Memoirs on Differential Equations and Mathematical Physics
Volume 41, 2007, 69–85

A. Lomtatidze, Z. Opluštil, and J. Šremr

ON A NONLOCAL BOUNDARY VALUE PROBLEM FOR FIRST ORDER LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

Abstract. Efficient sufficient conditions are established for the unique solvability of the problem

$$u'(t) = \ell(u)(t) + q(t), \quad u(a) = h(u) + c,$$

where $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ and $h : C([a,b];\mathbb{R}) \to \mathbb{R}$ are linear bounded operatos, $q \in L([a,b];\mathbb{R})$, and $c \in \mathbb{R}$.

2000 Mathematics Subject Classification. 34K06, 34K10.

Key words and phrases. Linear functional differential equation, non-local boundary value problem, existence, uniqueness.

რეზიუმე. ნაშრომში დადგენილია

$$u'(t) = \ell(u)(t) + q(t), \quad u(a) = h(u) + c$$

ამოცანის ცალსახად ამოხსნადობის ეფექტური საკმარისი პირობები, სადაც $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ და $h : C([a,b];\mathbb{R}) \to \mathbb{R}$ წრფივი შემოსაზღვრული ოპერატორებია, ხოლო $q \in L([a,b];\mathbb{R})$ და $c \in \mathbb{R}$.

1. Introduction

The following notation is used throughout the paper.

$\mathbb{R}$ is the set of all real numbers. $\mathbb{R}_+ = [0, +\infty[$.

If $x \in \mathbb{R}$, then

$$[x]_+ = \frac{1}{2}\,(|x| + x), \quad [x]_- = \frac{1}{2}\,(|x| - x).$$

$C([a,b];\mathbb{R})$ is the Banach space of continuous functions $v : [a,b] \longrightarrow \mathbb{R}$ with the norm

$$\|v\|_C = \max\big\{|v(t)| : \; t \in [a,b]\big\}.$$

$C([a,b];\mathbb{R}_+) = \big\{u \in C([a,b];\mathbb{R}) : \; u(t) \ge 0 \text{ for } t \in [a,b]\big\}$.

$L([a,b];\mathbb{R})$ is the Banach space of Lebesgue integrable functions $p : [a,b] \to \mathbb{R}$ with the norm

$$\|p|_L = \int_a^b |p(s)|\, ds.$$

$L([a,b];\mathbb{R}_+) = \big\{p \in L([a,b];\mathbb{R}) : \; p(t) \ge 0 \text{ for almost all } t \in [a,b]\big\}$.

$\mathcal{L}_{ab}$ is the set of linear bounded operators $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$.

P_{ab} is the set of operators $\ell \in \mathcal{L}_{ab}$ transforming the set $C([a,b];\mathbb{R}_+)$ into the set $L([a,b];\mathbb{R}_+)$.

F_{ab} is the set of linear bounded functionals $h : C([a,b];\mathbb{R}) \to \mathbb{R}$.

PF_{ab} is the set of functionals $h \in F_{ab}$ transforming the set $C([a,b];\mathbb{R}_+)$ into the set $\mathbb{R}_+$.

Throughout the paper, the equalities and inequalities with integrable functions are understood almost everywhere.

On the interval $[a,b]$, we consider the problem on the existence and uniqueness of a solution of the equation

$$u'(t) = \ell(u)(t) + q(t) \tag{1}$$

satisfying the boundary condition

$$u(a) = h(u) + c. \tag{2}$$

Here we suppose that $\ell \in \mathcal{L}_{ab}$, $q \in L([a,b];\mathbb{R})$, $h \in F_{ab}$, and $c \in \mathbb{R}$. Moreover, it is natural to assume that $\widetilde{h} \not\equiv 0$, where $\widetilde{h}(v) \overset{def}{=} v(a) - h(v)$.

By a solution of the equation (1) we understand an absolutely continuous function $u : [a,b] \to \mathbb{R}$ satisfying the equation (1) almost everywhere in $[a,b]$.

In [4] (see also [5, 9]), the efficient sufficient conditions are given for the unique solvability of the problem

$$u'(t) = \ell(u)(t) + q(t), \quad u(a) = \lambda u(b) + c. \tag{3}$$

It is clear that the problem (3) is a particular case of (1), (2) with $h(v) \overset{def}{=} \lambda v(b)$.

In this paper, the results from [4] are extended for the problem (1), (2) with

$$h(v) \stackrel{def}{=} \lambda v(b) + h_0(v) - h_1(v), \quad h_0,\, h_1 \in PF_{ab},$$

i.e., for the case where the boundary condition in (3) is perturbed by a linear continuous functional $h_0 - h_1$ (in general nonlocal).

The paper is organized as follows. In Section 2, we give conditions for the unique solvability of the problem (1), (2). These results are further concretized for the problem

$$u'(t) = p(t)u(\tau(t)) + q(t), \quad \int_a^b u(s)\, d\sigma(s) = c, \tag{4}$$

where $p,\, q \in L([a,b];\mathbb{R})$, $\tau : [a,b] \to [a,b]$ is a measurable function, $\sigma : [a,b] \to \mathbb{R}$ is an absolutely continuous function, $\sigma(a) > 0, \sigma(b) > 0$ and $c \in \mathbb{R}$. The assertions formulated in Section 2 are proved in Section 3.

