y$, $x = x_0 + \omega^{-1}(R)y$, $\omega^{-1}(R)y = (\omega_1^{-1}(R)y_1,\dots,\omega_n^{-1}(R)y_n)$. Then the parallelepiped $\Pi_R(x_0)$ is mapped into the cube $\{y \in E_n : |y| < 1\}$. The equation (1) transforms to

$$\sum_{i,j=1}^{n}\overline{a}_{ij}(y)\,\frac{\partial^2\overline{u}}{\partial y_i\,\partial y_j} = \overline{f}(y),\quad y \in D',$$

while the condition $u|_{\partial D} = 0$ becomes $\overline{u}|_{\partial D'} = 0$, where D' is the image of D. Here $\overline{a}_{ij}(y) = \frac{a_{ij}(x_0+\omega^{-1}(R)y)}{\omega_i^{-1}(R)\omega_j^{-1}(R)}$, $\overline{f} = f(x_0+\omega^{-1}(R)y)$, $\overline{u}(y) = u(x_0 + \omega^{-1}(R)y)$. Applying the Alexandrov inequality (see, e.g., [7, p. 105]) to the operator $L'_y = \sum\limits_{i,j=1}^{n}\overline{a}_{ij}(y)\frac{\partial^2\overline{u}}{\partial y_i\,\partial y_j}$ in the domain D', we have

$$\sup_{y\in D'}|\overline{u}(x)| \le C\Bigl(\int\limits_{D'}\frac{|L'_y\overline{u}|^n}{\det\|\overline{a}_{ij}\|}\,dy\Bigr)^{1/n},\quad C = C(n). \tag{10}$$

By the property of determinants we have

$$\det\|\overline{a}_{ij}(y)\| = \Bigl(\prod_{j=1}^{n}\omega_j^{-1}(R)\Bigr)^{-2}\det\|a_{ij}(x_0+\omega^{-1}(R)y)\|.$$

After making the reverse transformation of the variables, we obtain

$$\sup_{x\in D}|u(x)| \le C\Big(\prod_{j=1}^{n}\omega_j^{-1}(R)\Big)^{1/n}\Big(\int\limits_D \frac{|Lu|^n}{\det\|a_{ij}\|}\,dx\Big)^{1/n},$$

whence by Lemma 1 it follows that

$$\sup_{x\in D}|u(x)| \le C\Big(\prod_{j=1}^{n}\omega_j^{-1}(R)\Big)^{1/n}\Big(\int\limits_D \frac{|Lu|^n\,dx}{\prod\limits_{j=1}^{n}\lambda_j(x)}\Big)^{1/n}.$$

Let $u\in \dot W_\theta^{2,n}(D)$. We show that in this case the estimate (9) is valid. We have: $\exists u_m \in C^2(D)\cap C(\overline{D})$ such that $u_m(x)=0$, $x\in\partial D$, $m=1,2,\dots$,

$$\|u_m-u\|_{W_\theta^{2,n}(D)} \to 0 \quad (m\to\infty).$$

By virtue of (8), for the functions u_m and the operator L_x we have

$$\sup_{x\in D}|u_m(x)| \le C\Big(\prod_{j=1}^{n}\omega_j^{-1}(R)\Big)^{1/n}\|L_x u_m\|_{n,\theta}\,,$$

which implies

$$\sup_{x\in D}|u_m(x)| \le C\Big(\prod_{j=1}^{n}\omega_j^{-1}(R)\Big)^{1/n}\big(\|L_x u\|_{n,\sigma}+\|L_x(u_m-u)\|_{n\theta}\big).$$

Hence, since $m\to\infty$ and from (2)–(4) it follows

$$\|L_x(u_m-u)\|_{n,\theta} \le C\prod_{j=1}^{n}\Big(\frac{\omega_j^{-1}(R)}{R}\Big)^2\|u_m-u\|_{W_\theta^{2,n}} \to 0 \quad (m\to\infty),$$

we obtain the estimate (9). □

Lemma 3. *Let $u\in \dot W_\theta^{2,n}(D)$, $D\subset \Pi_R(0)$. Then the integral representation* (5) *is valid.*

Proof. Denote by $G_{y,m}(x)$ Green's function of the Dirichlet problem for the operator $L_{x,m}=\sum\limits_{i,j=1}^{n}\widetilde a_{ij}(x)\sqrt{\lambda_{i,m}\lambda_{j,m}}\,\frac{\partial^2}{\partial x_i\partial x_j}$, $\lambda_{i,m}(x)=\big(\frac{\omega_i^{-1}(1/m)}{1/m}\big)^2$ for $x\in\Pi_{1/m}(0)$, $\lambda_i^m(x)=\lambda_i(x)$ for $x\in D\setminus\Pi_{1/m}(0)$, $\widetilde a_{ij}=\frac{a_{ij}}{\sqrt{\lambda_i(x)\lambda_j(x)}}$ $(i,j=1,2,\dots,n)$.

Let $\forall f\in C^\infty(D)$, $v\in C^2(D)\cap C(\overline{D})$ be the solution of the Diriclet problem

$$L_{x,m}v(x)=f \quad \text{in} \quad D, \quad v|_{\partial D}=0.$$

Then

$$v(x)=\int\limits_D G_{y,m}(x)f(y)\,dy, \tag{11}$$

where $G_{y,m}(x)$ is Green's function of the operator $L_{x,m}$ and the Dirichlet problem for this operator in the domain D. Since the operator $L_{x,m}$ has smooth (Lipshitzian) coefficients and does not degenerate, so the Dirichlet problem is solvable and the representation (11) holds for it.

By virtue of Alexandrov's maximum principle (see, e.g., [7]) we have

$$\sup_{x\in D}|v(x)| \le C\biggl(\prod_{j=1}^{n}\omega_j^{-1}(R)\biggr)^{1/n}\|f\|_{n,\theta_m},$$

where $\theta_m = \Bigl(\prod\limits_{j=1}^{n}\lambda_{j,m}(x)\Bigr)^{-1}$, $C = C(n,\gamma)$. Hence, taking into account the representation (11), the inequality

$$\|f\|_{n,\theta_m} < \|f\|_{n,\theta}$$

and the fact the class of functions $C^\infty(D)$ is complete in $L_{n,\theta}(D)$, we obtain

$$\|G_{(\cdot),m}(x)\|_{n',\Lambda} \le C\biggl(\prod_{j=1}^{n}\omega_j^{-1}(R)\biggr)^{1/n}, \quad x\in D. \tag{12}$$

Now $\exists G_{(\cdot),m_k}(x)\to G_{(\cdot)}(x)$ weakly in $L_{n',\Lambda}(D)$, where $G_{(\cdot)}(x)$, $x\in D$, is a function from $L_{n',\Lambda}(D)$.

