Mem. Differential Equations Math. Phys. 36 (2005), 153–156

T. Kiguradze

ON LIDSTONE BOUNDARY VALUE PROBLEM FOR HIGHER ORDER NONLINEAR HYPERBOLIC EQUATIONS WITH TWO INDEPENDENT VARIABLES

(Reported on July 20, 2005)

Let m and n be positive integers, $a > 0$, $b > 0$ and $D = [0, a] \times [0, b]$. In the rectangle D consider the nonlinear hyperbolic equation

$$u^{(2m,2n)} = f\big(x, y, u, \dots, u^{(2m-1,0)}, \dots, u^{(0,2n-1)}, \dots, u^{(2m-1,2n-1)}\big) \tag{1}$$

with the boundary conditions

$$\begin{aligned} u^{(2i,0)}(0, y) = \varphi_{1i}(y), \quad u^{(2i,0)}(a, y) = \varphi_{2i}(y) \quad (i = 0, \dots, m-1),\\ u^{(2m,2k)}(x, 0) = \psi_{1k}(x), \quad u^{(2m,2k)}(x, b) = \psi_{2k}(x) \quad (i = 0, \dots, n-1), \end{aligned} \tag{2}$$

where

$$u^{(i,k)}(x, y) = \frac{\partial^{i+k} u(x, y)}{\partial x^i \partial y^k} \quad (i = 0, \dots, 2m;\ k = 0, \dots, 2n).$$

Moreover, below it will be assumed that the function $f : D \times \mathbb{R}^{4mn} \to \mathbb{R}$ is continuous, the functions $\varphi_{2i} : [0, b] \to \mathbb{R}$, $\varphi_{2i} : [0, b] \to \mathbb{R}$ $(i = 0, \dots, m-1)$ are $2n$–times continuously differentiable, and the functions $\psi_{2k} : [0, a] \to \mathbb{R}$, $\psi_{2k} : [0, a] \to \mathbb{R}$ $(i = 0, \dots, n-1)$ are continuous.

By $C^{2m,2n}(D)$ denote the space of continuous functions $u : D \to \mathbb{R}$ having the continuous partial derivatives $u^{(j,k)}$ $(j = 0, \dots, 2m;\ k = 0, \dots, 2n)$. By a solution of problem (1),(2) we will understand a *classical solution*, i.e., a function $u \in C^{2m,2n}(D)$ satisfying equation (1) and boundary conditions (2) everywhere in D.

By analogy with the problem

$$z^{(2m)} = g(x, z, \dots, z^{(2m-1)}), \tag{3}$$

$$z^{(2i)}(0) = c_{1i}, \quad z^{(2i)}(a) = c_{2i} \ \ (i = 1, \dots, n), \tag{4}$$

problem (1),(2) will be called the Lidstone problem.

Problem (3),(4) and its various generalizations were investigated by many authors (see, e.g., [1–8], [12]). As for the problem (1),(2), it was studied in the case, where $m = n = 1$ and (1) is a linear equation (see [9–11]).

The given below sufficient conditions of solvability and unique solvability of problem (1),(2) concern the case, where on the set $D \times \mathbb{R}^n$ the function f on satisfies either of the conditions

$$\begin{aligned} &\big|f(x, y, z_{00}, \dots, z_{2m-10}, \dots, z_{02n-1}, \dots, z_{2m-12n-1})\big|\\ &\qquad\qquad \le \sum_{i=0}^{2m-1} \sum_{k=0}^{2n-1} p_{ik}(x, y)|z_{ik}| + q\Big(x, y, \sum_{i=0}^{2m-1} \sum_{k=0}^{2n-1} |z_{ik}|\Big) \end{aligned} \tag{5}$$

2000 *Mathematics Subject Classification.* 35L35, 35B30.

Key words and phrases. Lidstone problem, higher order, nonlinear hyperbolic equation.

and

$$|f(x,y,z_{00},\dots,z_{2m-12n-1}) - f(x,y,\overline{z}_{00},\dots,\overline{z}_{2m-12n-1})| \\ \le \sum_{i=0}^{2m-1}\sum_{k=0}^{2n-1} p_{ik}(x,y)|z_{ik}-\overline{z}_{ik}|, \tag{6}$$

where $p_{ik} : D \to [0,+\infty)$ $(i=0,\dots,2m-1;\ k=0,\dots,2n-1)$ are continuous functions, and $q : D\times[0,+\infty) \to [0,+\infty)$ is a continuous function that is nondecreasing in the second argument and

$$\lim_{\rho\to+\infty} \frac{1}{\rho}\int_0^a\int_0^b q(x,y,\rho)\,dx\,dy = 0. \tag{7}$$

Along with (1),(2) we will consider the differential inequality

$$\left|u^{(2m,2n)}(x,y)\right| \le \sum_{i=0}^{2m-1}\sum_{k=0}^{2n-1} p_{ik}(x,y)\left|u^{(i,k)}(x,y)\right| \tag{8}$$

with the homogeneous boundary conditions

$$\begin{gathered} u^{(2i,0)}(0,y)=0, \quad u^{(2i,0)}(a,y)=0 \quad (i=0,\dots,m-1), \\ u^{(2m,2k)}(x,0)=0, \quad u^{(2m,2k)}(x,b)=0 \quad (i=0,\dots,n-1). \end{gathered} \tag{9}$$

By a solution of problem (8),(9) we will understand a function $u\in C^{2m,2n}(D)$ satisfying inequality (8) and boundary conditions (9) everywhere in D.

