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ON COMPARISON THEOREMS OF STABILITY BETWEEN LINEAR SYSTEMS OF ORDINARY, IMPULSIVE AND DIFFERENCE EQUATIONS AND LINEAR SYSTEMS OF GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

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In the present paper, for each of the linear homogeneous systems: of ordinary differential equations

$$\frac{dx}{dt}=P(t)x, \tag{1}$$

impulsive equations

$$\frac{dx}{dt}=Q(t)x, \tag{2}$$

$$x(t_j+)-x(t_j-)=G_jx(t_j-)\ \ (j=1,2,\dots) \tag{3}$$

and of difference equations

$$\Delta y(k-1)=G(k-1)y(k-1)\ \ (k\in\mathbb{N}) \tag{4}$$

we give some sufficient (among them effective) conditions. If these conditions are fulfilled, then the stability in Liapunov sense of the above-mentioned systems implies that of the linear homogeneous system of the generalized ordinary differential equations

$$dx(t)=dA(t)\cdot x(t) \tag{5}$$

in the same sense.

Here we assume that $P,\,Q:\mathbb{R}_+\to\mathbb{R}^{n\times n}$ ($\mathbb{R}_+=[0,+\infty[$) are matrix-functions with Lebesgue integrable components on every closed interval form $\mathbb{R}_+$; $G_j\in\mathbb{R}^{n\times n}$ ($j=1,2,\dots$) are constant matrices, $t_j\in\mathbb{R}_+$ ($j=1,2,\dots$), $0<t_1<t_2<\cdots$, $\lim\limits_{j\to\infty}t_j=+\infty$, $G:\{0,1,\dots\}\to\mathbb{R}^{n\times n}$ is a matrix-function, and $A:\mathbb{R}_+\to\mathbb{R}^{n\times n}$ is a matrix-function with bounded total variation components on every closed interval from $\mathbb{R}_+$.

To a considerable extent, the interest to the theory of generalized ordinary differential equations has also been stimulated by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from a unified point of view (see [1]–[3]).

The following notation and definitions will be used in the paper:

$\mathbb{R}=]-\infty,+\infty[$, $[a,b]$ and $]a,b[$ ($a,\,b\in\mathbb{R}$) are, respectively, closed and open intervals; $\mathbb{N}$ is the set of all natural numbers, $\widetilde{\mathbb{N}}=\{0,1,\dots\}$.

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$-matrices $X=(x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\|=\max_{j=1,\dots,m}\sum_{i=1}^{n}|x_{ij}|;\quad |X|=(|x_{ij}|)_{i,j=1}^{n,m}.$$

$O_{n\times m}$ (or O) is the zero $n\times m$-matrix.

$\mathbb{R}^n=\mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x=(x_i)_{i=1}^n$.

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If $X \in \mathbb{R}^{n\times n}$, then X^{-1} and $\det(X)$ are, respectively, the matrix inverse to X and the determinant of X. I_n is the identity $n \times n$-matrix, $\operatorname{diag}(\lambda_1,\dots,\lambda_n)$ is the diagonal matrix with diagonal elements $\lambda_1,\dots,\lambda_n$; δ_{ij} is the Kroneker symbol, i.e., $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$ $(i,j = 1,\dots,n)$.

$r(H)$ is the spectral radius of the matrix $H \in \mathbb{R}^{n\times n}$.

$\bigvee\limits_a^b(X)$ is the sum of total variations on $[a,b]$ of the components x_{ij} $(i = 1,\dots,n;$ $j = 1,\dots,m)$ of the matrix-function $X : [a,b] \to \mathbb{R}^{n\times m}$; $V(X)(t) = (v(x_{ij})(t))_{i,j=1}^{n,m}$, where $v(x_{ij})(0) = 0$ and $v(x_{ij})(t) = \bigvee\limits_0^t(x_{ij})$ for $0 < t < +\infty$ $(i = 1,\dots,n;\ j = 1,\dots,m)$.

$X(t-)$ and $X(t+)$ are the left and the right limits of the matrix-function $X : \mathbb{R}_+ \to \mathbb{R}^{n\times m}$ at the point t. $(X(0-) = X(0))$; $d_1X(t) = X(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its components is such.