Therefore

$$\int\limits_D G_{y,m_k}(x)L_{y,m_k}u(y)\,dy = \int\limits_D G_{y,m_k}(x)(L_{y,m_k}-L_y)u(y)\,dy + \\ + \int\limits_D G_{y,m_k}(x)L_y u(y)\,dy, \tag{13}$$

$\forall u\in W_\theta^{2,n}(D)$.

Since $L_y u \in L_{n,\theta}(D)$ and $G_{(\cdot),m_k}(x)\to G_{(\cdot)}(x)$ converges weakly in $L_{n',\Lambda}(D)$, we have

$$\int\limits_D G_{y,m_k}(x)L_y u(y)\,dy \to \int\limits_D G_y(x)L_y u(y)\,dt \quad \text{as} \quad m_k\to\infty. \tag{14}$$

Further,

$$\Bigl|\int\limits_D G_{y,m_k}(x)(L_{y,m}-L_y)u(y)\,dy\Bigr| \le \\ \le \sum_{i,j=1}^{n}\int\limits_{\Pi_{1/m}(0)}|\widetilde{a}_{ij}|\bigl(\sqrt{\lambda_{i,m}\lambda_{i,m}}+\sqrt{\lambda_i\lambda_j}\bigr)|u_{x_ix_j}|G_{y,m}(x)\,dy \le \\ \le C_1\sum_{i=1}^{n}\biggl(\frac{\omega_i^{-1}(1/m)}{1/m}\biggr)^2\int\limits_{\Pi_{1/m}(0)}|u_{xx}|G_{y,m}(x)\,dy \le \\ \le C_2\sum_{i=1}^{n}\biggl(\frac{\omega_i^{-1}(1/m)}{1/m}\biggr)^2\|G_{(\cdot),m}(x)\|_{n,\Lambda}\|u_{xx}\|_{n,\sigma}\to 0 \tag{15}$$

as $m \to \infty$, $x \in D$, where $|u_{xx}|^2 = \sum\limits_{i,j=1}^{n} u^2_{x_i x_j}$, $\|u_{xx}\|_{n,\theta} = \sum\limits_{i,j=1}^{n} \|u_{x_i x_j}\|_{n,\theta}$, $C_2 = C_2(n, \gamma, C_1)$.

Now, using (15), (14) in (13), we can pass to the limit as $m_k \to \infty$, whence we obtain the representation

$$u(x) = \int\limits_D G_y(x) L_y u(y)\, dy, \quad x \in D. \qquad \square$$

Theorem 1. *Let D be a bounded domain containing the parallelepiped $\Pi_{2R}(x_0)$ and let the conditions (2)–(4) be fulfilled. Then there exists a constant $C > 0$ depending on n, γ, η, α, β, A, σ such that for $q = \frac{\sigma n}{2+\sigma(n-1)}$ the inequality*

$$\Bigg(\,\fint\limits_{\Pi_{2R}(x_0)} G^q_y(x)\, dy \Bigg)^{1/q} \le C \Bigg(\,\fint\limits_{\Pi_{2R}(x_0)} G_y(x)\, dy \Bigg), \quad x \in \Pi_R(x_0), \tag{16}$$

is valid.

Proof. Denote, for brevity, $\Pi_{2R}(x_0)$ by Π_{2R}. Let $\operatorname{supp} f \subset \Pi_{2R}$. According to Theorem 1 and the maximum principle,

$$\sup_{x \in \Pi_R} \Bigg| \int\limits_{\Pi_{2R}} G_y(x) f(y)\, dy \Bigg| \le C \Bigg(\prod_{j=1}^{n} \omega_j^{-1}(2R) \Bigg)^{1/n} \Bigg\| f \Bigg(\prod_{j=1}^{n} \lambda_j \Bigg)^{-1/n} \Bigg\|_{L_n(\Pi_{2R})},$$

where $f(x) \ge 0$ a.e. in D, $\prod\limits_{j=1}^{n} \lambda_j(x) = \prod\limits_{j=1}^{n} (\omega_j^{-1}(\rho(x))/\rho(x))^2$.

The condition (3) implies $\omega_j^{-1}(2R) \le \eta^{\delta_0} \omega_j^{-1}(R)$, where δ_0 is the smallest natural integer for which $\alpha^{\delta_0} \ge 2$. Therefore

$$\sup_{x \in \Pi_R} \Bigg| \int\limits_{\Pi_{2R}} G_y(x) f(y)\, dy \Bigg| \le$$

$$\le C \Bigg(\prod_{j=1}^{n} \omega_j^{-1}(2R) \Bigg)^{1/n} \Bigg\| f \prod_{j=1}^{n} \big(\omega_j^{-1}(\rho(x))/\rho(x) \big)^{-2} \Bigg\|_{L_n(\Pi_{2R})},$$

whence by virtue of the conjugacy of weighted Lebesgue spaces we have

$$\Bigg(\int\limits_{\Pi_{2R}} G^{n'}_y(x) \Bigg(\prod_{j=1}^{n} \big(\omega_j^{-1}(\rho(y))/\rho(y) \big)^{\frac{2}{n} n'} \Bigg) dy \Bigg)^{1/n'} \le C \Bigg(\prod_{j=1}^{n} \omega_j^{-1}(R) \Bigg)^{1/n}.$$