2. Main Results

In what follows, we will assume that the functional h admits the representation

$$h(v) = \lambda v(b) + h_0(v) - h_1(v),$$

where $\lambda > 0$ and $h_0,\, h_1 \in PF_{ab}$.

Before the formulation of the results, we introduce some notation. Put

$$\alpha(h) = (1 - h_0(1)) \min\Big\{1, \frac{1}{\lambda}\Big\}, \tag{5}$$

$$\beta(h) = (\lambda - h_1(1)) \min\Big\{1, \frac{1}{\lambda}\Big\}, \tag{6}$$

and define the functions ω_0 and ω_1 by the formulas

$$\omega_0(h,x) = \begin{cases} \dfrac{(x + \frac{1}{\lambda}\, h_0(1))(1 - h_0(1))}{1 - h_0(1) - x} - \Big(\dfrac{1}{\lambda}\, h_1(1) + \dfrac{1-\lambda}{\lambda}\Big) \\ \qquad \text{if } \lambda \le 1,\ (1-\lambda+h_1(1))x < (1-h(1))(1-h_0(1)) \\ \dfrac{(x + h_0(1))(1 - h_0(1))}{1 - h_0(1) - x} - (h_1(1) + 1 - \lambda) \\ \qquad \text{if } \lambda \le 1,\ (1-\lambda+h_1(1))x \ge (1-h(1))(1-h_0(1)) \\ \dfrac{(x + \lambda - 1 + h_0(1))(1 - h_0(1))}{1 - h_0(1) - \lambda x} - h_1(1) \\ \qquad \text{if } \lambda > 1,\ \lambda h_1(1)x < (1 - h(1))(1 - h_0(1)) \\ \dfrac{(x + \frac{\lambda-1}{\lambda} + \frac{1}{\lambda} h_0(1))(1 - h_0(1))}{1 - h_0(1) - \lambda x} - \dfrac{1}{\lambda}\, h_1(1) \\ \qquad \text{if } \lambda > 1,\ \lambda h_1(1)x \ge (1 - h(1))(1 - h_0(1)) \end{cases}, \tag{7}$$

$$
\omega_1(h,x)=\begin{cases}
\dfrac{(y+h_1(1))(1-\frac{1}{\lambda}\,h_1(1))}{1-\frac{1}{\lambda}h_1(1)-y}-(h_0(1)+\lambda-1) \\
\quad \text{if } \lambda\geq 1,\ (\lambda-1+h_0(1))y<(h(1)-1)\Big(1-\frac{1}{\lambda}\,h_1(1)\Big) \\
\dfrac{(y+\frac{1}{\lambda}\,h_1(1))(1-\frac{1}{\lambda}\,h_1(1))}{1-\frac{1}{\lambda}\,h_1(1)-y}-\Big(\dfrac{1}{\lambda}\,h_0(1)+\dfrac{\lambda-1}{\lambda}\Big) \\
\quad \text{if } \lambda\geq 1,\ (\lambda-1+h_0(1))y\geq(h(1)-1)\Big(1-\frac{1}{\lambda}\,h_1(1)\Big) \\
\dfrac{(y+\frac{1-\lambda}{\lambda}+\frac{1}{\lambda}\,h_1(1))(\lambda-h_1(1))}{\lambda-h_1(1)-y}-\dfrac{1}{\lambda}\,h_0(1) \\
\quad \text{if } \ \lambda<1,\ \ h_0(1)y<(h(1)-1)(\lambda-h_1(1)) \\
\dfrac{(y+1-\lambda+h_1(1))(\lambda-h_1(1))}{\lambda-h_1(1)-y}-h_0(1) \\
\quad \text{if } \ \lambda<1,\ \ h_0(1)y\geq(h(1)-1)(\lambda-h_1(1))
\end{cases}. \tag{8}
$$

Theorem 2.1. *Let $\ell=\ell_0-\ell_1$ with $\ell_0,\,\ell_1\in P_{ab}$,*

$$h(1)\leq 1 \tag{9}$$

and

$$h_0(1)<1,\quad h_1(1)\leq\lambda. \tag{10}$$

Let, moreover,

$$\|\ell_0(1)\|_L<\alpha(h), \tag{11}$$

$$\|\ell_1(1)\|_L<1+\beta(h)+2\sqrt{\alpha(h)-\|\ell_0(1)\|_L} \tag{12}$$

and

$$\|\ell_1(1)\|_L>\omega_0(h,\|\ell_0(1)\|_L). \tag{13}$$

Then the problem (1), (2) *has a unique solution.*

The following theorem can be regarded as a supplement of the preceding one.