$\mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X : \mathbb{R}_+ \to \mathbb{R}^{n\times m}$ of the bounded total variation on every closed segment from $\mathbb{R}_+$.

$L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X : \mathbb{R}_+ \to \mathbb{R}^{n\times m}$ such that their components are measurable and Lebesgue integrable functions on every closed interval from $\mathbb{R}_+$.

$\widetilde{C}([a,b];\mathbb{R}^{n\times m})$ is the set of all absolutely continuous matrix-functions $X : [a,b] \to \mathbb{R}^{n\times m}$.

$\widetilde{C}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X : \mathbb{R}_+ \to \mathbb{R}^{n\times m}$ whose restrictions to any arbitrary closed interval $[a,b]$ from $\mathbb{R}_+$ belong to $\widetilde{C}([a,b];\mathbb{R}^{n\times m})$.

$\widetilde{C}_{loc}(\mathbb{R}_+ \setminus \{t_j\}_{j=1}^{+\infty};\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X : \mathbb{R}_+ \to \mathbb{R}^{n\times m}$ whose restrictions to any arbitrary closed interval $[a,b]$ from $\mathbb{R}_+ \setminus \{t_j\}_{j=1}^{+\infty}$ belong to $\widetilde{C}_{loc}([a,b];\mathbb{R}^{n\times m})$.

$s_0 : \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}) \to \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R})$ is the operator defined by

$$s_0(x)(t) \equiv x(t) - \sum_{0<\tau\le t} d_1x(\tau) - \sum_{0\le\tau<t} d_2x(\tau).$$

If $g : \mathbb{R}_+ \to \mathbb{R}$ is a nondecreasing function, $x : \mathbb{R}_+ \to \mathbb{R}$ and $0 \le s < t < +\infty$, then

$$\int_s^t x(\tau)\,dg(\tau) =$$

$$= \int_{]s,t[} x(\tau)\,dg_1(\tau) - \int_{]s,t[} x(\tau)\,dg_2(\tau) + \sum_{0<\tau\le t} x(\tau)d_1g(\tau) - \sum_{0\le\tau<t} x(\tau)d_2g(\tau),$$

where $g_j : \mathbb{R}_+ \to \mathbb{R}$ $(j = 1,2)$ are continuous nondecreasing functions such that $g_1(t) - g_2(t) \equiv s_0(g)(t)$, and $\int_{]s,t[} x(\tau)\,dg_j(\tau)$ is the Lebesgue-Stieltjes integral over the open interval $]s,t[$ with respect to the measure corresponding to the functions g_j $(j = 1,2)$ (if $s = t$, then $\int_s^t x(\tau)\,dg(\tau) = 0$).

If $G = (g_{ik})_{i,k=1}^{l,n} : \mathbb{R}_+ \to \mathbb{R}^{l\times n}$ is a nondecreasing matrix-function, $X = (x_{ik})_{i,k=1}^{n,m} : \mathbb{R}_+ \to \mathbb{R}^{n\times m}$, then

$$\int_s^t dG(\tau)\cdot X(\tau) = \bigg(\sum_{k=1}^n \int_s^t x_{kj}(\tau)\,dg_{ik}(\tau)\bigg)_{i,j=1}^{l,m} \quad \text{for } 0 \le s \le t < +\infty,$$

$$S_0(G)(t) \equiv \big(s_0(g_{ik})(t)\big)_{i,k=1}^{l,n}.$$

If $G_j : \mathbb{R}_+ \to \mathbb{R}^{l\times n}$ $(j = 1, 2)$ are nondecreasing matrix-functions, $G(t) \equiv G_1(t) - G_2(t)$ and $X : \mathbb{R}_+ \to \mathbb{R}^{n\times m}$, then

$$\int_s^t dG(\tau)\cdot X(\tau) = \int_s^t dG_1(\tau)\cdot X(\tau) - \int_s^t dG_2(\tau)\cdot X(\tau) \text{ for } 0 \le s \le t < +\infty.$$

$E(\widetilde{\mathbb{N}}; \mathbb{R}^{n\times l})$ is the set of all matrix-functions $Y : \widetilde{\mathbb{N}} \to \mathbb{R}^{n\times l}$.