Since $\operatorname{mes}\Pi_{2R} = \prod\limits_{j=1}^{n} \omega_j^{-1}(R)$, by means of the latter inequality we obtain

$$\left(\fint\limits_{\Pi_{2R}} G_y^{n'}(x) \left(\prod_{j=1}^{n} \left(\omega_j^{-1}(\rho(y))/\rho(y)\right)^{2n'/n} \right) dy \right)^{1/n'} \leq$$

$$\leq C \left(\prod_{j=1}^{n} \omega_j^{-1}(R) \right)^{2/n-1}. \quad (17)$$

Now for $x \in \Pi_R$ we have $1 \leq \frac{C}{R^2} \int\limits_{\Pi_{2R}} G_y(x)\, dy$. Indeed, the function $z(x) = 1 - \sum\limits_{j=1}^{n} (x_j - x_j^0)^2 (\omega_j^{-1}(2R))^{-2}$ is a solution of the equation $L_x z = -2 \sum\limits_{j=1}^{n} a_{jj}(x) (\omega_j^{-1}(2R))^{-2}$ in the domain $D_R = \{x \in \Pi_{2R} : z(x) > 0\}$ and $z|_{\partial D_R} = 0$. Therefore for $x \in \Pi_R$ we have

$$1 \leq C \int\limits_{\Pi_{2R}} G_y(x) \sum_{j=1}^{n} a_{jj}(x) (\omega_j^{-1}(2R))^{-2} dy,$$

whence by virtue of (1) we obtain

$$1 \leq C \int\limits_{\Pi_{2R}} G_y(x) \left(\sum_{j=1}^{n} \lambda_j(y) (\omega_j^{-1}(2R))^{-2} \right) dy \leq$$

$$\leq C \int\limits_{\Pi_{2R}} G_y(x) \sum_{j=1}^{n} \left(\omega_j^{-1}(\rho(y)) / \left(\rho(y) \omega_j^{-1}(2R) \right) \right)^{-2} dy \leq$$

$$\leq \frac{C}{R^2} \int\limits_{\Pi_{2R}} G_y(x)\, dy, \quad x \in \Pi_R. \quad (18)$$

We have used the monotonicity of $\omega_j^{-1}(t)/t$ for the convex functions $\{\omega_j(t)\}$ and the fact that $\rho(y) \leq 2R$ in Π_{2R}. From the estimates (17), (18) it follows that

$$\left(\fint\limits_{\Pi_{2R}} G_y(x)^{n'} \left(\prod_{j=1}^{n} \omega_j^{-1}(\rho(y))/\rho(y) \right)^{\frac{2}{n} n'} dy \right)^{1/n'} \leq$$

$$\leq \frac{C}{R^2} \left(\prod_{j=1}^{n} \omega_j^{-1}(R) \right)^{\frac{2}{n}} \left(\fint\limits_{\Pi_{2R}} G_y(x)\, dy \right), \quad x \in \Pi_R. \quad (19)$$

Using the Hölder inequality and (19), for $1 < q < n'$ we have

$$\Big(\fint_{\Pi_{2R}} G_y(x)^q\,dy\Big) \le$$

$$\le C\Big(\prod_{j=1}^{n}\omega_j^{-1}(R)\Big)^{\frac{2}{n}}\Bigg(\fint_{\Pi_{2R}}\Big(\prod_{j=1}^{n}\omega_j^{-1}(\rho(y))/\rho(y)\Big)^{-\frac{2}{n}\frac{n'q}{n'-q}}dy\Bigg)^{\frac{n'-q}{n'q}}\times$$

$$\times\frac{1}{R^2}\Big(\fint_{\Pi_{2R}} G_y(x)\,dy\Big) \le C\Big(\prod_{j=1}^{n}\omega_j^{-1}(R)\Big)^{\frac{1}{q'}+\frac{1}{n}}\frac{1}{R^2}\times$$

$$\times\Bigg(\fint_{\Pi_{2R}}\Big(\prod_{j=1}^{n}\omega_j^{-1}(\rho(y))/\rho(y)\Big)^{-\frac{2}{n}\frac{n'q}{n'-q}}dy\Bigg)^{\frac{n'-q}{n'q}}\Big(\int_{\Pi_{2R}} G_y(x)\,dy\Big). \quad (20)$$

Note that $\frac{2}{n}\frac{n'q}{n'-q} = \sigma$ and

$$\int_{\Pi_{2R}}\Big(\prod_{j=1}^{n}\omega_j^{-1}(\rho(y))/\rho(y)\Big)^{-\sigma}dy \le C\prod_{j=1}^{n}\int_0^{\omega_j^{-1}(R)}(\omega_j(|y_j|)/|y_j|)^{\sigma}\,dy_j. \quad (21)$$

Indeed, the left-hand side is equal to

$$\int_0^{\omega_1^{-1}(2R)}dy_1\int_0^{\omega_2^{-1}(2R)}dy_2\cdots\int_0^{\omega_n^{-1}(2R)}\Big(\prod_{j=1}^{n}\frac{\omega_j^{-1}(\rho(y))}{\rho(y)}\Big)^{-\sigma}dy_n.$$

The function $\omega_j^{-1}(t)/t$ increases (by virtue of the convexity of $\omega_j(t)$), $\rho(y)\ge\omega_j(|y_j|)$, and therefore $\frac{\omega_j^{-1}(\rho(y))}{\rho(y)}\ge\frac{|y_j|}{\omega_j(|y_j|)}$, $j=1,2,\ldots,n$. Then

$$\int_{\Pi_{2R}}\prod_{j=1}^{n}\Big(\frac{\omega_j^{-1}(\rho(y))}{\rho(y)}\Big)^{-\sigma}dy \le \prod_{j=1}^{n}\int_0^{\omega_j^{-1}(2R)}\Big(\frac{\omega_j(t)}{t}\Big)^{\sigma}dt \le$$

$$\le C\prod_{j=1}^{n}\int_0^{\omega_j^{-1}(R)}\Big(\frac{\omega_j(t)}{t}\Big)^{\sigma}dt.$$

The latter inequality follows from the condition (3). Indeed, (3) implies that

$$\omega_j^{-1}(\alpha t)\le\eta\omega_j^{-1}(t)\le\omega_j^{-1}(\beta t) \quad (22)$$

for sufficiently small values $t>0$.