Theorem 2.2. *Let $\ell=\ell_0-\ell_1$ with $\ell_0,\,\ell_1\in P_{ab}$. Let, moreover, the condition* (9) *hold,*

$$h_1(1)<\lambda, \tag{14}$$

$$\|\ell_1(1)\|_L<\beta(h), \tag{15}$$

$$\|\ell_0(1)\|_L<1+\alpha(h)+2\sqrt{\beta(h)-\|\ell_1(1)\|_L} \tag{16}$$

and

$$\|\ell_0(1)\|_L>\frac{\alpha(h)}{\beta(h)-\|\ell_1(1)\|_L}-1. \tag{17}$$

Then the problem (1), (2) *has a unique solution.*

Remark 2.1. Let $\ell=\ell_0-\ell_1$ with $\ell_0,\,\ell_1\in P_{ab}$. Define the operator $\psi:L([a,b];\mathbb{R})\to L([a,b];\mathbb{R})$ by setting

$$\psi(w)(t)\overset{def}{=}w(a+b-t)\ \text{ for }\ t\in[a,b].$$

Let φ be the restriction of ψ to the space $C([a,b];\mathbb{R})$ and

$$\widehat{\ell}_i(w)(t) \overset{def}{=} \psi(\ell(\varphi(w)))(t) \ \text{ for } \ t \in [a,b],$$
$$\widehat{h}_i(w) \overset{def}{=} \frac{1}{\lambda} h_i(\varphi(v)) \ \ (i = 0,1).$$

It is clear that if u is a solution of the problem (1), (2), then the function $v \overset{def}{=} \varphi(u)$ is a solution of the problem

$$v'(t) = \widehat{\ell}_1(v)(t) - \widehat{\ell}_0(v)(t), \quad v(a) = \frac{1}{\lambda}\, v(b) + \widehat{h}_1(v) - \widehat{h}_0(v), \tag{18}$$

and vice versa, if v is a solution of the problem (18), then the function $u \overset{def}{=} \varphi(v)$ is a solution of the problem (1), (2). Furthermore, $\omega_1(\widehat{h}, x) = \omega_0(h, x)$ and $\omega_o(\widehat{h}, x) = \omega_1(h, x)$, where

$$\widehat{h}(v) \overset{def}{=} \frac{1}{\lambda}\, v(b) + \widehat{h}_1(v) - \widehat{h}_0(v).$$

Mention also that $h(1) \geq 1$ if and only if $\widehat{h}(1) \leq 1$.

In view of Remark 2.1, Theorems 2.3 and 2.4 below can be obtained from Theorems 2.1 and 2.2. Note that in these theorems the condition

$$h(1) \geq 1 \tag{19}$$

is assumed instead of (9).

Theorem 2.3. *Let $\ell = \ell_0 - \ell_1$ with $\ell_0,\ \ell_1 \in P_{ab}$, the inequality* (19) *hold, and*

$$h_0(1) \leq 1, \quad h_1(1) < \lambda.$$

Let, moreover, the conditions (15) *and* (16) *be fulfilled and*

$$\|\ell_0(1)\|_L > \omega_1(h, \|\ell_1(1)\|_L),$$

where ω_1 is a function defined by (8)*. Then the problem* (1), (2) *has a unique solution.*

Theorem 2.4. *Let $\ell = \ell_0 - \ell_1$ with $\ell_0,\ \ell_1 \in P_{ab}$, the inequality* (19) *hold, and*

$$h_0(1) < 1.$$

Let, moreover, the conditions (11) *and* (12) *be fulfilled and*

$$\|\ell_1(1)\|_L > \frac{\beta(h)}{\alpha(h) - \|\ell_0(1)\|_L} - 1.$$

Then the problem (1), (2) *has a unique solution.*

Now we give several corollaries for the problem (4). Recall that in (4) $p,\ q \in L([a,b];\mathbb{R})$, $\tau : [a,b] \to [a,b]$ is a measurable function and $\sigma : [a,b] \to$

$\mathbb{R}$ is an absolutely continuous function such that $\sigma(a) > 0$, $\sigma(b) > 0$. We first introduce the following notation

$$p_0 = \int_a^b [p(s)\sigma(s)]_- \, ds, \tag{20}$$

$$p_1 = \int_a^b [p(s)\sigma(s)]_+ \, ds, \tag{21}$$

$$\alpha_0 = (\sigma(a) - p_0) \min\Big\{\frac{1}{\sigma(a)}, \frac{1}{\sigma(b)}\Big\}, \tag{22}$$

$$\beta_1 = (\sigma(b) - p_1) \min\Big\{\frac{1}{\sigma(a)}, \frac{1}{\sigma(b)}\Big\}, \tag{23}$$

$$\widetilde{\omega}_0(x) = \omega_0(h, x), \quad \widetilde{\omega}_1(x) = \omega_1(h, x), \tag{24}$$

where ω_0 and ω_1 are defined by (7) and (8), respectively, with $h(1) = \frac{\sigma(b)}{\sigma(a)} + p_0 - p_1$, $h_0(1) = p_0$ and $h_1(1) = p_1$.

Corollary 2.1. *Let*

$$0 \le \beta_0 \le \alpha_0, \quad \alpha_0 > 0,$$
$$\int_a^b [p(s)]_+ \, ds < \alpha_0, \tag{25}$$

and

$$\widetilde{\omega}_0\Big(\int_a^b [p(s)]_+\Big) < \int_a^b [p(s)]_- \, ds < 1 + \beta_0 + 2\sqrt{\alpha_0 - \int_a^b [p(s)]_+ \, ds}\,.$$

Then the problem (4) *has a unique solution.*

Corollary 2.2. *Let*

$$0 < \beta_0 \le \alpha_0,$$
$$\int_a^b [p(s)]_- \, ds < \beta_0, \tag{26}$$

and

$$\frac{\alpha_0}{\beta_0 - \int_a^b [p(s)]_- \, ds} - 1 < \int_a^b [p(s)]_+ \, ds < 1 + \alpha_0 + 2\sqrt{\beta_0 - \int_a^b [p(s)]_- \, ds}\,.$$