Δ is of the first order difference operator, i.e.,

$$\Delta Y(k-1) = Y(k) - Y(k-1) \text{ for } Y \in E(\widetilde{\mathbb{N}}; \mathbb{R}^{n\times l}),\ \ k \in \mathbb{N}.$$

By a solution of the system (5) we understand a vector-function $x \in \mathrm{BV}_{loc}(\mathbb{R}_+; \mathbb{R}^n)$ such that

$$x(t) = x(s) + \int_s^t dA(\tau)\cdot x(\tau) \text{ for } 0 \le s \le t < +\infty.$$

Under a solution of the impulsive system (2)–(3) we understand a left-continuous vector-function $x \in \widetilde{C}_{loc}(\mathbb{R}_+ \setminus \{t_j\}_{j=1}^{+\infty}; \mathbb{R}^n)$ satisfying the system (2) almost everywhere on $]t_j, t_{j+1}[$ and the relation (3) at the point t_j for every $j = \{1, 2, \dots\}$.

Note that each of the systems (1), (2)–(3) and (4) can be rewritten in the form of the system (5) if we assume, respectively, that

$$A(t) \equiv \int_0^1 P(\tau)\,d\tau, \quad A(t) \equiv \int_0^t Q(\tau)\,d\tau + \sum_{0\le t_j<t} G_j$$

and

$$A(0) = O_{n\times n}, \quad A(t) = \sum_{m=1}^{k} G(m-1) \text{ for } k-1 < t \le k \ \ (k = 1, 2, \dots).$$

Throughout the paper it will be assumed that

$$\det(I_n + G_j) \ne 0 \ \ (j = 1, 2, \dots), \quad \det(I_n + G(k)) \ne 0 \ \ (k = 0, 1, \dots)$$

and

$$\det(I_n + (-1)^j d_j A(t)) \ne 0 \text{ for } t \in \mathbb{R}_+ \ \ (j = 1, 2). \tag{6}$$

It should be noted that the inequalities (6) guarantee the unique solvability of the Cauchy problem for the systems (2)–(3), (4) and (5), respectively (see [1],[4]).

Definition 1. Let $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing function such that

$$\lim_{t\to+\infty} \xi(t) = +\infty.$$

Then a solution x_0 of the system (5) is called ξ-exponentially asymptotically stable if there exists a positive number η such that for every $\varepsilon > 0$ there exists a positive number $\delta = \delta(\varepsilon)$ such that an arbitrary solution x of the system (5) satisfying the inequality

$$\|x(t_0) - x_0(t_0)\| < \delta$$

for some $t_0 \in \mathbb{R}_+$ admits the estimate

$$\|x(t) - x_0(t)\| < \varepsilon \exp\big(-\eta(\xi(t) - \xi(t_0))\big) \text{ for } t \ge t_0.$$

Note that the exponential asymptotical stability is a particular case of ξ-exponentially asymptotical stability if we assume $\xi(t) \equiv t$.

Stability, uniformly stability and asymptotical stability are defined just in the same way as for the systems of ordinary differential equations (see, e.g., [5], [6]).

Definition 2. The system (5) is called stable in one or another sense if every solution of this system is stable in the same sense.

Definition 3. The matrix-function A is called stable in one or another sense if the system (5) is stable in the same sense.

Definition 4. The pair $(Q, \{G_j\}_{j=1}^{+\infty})$ (the matrix-function G) is called stable in one or another sense if the matrix-function A corresponding to the system (2)–(3) (to the system (4)) is stable in the same sense.

Theorem 1. *Let the matrix-functions* $P \in L_{loc}(\mathbb{R}_+; \mathbb{R}^{n\times n})$ *and* $A \in BV_{loc}(\mathbb{R}_+; \mathbb{R}^{n\times n})$ *satisfy the conditions*

$$\bigg\| \int\limits_0^{+\infty} |Z^{-1}(t)| \, dV(S_0(A) - B)(t) \bigg\| < +\infty \tag{7}$$

and

$$\Big\| \sum_{0\le t<+\infty} |Z^{-1}(t)| \cdot |d_j A(t)| \, \Big\| < +\infty \ \ (j = 1, 2),$$

where $B(t) \equiv \int\limits_0^t P(\tau)\, d\tau$, Z $(Z(0) = I_n)$ *is the fundamental matrix of the system* (1). *Then the stability of the matrix-function* P *in one or another sense guarantees that of the matrix-function* A *in the same sense.*