Theorem 1. *Let conditions* (5) *and* (7) (*condition* (6)) *hold, and let problem* (8), (9) *have only a trivial solution. Then problem* (1), (2) *has at least one* (*one and only one*) *solution.*

For arbitrary $s_0>0$, $s\in[0,s_0]$ and a positive integer j set

$$\begin{gathered} \lambda_1(s;s_0)=\frac{1}{s_0}, \quad \lambda_{2j+1}(s;s_0)=\frac{s(s_0-s)}{2s_0}\Big(\frac{s_0^2}{8}\Big)^{j-1}, \\ \lambda_2(s;s_0)=\frac{s(s_0-s)}{s_0}, \quad \lambda_{2j+2}(s;s_0)=\frac{s^2(s_0-s)^2}{2s_0}\Big(\frac{s_0^2}{8}\Big)^{j-1}. \end{gathered} \tag{10}$$

Theorem 2. *Let conditions* (5) *and* (7) (*condition* (6)) *hold, and*

$$\sum_{k=0}^{2n-1}\sum_{i=0}^{2m-1}\int_0^a\int_0^b p_{ik}(x,y)\lambda_{2m-i}(x;a)\lambda_{2n-k}(y;b)\,dx\,dy \le 1. \tag{11}$$

Then problem (1), (2) *has at least one* (*one and only one*) *solution.*

Let

$$\mu_{2j-1}(s_0)=\Big(\frac{s_0^2}{8}\Big)^{j-1}, \quad \mu_{2j}(s_0)=2\Big(\frac{s_0^2}{8}\Big)^{j} \quad (j=1,2,\dots).$$

Then by (10) we have

$$\lambda_k(s;s_0) \le \frac{1}{s_0}\mu_k(s_0) \quad \text{for} \quad 0\le s\le s \quad (k=1,2,\dots).$$

Therefore Theorem 2 implies the

Corollary 1. *Let conditions* (5) *and* (7) (*condition* (6) *hold, and let*

$$\sum_{k=0}^{2n-1}\sum_{i=0}^{2m-1}\mu_{2m-i}(a)\mu_{2n-k}(b)\int_0^a\int_0^b p_{ik}(x,y)\,dx\,dy \le ab. \tag{12}$$

Then problem (1), (2) *has at least one* (*one and only one*) *solution.*

Let us show that in Theorem 2 and Corollary 1, respectively, conditions (11) and (12) are unimprovable from the viewpoint that they cannot be replaced by the conditions

$$\sum_{k=0}^{2n-1}\sum_{i=0}^{2m-1}\int_0^a\int_0^b p_{ik}(x,y)\lambda_{2m-i}(x;a)\lambda_{2n-k}(y;b)\,dx\,dy \le 1+\varepsilon \tag{11_ε}$$

and

$$\sum_{k=0}^{2n-1}\sum_{i=0}^{2m-1}\mu_{2m-i}(a)\mu_{2n-k}(b)\int_0^a\int_0^b p_{ik}(x,y)\,dx\,dy \le (1+\varepsilon)ab, \tag{12_ε}$$

no matter how small $\varepsilon > 0$ is. Indeed, as it was shown in [6] (see Example 1.1), for an arbitrary $\varepsilon > 0$ there exist continuous functions $g_1 : [0,a] \to [0,+\infty)$ and $g_2 : [0,b] \to [0,+\infty)$ such that

$$4 < a\int_0^a g_1(x)\,dx < 4\sqrt{1+\varepsilon}, \quad 4 < b\int_0^b g_2(y)\,dy < 4\sqrt{1+\varepsilon},$$

and the boundary value problems

$$w'' = -g_1(x)w, \quad w(0) = w(a) = 0$$

and

$$w'' = -g_2(y)w, \quad w(0) = w(b) = 0$$

have nontrivial solutions w_1 and w_2. If $m > 1$ ($n > 1$), then by v_1 (by v_2) denote the solution of the problem

$$v^{(2m-2)} = w_1(x), \quad v^{(2i)}(0) = v^{(2i)}(a) = 0 \quad (i = 0,\dots,m-1)$$
$$\Big(v^{(2n-2)} = w_2(y), \quad v^{(2k)}(0) = v^{(2k)}(b) = 0 \quad (k = 0,\dots,n-1)\Big).$$