Let δ_0 be the smallest integer for which $\alpha^{\delta_0}\ge 2$. Then from (22) we obtain $\omega_j^{-1}(2t)\le\omega_j^{-1}(\alpha^{\delta_0}t)\le\eta^{\delta_0}\omega_j^{-1}(t)$. Further, if $\omega_j(t)\le\beta^{\delta_0}\omega_j(t/\eta^{\delta_0})$, then $\frac{\omega_j(t)}{t} \le \left(\frac{\beta}{\eta}\right)^{\delta_0} \frac{\omega_j(t/\eta^{\delta_0})}{t/\eta^{\delta_0}}$, whence it follows that

$$\int\limits_0^{\omega_j^{-1}(2R)} \left(\frac{\omega_j(t)}{t}\right)^{\sigma} dt \le \left(\frac{\beta}{\eta}\right)^{\delta_0\sigma} \int\limits_0^{\eta^{-\delta_0}\omega_j^{-1}(2R)} \left(\frac{\omega_j(t)}{t}\right)^{\sigma} dt \le$$

$$\le \left(\frac{\beta}{\eta}\right)^{\delta_0\sigma} \int\limits_0^{\omega_j^{-1}(R)} \left(\frac{\omega_j(t)}{t}\right)^{\sigma} dt,$$

$j = 1, \dots, n$. By virtue of (21), from (20) we obtain

$$\left(\fint_{\Pi_{2R}} G_y(x)^q dy\right)^{1/q} \le C(R)\left(\fint_{\Pi_{2R}} G_y(x)\, dy\right), \quad x \in \Pi_R, \tag{23}$$

where $q = \frac{\sigma n}{2+\sigma(n-1)}$, $1 < q < n'$,

$$C(R) = \left(\prod_{j=1}^n \omega_j^{-1}(2R)\right)^{\frac{1}{q'}+\frac{1}{n}} \frac{1}{R^2} \prod_{j=1}^n \left(\int\limits_0^{\omega_j^{-1}(R)} \left(\frac{\omega_j(y_j)}{y_j}\right)^{\sigma} dy_j\right)^{2/\sigma n},$$

which implies

$$C(R) = \prod_{j=1}^n \left[\left(\frac{\omega_j^{-1}(R)}{R}\right)^{\sigma-1} \frac{1}{R} \int\limits_0^{\omega_j^{-1}(R)} \left(\frac{\omega_j(t)}{t}\right)^{\sigma} dy_j\right]^{2/\sigma n}. \tag{24}$$

Now observe that the condition $1 < q < n'$ is equivalent to the condition $2 < \sigma < \infty$. Hence, using the condition (4), from (23) and (24) we obtain (16). □

Remark. The inequality (16) also holds for Green's function of the parallelepiped $\Pi_{2R}(x_0)$, i.e. if $G_y^R(x)$ is Green's function for $\Pi_{2R}(x_0)$, then for $q = \frac{\sigma n}{2+\sigma(n-1)}$ we have the inequality

$$\left(\fint_{\Pi_{2R}(x_0)} \left(G_y^R(x)\right)^q dy\right)^{1/q} \le C\left(\fint_{\Pi_{2R}(x_0)} G_y^R(x)\, dy\right), \quad x \in \Pi_R(x_0), \tag{25}$$

where the constant $C > 0$ depends on n, α, β, γ, η, A, σ.

Lemma 4 (increase lemma for narrow domains)**.** *Let $D \subset \Pi_R(x_0)$ be a domain having limiting points on the surface of the parallelepiped $\Pi_R(x_0)$, $x_0 \in D$. Assume that $u \in W_\theta^{2,n}(D)$ is a positive solution of the equation (1) in D that vanishes on ∂D. Then for any $Q > 1$ there exists $\delta > 0$, depending on Q, n, α, β, γ, η, A,σ, such that*

$$\frac{\operatorname{mes} D}{\operatorname{mes} \Pi_R} < \delta \tag{26}$$

implies $\sup\limits_{x\in D} u(x) \ge Q u(x_0)$.

Proof. Let us assume that $M = \sup\limits_{x\in D} u(x)$ and consider the auxiliary function $z(x) = u(x) - \sum\limits_{j=1}^{n}(x_j - x_j^0)^2(\omega_j^{-1}(R))^{-2}M$. Then

$$L_x z = -2M\sum_{j=1}^{n} a_{jj}(x)(\omega_j^{-1}(R))^{-2} \quad \text{in} \quad D,$$

and also $z(x_0) = u(x_0)$ and $z(x) \leq 0$ on ∂D. Indeed, on the part $\partial D \cap \partial\Pi_R(x_0)$ we have $z(x) \leq M - M. \inf\limits_{x\in\partial\Pi_R(x_0)} \sum\limits_{j=1}^{n}(x_j - x_j^0)^2(\omega_j^{-1}(R))^{-2}$. Assuming in (1) that $\xi = (0,\dots,1,0,\dots,0)$, where 1 stands as the j-th coordinate of ξ, we obtain $a_{jj} \leq \gamma^{-1}\lambda_j$, $j = 1,2,\dots,n$. Then

$$|L_x z| \leq 2M\gamma^{-1}\sum_{j=1}^{n}\lambda_j(x)(\omega_j^{-1}(R))^{-2},$$

whence, taking into account the form $\lambda_j(x) = (\omega_j^{-1}(\rho(x))/\rho(x))^2$, the monotonicity of the functions $\omega_j^{-1}(t)$, $\omega_j^{-1}(t)/t$ and the fact that $\omega_j(0) = 0$, for $j = 1,\dots,n$ we have

$$|L_x z| \leq 2\gamma^{-1}M\sum_{j=1}^{n}\left(\frac{\omega_j^{-1}(\rho(x))}{\omega_j^{-1}(R)}\right)^2\frac{1}{\rho(x)^2} \leq 2n\gamma^{-1}\frac{M}{R^2}\,. \tag{27}$$

Applying Lemma 2 to the function z in the domain D, we obtain

$$u(x_0)=z(x_0)\leq\sup_{x\in D} z\leq C\left(\prod_{j=1}^{n}\omega_j^{-1}(R)\right)^{1/n}\left\|L_x Z\left(\prod_{j=1}^{n}\lambda_j(x)\right)^{-1/n}\right\|_{L_n(D)},$$

whence, by virtue of (27), it follows that

$$u(x_0) \leq C\,\frac{M}{R^2}\left(\prod_{j=1}^{n}\omega_j^{-1}(R)\right)^{1/n}\left\|\left(\prod_{j=1}^{n}\lambda_j(x)\right)^{-1/n}\right\|_{L_n(D)}. \tag{28}$$