Corollary 1. *Let the matrix-function* $P \in L_{loc}(\mathbb{R}_+; \mathbb{R}^{n\times n})$ *satisfy the Lappo–Danilevskiĭ condition*

$$P(t) \int\limits_0^t P(\tau)\, d\tau = \int\limits_0^t P(\tau)\, d\tau \cdot P(t) \ \ \text{for} \ \ t \in \mathbb{R}_+.$$

Let, moreover, the matrix-function $A \in BV_{loc}(\mathbb{R}_+; \mathbb{R}^{n\times n})$ *be such that*

$$\bigg| \int\limits_0^{+\infty} \Big| \exp\Big(- \int\limits_0^t P(\tau)\, d\tau\Big) \Big| \, dV(S_0(A) - B)(t) \bigg| < +\infty$$

and

$$\bigg\| \sum_{0\le t<+\infty} \Big| \exp\Big(- \int\limits_0^t P(\tau)\, d\tau\Big) \Big| \cdot |d_j A(t)| \, \bigg\| < +\infty,$$

where $B(t) \equiv \int\limits_0^t P(\tau)\, d\tau$. *Then the stability of the matrix-function* P *in one or another sense guarantees that of the matrix-function* A *in the same sense.*

Corollary 2. *Let the components of the matrix-functions* $P = \mathrm{diag}(p_1, \dots, p_n) \in L_{loc}(\mathbb{R}_+; \mathbb{R}^{n\times n})$ *and* $A = (a_{ik})_{i,k=1}^n \in \mathrm{BV}_{loc}(\mathbb{R}_+; \mathbb{R}^{n\times n})$ *satisfy the conditions*

$$\bigg| \int\limits_0^{+\infty} \exp\Big(- \int\limits_0^t p_i(\tau)\, d\tau\Big) \, dv(b_{ik}(\tau) \bigg| < +\infty$$

and

$$\sum_{0\le t<+\infty} \exp\Big(- \int\limits_0^t p_i(\tau)\, d\tau\Big) \cdot |d_j a_{ik}(t)| < +\infty \ \ (j = 1, 2; \ \ i, k = 1, \dots, n),$$

where $b_{ik}(t) \equiv \delta_{ik} \int\limits_0^t p_i(\tau)\, d\tau - s_0(a_{ik})(t)$ $(i, k = 1, \dots, n)$. *Then the conclusion of Corollary* 1 *is true.*

Theorem 2. *Let the matrix-functions $Q \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ and $A \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ as well as the constant matrices $G_j \in \mathbb{R}^{n\times n}$ $(j=1,2,\dots)$ satisfy the conditions* (7) *and*

$$\Big\| \sum_{j=1}^{+\infty} \big|Z^{-1}(t_j)(I_n+G_j)^{-1}\big| \cdot \big|d_2A(t_j)-G_j\big| \Big\| < +\infty, \tag{8}$$

where $B(t) \equiv \int\limits_0^t Q(\tau)\,d\tau$, Z $(Z(0)=I_n)$ is the fundamental matrix of the impulsive system (2)–(3). *Then the stability of the pair $(Q,\{G_j\}_{j=1}^{+\infty})$ in one or another sense guarantees that of the matrix-function A in the same sense.*

Corollary 3. *Let the matrix-function $Q=(q_{ik})_{i,k=1}^n \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ and the constant matrices $G_j=(g_{jik})_{i,k=1}^n \in \mathbb{R}^{n\times n}$ $(j=1,2,\dots)$ satisfy the conditions*

$$1+g_{jii}\neq 0 \ \ (i=1,\dots,n;\ \ j=1,2,\dots),$$

$$\int\limits_{t^*}^{t} \exp\Big(\int\limits_{\tau}^{t} q_{ii}(s)\,ds\Big)|q_{ik}(\tau)| \prod_{\tau\le t_j<t} |1+g_{jii}|\,d\tau +$$
$$+\sum_{t^*\le t_l<t} \exp\Big(\int\limits_{t_l}^{t} q_{ii}(s)\,ds\Big)|g_{lik}(\tau)| \prod_{t_l<t_j<t} |1+g_{jii}| \le$$
$$\le h_{ik} \ \ (i\neq k;\ \ i,k=1,\dots,n;\ \ j=1,2,\dots)$$