For $m = 1$ ($n = 1$) set $v_1(x) = w_1(x)$ ($v_2(y) = w_2(y)$). Then the function

$$u(x,y) = v_1(x)v_2(y)$$

is a nontrivial solution of the homogeneous equation

$$u^{(2m,2n)} = g(x,y)u^{(2m-2,2n-2)}$$

subject to the boundary conditions (9), where

$$g(x,y) = g_1(x)g_2(y)$$

and

$$16 < ab\int_0^a\int_0^b g(x,y)\,dx\,dy < 16(1+\varepsilon). \tag{13}$$

On the other hand, the function

$$f(x,y,z_{00},\dots,z_{2m-1 2n-1}) \equiv g(x,y)z_{2m-2 2n-2}$$

satisfies condition (6), where

$$p_{ik}(x,y) \equiv 0 \quad \text{for} \quad i \ne 2m-2 \ \text{ or } \ k \ne 2n-2,$$
$$p_{ik}(x,y) \equiv g(x,y) \quad \text{for} \quad i = 2m-2,\ k = 2n-2.$$

Moreover, as it follows from inequality (13), conditions (11) and (12) are violated, while conditions (11_ε) and (12_ε) hold.

Theorem 3. *Let conditions* (5) *and* (7) (*condition* (6) *hold, where*

$$p_{ik}(x,y) \equiv p_{ik} \quad (i = 0,\dots,2m-1;\ k = 0,\dots,2n-1)$$

are nonnegative constants satisfying the inequality

$$\sum_{k=0}^{2n-1}\sum_{i=0}^{2m-1}\Big(\frac{a}{\pi}\Big)^{2m-i}\Big(\frac{b}{\pi}\Big)^{2n-k}p_{ik} < 1. \tag{14}$$

Then problem (1), (2) *has at least one* (*one and only one*) *solution.*

Let $i \in \{0, \dots, m-1\}$, $k \in \{0, \dots, n-1\}$. Then the differential equation

$$u^{(2m,2n)} = (-1)^{m+n+i+k} \left(\frac{\pi}{a}\right)^{2m-2i} \left(\frac{\pi}{b}\right)^{2n-2k} u^{(2i,2k)}$$

has a nontrivial solution

$$u(x,y) = \sin\left(\frac{\pi}{a}\,x\right) \sin\left(\frac{\pi}{b}\,y\right).$$

Consequently, in Theorem 3 the strict inequality (14) cannot be replaced by the unstrict inequality

$$\sum_{k=0}^{2n-1} \sum_{i=0}^{2m-1} \left(\frac{a}{\pi}\right)^{2m-i} \left(\frac{b}{\pi}\right)^{2n-k} p_{ik} \leq 1.$$

References

1. R. P. Agarwal, D. O'Regan, I. Rachůnkova, and S. Staněk, Two-point higher-order BVPs with singularities in phase variables. *Comput. Math. Appl.* **46**(2003), No. 12, 1799–1826.

2. R. P. Agarwal, D. O'Regan, and S. Staněk, Singular Lidstone boundary value problem with given maximal values for solutions. *Nonlinear Anal.* **55**(2003), No. 7–8, 859–881.

3. R. P. Agarwal and P. J. Y. Wong, Lidstone polynomials and boundary value problems. *Comput. Math. Appl.* **17**(1989), No. 10, 1397–1421.

4. P. Hartman, Ordinary differential equations. *John Wiley & Sons, Inc., New York–London–Sydney*, 1964.

5. I. Kiguradze, Concerning the solvability of two-point singular boundary value problems. (Russian) *Uspekhi Mat. Nauk* **53**(1998), No. 4, 153.

6. I. Kiguradze, Some optimal conditions for the solvability of two-point singular boundary value problems. *Funct. Differ. Equ.* **10**(2003), No. 1–2, 259–281.

7. I. Kiguradze and G. Tskhovrebadze, On the two-point boundary value problems for systems of higher order ordinary differential equations with singularities. *Georgian Math. J.* **1**(1994), No. 1, 31–45.

8. I. Kiguradze, B. Půža and I. P. Stavroulakis, On singular boundary value problems for functional differential equations of higher order. *Georgian Math. J.* **8**(2001), No. 4, 791–814.

9. T. Kiguradze, On the correctness of the Dirichlet problem in a characteristic rectangle for fourth order linear hyperbolic equations. *Georgian Math. J.* **6**(1999), No. 5, 447–470.

10. T. Kiguradze, On the Dirichlet problem in a characteristic rectangle for fourth order linear singular hyperbolic equations. *Georgian Math. J.* **6**(1999), No. 6, 537–552.

11. T. Kiguradze and V. Lakshmikantham, On the Dirichlet problem in a characteristic rectangle for higher order linear hyperbolic equations. *Nonlinear Anal.* **50**(2002), No. 8, *Ser. A: Theory Methods*, 1153–1178.

12. P. J. Y. Wong and R. P. Agarwal, Eigenvalues of Lidstone boundary value problems. *Appl. Math. Comput.* **104**(1999), No. 1, 15–31.

Author's address:

T. Kiguradze
Florida Institute of Technology
Department of Mathematical Sciences
150 W. University Blvd.
Melbourne, Fl 32901
USA
E-mail: tkigurad@fit.edu