Then the problem (4) *has a unique solution.*

Corollary 2.3. *Let*

$$\beta_0 \geq \alpha_0 \geq 0, \quad \beta_0 > 0,$$

the condition (26) *hold, and*

$$\widetilde{\omega}_1\Big(\int_a^b [p(s)]_-\,ds\Big) < \int_a^b [p(s)]_+\,ds < 1 + \alpha_0 + 2\sqrt{\beta_0 - \int_a^b [p(s)]_-\,ds}\,.$$

Then the problem (4) *has a unique solution.*

Corollary 2.4. *Let*

$$\beta_0 \geq \alpha_0 > 0,$$

the condition (25) *hold, and*

$$\frac{\beta_0}{\alpha_0 - \int_a^b [p(s)]_+\,ds} < \int_a^b [p(s)]_-\,ds < 1 + \beta_0 + 2\sqrt{\alpha_0 - \int_a^b [p(s)]_+\,ds}\,.$$

Then the problem (4) *has a unique solution.*

3. Proofs

It is well-known from the general theory of boundary value problems for functional differential equations that the problem (1), (2) has the so-called Fredholm property, i.e., the problem (1), (2) is uniquely solvable for arbitrary $q \in L([a,b];\mathbb{R})$ and $c \in \mathbb{R}$ if and only if the corresponding homogeneous problem

$$u'(t) = \ell(u)(t), \tag{27}$$

$$u(a) = h(u) \tag{28}$$

has only the trivial solution (see, e.g., [1], [2], [7], [8], [6], [3]). Therefore, to prove the theorems, it is suficient to show that the homogeneous problem (27), (28) has only the trivial solution.

First, we prove the following lemma.

Lemma 3.1. *Assume that* $\ell = \ell_0 - \ell_1$ *with* $\ell_0,\, \ell_1 \in P_{ab}$ *and*

$$h_0(1) \leq 1, \quad h_1(1) \leq \lambda. \tag{29}$$

Let, moreover, either the conditions (11) *and* (12) *or the conditions* (15) *and* (16) *be satisfied. If* u *is a solution of the homogeneous problem* (27), (28), *then there exists* $\delta \in \{-1, 1\}$ *such that*

$$\delta u(t) \geq 0 \ \ for \ \ t \in [a,b].$$

Proof. Assume that u is a solution of the problem (27), (28) and there exist t_1, $t_2 \in [a,b]$ such that $u(t_1)u(t_2) < 0$. Put

$$M = \max\big\{u(t) :\ t \in [a,b]\big\}, \quad m = -\min\big\{u(t) :\ t \in [a,b]\big\}, \tag{30}$$

and choose $t_M, t_m \in [a,b]$ such that

$$u(t_M) = M, \quad u(t_m) = -m. \tag{31}$$

It is clear that

$$M > 0, \quad m > 0. \tag{32}$$

Without loss of generality, we can suppose that $t_m < t_M$.

The integration of (27) from a to t_m, from t_m to t_M, and from t_M to b, in view of (30), (31), and the assumption that ℓ_0, $\ell_1 \in P_{ab}$, yields

$$\begin{aligned} u(a) + m &= \int\limits_a^{t_m} \ell_1(u)(s)\,ds - \int\limits_a^{t_m} \ell_0(u)(s)\,ds \le \\ &\le M \int\limits_a^{t_m} \ell_1(1)(s)\,ds + m \int\limits_a^{t_m} \ell_0(1)(s)\,ds, \end{aligned} \tag{33}$$

$$\begin{aligned} M + m &= \int\limits_{t_m}^{t_M} \ell_0(u)(s)\,ds - \int\limits_{t_m}^{t_M} \ell_1(u)(s)\,ds \le \\ &\le M \int\limits_{t_m}^{t_M} \ell_0(1)(s)\,ds + m \int\limits_{t_m}^{t_M} \ell_1(1)(s)\,ds, \end{aligned} \tag{34}$$

$$\begin{aligned} M - u(b) &= \int\limits_{t_M}^{b} \ell_1(u)(s)\,ds - \int\limits_{t_M}^{b} \ell_0(u)(s)\,ds \le \\ &\le M \int\limits_{t_M}^{b} \ell_1(1)(s)\,ds + m \int\limits_{t_M}^{b} \ell_0(1)(s)\,ds. \end{aligned} \tag{35}$$

On the other hand, the condition (28), in view of (30) and the assumption that $h_0, h_1 \in PF_{ab}$, yields

$$u(a) - \lambda u(b) = h_0(u) - h_1(u) \ge -mh_0(1) - Mh_1(1).$$

It follows from (33) and (35) that

$$\begin{aligned} M(\lambda - h_1(1)) + m(1 - h_0(1)) \le M\bigg(\int\limits_a^{t_m} \ell_1(1)(s)\,ds + \lambda \int\limits_{t_M}^{b} \ell_1(1)(s)\,ds \bigg) + \\ + m\bigg(\int\limits_a^{t_m} \ell_0(1)(s)\,ds + \lambda \int\limits_{t_M}^{b} \ell_0(1)(s)\,ds \bigg), \end{aligned}$$

i.e.,

$$M\beta(h) + m\alpha(h) \le MB_1 + mA_1, \tag{36}$$

where $\alpha(h)$ and $\beta(h)$ are defined by (5) and (6),

$$A_1 = \int_J \ell_0(1)(s)\,ds, \quad B_1 = \int_J \ell_1(1)(s)\,ds \tag{37}$$

and $J = [a, t_m] \cup [t_M, b]$. Furthermore, (34) results in

$$M + m \le MA_2 + mB_2, \tag{38}$$

where

$$A_2 = \int_{t_m}^{t_M} \ell_0(1)(s)\,ds, \quad B_2 = \int_{t_m}^{t_M} \ell_1(1)(s)\,ds. \tag{39}$$