and

$$\sup\Big\{\int\limits_0^t q_{ii}(s)\,ds+\sum_{0\le t_j<t}\ln|1+g_{jii}|:\ t\ge t^*\Big\}<+\infty \ \ (i=1,\dots,n),$$

where $t^*\in\mathbb{R}_+$, $h_{ik}\in\mathbb{R}_+$ $(i\neq k;\ i,k=1,\dots,n)$, and the constant matrix $H=(h_{ik})_{i,k=1}^n$, $h_{ii}=0$ $(i=1,\dots,n)$ is such that

$$r(H)<1. \tag{9}$$

Let, moreover, the matrix-function $A\in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be such that the conditions (7) *and* (8) *hold, where $B(t)\equiv\int\limits_0^t Q(\tau)\,d\tau$. Then the matrix-function A is stable.*

Theorem 3. *Let the matrix-functions $G\in E(\widetilde{\mathbb{N}};\mathbb{R}^{n\times n})$ and $A\in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ satisfy the conditions*

$$\Big\|\sum_{k=1}^{+\infty}\Big|\prod_{m=0}^{k-1}(I_n+G(m))^{-1}\Big|\cdot\big|d_2A(k-1)-G(k-1)\big|\Big\|<+\infty \tag{10}$$

and

$$\Big\|\sum_{k=1}^{+\infty}\Big|\prod_{m=0}^{k-1}(I_n+G(m))^{-1}\Big|\cdot\Big(V(A)(k)-V(A)(k-1)-|d_2A(k-1)|\Big)\Big\|<+\infty \tag{11}$$

Then the stability of the matrix-function G in one or another sense guarantees that of the matrix-function A in the same sense.

Corollary 4. *Let the components of the matrix-function $G=(g_{ij})_{i,j=1}^n\in E(\widetilde{\mathbb{N}};\mathbb{R}^{n\times n})$ satisfy the conditions*

$$1+g_{ii}(k)\neq 0 \ \ \text{for}\ \ k\ge k^* \ \ (i=1,\dots,n),$$

$$\sum_{l=k^*}^{k}|g_{ij}(l-1)|\prod_{m=l+2}^{k}\big|1+g_{ii}(m-1)\big|<h_{ij} \ \ \text{for}\ \ k\ge k^* \ \ (i\neq j;\ \ i,j=1,\dots,n)$$

and

$$\sup\Big\{\sum_{m=l}^{k-1}\ln|1+g_{ii}(m)|:\ k>l\Big\}<+\infty\ \ (i=1,\dots,n),$$

where $k^*\in\widetilde{\mathbb{N}}$ *and* $h_{ij}\in\widetilde{\mathbb{N}}$ $(i\neq j;\ i,j=1,\dots,n)$*, and* $H=(h_{ij})_{i,j=1}^n$*,* $h_{ii}=0$ $(i=1,\dots,n)$ *is a constant matrix satisfying the condition* (9)*. Let, moreover, the matrix-function* $A\in\mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ *be such that the conditions* (10) *and* (11) *hold. Then the matrix-function* A *is uniformly stable.*

References

1. Š. Schwabik, M. Tvrdy, and O. Vejvoda, Differential and integral equations. Boundary value problems and adjoints. *D. Reidel Publishing Co., Dordrecht-Boston, Mass.-London*, 1979.
2. M. Ashordia and N. Kekelia, On the ξ-exponentially asymptotic stability of linear systems of generalized ordinary differential equations. *Georgian Math. J.* **8**(2001), No. 4, 645–664.
3. M. Ashordia, On the general and multipoint boundary value problems for linear systems of generalized ordinary differential equations, linear impulse and linear difference systems. *Mem. Differential Equations Math. Phys.* **36**(2005), 1–80.
4. A. M. Samoilenko and N. A. Perestyuk, Differential equations with impulse action. (Russian) *Vysshaja Shkola, Kiev*, 1987.
5. B. P. Demidovich, Lectures on mathematical theory of stability. (Russian) *Nauka, Moscow*, 1967.
6. I. T. Kiguradze, The initial value problem and boundary value problems for systems of ordinary differential equations. Vol. I. Linear theory. (Russian) *Metsniereba, Tbilisi*, 1997.

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