On the other hand,

$$\int\limits_D \frac{dy}{\prod\limits_{j=1}^{n}\lambda_j(y)} = \int\limits_D \prod_{j=1}^{n}\left(\frac{\rho(y)}{\omega_j^{-1}(\rho(y))}\right)^2 dy. \tag{29}$$

For $y \in D$ we have $\rho(y) = \sum\limits_{j=1}^{n}\omega_j(|y_j|)$, which implies that $\omega_j(|y_j|) < \rho(y)$ for any $j = 1,\dots,n$. Since the functions $\omega_j^{-1}(t)/t$ are monotone, we have

$$\frac{\omega_j^{-1}(\rho(y))}{\rho(y)} \geq \frac{|y_j|}{\omega_j(|y_j|)}\,.$$

Therefore (29) implies

$$\int\limits_D \frac{dy}{\prod\limits_{j=1}^n \lambda_j(y)} = \int\limits_D \prod_{j=1}^n \left(\frac{\omega_j(|y_j|)}{|y_j|}\right)^2 dy. \tag{30}$$

If we apply the Hölder inequality to the right-hand side of (30) and assume that $\sigma = 2n'q/(n(n'-q))$, then we will have

$$\int\limits_D \frac{dy}{\prod\limits_{j=1}^n \lambda_j(y)} = \left(\int\limits_D \prod_{j=1}^n \left(\frac{\omega_j(|y_j|)}{|y_j|}\right)^\sigma dy\right)^{2/\sigma} (\operatorname{mes} D)^{1-\frac{2}{\sigma}}. \tag{31}$$

Now, from (31) we derive

$$\int\limits_D \frac{dy}{\prod\limits_{j=1}^n \lambda_j(y)} \le \left(\frac{\operatorname{mes} D}{\operatorname{mes}\Pi_R}\right)^{1-\frac{2}{\sigma}} (\operatorname{mes}\Pi_R)^{1-\frac{2}{\sigma}} \left(\int\limits_{\pi_R} \left(\prod_{j=1}^n \frac{\omega_j(|y_j|)}{|y_j|}\right)^\sigma dy\right)^{2/\sigma} \le$$

$$\le C\left(\prod_{j=1}^n \omega_j^{-1}(R)\right)^{1-\frac{2}{\sigma}} \prod_{j=1}^n \left(\int\limits_0^{\omega_j^{-1}(R)} \left(\frac{\omega_j(t)}{t}\right)^\sigma dt\right)^{2/\sigma} \left(\frac{\operatorname{mes} D}{\operatorname{mes}\Pi_R}\right)^{1-\frac{2}{\sigma}},$$

whence

$$u(x_0) \le C\,\frac{M}{R^2}\left(\prod_{j=1}^n \omega_j^{-1}(R)\right)^{\frac{2}{n}(1-\frac{1}{\sigma})} \times$$

$$\times \prod_{j=1}^n \left(\int\limits_0^{\omega_j^{-1}(R)} \left(\frac{\omega_j(t)}{t}\right)^\sigma dt\right)^{2/\sigma n} \left(\frac{\operatorname{mes} D}{\operatorname{mes}\Pi_R}\right)^{\frac{2}{n}(1-\frac{2}{\sigma})}. \tag{32}$$

By virtue of (4) and (32) we obtain $u(x_0) \le CM\big(\frac{\operatorname{mes} D}{\operatorname{mes}\Pi_R}\big)^{\frac{1}{n}(1-\frac{2}{\sigma})}$, whence, using (2), we find $u(x_0) \le CM\delta^{\frac{1}{n}(1-\frac{2}{\sigma})}$. Putting $C\delta^{\frac{1}{n}(1-\frac{2}{\sigma})} = Q^{-1}$ and using the condition $2<\sigma<\infty$, we obtain $M \ge QU(x_0)$. □

Lemma 5 (Moser type inequality)**.** *Let* $0<p<\infty$, $u(x)\in W_\theta^{2,n}(\Pi_{2R}(x_0))$ *be a positive solution of the equation* (1) *in* $\Pi_{2R}(x_0)$*. Then the estimate*

$$\sup_{\Pi_R(x_0)} u(x) \le C\left(\ \fint\limits_{\Pi_{2R}(x_0)} u(x)^p\,dx\right)^{1/p} \tag{33}$$

holds, where the constant $C>0$ *depends on* n, α, β, γ, η, A, σ *and also on* p.

Proof. We will follow the scheme from [7] to obtain the estimate (33) from Lemma 4.

It is obvious that for the functions $\omega_j(t)$ satisfying the condition (3) we have the estimate

$$\omega_j(kt) \geq k^\mu \omega_j(t), \quad t > 0, \quad k \geq 1. \tag{34}$$

Putting $Q = 2^{\mu+1}$ in Lemma 4, let us find the corresponding δ. Assume $\sup\limits_{\Pi_R(x_0)} u(x) = u(x_1) = 2M$. Let

$$u_1 = u - M \quad \text{and} \quad D_1 = \{x \in \Pi_{R/2}(x_0) : u_1 > 0\}.$$

If $\operatorname{mes} D_1 > \delta \operatorname{mes} \Pi_{R/2}(x_1)$, then

$$\int\limits_{\Pi_{2R}(x_0)} u^p\,dx \geq \int\limits_{\Pi_{R/2}(x_1)} u^p\,dx \geq \int\limits_{D_1} u^p\,dx \geq$$
$$\geq \delta \operatorname{mes} \Pi_{R/2}(x_1) M^p \geq \delta_1 \operatorname{mes} \Pi_R(x_1) M^p$$