First suppose that the conditions (11) and (12) hold. In that case, we have $h_0(1) < 1$ (see (11) and (5)). According to (11), it is clear that

$$A_1 < \alpha(h), \quad A_2 < 1.$$

Thus, it follows from (32), (36) and (38) that

$$B_1 > \beta(h), \quad B_2 > 1 \tag{40}$$

and

$$(\alpha(h) - A_1)(1 - A_2) \le (B_1 - \beta(h))(B_2 - 1). \tag{41}$$

Obviously,

$$\begin{aligned} &(\alpha(h) - A_1)(1 - A_2) \ge \alpha(h) - (A_1 + A_2), \\ &(B_1 - \beta(h))(B_2 - 1) \le \frac{1}{4}\big(B_1 + B_2 - 1 - \beta(h)\big)^2. \end{aligned} \tag{42}$$

By virtue of (11), (37), (39) and (42), the inequality (41) yields

$$0 < 4(\alpha(h) - \|\ell_0(1)\|_L) \le \big(\|\ell_1(1)\|_L - 1 - \beta(h)\big)^2,$$

which, in view of (40), contradicts (12).

Now suppose that the conditions (15) and (16) are satisfied. In that case, we have $h_1(1) < \lambda$ (see (15) and (6)). According to (15), it is clear that

$$B_1 < \beta(h), \quad B_2 < 1.$$

Thus, it follows from (32), (36) and (38) that

$$A_1 > \alpha(h), \quad A_2 > 1, \tag{43}$$

and

$$(\beta(h) - B_1)(1 - B_2) \le (A_1 - \alpha(h))(A_2 - 1). \tag{44}$$

Obviously,

$$\begin{aligned} &(\beta(h) - B_1)(1 - B_2) \ge \beta(h) - (B_1 + B_2), \\ &(A_1 - \alpha(h))(A_2 - 1) \le \frac{1}{4}\big(A_1 + A_2 - 1 - \alpha(h)\big)^2. \end{aligned} \tag{45}$$

By virtue of (15), (37), (39) and (45), the inequality (44) implies that

$$0 < 4(\beta(h) - \|\ell_1(1)\|_L) \le (\|\ell_0(1)\|_L - 1 - \alpha(h))^2,$$

which, in view of (43), contradicts (16). The contradictions obtained prove the validity of the lemma. □

Proof of Theorem 2.1. As it has been mentioned above, it is sufficient to show that the homogeneous problem (27), (28) has only the trivial solution.

Assume the contrary, i.e., the problem (27), (28) has a nontrivial solution u. According to Lemma 3.1, without loss of generality we can assume that

$$u(t) \ge 0 \ \text{ for } \ t \in [a,b]. \tag{46}$$

Put

$$M = \max\big\{u(t): \ t \in [a,b]\big\}, \quad m = \min\big\{u(t): \ t \in [a,b]\big\}, \tag{47}$$

and choose $t_M, t_m \in [a,b]$ such that

$$u(t_M) = M, \quad u(t_m) = m. \tag{48}$$

Obviously,

$$M > 0, \quad m \ge 0, \tag{49}$$

and either

$$t_M \le t_m \tag{50}$$

or

$$t_M > t_m. \tag{51}$$

First suppose that (50) holds. The integration of (27) from a to t_M and from t_m to b, in view of (47)–(49) and the assumption that $\ell_0, \ell_1 \in P_{ab}$, yields

$$M - u(a) = \int_a^{t_M} \ell_0(u)(s)\,ds - \int_a^{t_M} \ell_1(u)(s)\,ds \le M \int_a^{t_M} \ell_0(1)(s)\,ds, \tag{52}$$

$$u(b) - m = \int_{t_m}^{b} \ell_0(u)(s)\,ds - \int_{t_m}^{b} \ell_1(u)(s)\,ds \le M \int_{t_m}^{b} \ell_0(1)(s)\,ds. \tag{53}$$

On account of (47) and the assumption that $h_0, h_1 \in PF_{ab}$, the condition (28) gives

$$\lambda u(b) - u(a) = h_1(u) - h_0(u) \ge m h_1(1) - M h_0(1). \tag{54}$$

Now from (52)–(54) we get

$$M(1 - h_0(1)) - m(\lambda - h_1(1)) \le M\bigg(\int_a^{t_M} \ell_0(1)(s)\,ds + \lambda \int_{t_m}^{b} \ell_0(1)(s)\,ds\bigg) \le$$

$$\le M\|\ell_0(1)\|_L \, \frac{1}{\min\{1, \frac{1}{\lambda}\}},$$

whence, in view of (5), (9) and (49), it follows that

$$M(\alpha(h) - \|\ell_0(1)\|_L) \leq m\alpha(h). \tag{55}$$

Now suppose that (51) holds. The integration of (27) from t_m to t_M, on account of (47)–(49) and the assumption that $\ell_0,\ \ell_1 \in P_{ab}$, results in

$$M - m = \int_{t_m}^{t_M} \ell_0(u)(s)\,ds - \int_{t_m}^{t_M} \ell_1(u)(s)\,ds \leq M \int_a^b \ell_0(1)(s)\,ds. \tag{56}$$

By virtue of (9), (10) and (56), it is not difficult to verify that the inequality (55) is fulfilled.

Therefore, in both cases (50) and (51), the inequality (55) is satisfied. On the other hand, the integration of (27) from a to b, in view of (47)–(49) and the assumption that $\ell_0,\ \ell_1 \in P_{ab}$, yields

$$\begin{aligned} u(b) - u(a) = \int_a^b \ell_0(u)(s)\,ds - \int_a^b \ell_1(u)(s)\,ds \leq \\ \leq M\|\ell_0(1)\|_L - m\|\ell_1(1)\|_L, \end{aligned}$$

i.e,

$$m\|\ell_1(1)\|_L \leq M\|\ell_0(1)\|_L + u(a) - u(b). \tag{57}$$

Furthermore, the condition (28) implies

$$u(a) - u(b) = (\lambda - 1)u(b) + h_0(u) - h_1(u), \tag{58}$$

$$u(a) - u(b) = \Big(1 - \frac{1}{\lambda}\Big)u(a) + \frac{1}{\lambda}\,h_0(u) - \frac{1}{\lambda}\,h_1(u). \tag{59}$$

Suppose first that

$$\lambda \leq 1, \quad (1 - \lambda + h_1(1))\|\ell_0(1)\|_L < (1 - h(1))(1 - h_0(1)).$$

The inequalities (57) and (59), together with (47) and the assumption that $h_0,\ h_1 \in PF_{ab}$, result in

$$m\|\ell_1(1)\|_L \leq M\|\ell_0(1)\|_L - m\,\frac{1-\lambda}{\lambda} + M\,\frac{1}{\lambda}\,h_0(1) - m\,\frac{1}{\lambda}\,h_1(1). \tag{60}$$

Hence, from (55) and (60), by virtue of (11) and (49), we get

$$\begin{aligned} \big(1 - h_0(1) - \|\ell_0(1)\|_L\big)\Big(\|\ell_1(1)\|_L + \frac{1-\lambda}{\lambda} + \frac{1}{\lambda}\,h_1(1)\Big) \leq \\ \leq \Big(\|\ell_0(1)\|_L + \frac{1}{\lambda}\,h_0(1)\Big)(1 - h_0(1)), \end{aligned}$$

which, in view of (11) and (7), contradicts (13).

Suppose that

$$\lambda \leq 1, \quad \big(1 - \lambda + h_1(1)\big)\|\ell_0(1)\|_L \geq (1 - h(1))(1 - h_0(1)).$$

The inequalities (57) and (58) together with (47) and the assumption that $h_0, h_1 \in PF_{ab}$ result in

$$m\|\ell_1(1)\|_L \leq M\|\ell_0(1)\|_L - m(1 - \lambda) + Mh_0(1) - mh_1(1). \tag{61}$$

Hence, from (55) and (61), by virtue of (11) and (49), we get

$$\begin{gathered}\big(1-h_0(1)-\|\ell_0(1)\|_L\big)\big(\|\ell_1(1)\|_L+1-\lambda+h_1(1)\big)\leq\\ \leq\big(\|\ell_0(1)\|_L+h_0(1)\big)(1-h_0(1)),\end{gathered}$$

which, in view of (11) and (7), contradicts (13).

Now suppose that

$$\lambda>1,\quad \lambda h_1(1)\|\ell_0(1)\|_L<(1-h(1))(1-h_0(1)).$$

The inequalities (57) and (58) together with (47) and the assumption that $h_0, h_1 \in PF_{ab}$ result in

$$m\|\ell_1(1)\|_L\leq M\|\ell_0(1)\|_L+M(\lambda-1)+Mh_0(1)-mh_1(1). \tag{62}$$

Hence, from (55) and (62), by virtue of (11) and (49), we get

$$\begin{gathered}\big(1-h_0(1)-\lambda\|\ell_0(1)\|_L\big)\big(\|\ell_1(1)\|_L+h_1(1)\big)\leq\\ \leq\big(\|\ell_0(1)\|_L+\lambda-1+h_0(1)\big)(1-h_0(1)),\end{gathered}$$

which, in view of (11) and (7), contradicts (13).

Finally, suppose that

$$\lambda>1,\quad \lambda h_1(1)\|\ell_0(1)\|_L\geq(1-h(1))(1-h_0(1)).$$

The inequalities (57) and (59) together with (47) and the assumption that h_0, $h_1 \in PF_{ab}$ result in

$$m\|\ell_1(1)\|_L\leq M\|\ell_0(1)\|_L+M\,\frac{\lambda-1}{\lambda}+M\,\frac{1}{\lambda}\,h_0(1)-m\,\frac{1}{\lambda}\,h_1(1). \tag{63}$$

Hence, from (55) and (63), by virtue of (11) and (49), we get

$$\begin{gathered}\big(1-h_0(1)-\lambda\|\ell_0(1)\|_L\big)\Big(\|\ell_1(1)\|_L+\frac{1}{\lambda}\,h_1(1)\Big)\leq\\ \leq\Big(\|\ell_0(1)\|_L+\frac{\lambda-1}{\lambda}+\frac{1}{\lambda}\,h_0(1)\Big)(1-h_0(1)),\end{gathered}$$

which, in view of (11) and (7), contradicts (13).