and the assertion is proved with $C = \delta_1^{-1}$, where δ_1 is a number smaller than δ. If however $\operatorname{mes} D_1 < \delta \operatorname{mes} \Pi_{R/2}(x_1)$, then there exists $\rho_1 > 0$ such that $\operatorname{mes} D_1 \cap \Pi_{\rho_1}(x_1) = \delta \operatorname{mes} \Pi_{\rho_1}(x_1)$. Apply Lemma 4 to the function u_1 in the domain $D_1 \cap \Pi_{\rho_1}(x_1)$. Then there exists a point $x_2 \in \partial\Pi_{\rho_1}(x_1)$ such that $u(x_2) > 2^{\mu+1}M$. Assume $u_2 = u - 2^\mu M$, $D_2 = \{x \in \Pi_{R/2}(x_2) : u_2 > 0\}$. If $\operatorname{mes}(D_2 \cap \Pi_{R/4}(x_2)) \geq \delta \operatorname{mes} \Pi_{R/4}(x_2)$, then the statement is proved. If $\operatorname{mes}(D_2 \cap \Pi_{R/4}(x_2)) < \delta \operatorname{mes} \Pi_{R/4}(x_2)$, then there exists $0 < \rho_2 < R/4$ such that $\operatorname{mes}(D_2 \cap \Pi_{\rho_2}(x_2)) = \delta \operatorname{mes} \Pi_{\rho_2}(x_2)$. Applying Lemma 4 in $D_2 \cap \Pi_{\rho_2}(x_2)$ to the function u_2, we find a point $x_3 \in \partial\Pi_{\rho_2}(x_2)$ such that $u(x_3) \geq 2^{2\mu+1}M$. Continuing this process, we come to the sequence $\rho_1, \rho_2, \dots, \rho_k, \dots$.

Let ρ_k be a number such that $\rho_1 + \rho_2 + \cdots + \rho_k > R/2$. This number exists because otherwise by virtue of the condition on the functions $\{\omega_i\}$, $i = 1, \dots, n$, we would have $\omega_i(|x_k^i - x_1^i|) \leq \omega_i\Big(\sum\limits_{j=2}^{k} |x_j^i - x_{j-1}^i|\Big) \leq \sum\limits_{j=2}^{\infty} \rho_j < \frac{R}{2}$, whence $\sum\limits_{i=1}^{n} \omega_i(|x_k^i - x_0^i|) \leq \frac{R}{2}$, i.e. all x_k belong to $\Pi_{R/2}(x_1)$. On the other hand, $u(x_k) \to \infty$ as $k \to \infty$, which contradicts the boundedness of $u(x)$ in $\Pi_{R/2}(x_1) \subset \Pi_{3R/2}(x_0)$. Therefore there exists i_0, $1 \leq i_0 < k$, such that $\rho_{i_0} > \frac{R}{2^{i_0}}$. On the set D_{i_0} we have $u \geq 2^{i_0\mu}M$ and $\operatorname{mes} D_{i_0} \geq C\prod\limits_{j=1}^{n} \omega_j^{-1}(R/2^{i_0})$. Therefore

$$\int\limits_{\Pi_{2R}(x_0)} u^p\,dx \geq CM^p\left(\prod_{j=1}^{n} \omega_j^{-1}\left(\frac{R}{2^{i_0}}\right) \cdot 2^{i_0\mu}\right) \geq CM^p \prod_{j=1}^{n} \omega_j^{-1}(R)$$

by virtue of (34), so we come to the inequality (33). □

Lemma 6. *Let $\Pi_R(x_0) \subset D$, and the conditions (2)–(4) be fulfilled. Then there exists a constant $C > 0$, depending on n α, β, γ, η, A, σ, such* *that*

$$\inf_{x\in\Pi_R(x_0)} \int\limits_{\Pi_{2R}(x_0)} G_y(x)\,dy \ge CR^2.$$

Proof. As a matter of fact, this statement has been proved in Theorem 1. Consider the function $w = 1 - \sum\limits_{j=1}^{n} \frac{(x_j - x_j^0)^2}{(\omega_j^{-1}(R))^2}$. Then $L_x w = 2 - \sum\limits_{i=1}^{n} \frac{a_{ii}(x)}{(\omega_i^{-1}(R))^2}$.

Assume that $x \in \Pi_R(x_0)$. Then we have

$$1 - \sum_{j=1}^{n} \left(\frac{\omega_j^{-1}(R)}{\omega_j^{-1}(2R)}\right)^2 \le C \int\limits_{\Pi_{2R}(x_0)} G_y(x) \sum_{j=1}^{n} \frac{a_{jj}(y)}{(\omega_j^{-1}(2R))^2}\,dy.$$

The condition (3) implies that $\omega_j^{-1}(R) \ge \eta^{-\delta_0}\omega_j^{-1}(2R)$. Now

$$1 - \eta^{-\delta_0} \le C\gamma^{-1} \int\limits_{\Pi_{2R}(x_0)} G_y(x) \sum_{j=1}^{n} \frac{\lambda_j(y)}{(\omega_j^{-1}(2R))^2}\,dy \le$$

$$\le C\gamma^{-1} \int\limits_{\Pi_{2R}(x_0)} G_y(x) \sum_{j=1}^{n} \left(\frac{\omega_j^{-1}(\rho(y))}{\omega_j^{-1}(R)}\right)^2 \frac{dy}{\rho(y)^2} \le \frac{C}{R^2} \int\limits_{\Pi_{2R}(x_0)} G_y(x)\,dy,$$

whence $CR^2 \le \int\limits_{\Pi_{2R}(x_0)} G_y(x)\,dy$, $x \in \Pi_R(x_0)$. □

Lemma 7. *Let* $\Pi_R(x_0) \subset D$ *and the conditions* (2)–(4) *be fulfilled. Then*

$$\int\limits_{\Pi_R(x_0)} G_y(x)\,dy \le CR^2, \quad x \in \Pi_R(x_0),$$

where the constant $C > 0$ *depends on* n, α, β, γ, η, A, σ.

Proof. By virtue of the Hölder inequality we have

$$\int\limits_{\Pi_R(x_0)} G_y(x)\,dy \le \operatorname{mes}\Pi_R(x_0) \fint\limits_{\Pi_R(x_0)} G_y(x)\,dy \le$$

$$\le \operatorname{mes}\Pi_R(x_0)\Bigg(\fint\limits_{\Pi_R(x_0)} G_y^q(x)\,dy\Bigg)^{1/q} \le$$

$$\le \operatorname{mes}\Pi_R(x_0)\Bigg(\fint\limits_{\Pi_R(x_0)} G_y^{n'}(x) \prod_{j=1}^{n} \left(\frac{\omega_j^{-1}(\rho(y))}{\rho(y)}\right)^{\frac{2}{n}n'} dy\Bigg)^{1/n'} \times$$

$$\times \Bigg(\fint\limits_{\Pi_R(x_0)} \left(\prod_{j=1}^{n} \frac{\omega_j^{-1}(\rho(y))}{\rho(y)}\right)^{-\frac{2}{n}\cdot\frac{n'q}{n'-q}} dy\Bigg)^{\frac{n'-q}{n'q}}$$

By virtue of the estimate (17) the latter inequality implies

$$\int\limits_{\Pi_R(x_0)} G_y(x)\,dy \le (\operatorname{mes}\Pi_R(x_0))^{\frac{2}{n}-\frac{1}{q}+\frac{1}{n'}} \times$$

$$\times\Bigg(\ \fint\limits_{\Pi_R(x_0)} \Bigg(\prod_{j=1}^{n} \frac{\omega_j^{-1}(\rho(y))}{\rho(y)}\Bigg)^{-\frac{2}{n}\cdot\frac{n'q}{n'-q}} dy\Bigg)^{\frac{n'-q}{n'q}} =$$

$$= C\Bigg(\prod_{j=1}^{n}\omega_j^{-1}(R)\Bigg)^{\frac{1}{q'}+\frac{1}{n}} \prod_{j=1}^{n}\Bigg(\int\limits_0^{\omega_j^{-1}(R)} \Bigg(\frac{\omega_j(t)}{t}\Bigg)^{\frac{2}{n}\cdot\frac{n'q}{n'-q}} dt\Bigg)^{\frac{n'-q}{n'q}} \tag{35}$$

from which (assuming that $\sigma = \frac{2}{n}\cdot\frac{n'q}{n'-q}$) we derive, by means of the condition (4), that $\int\limits_{\Pi_R(x_0)} G_y(x)\,dy \le CR.$ $\square$

Theorem 2. *Let D be a bounded domain and $\Pi_{2R}(x_0) \subset D$. Assume that $u(x) \in W_\theta^{2,n}(D)$ is a positive solution of the equation (1) for which the conditions (2)–(4) are fulfilled. Then there exists a constant $C > 0$, depending on n, α, β, γ, η, A, σ, such that*

$$\sup_{x\in\Pi_R(x_0)} u(x) \le C \inf_{x\in\Pi_R(x_0)} u(x). \tag{36}$$

Proof. Let $1 = \inf\limits_{x\in\Pi_R(x_0)} u(x) = u(x_1)$. Denote $E_t = \{x \in \Pi_{2R}(x_0) : u(x) > t\}$, $t > 1$. Then by the maximum principle $G_y^R(x) \le G_y(x)$, where $G_y^R(x)$ is Green's function for $\Pi_{2R}(x_0)$. Then

$$\frac{u(x)}{t} \ge \frac{C}{R^2}\int\limits_{E_t} G_y^R(x)\,dy, \quad x \notin E_t.$$

Indeed, on ∂E_t we have $\frac{u(x)}{t} \ge 1$, $\frac{1}{CR^2}\int\limits_{E_t} G_y^R(x)\,dy \le 1$. On $\partial\Pi_{2R}(x_0)$ we have $\frac{u(x)}{t} > 0$, $\frac{1}{CR^2}\int\limits_{E_t} G_y^R(x)\,dy = 0$. In $\Pi_{2R}(x_0)\setminus E_t$ both functions $\frac{u(x)}{t}$ and $\frac{1}{CR^2}\int\limits_{E_t} G_y^R(x)\,dy$ are solutions of the equation (1), and $\frac{u(x)}{t} \ge \frac{1}{CR^2}\int\limits_{E_t} G_y^R(x)\,dy$ on the boundary $\Pi_{2R}(x_0)\setminus E_t$. We have made use of Lemma 7 to obtain

$$\frac{1}{CR^2}\int\limits_{E_t} G_y^R(x)\,dy \le \frac{1}{CR^2}\int\limits_{\Pi_R(x_0)} G_y^R(x)\,dy \le 1.$$

By the maximum principle we have $\frac{u(x)}{t} \geq \frac{1}{CR^2} \int\limits_{E_t} G_y^R(x)\,dy$, $x \in \pi_{2R}(x_0) \setminus E_t$. Putting in this inequality $x = x_0$, we obtain

$$\frac{1}{t} \geq \frac{1}{CR^2} \int\limits_{E_t} G_y^R(x)\,dy. \tag{37}$$

Assuming $E = E_t$, from the inequality (25) we obtain for the function $G_y^R(x)$

$$\int\limits_{E_t} G_y^R(x)\,dy \geq \frac{1}{C} \left(\frac{\operatorname{mes} E_t}{\operatorname{mes} \Pi_{2R}(x_0)} \right)^{\tau} \int\limits_{\Pi_{2R}(x_0)} G_y^R(x)\,dy, \tag{38}$$

$x \in \Pi_R(x_0) \setminus E_t$. The inequality (38) is a corollary of the inequality (25) (see [8]). Setting in (38) $x = x_1$ we have $\frac{1}{t} \geq C\big(\frac{\operatorname{mes} E_t}{\operatorname{mes} \Pi_{2R}(x_0)}\big)^{\tau} \frac{1}{R^2} \int\limits_{\Pi_{2R}(x_0)} G_y^R(x_1)\,dy$, from which by virtue of Lemma 6 we obtain

$$\frac{C}{t} \geq C \left(\frac{\operatorname{mes} E_t}{\operatorname{mes} \Pi_{2R}} \right)^{\tau}, \tag{39}$$

where $C, \tau > 0$ are some numbers depending on n, σ, A, α, β, η, γ. From (39) it follows that

$$\operatorname{mes} E_t \leq C \, \frac{\operatorname{mes} \Pi_{2R}}{t^p}\,, \quad p = \frac{1}{\tau}\,.$$