The contradictions obtained above prove that under the assumptions of theorem the problem (27), (28) has only the trivial solution. $\square$

Proof of Theorem 2.2. Suppose that the problem (27), (28) has a nontrivial solution u. According to Lemma 3.1, without loss of generality we can assume that (46) holds. Define the numbers M and m by (47), and choose t_M, $t_m \in [a,b]$ such that (48) holds. Obviously, (49) is true and either (50) or (51) is satisfied.

First suppose that (51) holds. The integration of (27) from a to t_m and from t_M to b, in view of (47)–(49) and the assumption that ℓ_0, $\ell_1 \in P_{ab}$, yields

$$u(a) - m = \int_a^{t_m} \ell_1(u)(s)\,ds - \int_a^{t_m} \ell_0(u)(s)\,ds \leq M \int_a^{t_m} \ell_1(1)(s)\,ds, \tag{64}$$

$$M - u(b) = \int_{t_M}^{b} \ell_1(u)(s)\,ds - \int_{t_M}^{b} \ell_0(u)(s)\,ds \leq M \int_{t_M}^{b} \ell_1(1)(s)\,ds. \tag{65}$$

The condition (28), in view of (47) and the assumption that $h_0,\ h_1 \in PF_{ab}$, implies

$$u(a) - \lambda u(b) = h_0(u) - h_1(u) \geq m h_0(1) - M h_1(1). \tag{66}$$

From (64)–(66), we get

$$M(\lambda - h_1(1)) - m(1 - h_0(1)) \leq M\Big(\int_a^{t_m} \ell_1(1)(s)\,ds + \lambda \int_{t_M}^{b} \ell_1(1)(s)\,ds\Big) \leq$$
$$\leq M\|\ell_1(1)\|_L \,\frac{1}{\min\{1, \frac{1}{\lambda}\}}\,,$$

i.e.,

$$M(\beta(h) - \|\ell_1(1)\|_L) \leq m\alpha(h). \tag{67}$$

Now suppose that (50) holds. The integration of (27) from t_M to t_m, in view of (47)–(49) and the assumption that $h_0, h_1 \in PF_{ab}$, results in

$$M - m = \int_{t_M}^{t_m} \ell_1(u)(s)\,ds - \int_{t_M}^{t_m} \ell_0(u)(s)\,ds \leq M \int_a^{b} \ell_1(1)(s)\,ds. \tag{68}$$

Using (9), (14) and (68), one can show that (67) is true.

Therefore, the inequality (67) is satisfied in both cases (50) and (51). On the other hand, the integration of (27) from a to b, in view of (47)–(49) and the assumption that $\ell_0,\ \ell_1 \in P_{ab}$, yields

$$u(a) - u(b) = \int_a^{b} \ell_1(u)(s)\,ds - \int_a^{b} \ell_0(u)(s)\,ds \leq$$
$$\leq M\|\ell_1(1)\|_L - m\|\ell_0(1)\|_L,$$

i.e.,

$$m\|\ell_0(1)\|_L \leq M\|\ell_1(1)\|_L + u(b) - u(a). \tag{69}$$

The condition (28) implies

$$u(b) - u(a) = (1 - \lambda)u(b) - h_0(u) + h_1(u), \tag{70}$$

$$u(b) - u(a) = \Big(\frac{1}{\lambda} - 1\Big)u(a) - \frac{1}{\lambda}\,h_0(u) + \frac{1}{\lambda}\,h_1(u). \tag{71}$$

Suppose first that $\lambda \leq 1$. The inequalities (69) and (70) together with (47) and the assumption that $h_0, h_1 \in PF_{ab}$ result in

$$m\|\ell_0(1)\|_L \leq M\|\ell_1(1)\|_L + M(1-\lambda) - mh_0(1) + Mh_1(1). \tag{72}$$

Hence, from (67) and (72), by virtue of (15) and (49), we get

$$\begin{aligned}&\big(\lambda - h_1(1) - \|\ell_1(1)\|_L\big)\big(\|\ell_0(1)\|_L + h_0(1)\big) \leq \\ &\leq \big(\|\ell_1(1)\|_L + 1 - \lambda + h_1(1)\big)(1 - h_0(1)),\end{aligned}$$

which, in view of (15), contradicts (17).

Let $\lambda > 1$. The inequalities (69) and (71) together with (47) and the assumption that $h_0, h_1 \in PF_{ab}$ imply

$$m\|\ell_0(1)\|_L \leq M\|\ell_1(1)\|_L - m\,\frac{\lambda-1}{\lambda} - m\,\frac{1}{\lambda}\,h_0(1) + M\,\frac{1}{\lambda}\,h_1(1). \tag{73}$$

Hence, from (67) and (73), by virtue of (15) and (49), we get

$$\begin{aligned}&\Big(1 - \frac{1}{\lambda}\,h_1(1) - \|\ell_1(1)\|_L\Big)\Big(\|\ell_0(1)\|_L + \frac{\lambda-1}{\lambda} + \frac{1}{\lambda}\,h_0(1)\Big) \leq \\ &\leq \Big(\|\ell_1(1)\|_L + \frac{1}{\lambda}\,h_1(1)\Big)\,\frac{1-h_0(1)}{\lambda}\,,\end{aligned}$$

which, in view of (15), contradicts (17).