Now, by Lemma 5, for $p_1 = p/2$ we have

$$\begin{aligned} \sup_{x \in \Pi_R(x_0)} (u(x))^{p_1} &\leq C \quad \fint\limits_{\Pi_{2R}(x_0)} u^{p_1}\,dx \leq \\ &\leq \frac{1}{\operatorname{mes} \Pi_{2R}} \left(\int\limits_1^{\infty} t^{p_1 - 1} \operatorname{mes} E_t\,dt + \int\limits_0^1 t^{p_1 - 1} |E_t|\,dt \right) \leq \\ &\leq C \left(\int\limits_1^{\infty} t^{p_1 - p - 1} dt + C_1 \right) \leq C_2(\sigma, A, \alpha, \beta, \gamma, \eta, n). \end{aligned}$$

Theorem 2 is proved. □

Lemma 8. *Let D be a bounded domain, $\Pi_{2R}(x_0) \subset D$, $u(x) \in W_{\theta}^{2,n}(D)$ be a solution of the equation (1). Then there exists a number $Q > 1$, depending on n, σ, A, α, β, η, γ, such that*

$$\operatorname*{osc}_{\Pi_{2R}(x_0)} u \geq Q \operatorname*{osc}_{\Pi_R(x_0)} u,$$

where $\operatorname*{osc}_E u = \sup_E u - \inf_E u$.

Proof. Apply Theorem 2 to the functions $u(x) - m_{2R}$ and $M_{2R} - u(x)$ in $\Pi_{2R}(x_0)$, where $m_{2R} = \inf\limits_{\Pi_{2R}(x_0)} u(x)$, $M_{2R} = \sup\limits_{\Pi_{2R}(x_0)} u(x)$. Then

$$M_R - m_{2R} \le C(m_R - m_{2R}) \quad \text{and} \quad M_{2R} - m_R \le C(M_{2R} - M_R).$$

The summation of these inequalities gives

$$(1 + C) \operatorname*{osc}_{\Pi_R(x_0)} u \le (C - 1) \operatorname*{osc}_{\pi_{2R}(x_0)} u,$$

whence

$$\operatorname*{osc}_{\Pi_{2R}(x_0)} u \ge \frac{C+1}{C-1} \operatorname*{osc}_{\Pi_R(x_0)} u,$$

where the constant $C > 0$ of Harnack's inequality depends on n, σ, A, α, β, η, γ. □

Theorem 3. *Let D be a bounded domain in E_n, $u(x) \in W^{2,n}_\theta(D)$ be a solution of the equation* (1)*, where the coefficients satisfy the conditions* (2)–(4)*. Then for any $\rho > 0$ there exist $\mu = \mu(\alpha, n, A, \beta, \gamma, \eta)$ and $H = H(\alpha, n, A, \beta, \gamma, \eta)$ such that for any $x, y \in D_\rho = \{x \in D : \operatorname{dist}(x, R^n \setminus D) > \rho\}$ we have the estimate*

$$|u(x) - u(y)| \le H|x - y|^\mu \sup_D |u|.$$

Proof. Fix $y \in D_\rho$. There exists R_0 such that $\Pi_{2R_0}(y) \subset D$. For this it is sufficient to take $R_0 = \frac{\omega^-(\rho)}{2}$, where $\omega^-(\rho) = \min\{\omega_1(\rho), \omega_2(\rho), \ldots, \omega_n(\rho)\}$. For $k = 0, 1, 2, \ldots,$ we denote $\rho_k = 2^{-k+1}R_0$, $\Pi^k = \Pi_{\rho_k}(y)$. By virtue of Theorem 2,

$$\operatorname*{osc}_{\Pi_{\rho_k}(y)} u \le \frac{1}{Q} \operatorname*{osc}_{\Pi_{\rho_{k-1}}(y)} u \le \cdots \le \frac{1}{Q^k} \operatorname*{osc}_{\Pi_{\rho_0}(y)} u.$$

Let R be any number from $(0, R_0]$. Then there is a natural number k such that $\rho_k \le R \le \rho_{k-1}$. In that case,

$$\operatorname*{osc}_{\Pi_R(y)} u \le \frac{1}{Q} \left(\frac{\rho_k}{R_0}\right)^\nu \operatorname*{osc}_{\Pi_{R_0}(y)} u \le 2 \operatorname*{osc}_{\Pi_{2R_0}(y)} |u| \left(\frac{R}{R_0}\right)^\nu, \tag{40}$$

where $\nu = \log_2 Q$. Let $x \in D_\rho$, $x \ne y$, be any point.

Two cases are possible: i) $\omega^+(|x - y|) < R_0$, ii) $\omega^+(|x - y|) \ge R_0$, where $\omega^+(\rho) = \max\{\omega_1(\rho), \ldots, \omega_n(\rho)\}$. In the case i) we have $x \in \Pi_{\omega^+(|x-y|)}(y)$, and therefore (40) implies

$$|u(x) - u(y)| \le 2 \sup_D |u| \, R_0^{-\nu} \left(\omega^+(|x - y|)\right)^\nu. \tag{41}$$

Let $t_2 \geq t_1 > 0$, k be a natural integer for which $\eta^k \leq \frac{t_2}{t_1} \leq \eta^{k+1}$. Then by virtue of the condition (3)

$$\omega_j(t_2) = \omega_j\left(\frac{t_2}{t_1} \cdot t_1\right) \geq \omega_j(\eta^k t_1) > \alpha^k \omega_j(t_1) \geq \alpha^{\log_\eta \frac{t_2}{t_1} - 1} = \\ = \frac{1}{\alpha}\left(\frac{t_2}{t_1}\right)^{\log_\eta \alpha} \omega_j(t_1), \tag{42}$$

where $j \in \{1, 2, \ldots, n\}$. By virtue of the condition (42) $\omega_j(t)/t^{-\xi}$ is bounded for sufficiently small t, where $\xi = \log_\eta \alpha$. Then from (41) we obtain

$$|u(x) - u(y)| \leq \frac{\sup\limits_D |u| \cdot 2^{\nu+1}}{(\overline{\omega}(\rho))^\nu} |x - y|^{\xi\nu}, \quad x \in D_\rho.$$

In the case ii) we have $|u(x) - u(y)| \leq 2 \sup\limits_D |u| \leq 2 \sup\limits_D |u| \frac{(|x-y|)^{\xi\nu}}{R_0^\nu}$, $x \in D_\rho$. □

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