The contradiction obtained above proves that under the assumptions of the theorem, the problem (27), (28) has only the trivial solution. □

Theorems 2.3 and 2.4 follow immediately from Remark 2.1 and Theorems 2.1 and 2.2.

Proof of Corollary 2.1. Let u be a solution of the problem

$$\begin{aligned}&u'(t) = p(t)u(\tau(t)),\\ &\int_a^b u(s)\,d\mu(s) = 0.\end{aligned} \tag{74}$$

It is clear that

$$\begin{aligned}\int_a^b u(s)\,d\sigma(s) &= u(b)\sigma(b) - u(a)\sigma(a) - \int_a^b u'(s)\sigma(s)\,ds = \\ &= u(b)\sigma(b) - u(a)\sigma(a) - \int_a^b p(s)\sigma(s)u(\tau(s))\,ds.\end{aligned}$$

Hence, in view of (74), we get

$$u(a) = \frac{\sigma(b)}{\sigma(a)}\,u(b) - \int_a^b p(s)\sigma(s)u(\tau(s))\,ds.$$

Thus u satisfies (28) with

$$h(v) \stackrel{def}{=} \lambda v(b) + h_0(v) - h_1(v),$$

$$h_0(v) \stackrel{def}{=} \int_a^b [p(s)\sigma(s)]_- v(\tau(s))\, ds,$$

$$h_1(v) \stackrel{def}{=} \int_a^b [p(s)\sigma(s)]_+ v(\tau(s))\, ds,$$

$$\lambda = \frac{\sigma(b)}{\sigma(a)}.$$

Therefore, in view of (20)–(24), the validity of the corollary follows from Theorem 2.1. □

Corollaries 2.2-2.4 can be proved analogously.

Acknowledgement

For the first author, the research was supported by the Ministry of Education of the Czech Republic under the project MSM0021622409. For the second author, the published results were acquired using the subsidization of the Ministry of Education, Youth and Sports of the Czech Republic, research plan MSM 0021630518 "Simulation modelling of mechatronic systems". For the third author, the research was supported by the grant No. 201/04/P183 of the Grant Agency of the Czech Republic and by the Academy of Sciences of the Czech Republic, Institutional Research Plan No. AV0Z10190503.

References

1. N. V. Azbelev, V. P. Maksimov, and L. F. Rakhmatulina, Introduction to the theory of functional-differential equations. (Russian) *Nauka, Moscow*, 1991.
2. E. Bravyi, A note on the Fredholm property of boundary value problems for linear functional differential equations. *Mem. Differential Equations Math. Phys.* **20**(2000), 133–135.
3. R. Hakl, A. Lomtatidze, and I. P. Stavroulakis, On a boundary value problem for scalar linear functional differential equations. *Abstr. Appl. Anal.* **2004**, No. 1, 45–67.
4. R. Hakl, A. Lomtatidze, and J. Šremr, On a periodic type boundary-value problem for first order linear functional differential equations. *Nelinijni Koliv.* **5**(2002), No. 3, 416–433; English transl.: *Nonlinear Oscil. (N. Y.)* **5**(2002), No. 3, 408–425
5. R. Hakl, A. Lomtatidze, and J.Šremr, Some boundary value problems for first order scalar functional differential equations. *Folia Facultatis Scientiarum Naturalium Universitatis Masarykianae Brunensis. Mathematica* 10. *Brno: Masaryk University*, 2002.
6. I. Kiguradze and B. Půža, Boundary value problems for systems of linear functional differential equations. *Folia Facultatis Scientiarium Naturalium Universitatis Masarykianae Brunensis. Mathematica*, 12. *Masaryk University, Brno*, 2003.

7. I. Kiguradze and B. Půža, On boundary value problems for systems of linear functional-differential equations. *Czechoslovak Math. J.* **47(122)**(1997), No. 2, 341–373.
8. Š. Schwabik, M. Tvrdý, and O. Vejvoda, Differential and integral equations. Boundary value problems and adjoints. *D. Reidel Publishing Co., Dordrecht-Boston, Mass.-London*, 1979.
9. J. Šremr and P. Šremr, On a two point boundary problem for first order linear differential equations with a deviating argument. *Mem. Differential Equations Math. Phys.* **29**(2003), 75–124.

(Received 14.02.2007)

Authors' addresses:

A. Lomtatidze
Department of Mathematical Analysis Faculty of Sciences
Masaryk University
Janáčkovo nám. 2a, 662 95 Brno
Czech Republic
E-mail: bacho@math.muni.cz

Z. Opluštil
Institute of Mathematics
Faculty of Mechanical Engineering
Technická 2, 616 69 Brno
Czech Republic
E-mail: oplustil@fme.vutbr.cz

J. Šremr
Institute of Mathematics
Academy of Sciences of the Czech Republic
Žižkova 22, 616 62 Brno
Czech Republic
E-mail: sremrl@ipm.cz